Fluctuations of Conserved Charges as Probes of QCD Phase Structure

Masakiyo Kitazawa
(Osaka U.)

MK, Asakawa, PRC85, 021901C(2012); PRC86, 024904(2012)
MK, Asakawa, Ono, PLB728, 386(2014)

JHI2014, J-PARC, 17/Mar./2014
Beam-Energy Scan

Hadrons

Color SC
Beam-Energy Scan

J-PARC
FAIR, NICA

Hadrons

Color SC

Grand Canonical Ensemble

Au+Au

Cleymans
Andronic

STAR Preliminary
Fluctuations

- Fluctuations reflect properties of matter.
- Enhancement near the critical point
  Stephanov, Rajagopal, Shuryak ('98); Hatta, Stephanov ('02); Stephanov ('09); …
- Ratios between cumulants of conserved charges
  Asakawa, Heinz, Muller ('00); Jeon, Koch ('00); Ejiri, Karsch, Redlich ('06)
- Signs of higher order cumulants
  Asakawa, Ejiri, MK ('09); Friman, et al. ('11); Stephanov ('11)
Definite definition for operators

- as a Noether current
- calculable on any theory

ex: on the lattice

\[ 12 \frac{c^B_4}{c^B_2} = \frac{\langle B^4 \rangle - 3 \langle B^2 \rangle^2}{\langle B^2 \rangle} \]
Conserved Charges: Theoretical Advantage

- **Definite definition for operators**
  - as a Noether current
  - calculable on any theory
  ex: on the lattice

- **Simple thermodynamic relations**

\[
\langle \delta N_c^n \rangle = \frac{1}{VT^{n-1}} \frac{\partial^n \Omega}{\partial \mu_c^n}
\]

- Intuitive interpretation for the behaviors of cumulants

ex: \[
\langle \delta N_B^3 \rangle = \frac{1}{VT^2} \frac{\partial \langle \delta N_B^2 \rangle}{\partial \mu_B}
\]

Asakawa, Ejiri, MK, 2009
Recent Progress in Lattice Community

From LATTICE2013 presentations
Fluctuations

Free Boltzmann $\rightarrow$ Poisson

$$\langle \delta N^n \rangle_c = \langle N \rangle$$

$$\langle \delta N_q^n \rangle_c = \langle N_q \rangle$$

$3N_B = N_q$

$$\langle \delta N^n_B \rangle_c = \frac{1}{3^{n-1}} \langle N_B \rangle$$

$$\langle \delta N^n_B \rangle_c = \langle N_B \rangle$$
Fluctuations

\[ 3N_B = N_q \]

Free Boltzmann \( \rightarrow \) Poisson

\[ \langle \delta N^n \rangle_c = \langle N \rangle \]

\[ \langle \delta N^n_q \rangle_c = \langle N_q \rangle \]

\[ \langle \delta N^n_{B} \rangle_c = \frac{1}{3n-1} \langle N_B \rangle \]

\[ 12 \frac{c^B_4}{c^B_2} = \frac{\langle B^4 \rangle - 3 \langle B^2 \rangle^2}{\langle B^2 \rangle} \]

RBC-Bielefeld ’09

\begin{align*}
n_t &= 2+1, \quad m_\pi = 220 \text{ MeV} \\
n_t &= 2, \quad m_\pi = 770 \text{ MeV} \\
\text{Resonance gas} \\
\text{filled: } n_t=4 \\
\text{open: } n_t=6
\end{align*}
Proton # Cumulants @ STAR-BES

\[ \frac{C_4}{C_2} \]

\[ \frac{C_3}{C_1} = \frac{C_3/C_2}{\text{Poissonian}} \]

**CAUTION!**

proton number ≠ baryon number

MK, Asakawa, 2011;2012
Charge Fluctuation @ LHC

ALICE, PRL110,152301(2013)

\[ D = 4 \frac{\langle \delta N_Q^2 \rangle}{\langle N_Q^+ + N_Q^- \rangle} \]

- \( D \sim 3-4 \) Hadronic
- \( D \sim 1-1.5 \) Quark

significant suppression from hadronic value at LHC energy!

