Fluctuations of Conserved Charges as Probes of QCD Phase Structure

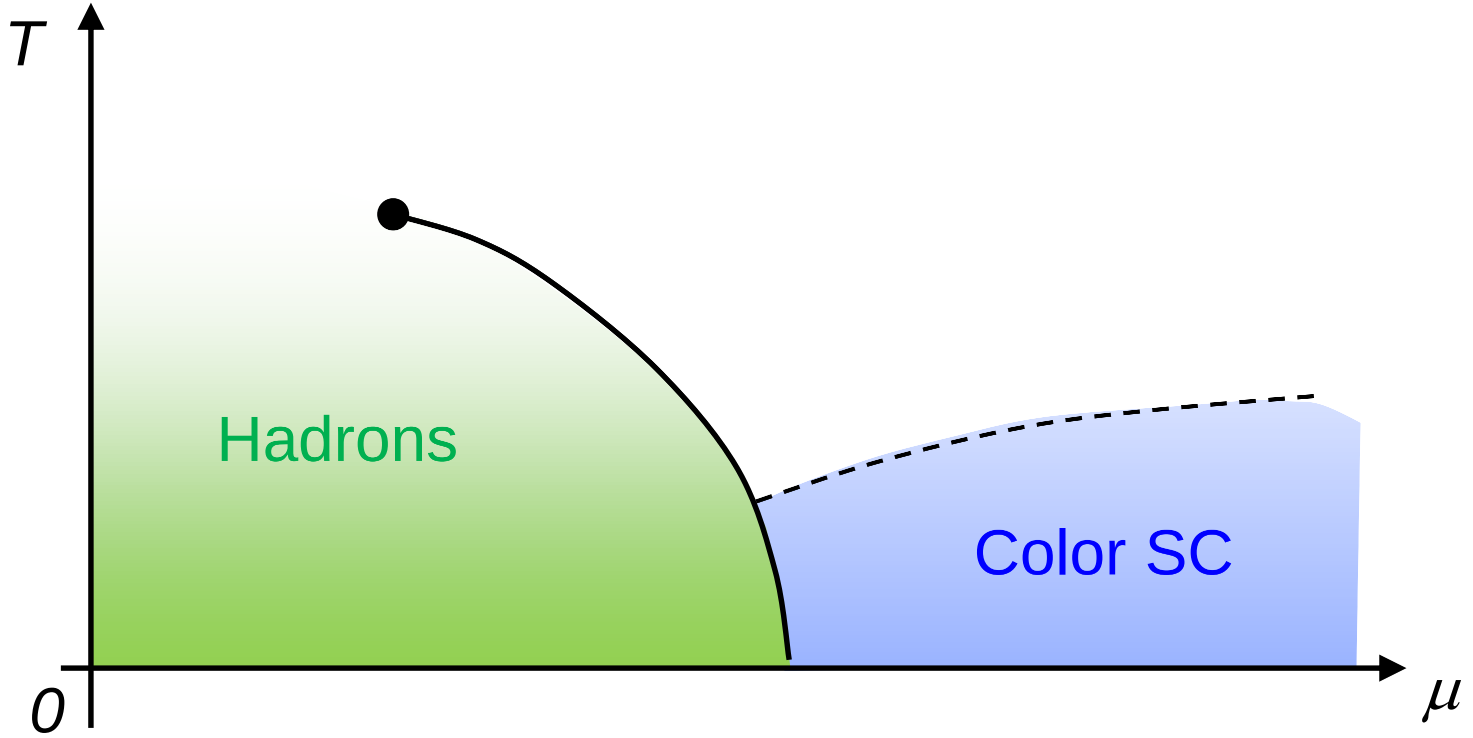
Masakiyo Kitazawa

(Osaka U.)

MK, Asakawa, PRC85,021901C(2012); PRC86, 024904(2012)
MK, Asakawa, Ono, PLB728,386(2014)

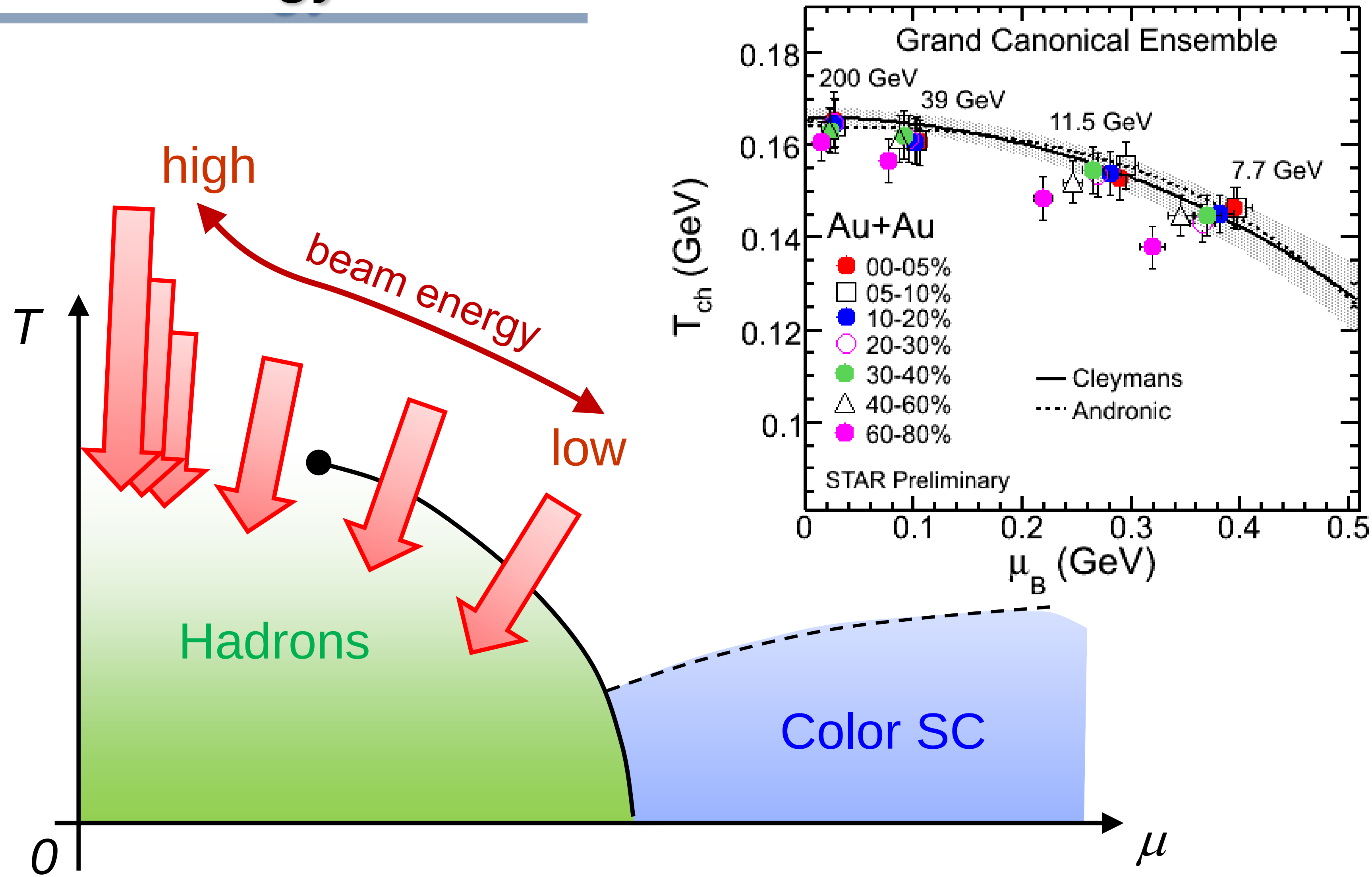
JHI2014, J-PARC, 17/Mar./2014

Beam-Energy Scan



Beam-Energy Scan

STAR 2012



Beam-Energy Scan

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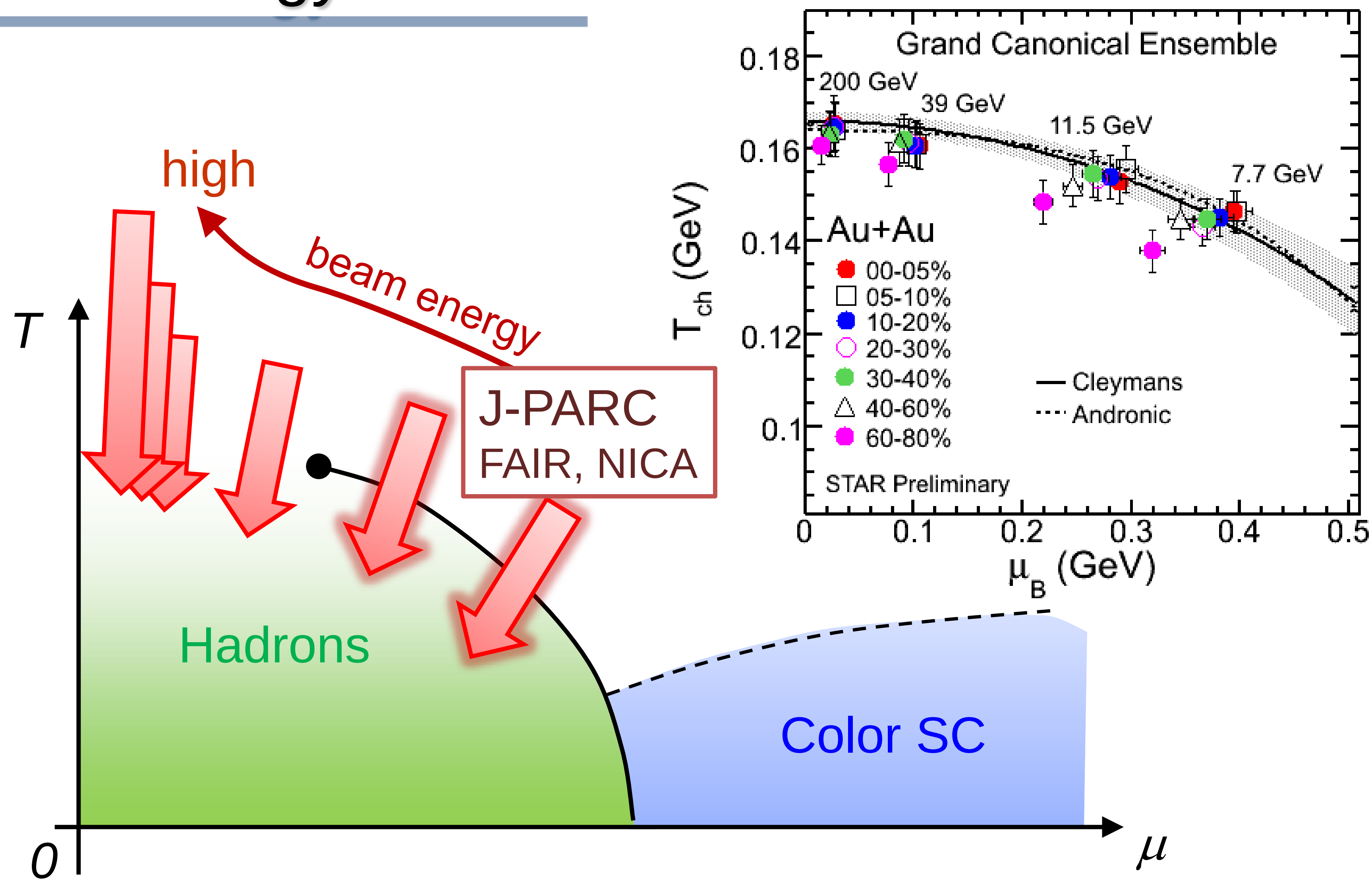


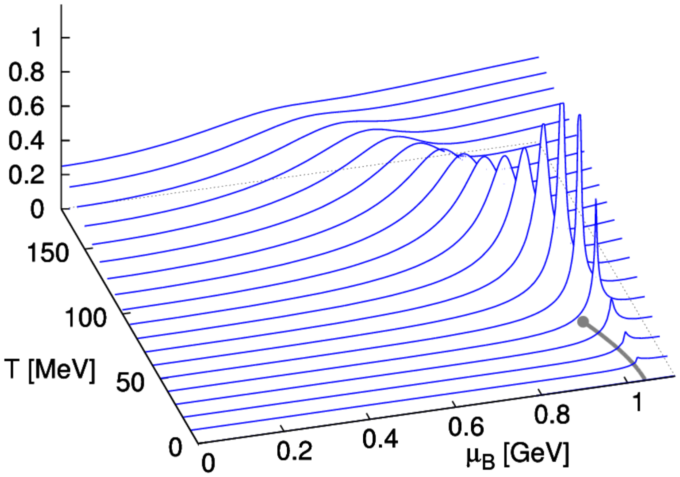
Fluctuations

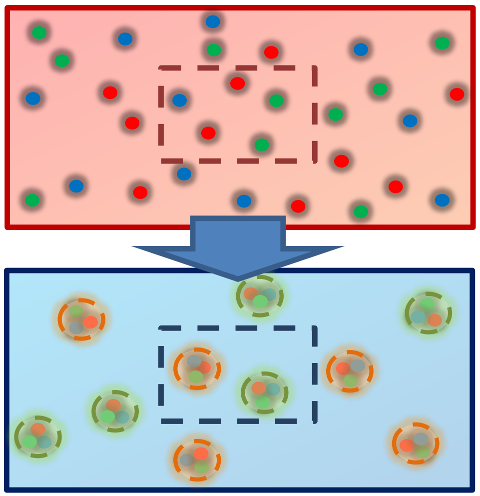
- Fluctuations reflect properties of matter.
  - Enhancement near the critical point

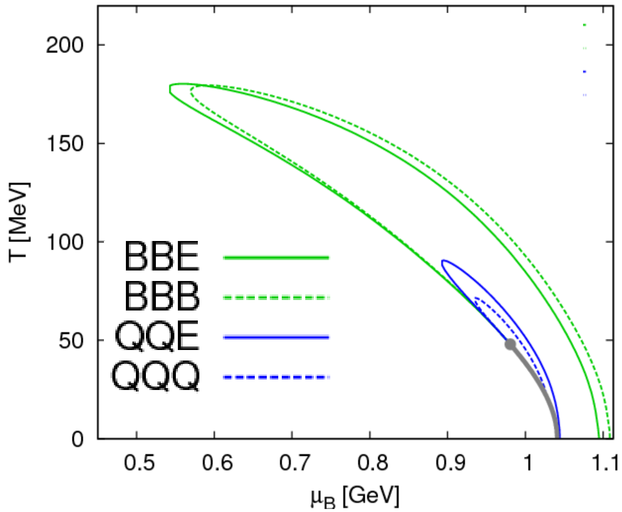
    Stephanov,Rajagopal,Shuryak('98); Hatta,Stephanov('02); Stephanov('09);…
  - Ratios between cumulants of conserved charges

    Asakawa,Heinz,Muller('00); Jeon, Koch('00); Ejiri,Karsch,Redlich('06)
  - Signs of higher order cumulants

    Asakawa,Ejiri,MK('09); Friman,et al.('11); Stephanov('11)



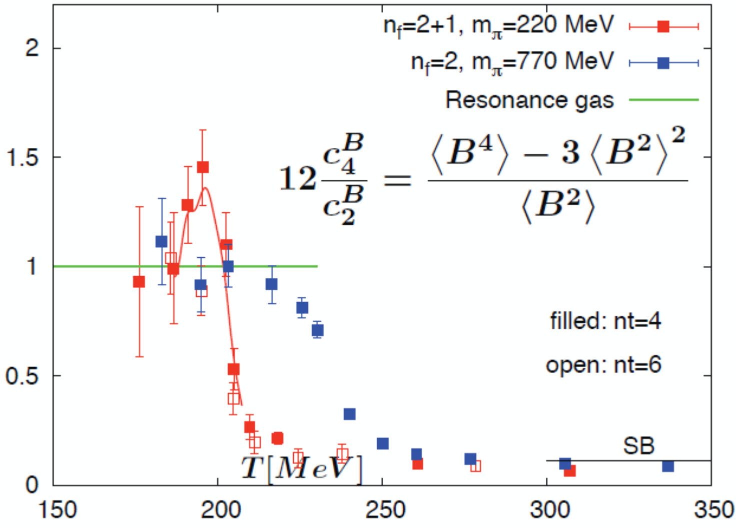




Conserved Charges : Theoretical Advantage

- Definite definition for operators
  - as a Noether current
  - calculable on any theory

ex: on the lattice



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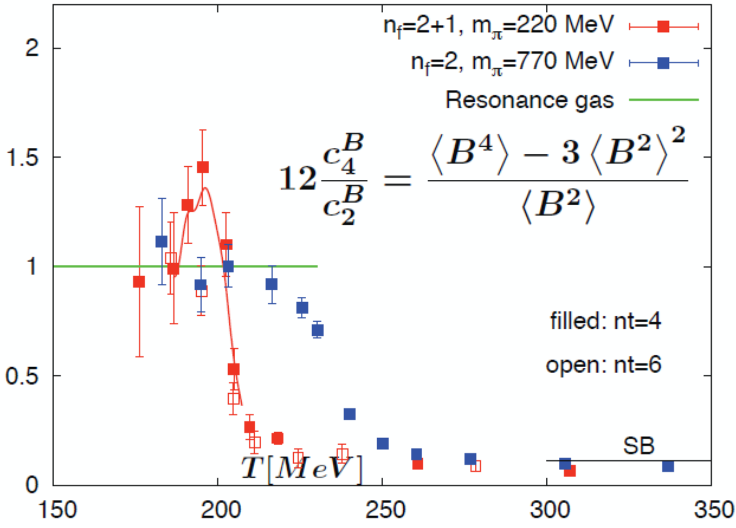


- Simple thermodynamic relations

$$\langle \delta N_c^n \rangle = \frac{1}{VT^{n-1}} \frac{\partial^n \Omega}{\partial \mu_c^n}$$

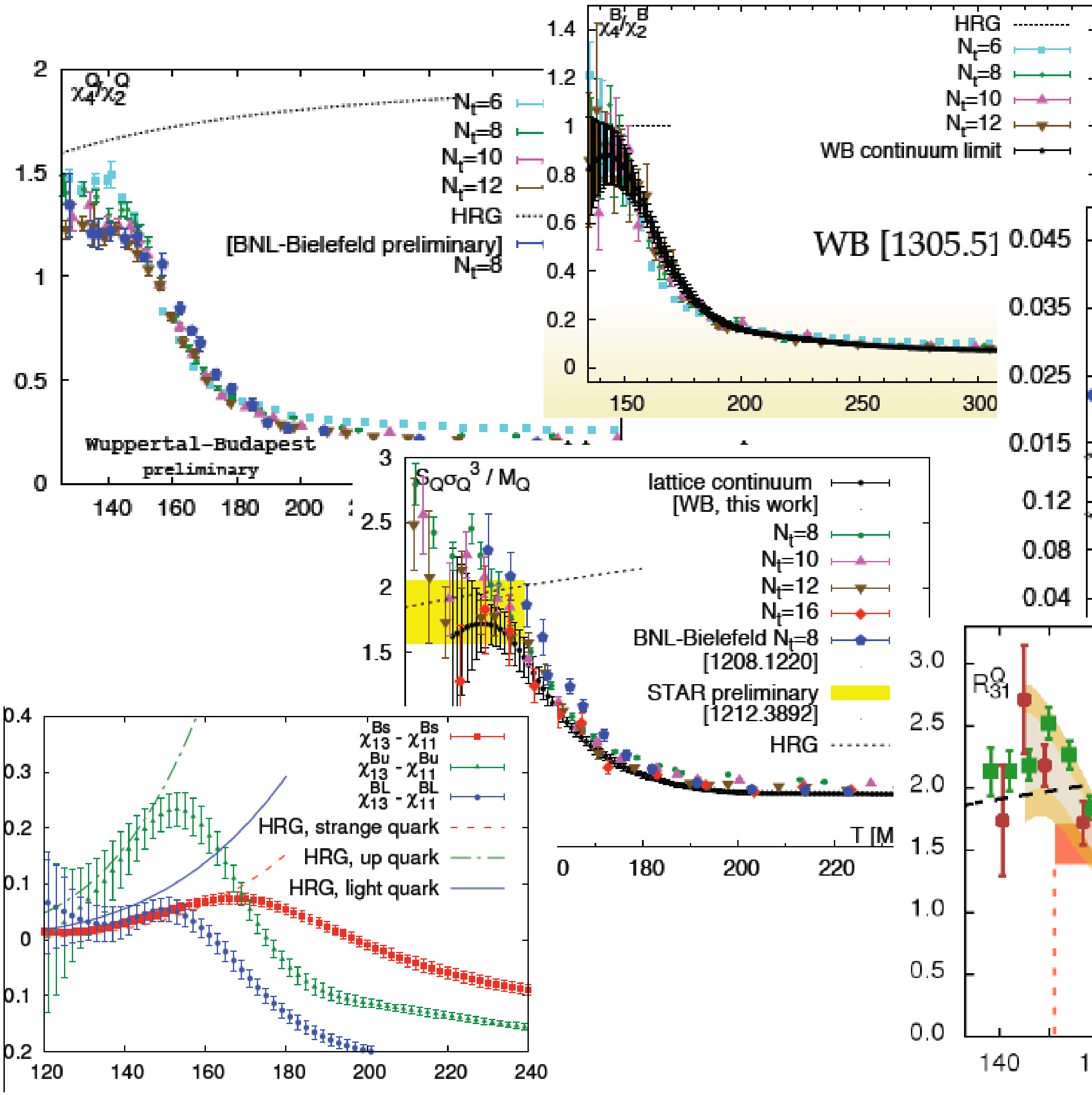
  - Intuitive interpretation for the behaviors of cumulants