\( \langle \delta N_Q^2 \rangle \) is not equilibrated at freeze-out at LHC energy!
$\Delta \eta$ Dependence @ ALICE

ALICE
PRL 2013

rapidity window
Variation of a conserved charge is achieved only through diffusion.

The larger $\Delta \eta$, the slower diffusion.
$\Delta \eta$ Dependence @ ALICE

$\Delta \eta$ dependences of fluctuation observables encode history of the hot medium!
$\langle \delta N_B^2 \rangle$ and $\langle \delta N_p^2 \rangle @ LHC$?

$\langle \delta N_Q^2 \rangle$, $\langle \delta N_B^2 \rangle$, $\langle \delta N_p^2 \rangle$

should have different $\Delta \eta$ dependence.

Baryon # cumulants are experimentally observable! MK, Asakawa, 2011;2012
$<\delta N_B^2>$ and $<\delta N_p^2>$ at LHC?

$\langle \delta N_Q^2 \rangle$, $\langle \delta N_B^2 \rangle$, $\langle \delta N_p^2 \rangle$

should have different $\Delta \eta$ dependence.

Baryon # cumulants are experimentally observable! MK, Asakawa, 2011;2012
How does $\langle \delta N_Q^4 \rangle_c$ behave as a function of $\Delta \eta$? 

- suppression
- enhancement
Baryon vs Proton Number Fluctuations

MK, Asakawa, PRC85, 021901C (2012); PRC86, 024904 (2012)

\[
\frac{\langle \delta N_n^B \rangle_c}{\langle \delta N_m^B \rangle_c} \neq \frac{\langle \delta N_n^p \rangle_c}{\langle \delta N_m^p \rangle_c}
\]

\[
\langle \delta N_n^B \rangle_c \text{ are experimentally observable}
\]
Nucleon Isospin as Two Sides of a Coin

Nucleons have two isospin states.
Nucleon Isospin as Two Sides of a Coin

Nucleons have two isospin states.

Coins have two sides.

MK, Asakawa, 2012
Slot Machine Analogy

\[ P(N) = \sum \mathcal{N} + \mathcal{N} \]
Extreme Examples

Fixed # of coins

Constant probabilities
Reconstructing Total Coin Number

\[ P_{\circ} (N_{\circ}) = \sum P_{\circ} (N_{\circ}) B_{1/2}(N_{\circ}; N_{\circ}) \]

\[ B_p(k; N) = p^k (1 - p)^{N-k} k C_N \text{ : binomial distr. func.} \]
Reconstructing Baryon Number Cumulants

\[ \mathcal{P}(N_p, N_n, N_{\bar{p}}, N_{\bar{n}}) = F(N_N, N_{\bar{N}})B(N_p; N_N)B(N_{\bar{p}}; N_{\bar{N}}) \]

- for any phase space in the final state.

\[ \begin{align*}
\langle (\delta N_p^{(\text{net})})^2 \rangle &= \frac{1}{4} \langle (\delta N_N^{(\text{net})})^2 \rangle + \frac{1}{4} \langle N_N^{(\text{tot})} \rangle \\
\langle (\delta N_N^{(\text{net})})^2 \rangle &= 4 \langle (\delta N_p^{(\text{net})})^2 \rangle - 2 \langle N_p^{(\text{tot})} \rangle
\end{align*} \]

- for isospin symmetric medium
- effect of isospin density \(<10\%\) for \(\sqrt{s}>10\text{GeV}\)
- Similar formulas up to any order!
The decay rates and cross sections of the \( \Delta(1232) \) particle are shown in the diagram. The reactions are:

- \( p + \pi^+ \rightarrow \Delta^{++} \rightarrow p + \pi^+ \) (cross sections)
- \( p + \pi^0 \rightarrow \Delta^+ \rightarrow p + \pi^0 \) (decay rates)
- \( n + \pi^+ \rightarrow \Delta^0 \rightarrow n + \pi^0 \)
- \( p + \pi^- \rightarrow \Delta^0 \rightarrow p + \pi^- \)
- \( n + \pi^0 \rightarrow \Delta^- \rightarrow n + \pi^- \)
- \( n + \pi^- \rightarrow \Delta^- \rightarrow n + \pi^- \)
$\Delta(1232)$

Cross sections of $p$

$\Delta^+\rightarrow p + \pi^+$

$\Delta^0\rightarrow p + \pi^-$

$\Delta^-\rightarrow n + \pi^-$

$\Delta^+\rightarrow p + \pi^0$

$\Delta^0\rightarrow n + \pi^0$

$\Delta^-\rightarrow n + \pi^0$

$p + \pi \rightarrow \Delta^+,^0$

$\rightarrow p : n$

$= 5 : 4$

Lifetime to create $\Delta^+$ or $\Delta^0$

$\tau \simeq 4[\text{fm}]$

Hadronic stage

$\simeq 20[\text{fm}]$
Nucleons in Hadronic Phase

\[ m_\pi \approx T \ll m_N - \mu_N \]

\[ n_N \ll 1 \]

- rare NN collisions
- no quantum corr.

\[ n_N \ll n_\pi \]

- many pions
Difference btw Baryon and Proton Numbers

(1) \(N_B^{(\text{net})} = N_B - N_{\bar{B}}\) deviates from the equilibrium value.

(2) Boltzmann (Poisson) distribution for \(N_B, N_{\bar{B}}\).

\[
\begin{align*}
2\langle (\delta N_p^{(\text{net})})^2 \rangle & = \frac{1}{2} \langle (\delta N_B^{(\text{net})})^2 \rangle + \frac{1}{2} \langle (\delta N_B^{(\text{net})})^2 \rangle_{\text{free}} \\
2\langle (\delta N_p^{(\text{net})})^3 \rangle & = \frac{1}{4} \langle (\delta N_B^{(\text{net})})^3 \rangle + \frac{3}{4} \langle (\delta N_B^{(\text{net})})^3 \rangle_{\text{free}} \\
2\langle (\delta N_p^{(\text{net})})^4 \rangle_c & = \frac{1}{8} \langle (\delta N_B^{(\text{net})})^4 \rangle_c + \cdots
\end{align*}
\]

genuine info. Poissonian noise

Difference from Poisson (thermal) distribution is suppressed in proton number fluctuations.
Difference btw Baryon and Proton Numbers

(1) \( N_B^{(\text{net})} = N_B - N_{\bar{B}} \) deviates from the equilibrium value.

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& \quad + \cdots
\end{align*}
\]

\( \kappa \sigma^2 \)

from Poisson (thermal) distribution suppressed in proton number fluctuations.
Time Evolution of Higher Order Cumulants

MK, Asakawa, Ono, PLB728, 386 [arXiv:1307.2978]
How does $<\delta N_Q^4>_c$ behave as a function of $\Delta \eta$?

- suppression
- or
- enhancement
Hydrodynamic Fluctuations

**Stochastic diffusion equation**

\[ \partial_\tau n = D \partial_\eta^2 n + \partial_\eta \xi(\eta, \tau) \]

- Fluctuation of \( n \) is Gaussian in equilibrium
- Gaussian noise
- Markov (white noise) + continuity

**References**
- Landau, Lifshitz, Statistical Mechanics II
- Kapusta, Muller, Stephanov, 2012
- Stephanov, Shuryak, 2001
- cf) Gardiner, “Stochastic Methods”
How to Introduce Non-Gaussianity?