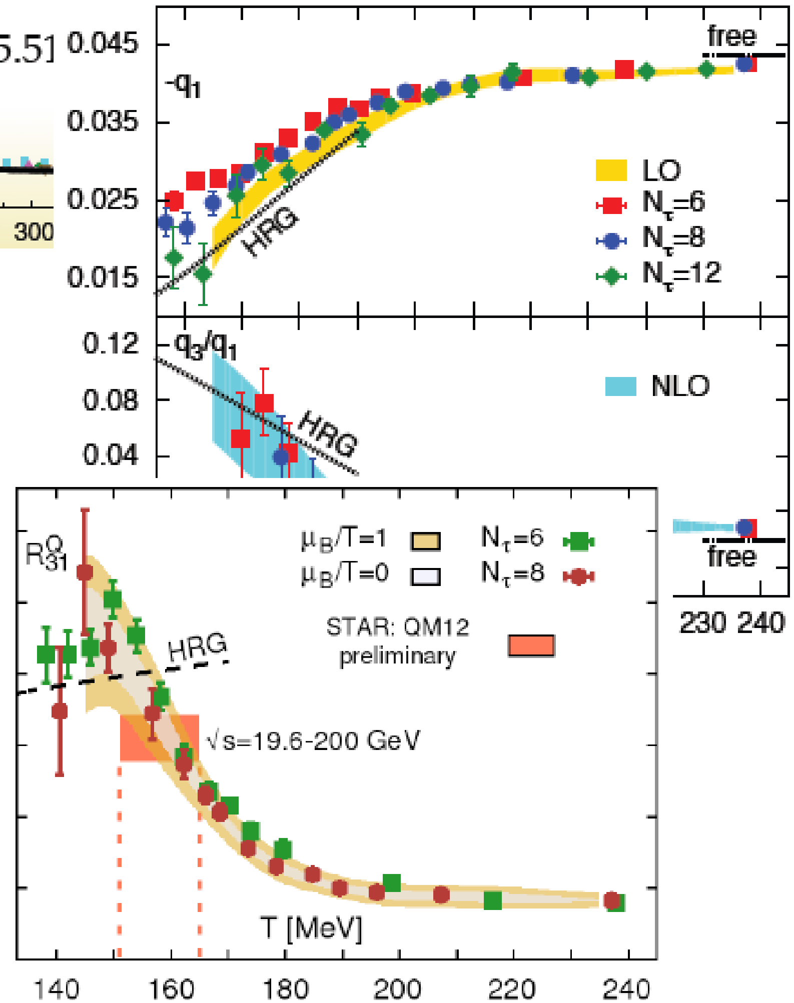
ex: $\langle \delta N_B^3 \rangle = \frac{1}{VT^2} \frac{\partial \langle \delta N_B^2 \rangle}{\partial \mu_B}$

Asakawa, Ejiri, MK, 2009

Recent Progress in Lattice Community

From LATTICE2013 presentations

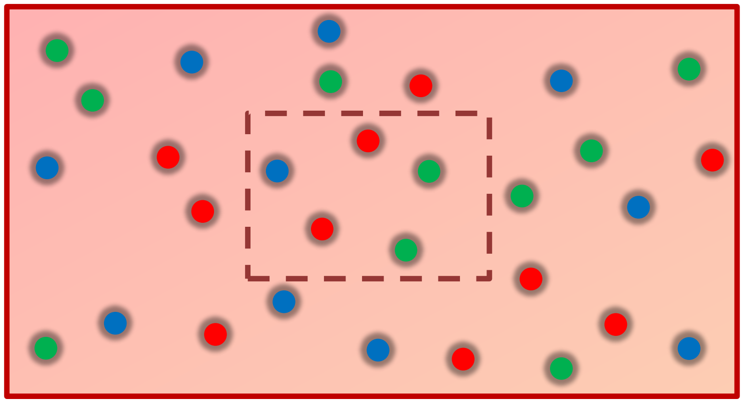




Fluctuations

Free Boltzmann → Poisson

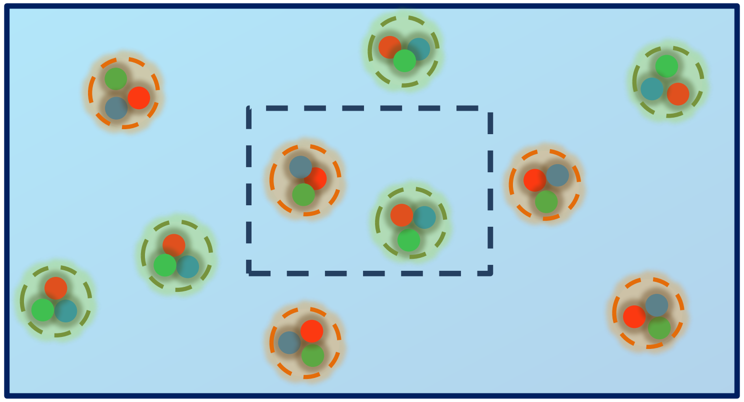
$$\langle \delta N^n \rangle_c = \langle N \rangle$$



$$\langle \delta N_q^n \rangle_c = \langle N_q \rangle$$

$$\Rightarrow \langle \delta N_B^n \rangle_c = \frac{1}{3^{n-1}} \langle N_B \rangle$$

$$3N_B = N_q$$



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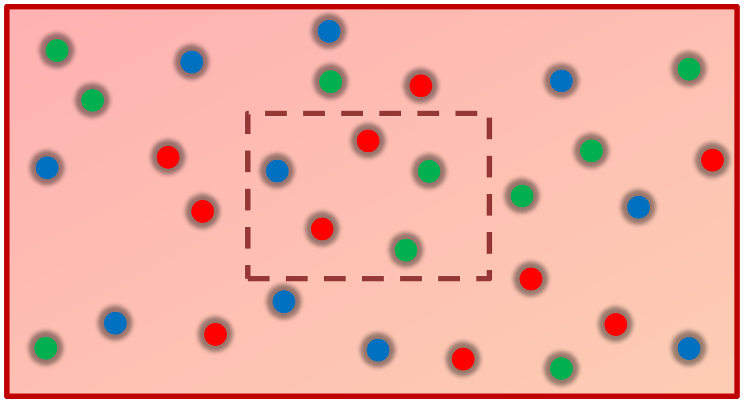

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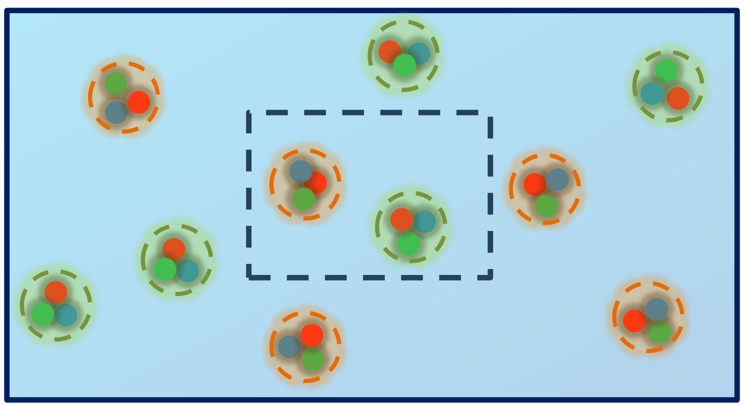

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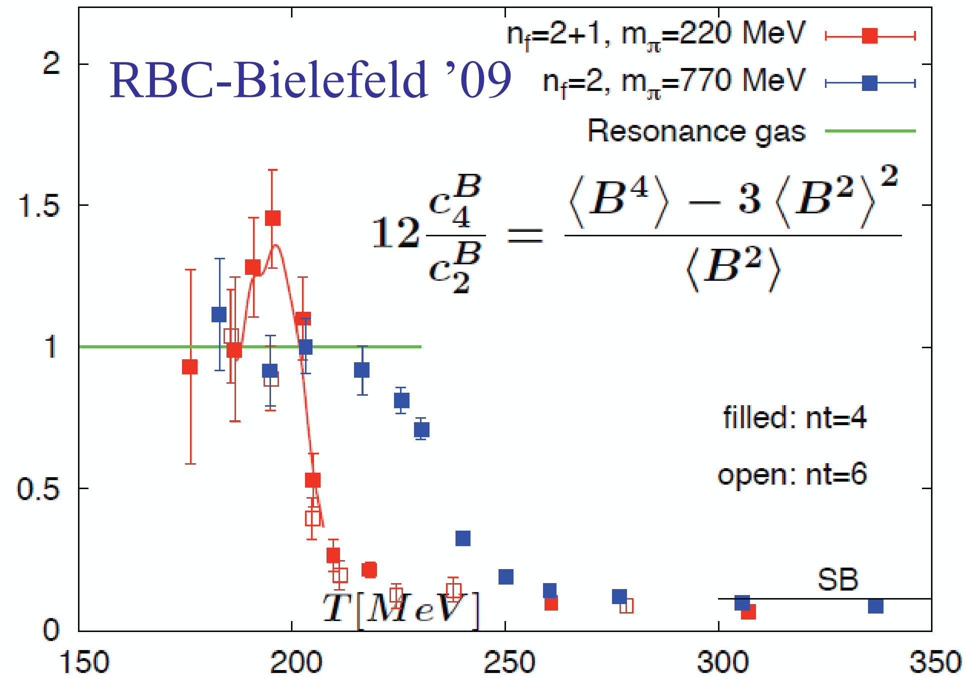
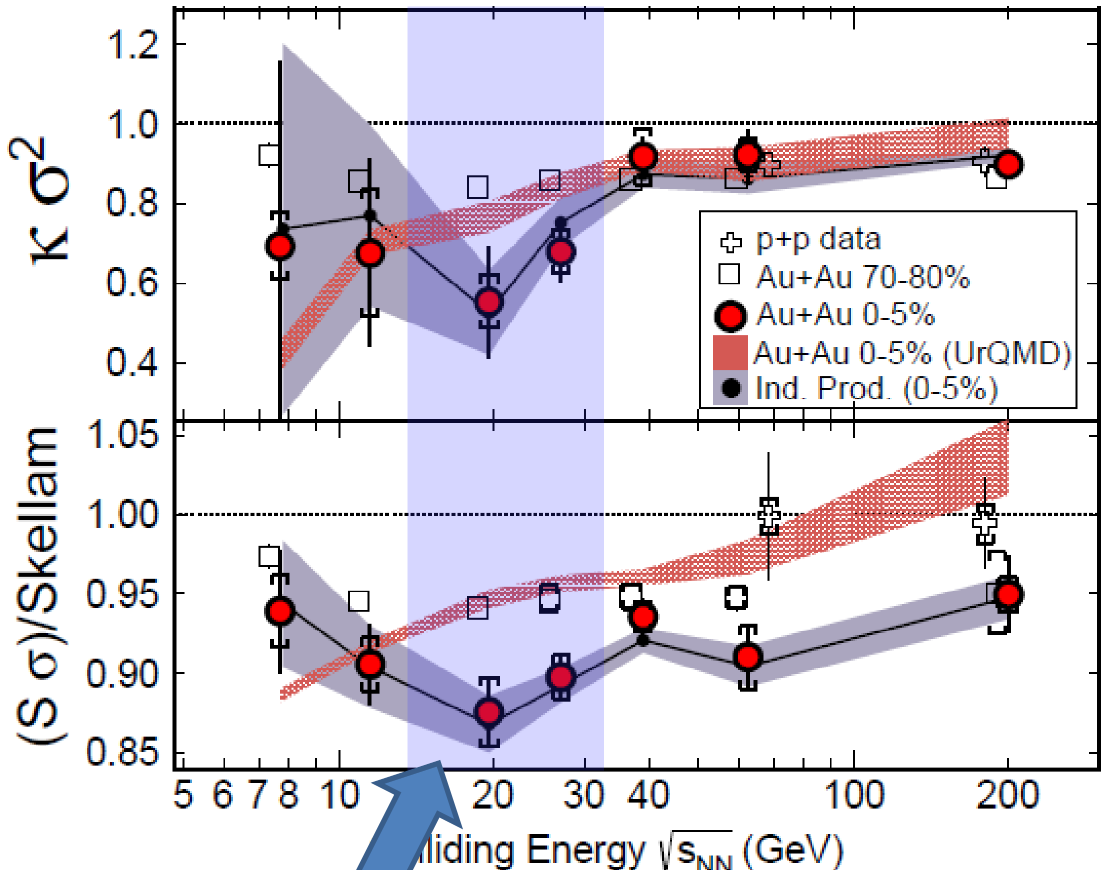


Proton # Cumulants @ STAR-BES



STAR,1309.5681

$$\frac{C_4}{C_2}$$

$$\frac{C_3}{C_1} = \frac{C_3/C_2}{\text{Poissonian}}$$

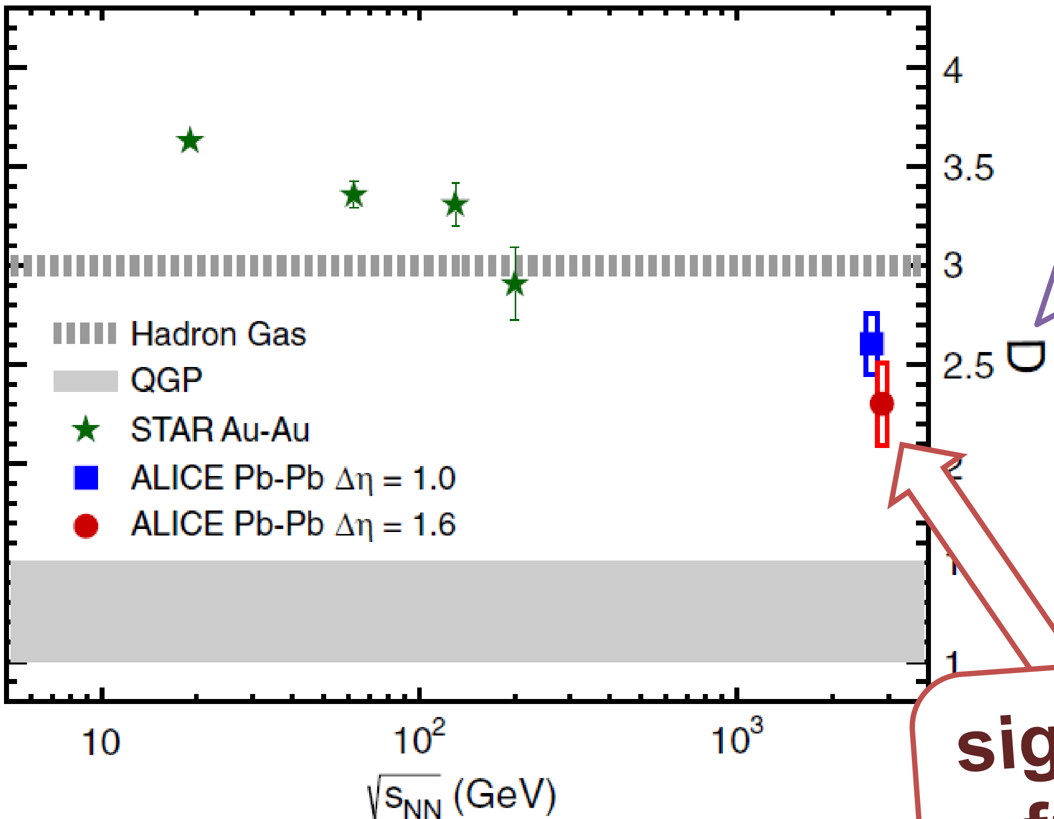
Something interesting??

CAUTION!

proton number $\neq$ baryon number

MK, Asakawa, 2011;2012

Charge Fluctuation @ LHC



ALICE, PRL110,152301(2013)

D-measure

$$D = 4\frac{\langle \delta N_Q^2 \rangle}{\langle N_Q^+ + N_Q^- \rangle}$$

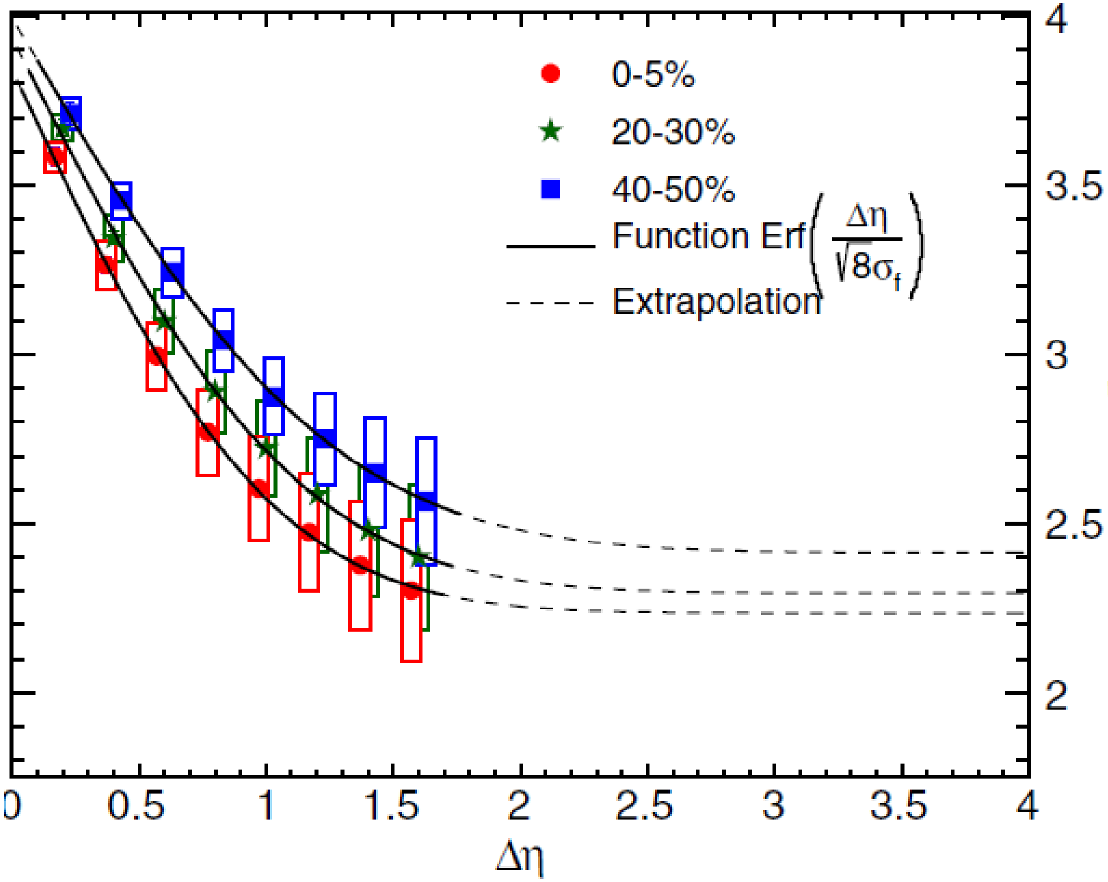
- *D* ~ 3-4 Hadronic
- *D* ~ 1-1.5 Quark

significant suppression from hadronic value at LHC energy!

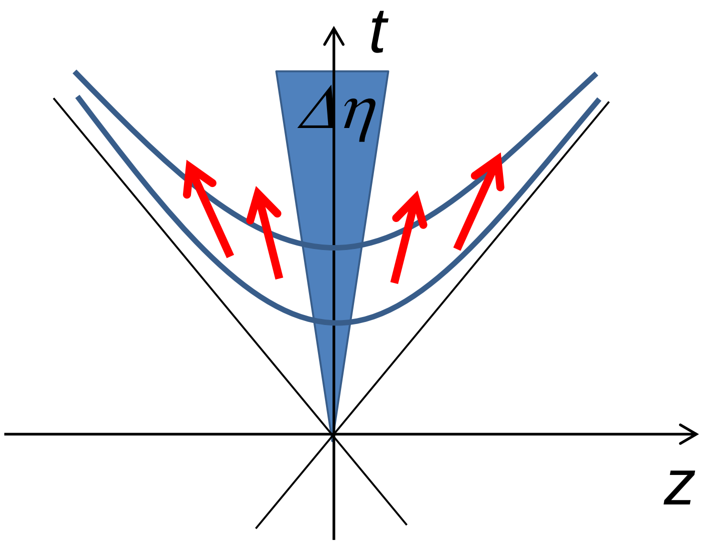
$\langle \delta N_Q^2 \rangle$ is not equilibrated at freeze-out at LHC energy!

Δη Dependence @ ALICE

ALICE
PRL 2013

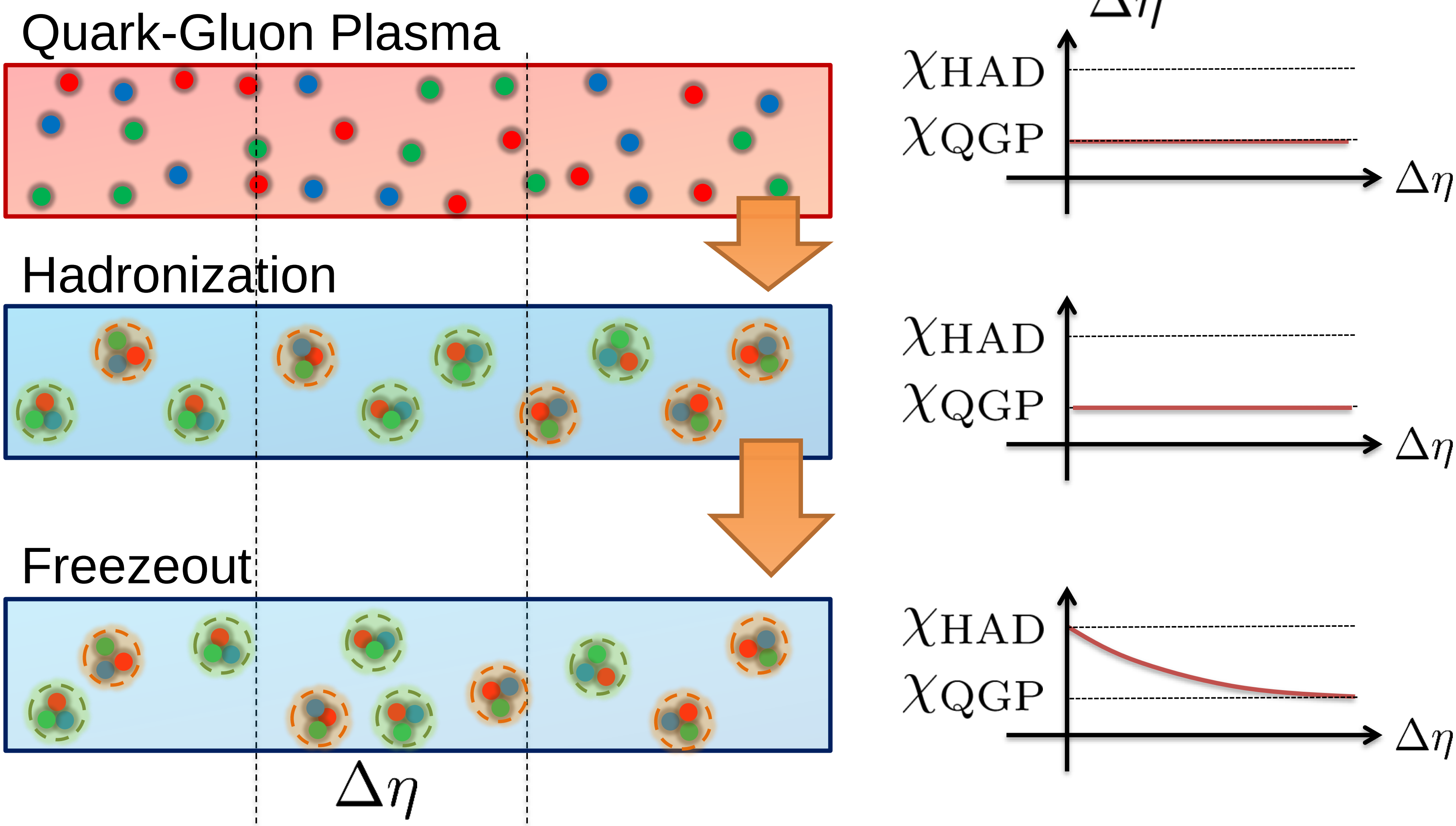


rapidity window



Time Evolution of Fluctuations

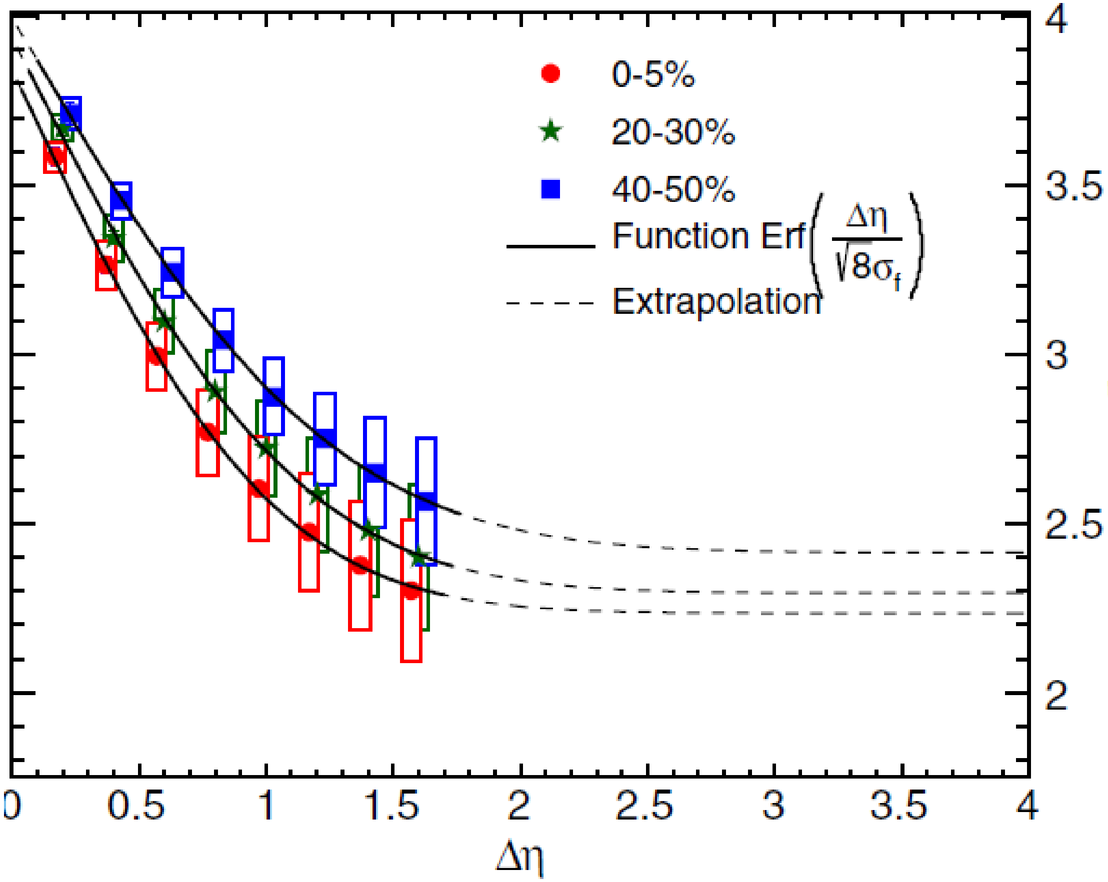
$$\frac{\langle \Delta N^2 \rangle}{\Delta \eta}$$

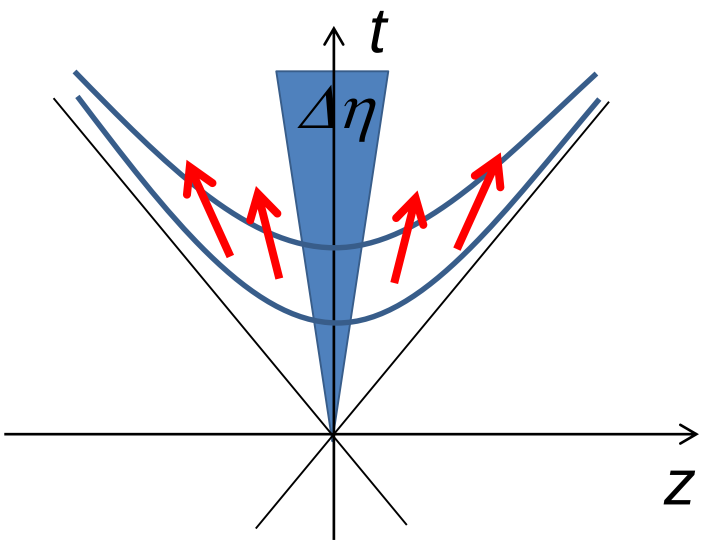


Variation of a conserved charge is achieved only through diffusion. ⇨ The larger $\Delta\eta$, the slower diffusion

Δη Dependence @ ALICE

ALICE
PRL 2013

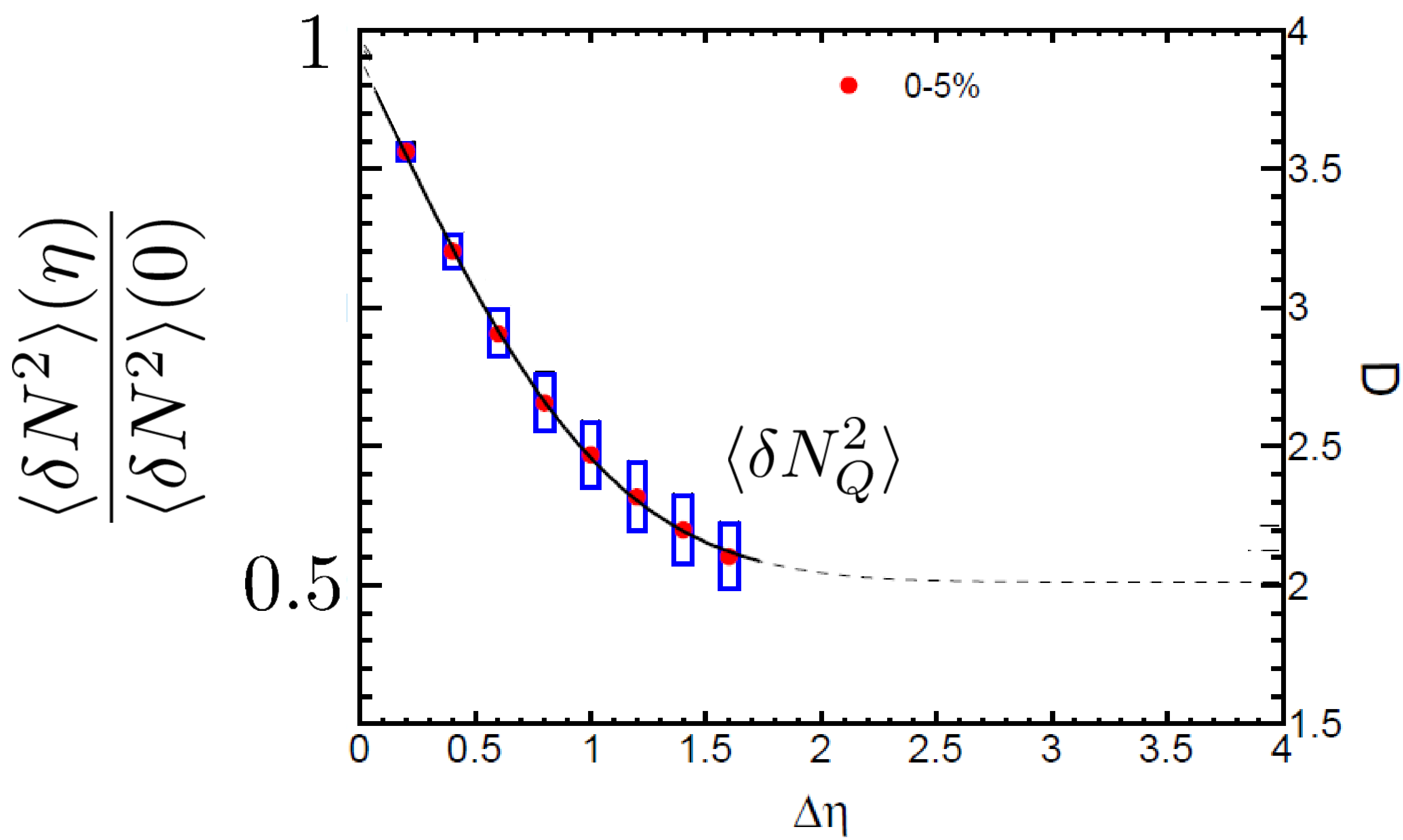




Δη dependences of fluctuation observables encode history of the hot medium!

$<\delta N_B^2>$ and $< \delta N_p^2 >$ @ LHC ?

$\langle \delta N_Q^2 \rangle, \langle \delta N_B^2 \rangle, \langle \delta N_p^2 \rangle$
should have different Δη dependence.



Baryon # cumulants are experimentally observable! MK, Asakawa, 2011;2012

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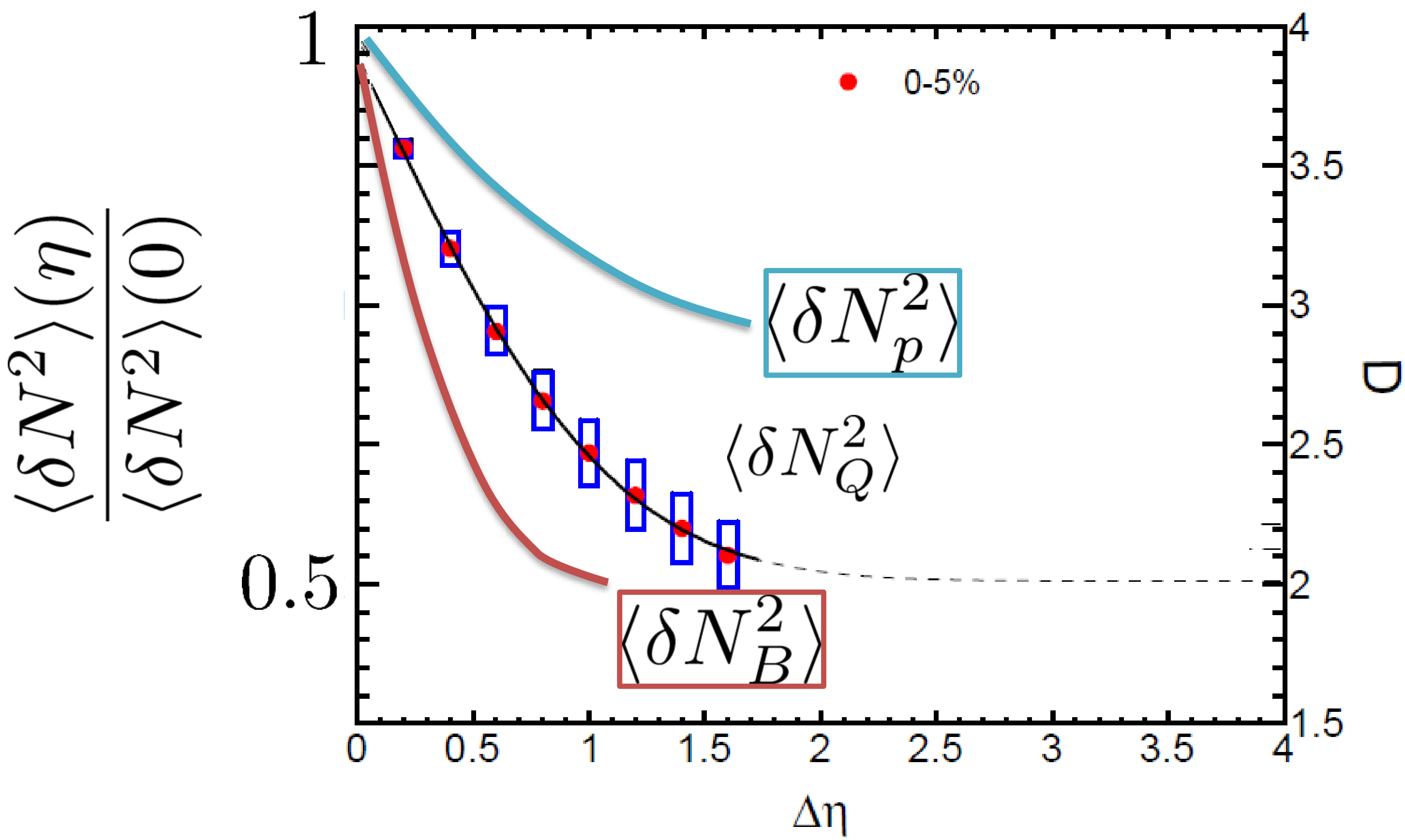
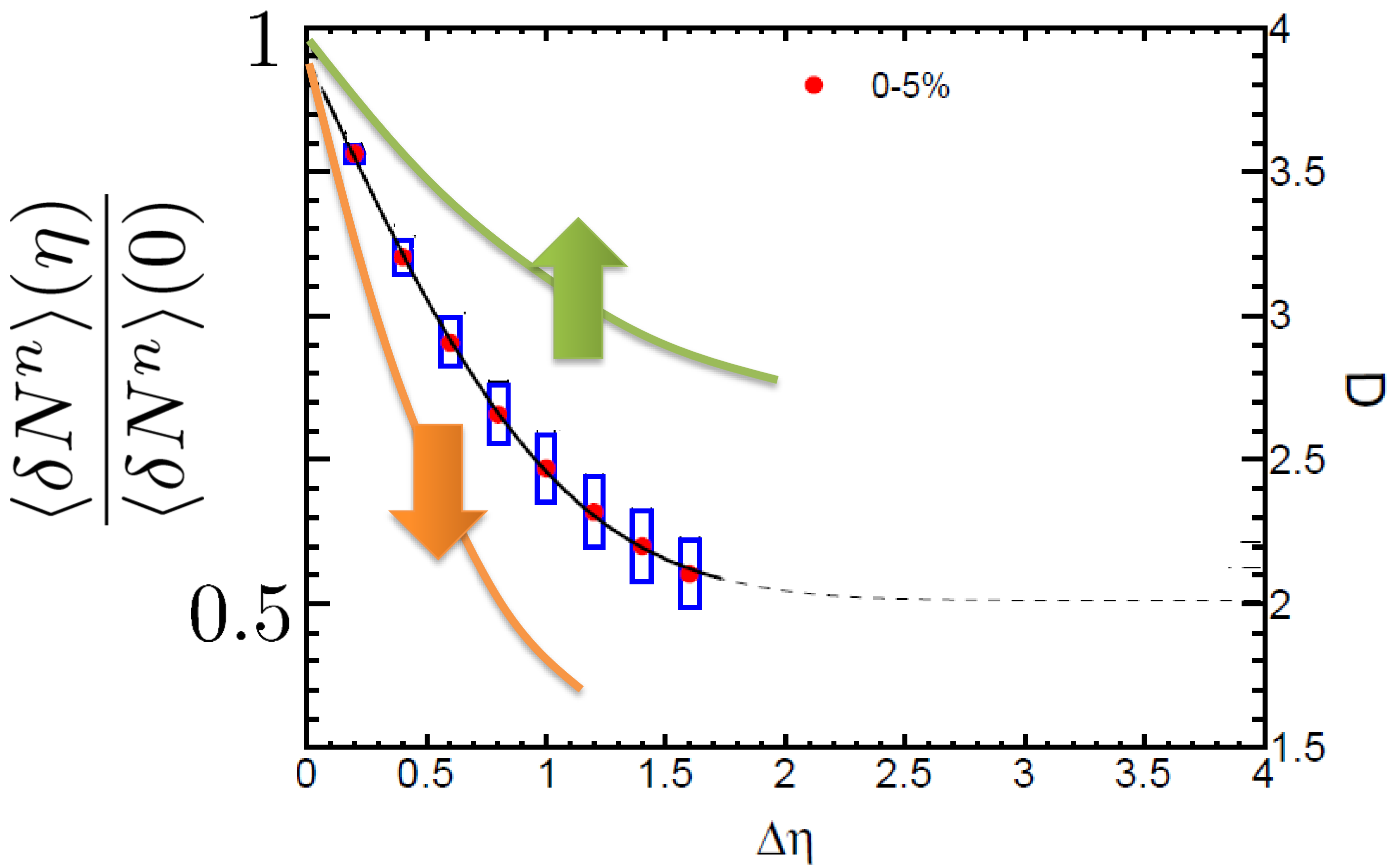


Baryon # cumulants are experimentally observable! MK, Asakawa, 2011;2012

$\langle \delta N_Q^4 \rangle$ @ LHC ?

How does $\langle \delta N_Q^4 \rangle_c$ behave as a function of $\Delta\eta$?

suppression or enhancement

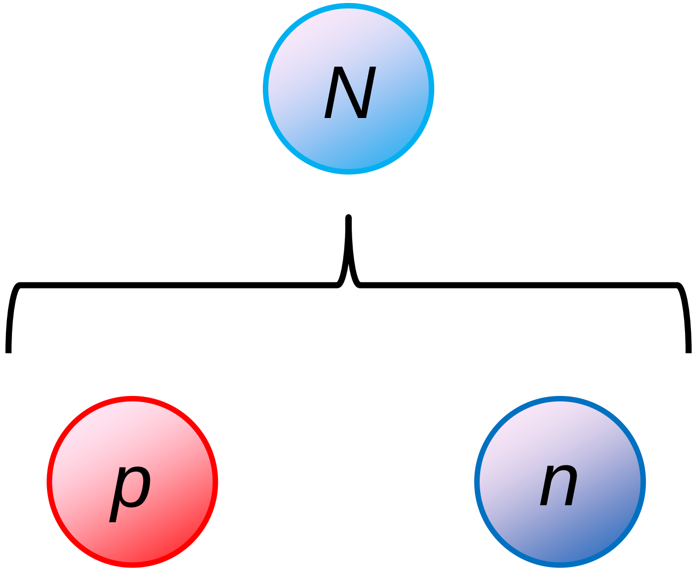


Baryon vs Proton Number Fluctuations

MK, Asakawa, PRC85,021901C(2012); PRC86, 024904(2012)

- $$\frac{\langle \delta N_B^n \rangle_c}{\langle \delta N_B^m \rangle_c} \neq \frac{\langle \delta N_p^n \rangle_c}{\langle \delta N_p^m \rangle_c}$$
- $\langle \delta N_B^n \rangle_c$ are experimentally observable

Nucleon Isospin as Two Sides of a Coin



Nucleons have
two isospin states.