**Stochastic** diffusion equation

\[ \partial_\tau n = D \partial_\eta^2 n + \partial_\eta \xi(\eta, \tau) \]

- Choices to introduce non-Gaussianity in equil.:
  - \( n \) dependence of diffusion constant \( D(n) \)
  - colored noise
  - discretization of \( n \)
How to Introduce Non-Gaussianity?

**Stochastic diffusion equation**

\[ \partial_\tau n = D \partial_\eta^2 n + \partial_\eta \xi(\eta, \tau) \]

- Choices to introduce non-Gaussianity in equil.:
  - \( n \) dependence of diffusion constant \( D(n) \)
  - Colored noise
  - Discretization of \( n \) **our choice**

**REMARK:** Fluctuations measured in HIC are almost Poissonian.
Diffusion Master Equation

Divide spatial coordinate into discrete cells

\[ P(n) \]

\( x \)

\( n_{x-1} \quad n_x \quad n_{x+1} \quad n_{x+2} \quad \ldots \)

Probability transitions are indicated by \( \gamma \) and \( \alpha \).
Diffusion Master Equation

Divide spatial coordinate into discrete cells

Master Equation for $P(n)$

$$\frac{\partial}{\partial t} P(n) = \gamma \sum_x [(n_x + 1) \{P(n + e_x - e_{x+1}) + P(n + e_x - e_{x-1})\}$$

$$- 2n_x P(n)]$$

Solve the DME **exactly**, and take $a \to 0$ limit

No approx., ex. van Kampen’s system size expansion
Baryons in Hadronic Phase

Baryons behave like Brownian pollens in water

\[ p, \bar{p}, n, \bar{n}, \Delta(1232) \]

mesons

baryons
Solution of DME in $a \to 0$ Limit

1st order (deterministic) $\langle n \rangle$
- consistent with diffusion equation with $D = \gamma a^2$
- Continuum limit with fixed $D = \gamma a^2$

2nd order $\langle \delta n^2 \rangle$
- consistent with stochastic diffusion eq.
  (for sufficiently smooth initial conditions)

Nontrivial results for non-Gaussian fluctuations

Shuryak, Stephanov, 2001
Net Charge Number

Prepare 2 species of (non-interacting) particles

\[ \bar{Q}(\tau) = \int_0^{\Delta \eta} d\eta \left( n_1(\eta, \tau) - n_2(\eta, \tau) \right) \]

Let us investigate

\[ \langle \bar{Q}^2 \rangle_c \quad \langle \bar{Q}^4 \rangle_c \]

at freezeout time $t$
Dependence at Freezeout

Initial fluctuations:

$$\langle \bar{Q}^2 \rangle_c = \langle \bar{Q}^4 \rangle_c = \langle \bar{Q}^2 Q_{(tot)} \rangle_c = 0$$

\[
c = \frac{\langle Q^2_{(tot)} \rangle_c}{\langle Q_{(tot)} \rangle_c}
\]

parameter sensitive to hadronization
In recombination model,

- $N_B^{(\text{net})} = 0$
- $N_B^{(\text{tot})} = 4$
- $N_B^{(\text{net})} = 0$
- $N_B^{(\text{tot})} = 0$

- $N_B^{(\text{tot})}$ can fluctuate, while $N_B^{(\text{net})}$ does not.
\[ \langle \delta N_Q^4 \rangle \text{ @ LHC} \]

Assumptions:
- boost invariant system
- small fluctuations of CC at hadronization
- short correlation in hadronic stage

4\textsuperscript{th}-order cumulant will be suppressed at LHC energy!

\( \Delta \eta \) dependences encode various information on the dynamics of HIC!
$\frac{\langle \delta N_{p}^{4} \rangle_c}{\langle \delta N_{p}^{2} \rangle}$ decreases as $\Delta \eta$ becomes larger at RHIC energy.
OUR SUGGESTIONS:

- Plot $\langle \delta N^2 \rangle$ and $\langle \delta N^4 \rangle$ separately
- Plot baryon number cumulants
Fluctuations @ J-PARC Energy

No suppression of $\langle \delta Q^2 \rangle$ at low collision energy.