MK, Asakawa,2012

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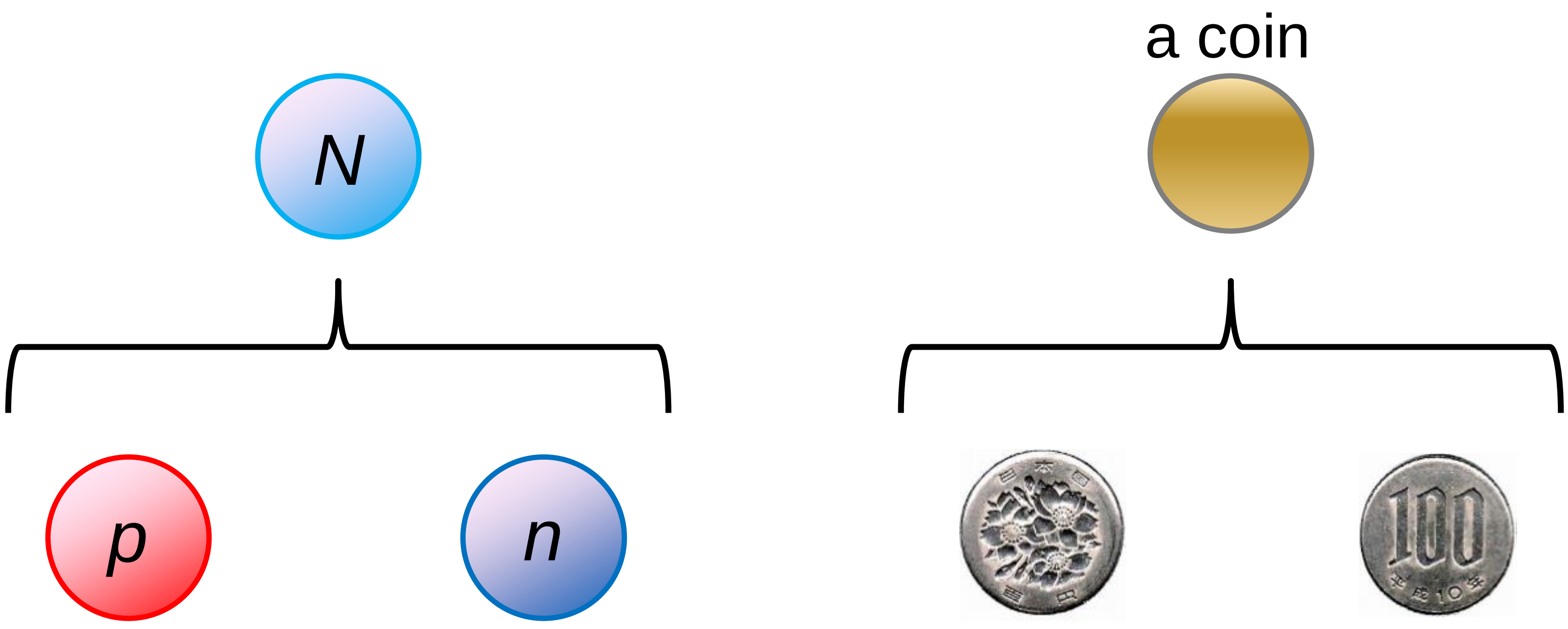
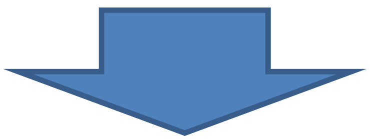


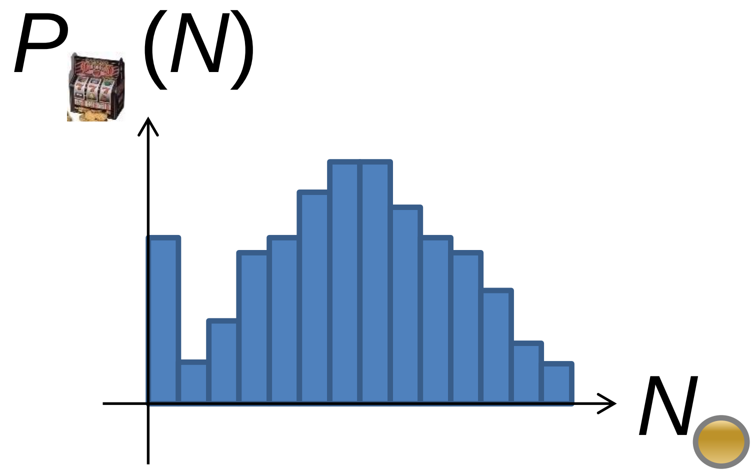
Nucleons have
two isospin states.

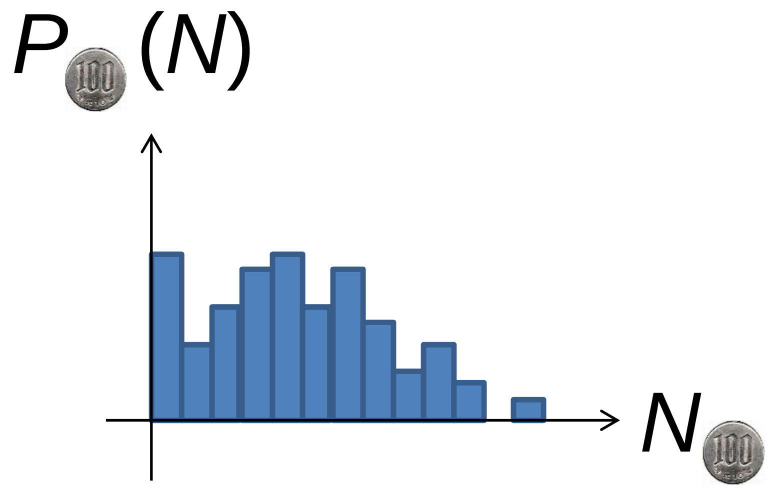
Coins have two sides.

MK, Asakawa,2012

Slot Machine Analogy



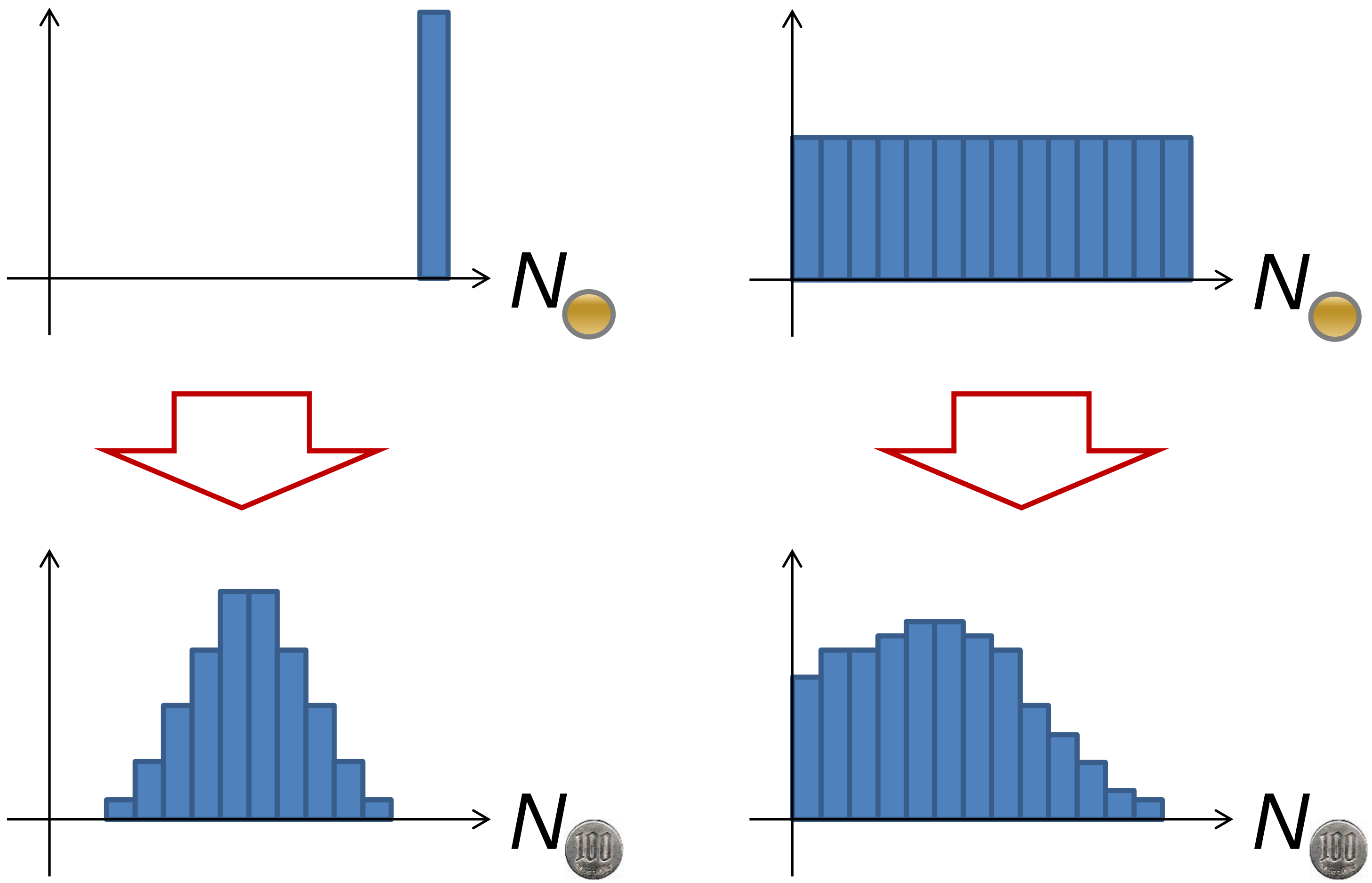




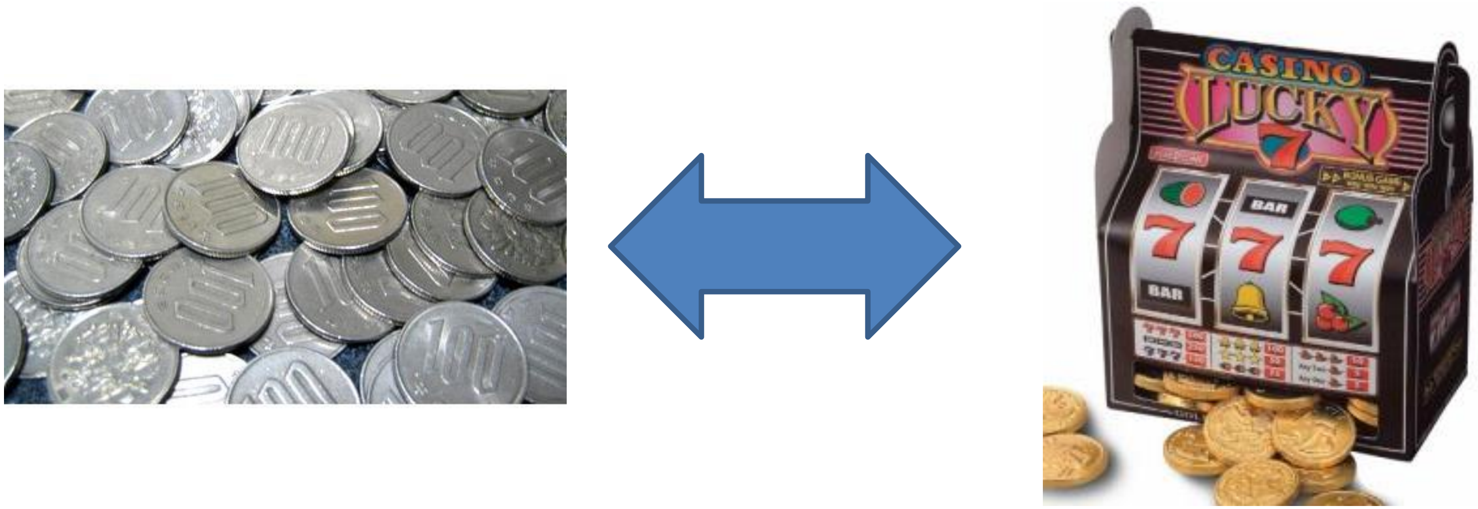
Extreme Examples

Fixed # of coins

Constant probabilities



Reconstructing Total Coin Number



$$B_p(k;N) = p^k(1-p)^{N-k}\ {}_kC_N$$:binomial distr. func.

Reconstructing Baryon Number Cumulants

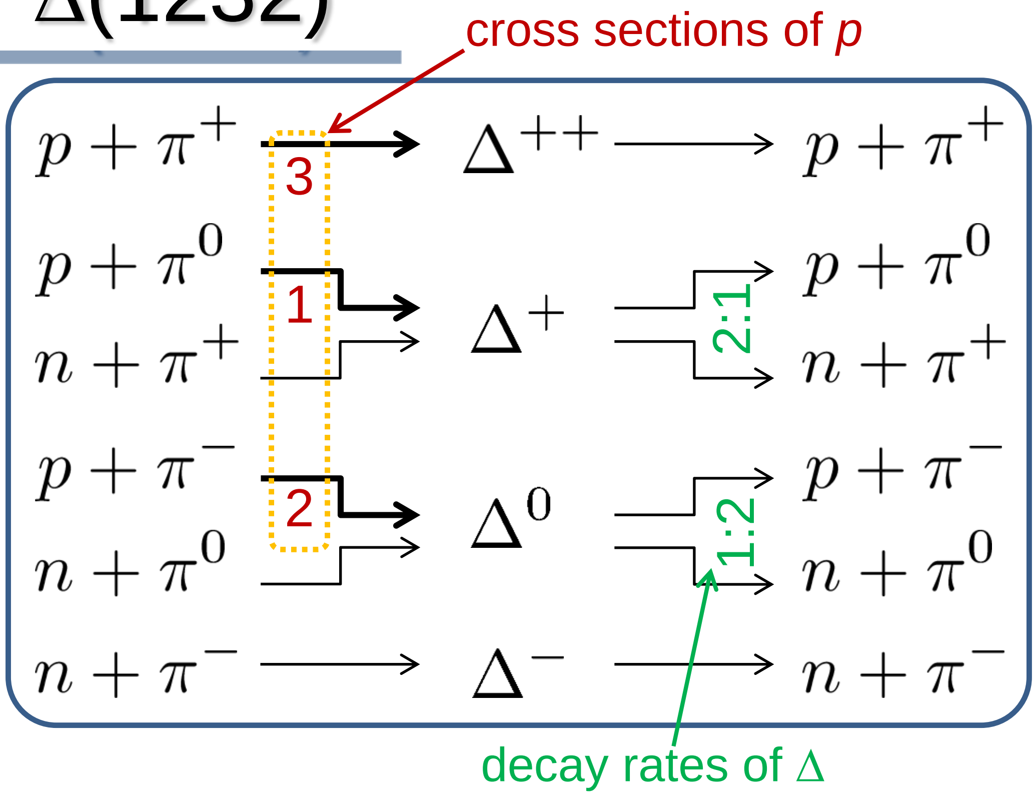
$$\mathcal{P}(N_p, N_n, N_{\bar{p}}, N_{\bar{n}}) = F(N_N, N_{\bar{N}})B(N_p; N_N)B(N_{\bar{p}}; N_{\bar{N}})$$

- for any phase space in the final state.

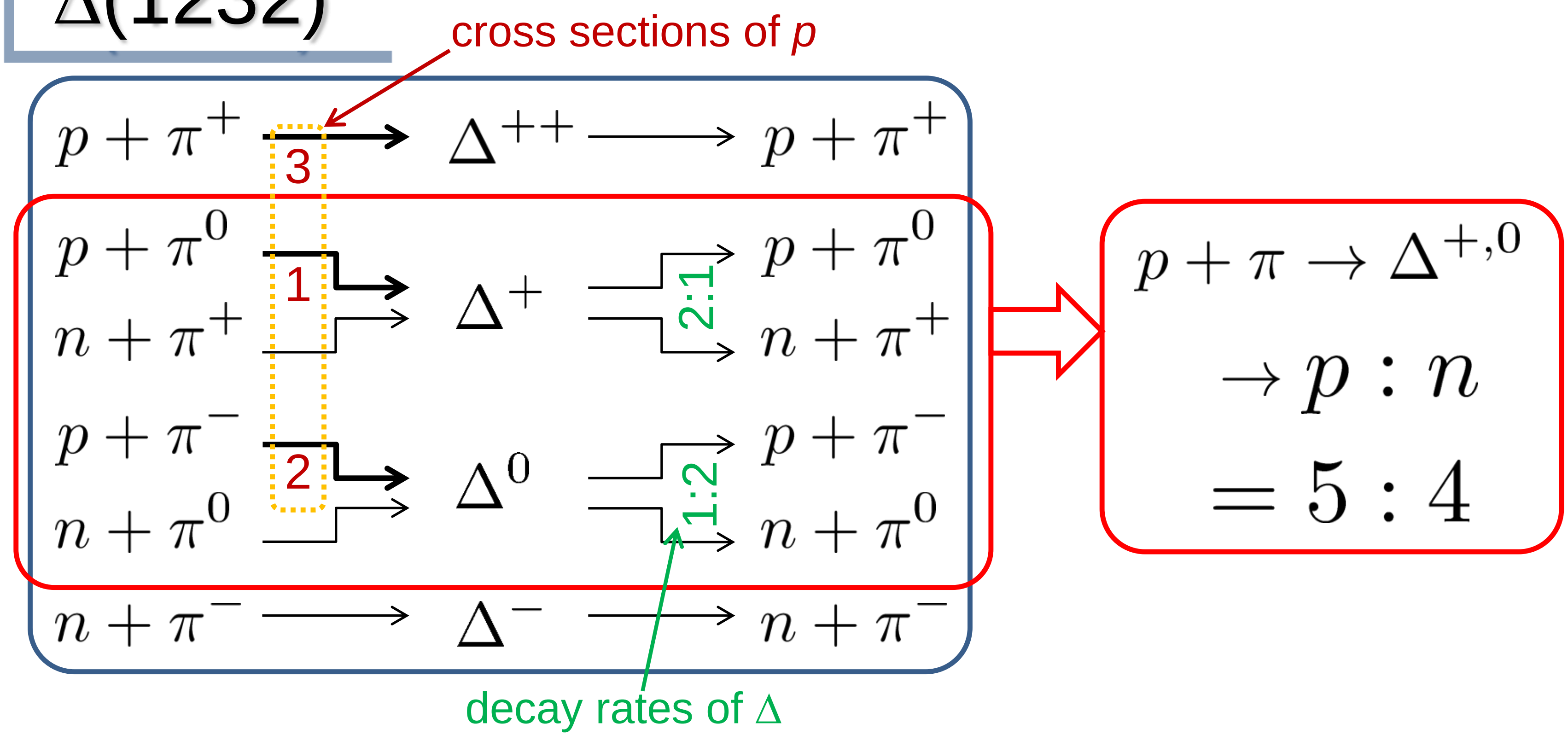
$$\begin{cases} \langle(\delta N_p^{(\mathrm{net})})^2\rangle = \frac{1}{4}\langle(\delta N_{\mathrm{N}}^{(\mathrm{net})})^2\rangle + \frac{1}{4}\langle N_{\mathrm{N}}^{(\mathrm{tot})}\rangle \\ \langle(\delta N_{\mathrm{N}}^{(\mathrm{net})})^2\rangle = 4\langle(\delta N_p^{(\mathrm{net})})^2\rangle - 2\langle N_p^{(\mathrm{tot})}\rangle \end{cases}$$

- for isospin symmetric medium
- effect of isospin density <10% for $\sqrt{s}$>10GeV
- Similar formulas up to any order!

Δ(1232)


cross sections of p
$p+\pi^+$
$p+\pi^0$
$n+\pi^+$
$p+\pi^-$
$n+\pi^0$
$n+\pi^-$
3
1
2
Δ^{++}
Δ^+
Δ^0
Δ^-
2:1
1:2
$p+\pi^+$
$p+\pi^0$
$n+\pi^+$
$p+\pi^-$
$n+\pi^0$
$n+\pi^-$
decay rates of Δ

Δ(1232)



Lifetime to create Δ^+ or Δ^0

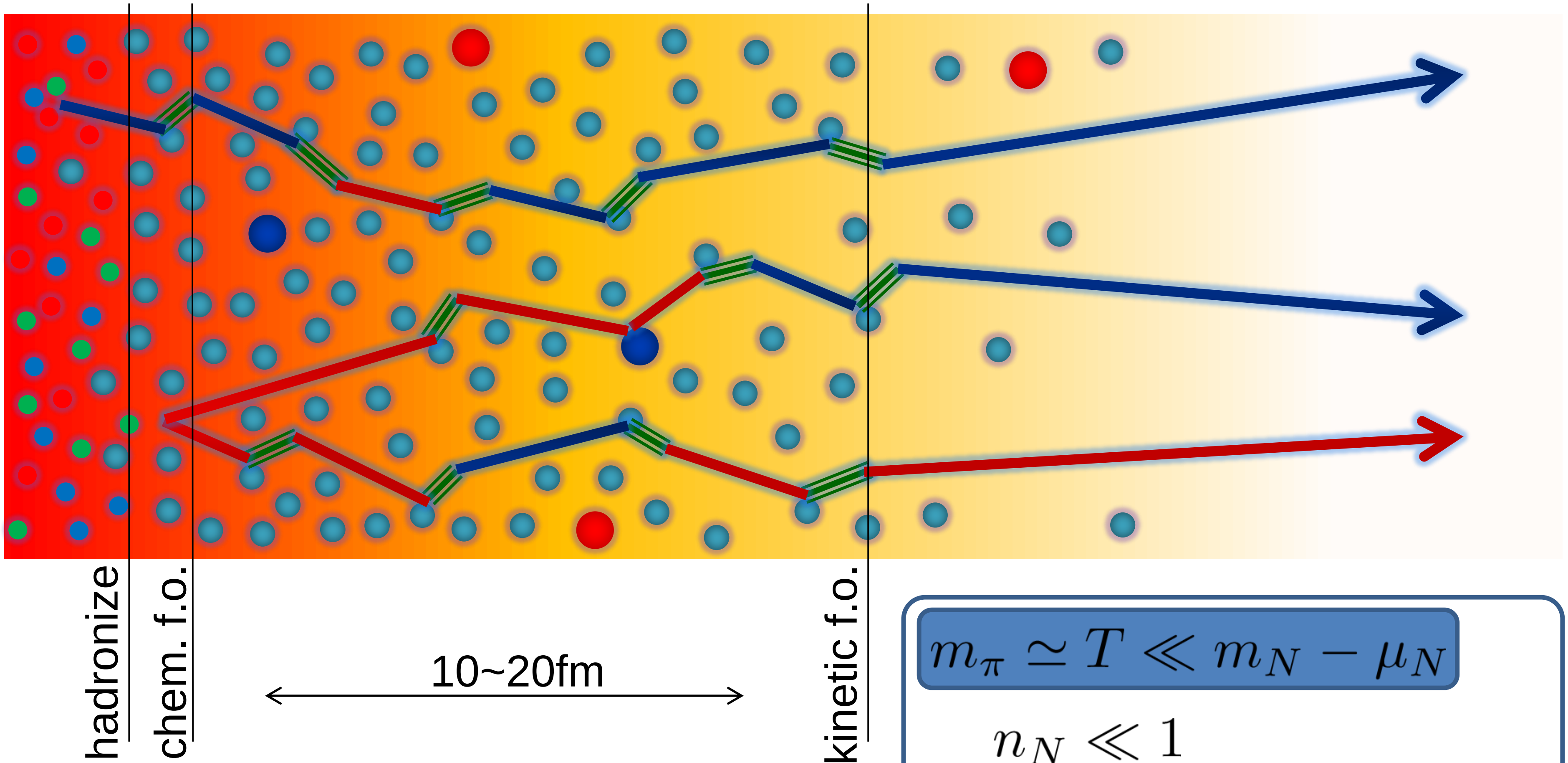
$$\tau \simeq 4[\mathrm{fm}]$$

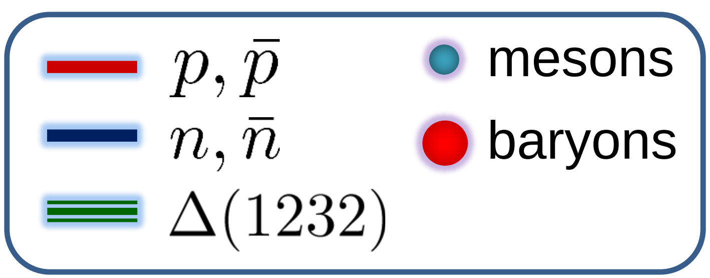
Hadronic stage

$$\simeq 20[\mathrm{fm}]$$

Nucleons in Hadronic Phase

time





$$m_\pi \simeq T \ll m_N - \mu_N$$

$n_N \ll 1$

- rare NN collisions
- no quantum corr.

$n_N \ll n_\pi$

- many pions

Difference btw Baryon and Proton Numbers

(1) $N_B^{(\mathrm{net})} = N_B - N_{\bar{B}}$ deviates from the equilibrium value.

(2) Boltzmann (Poisson) distribution for $N_B, N_{\bar{B}}$.

$$\begin{cases} 2\langle(\delta N_p^{(\mathrm{net})})^2\rangle = \frac{1}{2}\langle(\delta N_{\mathrm{B}}^{(\mathrm{net})})^2\rangle + \frac{1}{2}\langle(\delta N_{\mathrm{B}}^{(\mathrm{net})})^2\rangle_{\mathrm{free}} \\ 2\langle(\delta N_p^{(\mathrm{net})})^3\rangle = \frac{1}{4}\langle(\delta N_{\mathrm{B}}^{(\mathrm{net})})^3\rangle + \frac{3}{4}\langle(\delta N_{\mathrm{B}}^{(\mathrm{net})})^3\rangle_{\mathrm{free}} \\ 2\langle(\delta N_p^{(\mathrm{net})})^4\rangle_c = \frac{1}{8}\langle(\delta N_{\mathrm{B}}^{(\mathrm{net})})^4\rangle_c + \cdots \end{cases}$$

genuine info. Poissonian noise

Difference from Poisson (thermal) distribution is suppressed in proton number fluctuations.

Difference btw Baryon and Proton Numbers

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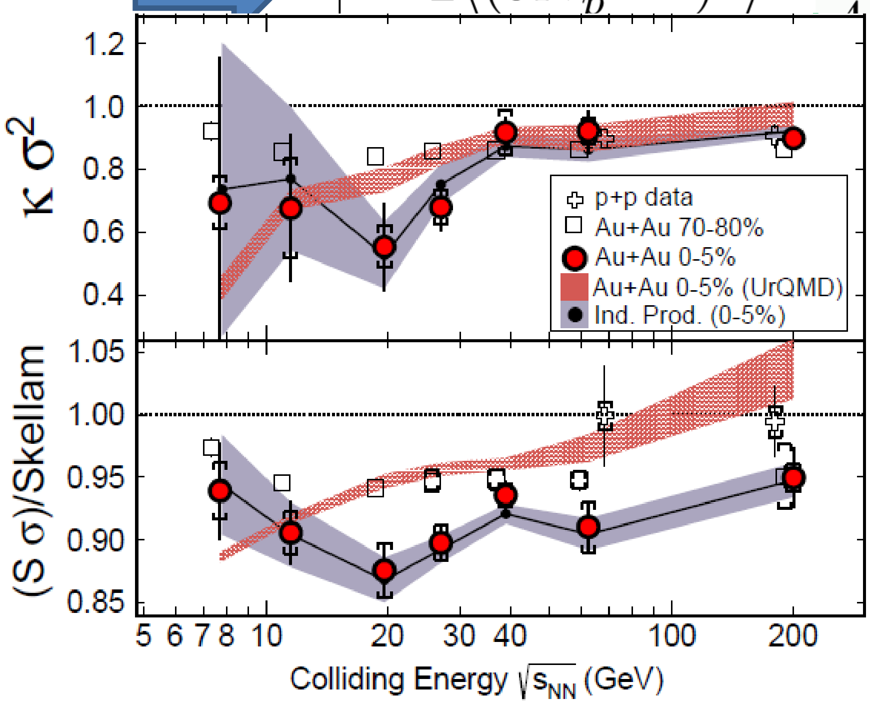
$$2\langle(\delta N_p^{(\mathrm{net})})^3\rangle = \frac{1}{4}\langle(\delta N_\mathrm{B}^{(\mathrm{net})})^3\rangle + \frac{3}{4}\langle(\delta N_\mathrm{B}^{(\mathrm{net})})^3\rangle_\mathrm{free}$$

$$\langle(\delta N_\mathrm{B}^{(\mathrm{net})})^4\rangle_c + \cdots$$

enuine info. Poissonian noise



from Poisson (thermal) distribution
ssed in proton number fluctuations.

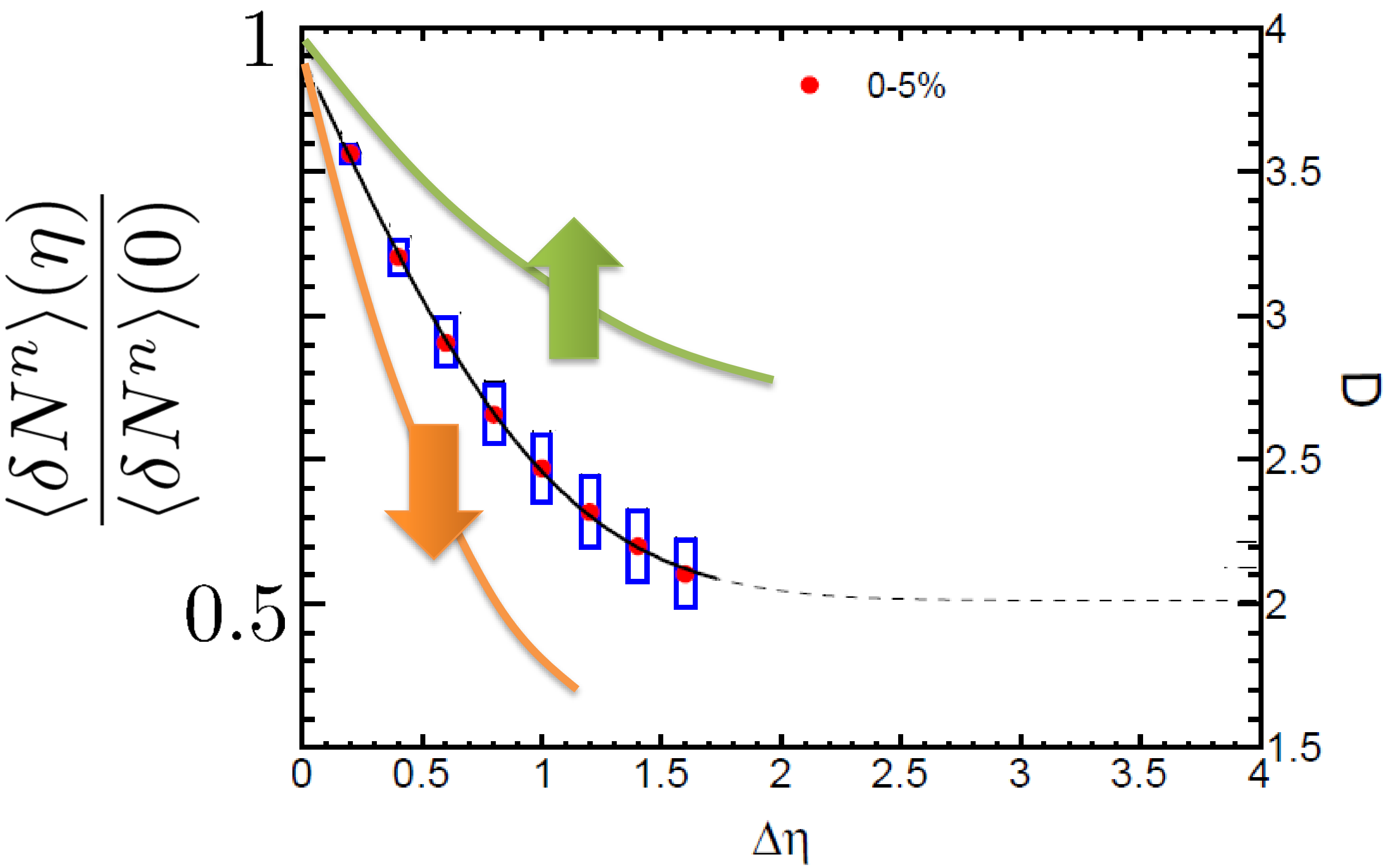
Time Evolution of Higher Order Cumulants

MK, Asakawa, Ono, PL**B728**, 386 [arXiv:1307.2978]

$<\delta N_Q^4>$ @ LHC ?

How does $\langle \delta N_Q^4 \rangle_c$ behave as a function of $\Delta\eta$?

suppression or enhancement



ALICE 2013

Hydrodynamic Fluctuations

Landau, Lifshitz, Statistical Mechaniqs II
Kapusta, Muller, Stephanov, 2012
Stephanov, Shuryak, 2001

Stochastic diffusion equation

$$\partial_\tau n = D\partial_\eta^2 n + \partial_\eta \xi(\eta,\tau)$$

Markov (white noise)
+
continuity

⇓

Gaussian noise

⇓

Fluctuation of *n* is Gaussian in equilibrium

cf) Gardiner, "Stochastic Methods"

How to Introduce Non-Gaussianity?

Stochastic diffusion equation

$$\partial_\tau n = D\partial_\eta^2 n + \partial_\eta \xi(\eta,\tau)$$

- Choices to introduce non-Gaussianity in equil.:
  - *n* dependence of diffusion constant *D*(*n*)
  - colored noise
  - discretization of *n*

How to Introduce Non-Gaussianity?

Stochastic diffusion equation

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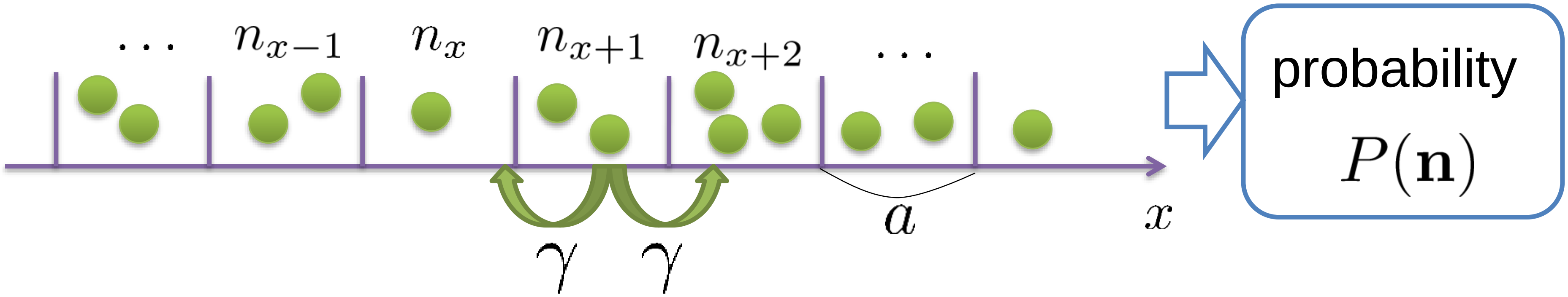
- Choices to introduce non-Gaussianity in equil.:
  - *n* dependence of diffusion constant *D*(*n*)
  - colored noise
  - discretization of *n* ⟸ **our choice**

REMARK: Fluctuations measured in HIC are almost Poissonian.

Diffusion Master Equation

Divide spatial coordinate into discrete cells



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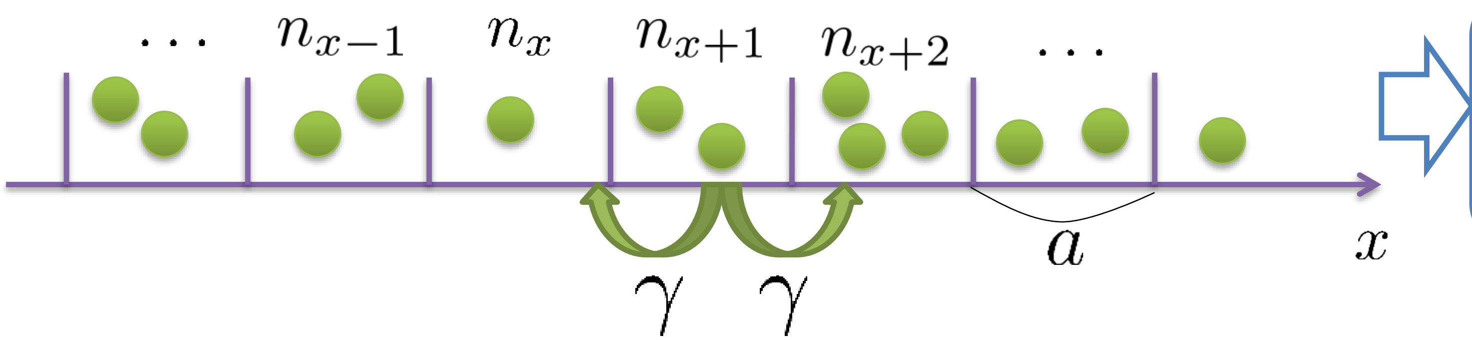


probability

$P(\mathbf{n})$

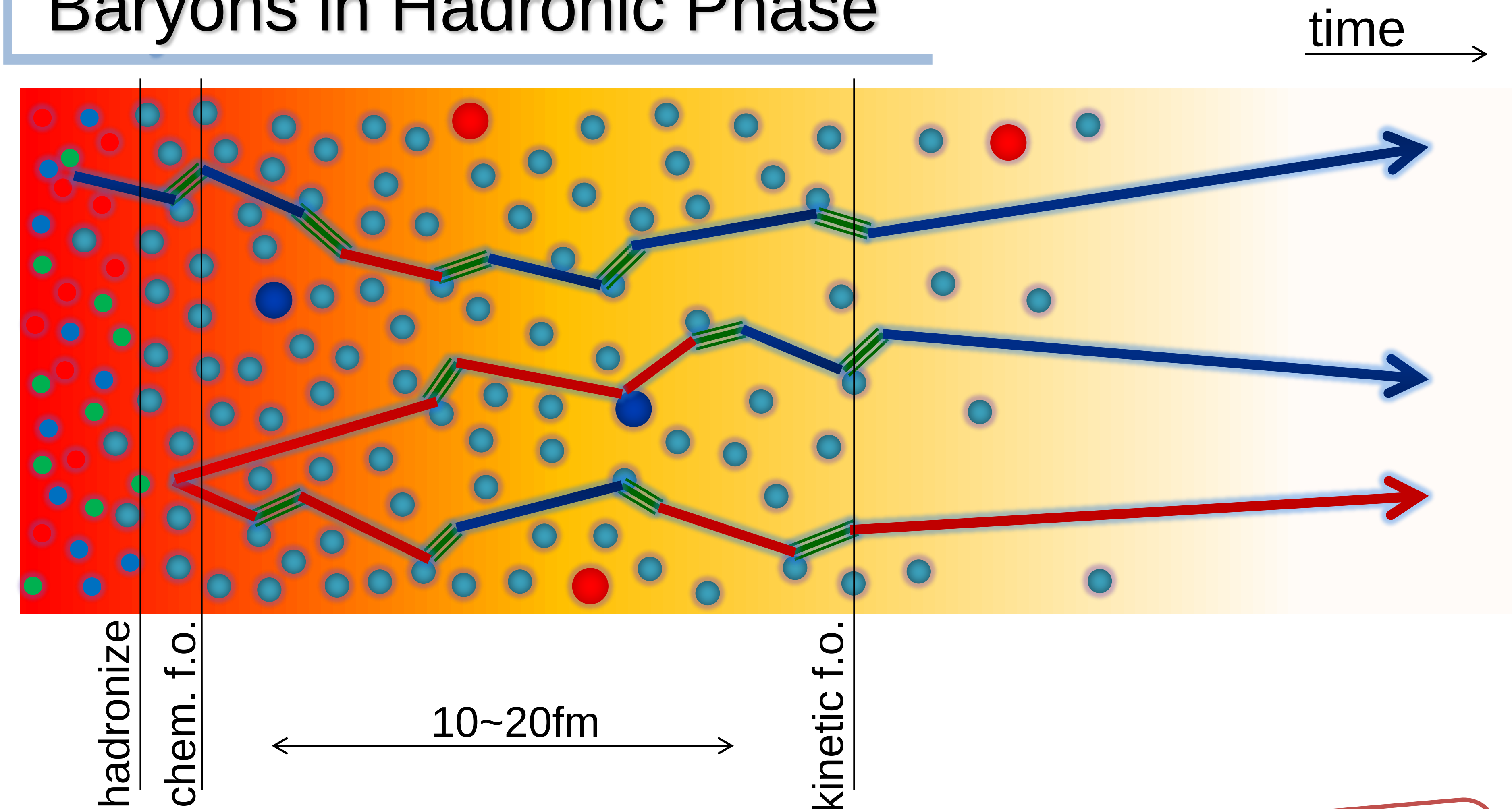
Master Equation for *P*(*n*)

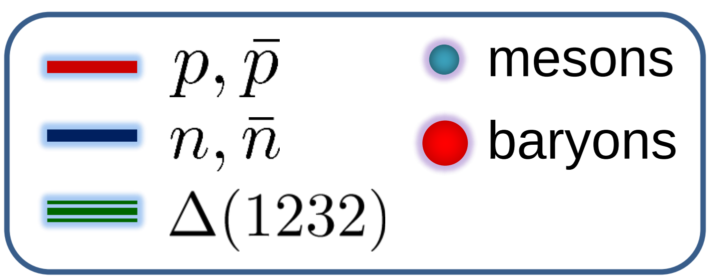
$$\frac{\partial}{\partial t} P(\mathbf{n}) = \gamma \sum_x [(n_x + 1) \{P(\mathbf{n} + \mathbf{e}_x - \mathbf{e}_{x+1}) + P(\mathbf{n} + \mathbf{e}_x - \mathbf{e}_{x-1})\} - 2 n_x P(\mathbf{n})]$$

Solve the DME **exactly**, and take *a*→0 limit

No approx., ex. van Kampen's system size expansion

Baryons in Hadronic Phase





Baryons behave like Brownian pollens in water

Solution of DME in a→0 Limit

1st order (deterministic) $\langle n \rangle$

- consistent with diffusion equation with $D=\gamma a^2$

➡ Continuum limit with fixed $D=\gamma a^2$

2nd order $\langle \delta n^2 \rangle$

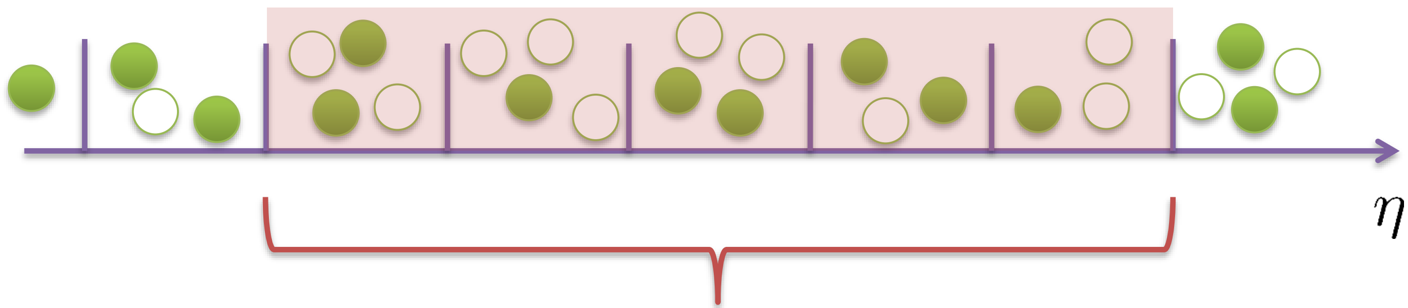
- consistent with stochastic diffusion eq.
  (for sufficiently smooth initial conditions)

Shuryak, Stephanov, 2001

Nontrivial results for non-Gaussian fluctuations

Net Charge Number

Prepare 2 species of (non-interacting) particles



$$\bar{Q}(\tau) = \int_0^{\Delta\eta} d\eta \, (n_1(\eta,\tau) - n_2(\eta,\tau))$$

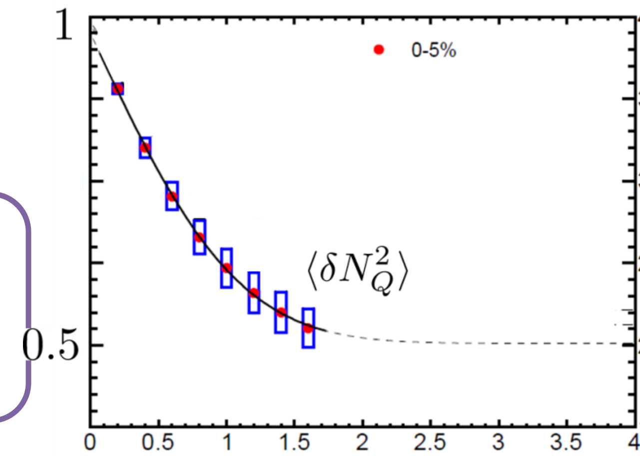
Let us investigate

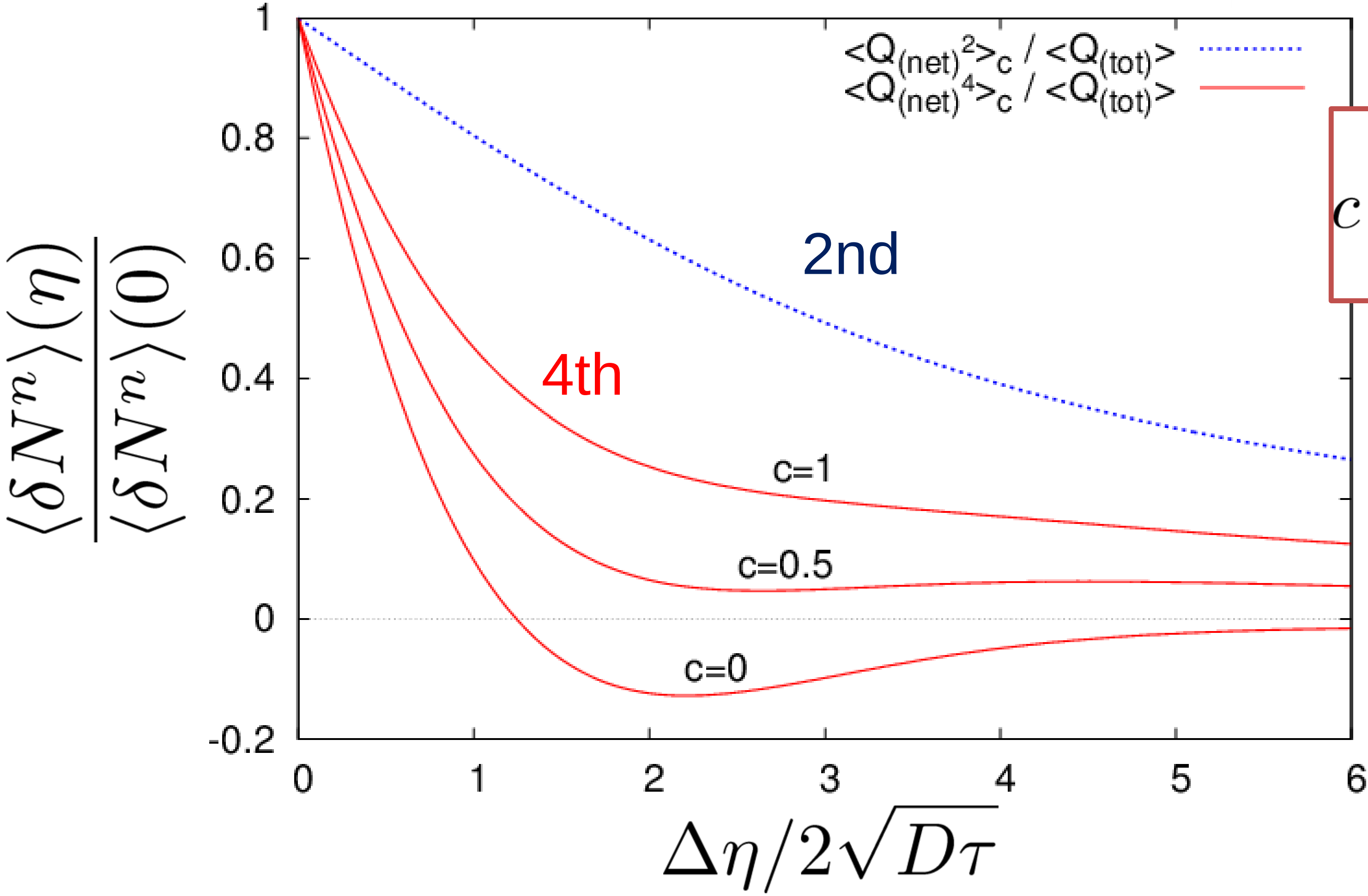
$\langle \bar{Q}^2 \rangle_c \quad \langle \bar{Q}^4 \rangle_c$ at freezeout time t

$\Delta\eta$ Dependence at Freezeout

Initial fluctuations:

$$\langle \bar{Q}^2 \rangle_c = \langle \bar{Q}^4 \rangle_c = \langle \bar{Q}^2 Q_{(\mathrm{tot})} \rangle_c = 0$$



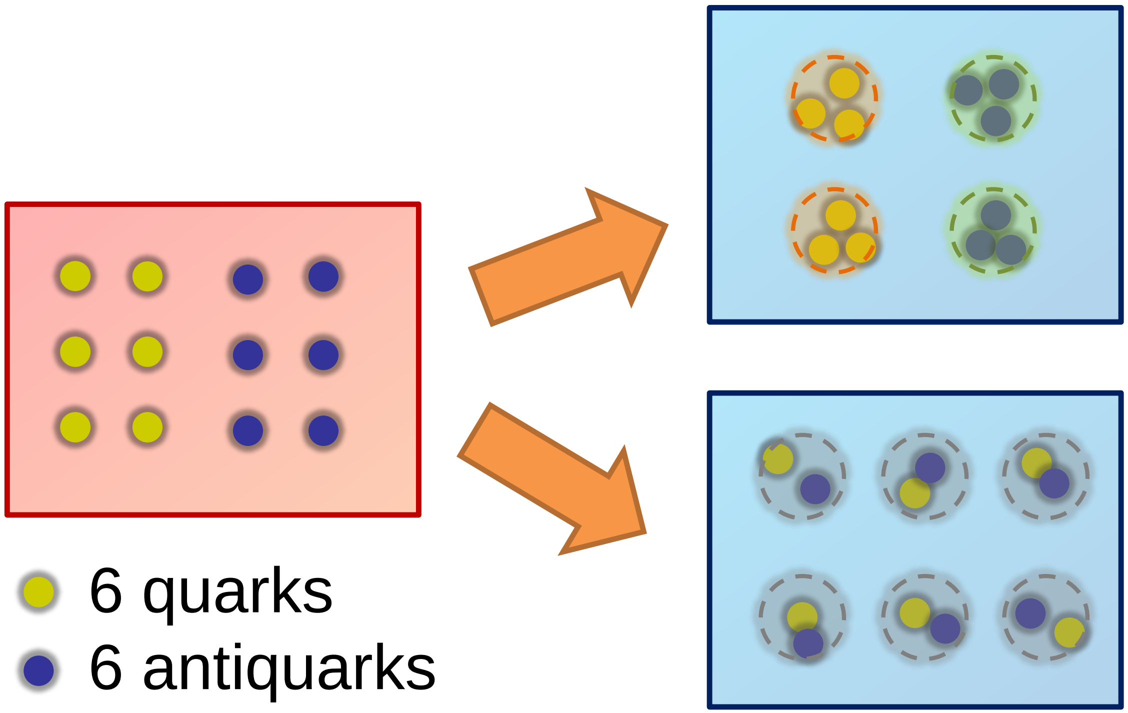


$$c = \frac{\langle Q_{(\mathrm{tot})}^2 \rangle_c}{\langle Q_{(\mathrm{tot})} \rangle}$$

parameter sensitive to hadronization

Total Charge Number

In recombination model,



$$N_B^{(\mathrm{net})} = 0 \qquad N_B^{(\mathrm{tot})} = 4$$

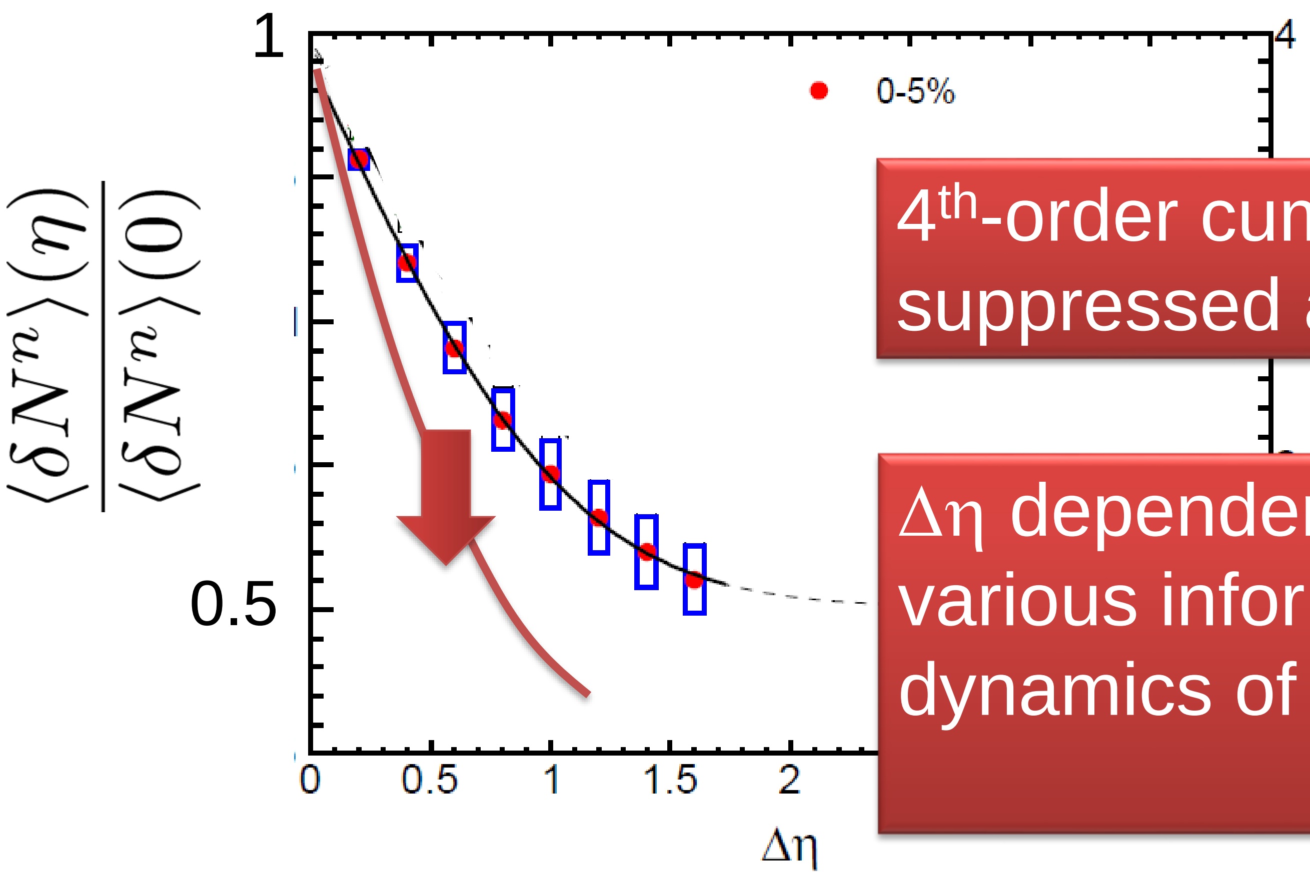
$$N_B^{(\mathrm{net})} = 0 \qquad N_B^{(\mathrm{tot})} = 0$$

- $N_B^{(\mathrm{tot})}$ can fluctuate, while $N_B^{(\mathrm{net})}$ does not.

$<\delta N_Q^4>$ @ LHC

Assumptions

- boost invariant system
- small fluctuations of CC at hadronization
- short correlation in hadronic stage



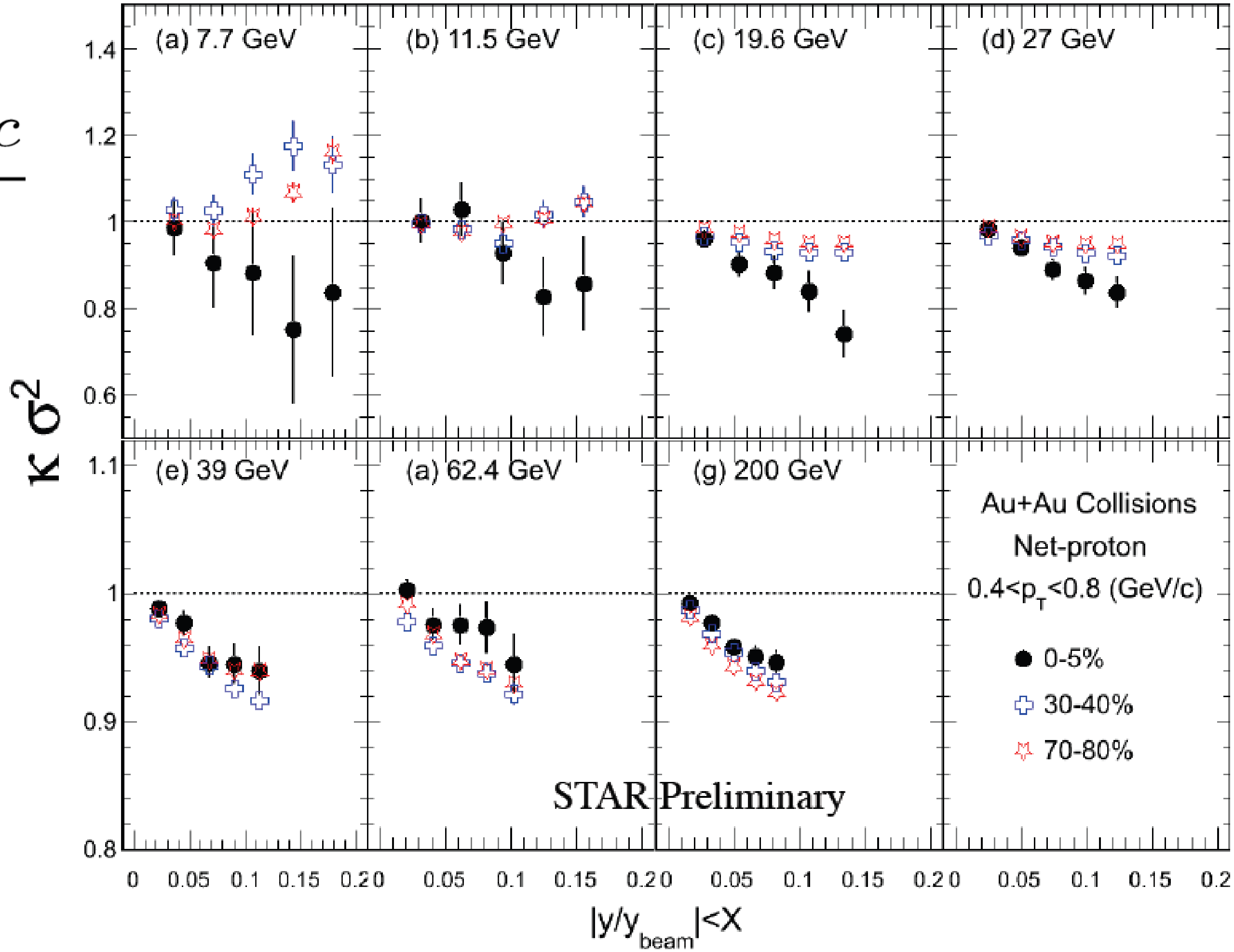
4^{th}-order cumulant will be suppressed at LHC energy!

$\Delta\eta$ dependences encode various information on the dynamics of HIC!

Δη Dependence at STAR

STAR, QM2012

$$\frac{\langle \delta N_p^4 \rangle_c}{\langle \delta N_p^2 \rangle}$$

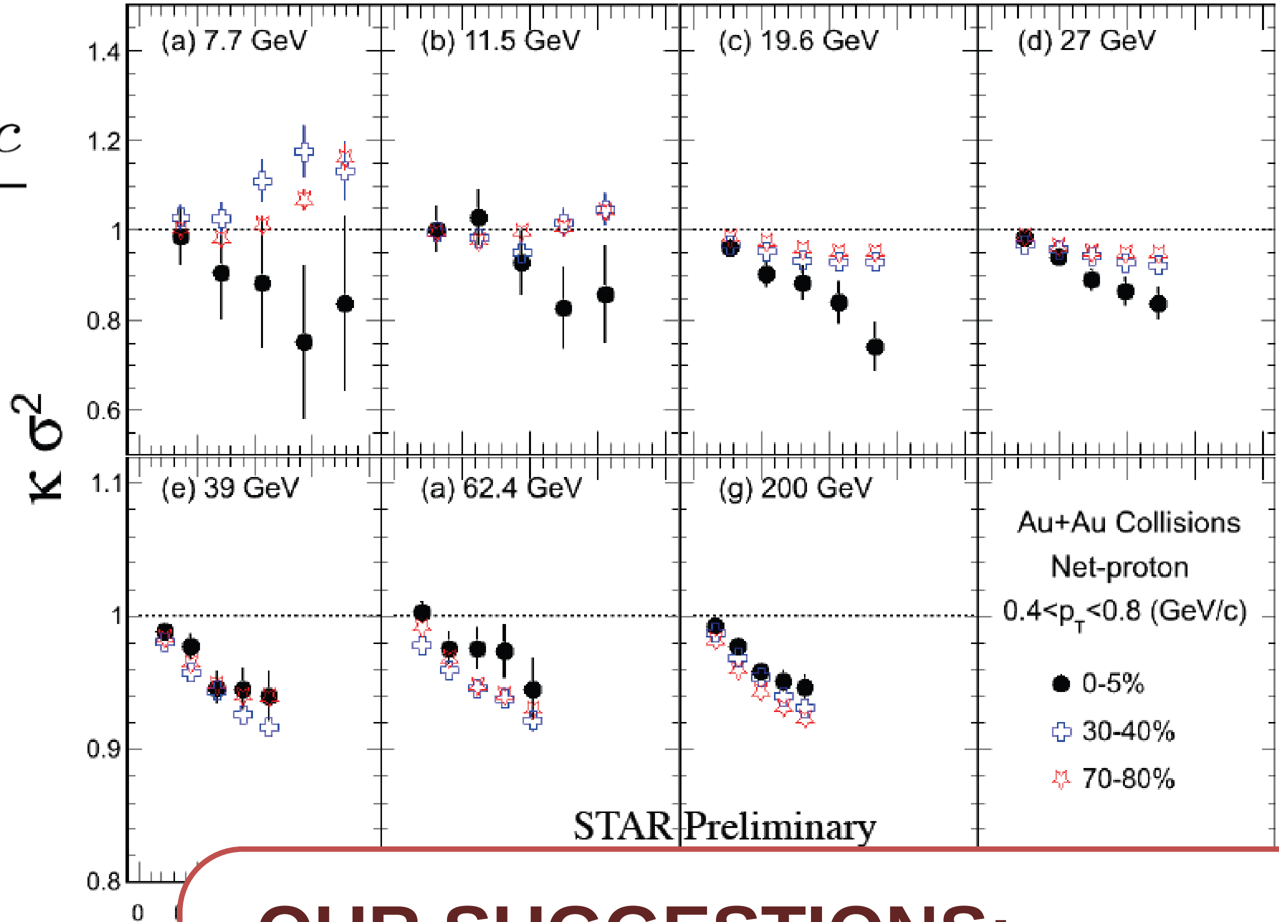


$\frac{\langle \delta N_p^4 \rangle_c}{\langle \delta N_p^2 \rangle}$ decreases as Δη becomes larger at RHIC energy.

Δη Dependence at STAR

STAR, QM2012

$$\frac{\langle \delta N_p^4 \rangle_c}{\langle \delta N_p^2 \rangle}$$



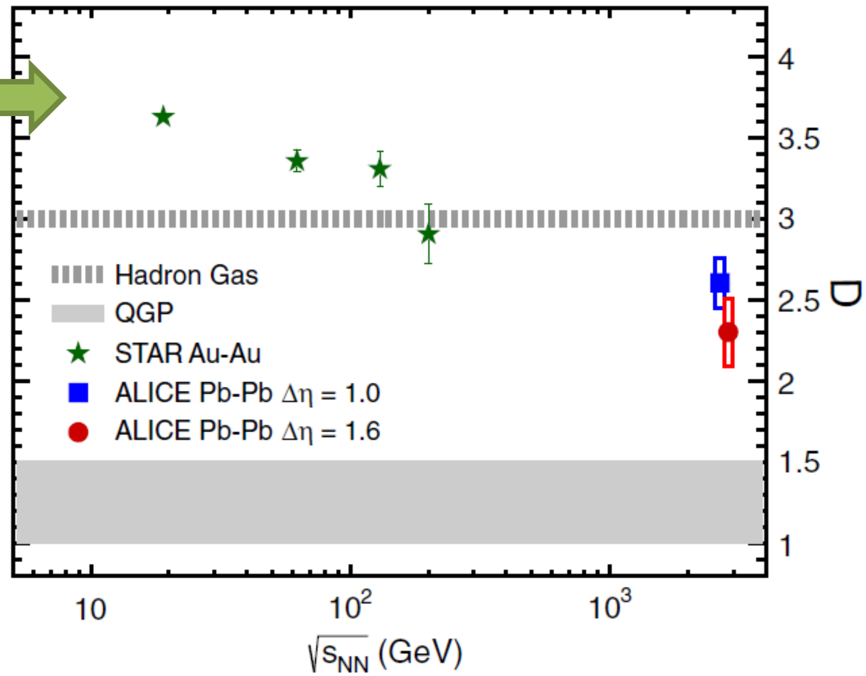
OUR SUGGESTIONS:

- Plot $<\delta N^2>$ and $<\delta N^4>$ separately
- Plot baryon number cumulants

Fluctuations @ J-PARC Energy

D-measure

No suppression of $\langle \delta Q^2 \rangle$ at low collision energy.



- Ingredients to be considered:
  - ✓ Bjorken expansion
  - ✓ pseudo rapidity vs coordinate-space rapidity
  - ✓ finite volume effect (global charge conservation)

Summary

Fluctuations in HIC are nonthermal!

Plenty of physics in $\Delta\eta$ dependences of various cumulants

$$\langle N_Q^2 \rangle_c, \ \langle N_B^2 \rangle_c, \ \langle N_Q^4 \rangle_c, \ \langle N_B^4 \rangle_c, \ \langle N_{ch}^2 \rangle_c, \cdots$$

Physical meanings of fluctuation obs. in experiments.

Diagnosing dynamics of HIC

- history of hot medium
- mechanism of hadronization
- diffusion constant

Summary

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Physical meanings of fluctuation obs. in experiments.

Diagnosing dynamics of HIC

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- mechanism of hadronization
- diffusion constant

Search of QCD Phase Structure in HIC

Open Questions & Future Work

- Why the primordial fluctuations are observed only at LHC, and not RHIC ?
- Extract more information on each stage of fireballs using fluctuations

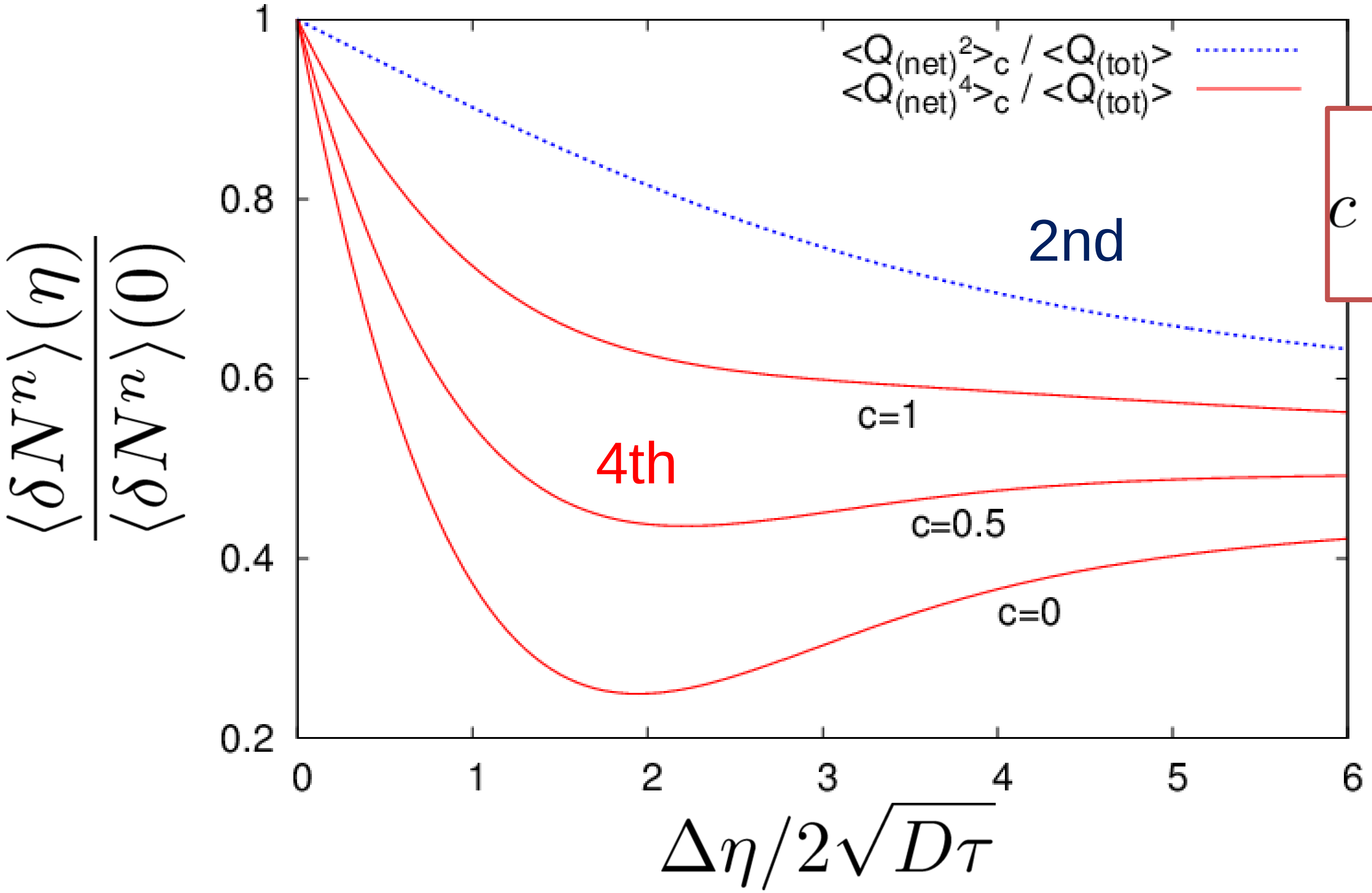
- Model refinement
  - Including the effects of
    nonzero correlation length / relaxation time
    global charge conservation

  - Non Poissonian system ← interaction of particles

$\Delta\eta$ Dependence at Freezeout

Initial fluctuations:

$$\langle \bar{Q}^2 \rangle_c = \langle \bar{Q}^4 \rangle_c = \langle \bar{Q}^2 Q_{(\mathrm{tot})} \rangle_c = 0.5 \langle Q_{(\mathrm{tot})} \rangle$$



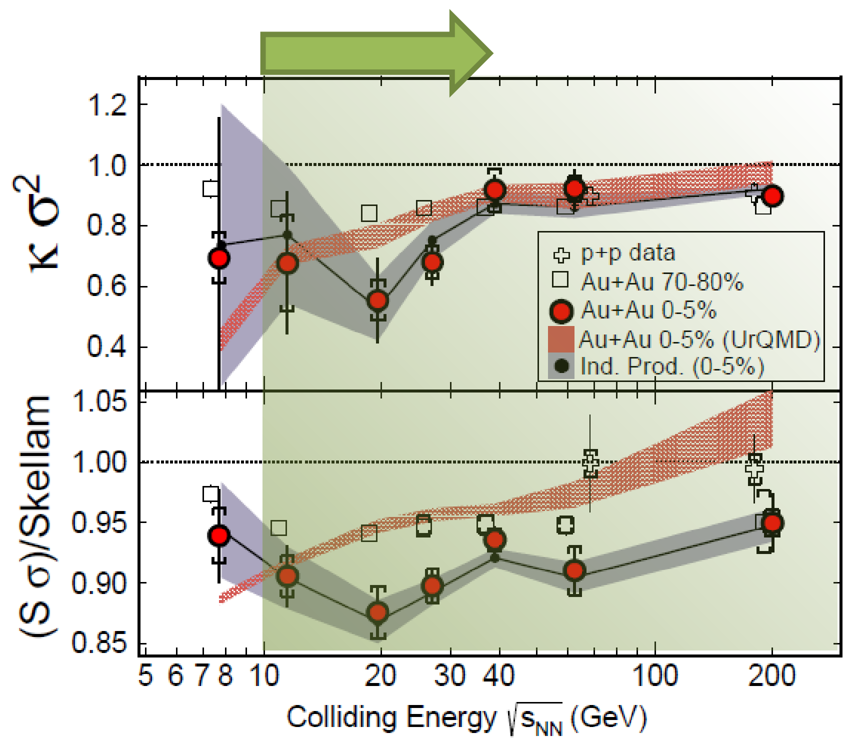
$$c = \frac{\langle Q^2_{(\mathrm{tot})} \rangle_c}{\langle Q_{(\mathrm{tot})} \rangle}$$

parameter sensitive to hadronization

Price of Baryon Number Reconstruction

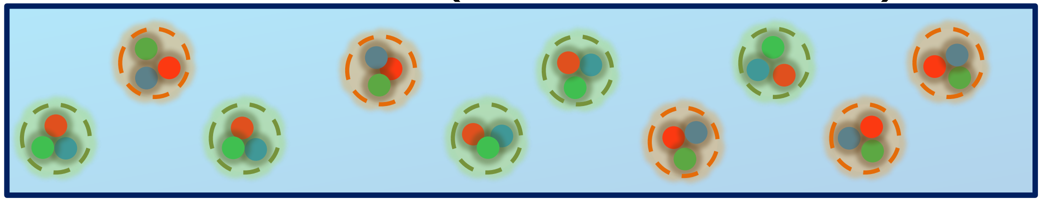
- Statistical error will be large.
- Applicable only to high-energy collisions.
  - Sufficiently large number of pions
    - $T_c > m_\pi$
  - Approximate isospin symmetry
    - $\frac{\langle N_p \rangle}{\langle N_N \rangle} = \frac{1}{2}$

$$\sqrt{s_{NN}} > 10\text{GeV}$$



Time Evolution in Hadronic Phase

Hadronization (initial condition)



- Boost invariance / infinitely long system
- Local equilibration / local correlation

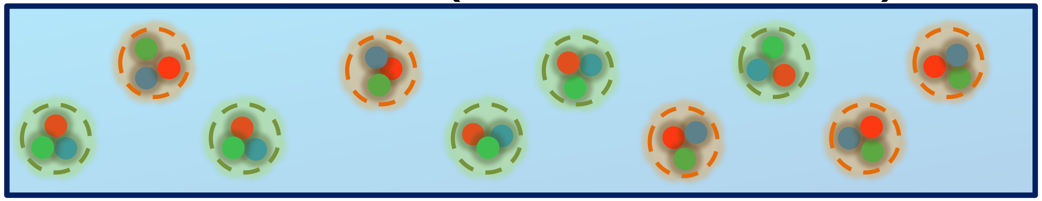
$\langle \bar{Q}^2 \rangle_c \quad \langle \bar{Q}^4 \rangle_c$ $\quad \langle \bar{Q}^2 Q_{(\mathrm{tot})} \rangle_c$ $\quad \langle Q_{(\mathrm{tot})}^2 \rangle_c$

suppression owing to local charge conservation

strongly dependent on hadronization mechanism

Time Evolution in Hadronic Phase

Hadronization (initial condition)



Time evolution via DME

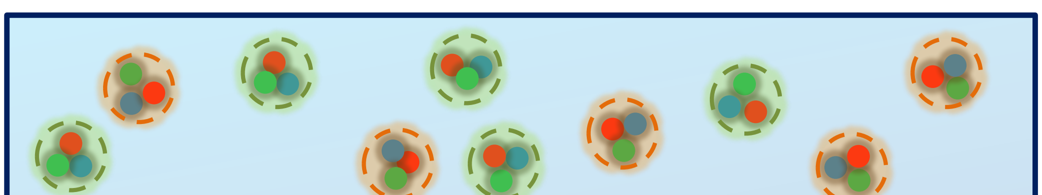
- Boost invariance / infinitely long system
- Local equilibration / local correlation

$\langle \bar{Q}^2 \rangle_c \quad \langle \bar{Q}^4 \rangle_c$ — suppression owing to local charge conservation

$\langle \bar{Q}^2 Q_{(\mathrm{tot})} \rangle_c$

$\langle Q^2_{(\mathrm{tot})} \rangle_c$ — strongly dependent on hadronization mechanism

Freezeout



Chemical Reaction 1

$$X \underset{k_2}{\overset{k_1}{\rightleftharpoons}} A$$

x: # of X
a: # of A (**fixed**)

Master eq.:

$$\frac{\partial}{\partial t} P(x,t) = k_2 a P(x-1,t) + k_1 (x+1) P(x+1,t) - (k_1 x + k_2 a) P(x,t)$$

Cumulants with fixed initial condition $P(x,0) = \delta_{x,N_0}$

$$\langle x(t) \rangle = N_0 e^{-k_1 t} + N_{eq}(1 - e^{-k_1 t})$$

$$\langle \delta x(t)^2 \rangle = N_0 (e^{-k_1 t} - e^{-2k_1 t}) + N_{eq}(1 - e^{-k_1 t})$$

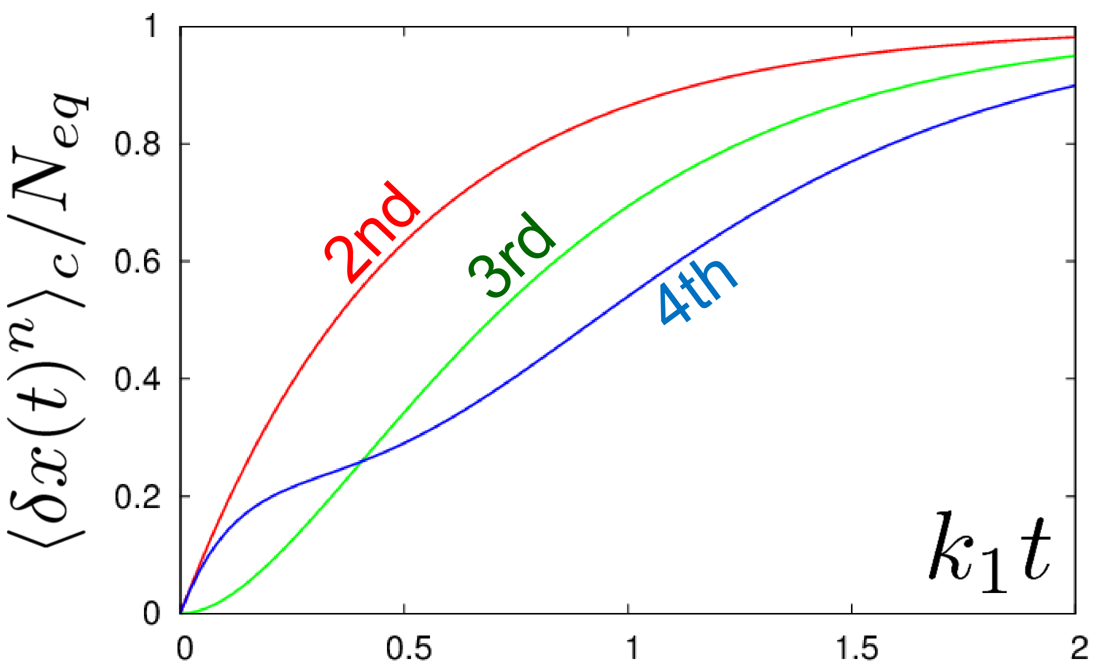
$$\langle \delta x(t)^3 \rangle = N_0 (e^{-k_1 t} - 3e^{-2k_1 t} + 2e^{-3k_1 t}) + N_{eq}(1 - e^{-k_1 t})$$

initial equilibrium

Chemical Reaction 2

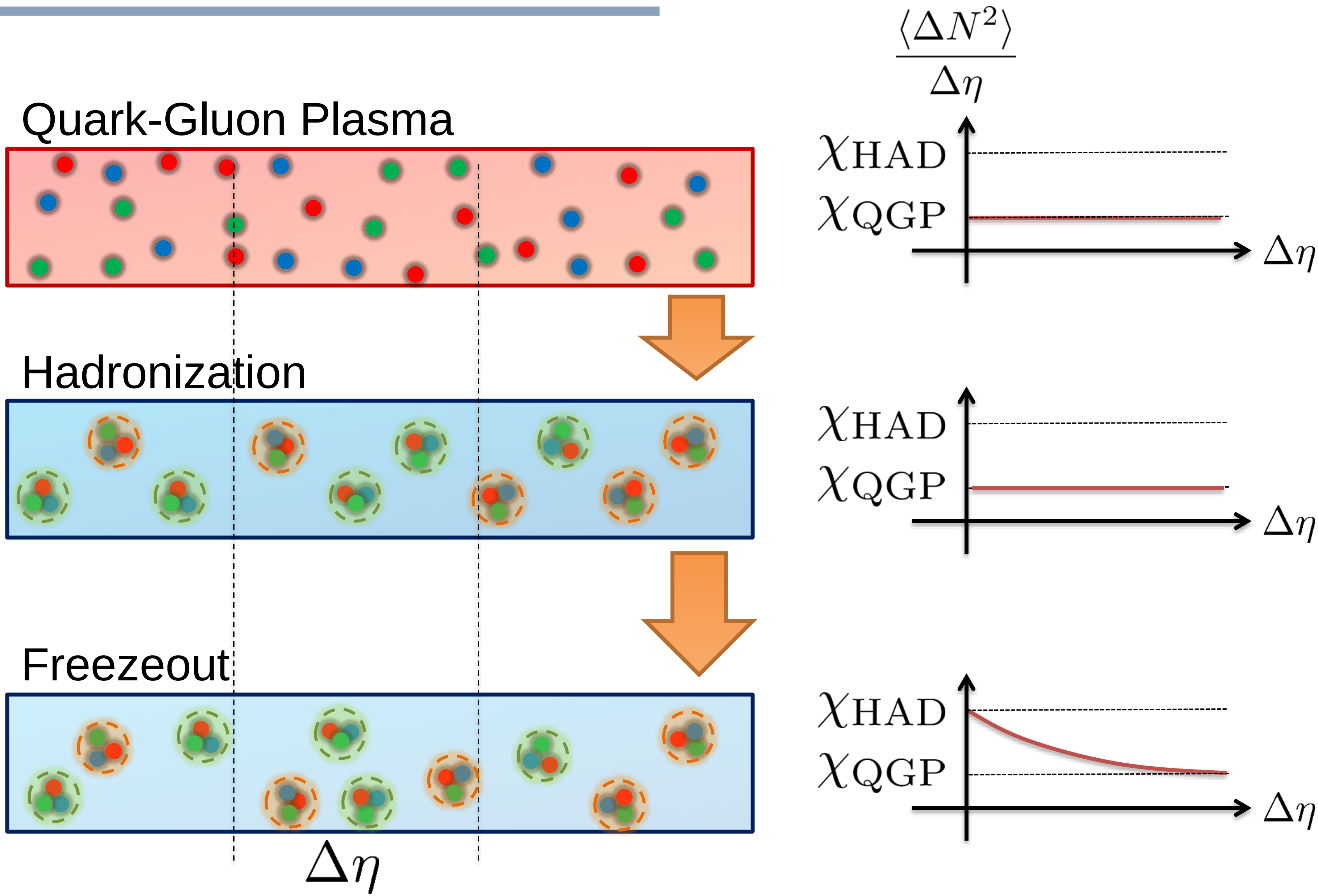
$$X \underset{k_2}{\overset{k_1}{\rightleftharpoons}} A$$

$$N_0 = N_{eq} \Rightarrow \begin{cases} \langle x(t) \rangle = N_{eq} \\ \langle \delta x(t)^2 \rangle = N_{eq}(1 - e^{-2k_1 t}) \\ \langle \delta x(t)^3 \rangle = N_{eq}(1 - 3e^{-2k_1 t} + 2e^{-3k_1 t}) \end{cases}$$



Higher-order cumulants grow slower.

Time Evolution in HIC



Hydrodynamic Fluctuations

Landau, Lifshitz, Statistical Mechaniqs II
Kapusta, Muller, Stephanov, 2012

Diffusion equation

$$\partial_\tau n = D\partial_\eta^2 n$$

Stochastic diffusion equation

$$\partial_\tau n = D\partial_\eta^2 n + \partial_\eta \xi(\eta, \tau)$$

Stochastic Force

determined by fluctuation-dissipation relation

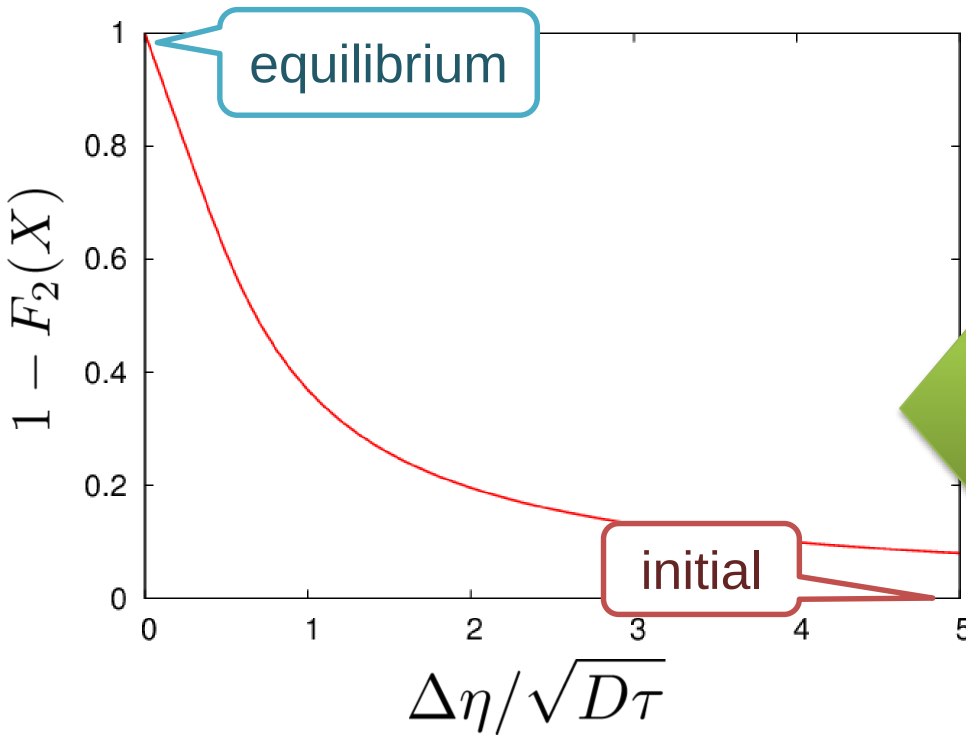
$\Delta\eta$ Dependence

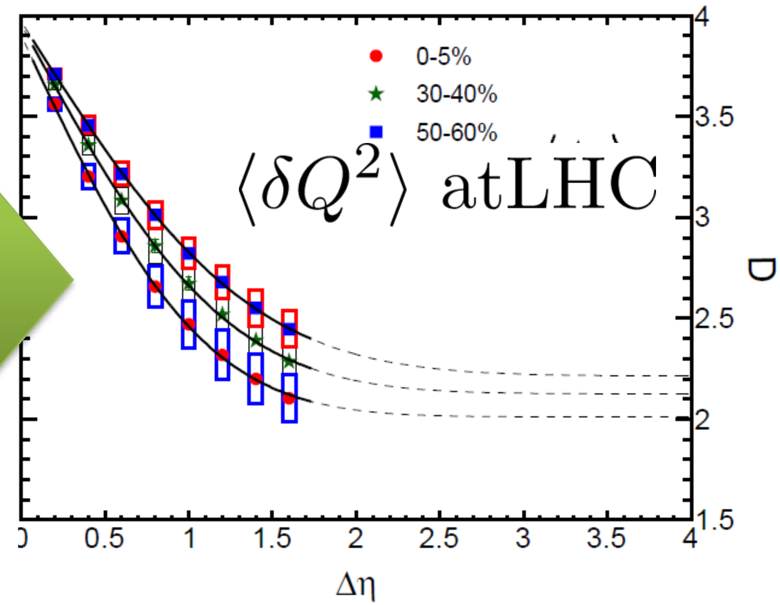
Shuryak, Stephanov, 2001

- Initial condition: $\langle \delta n(\eta_1, 0)\delta n(\eta_2, 0)\rangle = \sigma_2 \delta(\eta_1 - \eta_2)$
- Translational invariance

$$Q(\tau) = \int_0^{\Delta\eta} d\eta n(\eta, \tau)$$

$$\langle \delta Q(\tau)^2 \rangle = \underbrace{\sigma_2 F_2(X)}_{\text{initial}} + \underbrace{\chi_2(1 - F_2(X))}_{\text{equilibrium}}$$





Non-Gaussianity in Fluctuating Hydro?

It is **impossible** to directly extend the theory of hydro fluctuations to treat higher orders.

- No a priori extension of FD relations to higher orders
- **Theorem**

  Markov process + continuous variable

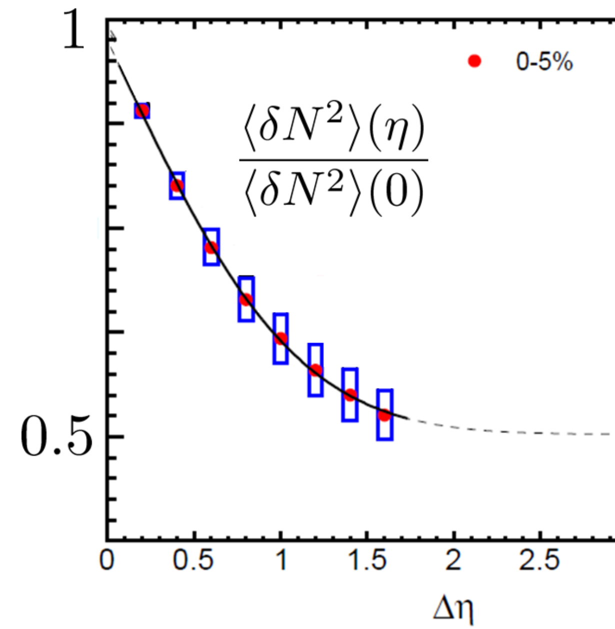
  → Gaussian random force

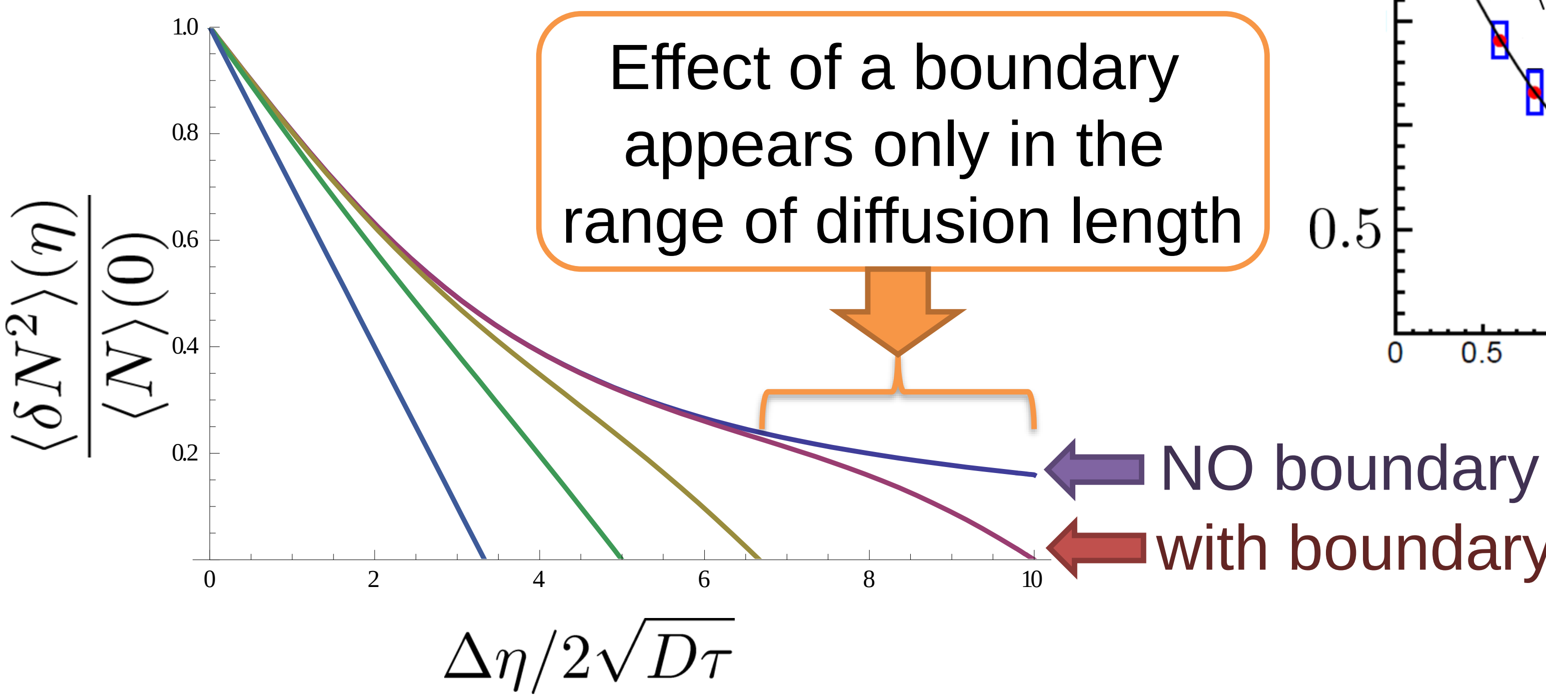
cf) Gardiner, "Stochastic Methods"

Global Charge Conservation

Sakaida,
poster session (3rd week)

Solve SDE or DME in a finite volume





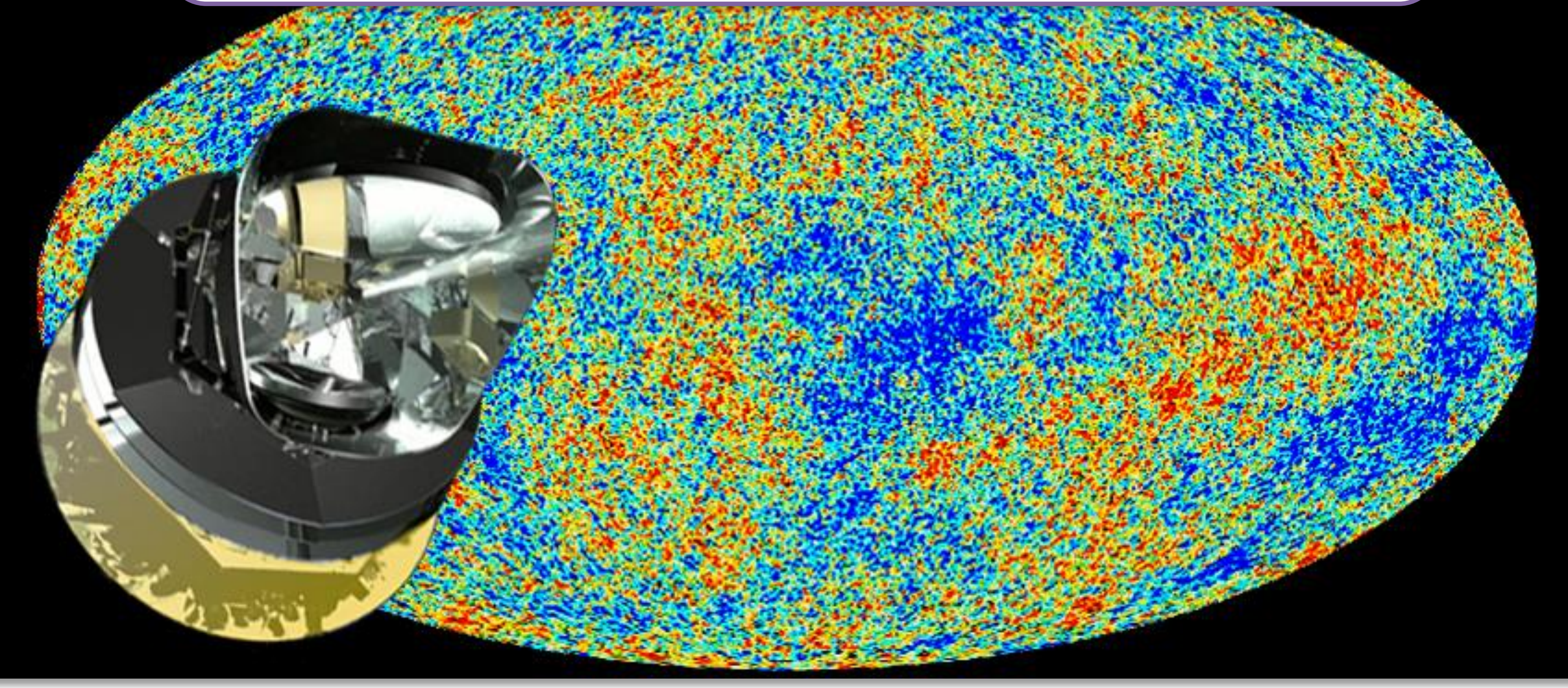
- Effect of GCC can be read off from Δη dependence.
- No GCC effect in ALICE experiments!

Non-Gaussianity

fluctuations (correlations)

$$\langle \delta n_1 \delta n_2 \rangle, \langle \delta n_1 \delta n_2 \delta n_3 \rangle, \langle \delta n_1 \delta n_2 \delta n_3 \delta n_4 \rangle_c, \cdots$$

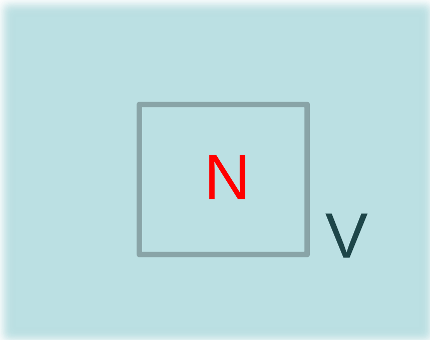
→ Non-Gaussianity

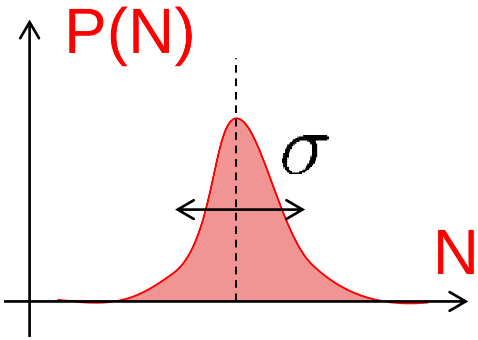


PLANCK : statistics insufficient to see non-Gaussianity…(2013)

Fluctuations

Observables in equilibrium are fluctuating.





Fluctuations

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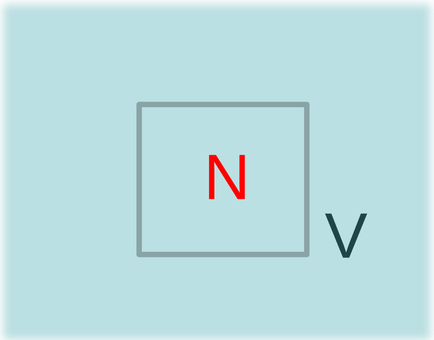
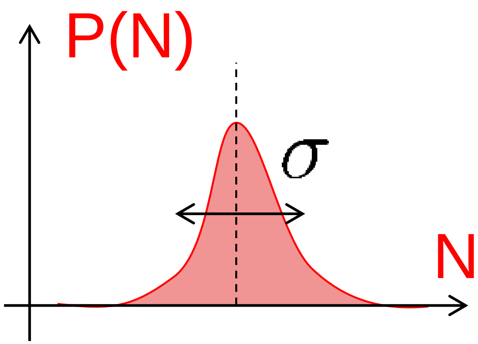




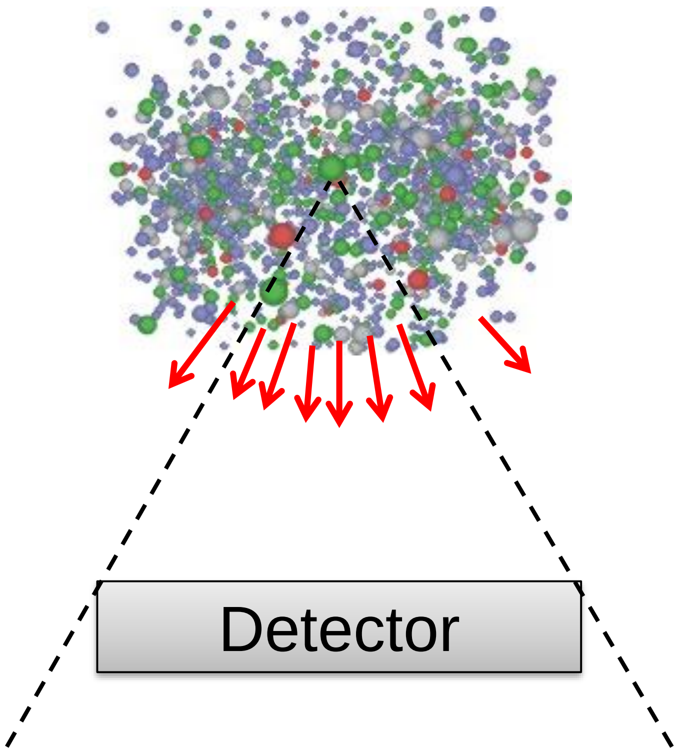
$\delta N = N - \langle N \rangle$

- Variance: $\langle \delta N^2 \rangle = V\chi_2 = \sigma^2$
- Skewness: $S = \dfrac{\langle \delta N^3 \rangle}{\sigma^3}$
- Kurtosis: $\kappa = \dfrac{\langle \delta N^4 \rangle - 3\langle \delta N^2 \rangle^2}{\chi_2 \sigma^2}$

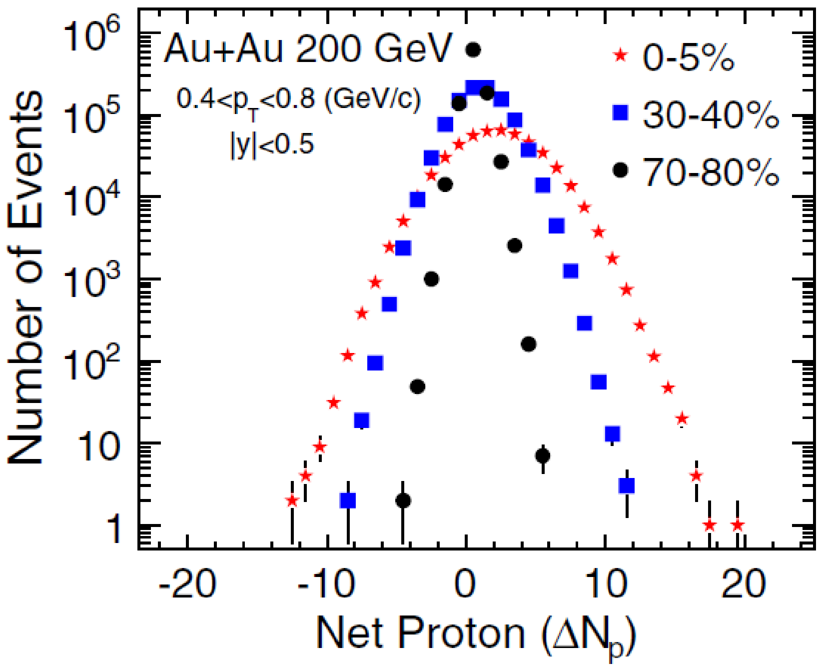
Non-Gaussianity

Event-by-Event Analysis @ HIC

Fluctuations can be measured by e-by-e analysis in HIC.



STAR, PRL**105** (2010)



$$\langle \delta N^2 \rangle, \langle \delta N^3 \rangle, \langle \delta N^4 \rangle_c, \cdots$$