Ingredients to be considered:

- Bjorken expansion
- Pseudo rapidity vs coordinate-space rapidity
- Finite volume effect (global charge conservation)
Summary

Diagnosing dynamics of HIC

Fluctuations in HIC are nonthermal!

Plenty of physics in $\Delta \eta$ dependences of various cumulants

$\langle N_Q^2 \rangle_c, \langle N_B^2 \rangle_c, \langle N_Q^4 \rangle_c, \langle N_B^4 \rangle_c,$

$\langle N_{ch}^2 \rangle_c, \cdots$

Physical meanings of fluctuation obs. in experiments.

Diagnosing dynamics of HIC

- history of hot medium
- mechanism of hadronization
- diffusion constant
Summary

Fluctuations in HIC are nonthermal!

Plenty of physics in $\Delta \eta$ dependences of various cumulants

Physical meanings of fluctuation obs. in experiments.

Diagnosing dynamics of HIC
- history of hot medium
- mechanism of hadronization
- diffusion constant

Search of QCD Phase Structure in HIC

$\langle N_Q^2 \rangle_c, \langle N_B^2 \rangle_c, \langle N_Q^4 \rangle_c, \langle N_B^4 \rangle_c,$
$\langle N_{ch}^2 \rangle_c, \ldots$
Open Questions & Future Work

- Why the primordial fluctuations are observed only at LHC, and not RHIC?
- Extract more information on each stage of fireballs using fluctuations

Model refinement
- Including the effects of nonzero correlation length / relaxation time
- Global charge conservation

- Non Poissonian system ← interaction of particles
Dependence at Freezeout

Initial fluctuations:

\[ \langle \bar{Q}^2 \rangle_c = \langle \bar{Q}^4 \rangle_c = \langle \bar{Q}^2 Q_{(tot)} \rangle_c = 0.5 \langle Q_{(tot)} \rangle_c \]

\[ c = \frac{\langle Q_{(net)}^2 \rangle_c}{\langle Q_{(tot)} \rangle_c} \]

Parameter sensitive to hadronization
- Statistical error will be large.
- Applicable only to high-energy collisions.
  - Sufficiently large number of pions
    - $T_c > m_\pi$
  - Approximate isospin symmetry
    - $\frac{\langle N_p \rangle}{\langle N_N \rangle} = \frac{1}{2}$

\[
\sqrt{s_{NN}} > 10\text{GeV}
\]
Time Evolution in Hadronic Phase

Hadronization (initial condition)

- Boost invariance / infinitely long system
- Local equilibration / local correlation

\[ \langle Q^2 \rangle_c \quad \langle Q^4 \rangle_c \quad \langle Q^2 Q_{(tot)} \rangle_c \quad \langle Q_{(tot)}^2 \rangle_c \]

suppression owing to local charge conservation

strongly dependent on hadronization mechanism
Time Evolution in Hadronic Phase

Hadronization (initial condition)

- Boost invariance / infinitely long system
- Local equilibration / local correlation

\( \langle \bar{Q}^2 \rangle_c \quad \langle \bar{Q}^4 \rangle_c \quad \langle \bar{Q}^2 Q_{(tot)} \rangle_c \quad \langle Q^2_{(tot)} \rangle_c \)

suppression owing to local charge conservation

strongly dependent on hadronization mechanism

Freezeout
Chemical Reaction 1

\[ \begin{array}{c}
\begin{array}{c}
X \\
\xrightarrow{k_1}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
A \\
k_2
\end{array}
\end{array} \]

x: # of X
a: # of A (fixed)

Master eq.:
\[ \frac{\partial}{\partial t} P(x, t) = k_2 a P(x - 1, t) + k_1 (x + 1) P(x + 1, t) - (k_1 x + k_2 a) P(x, t) \]

Cumulants with fixed initial condition
\[ P(x, 0) = \delta_{x,N_0} \]

\[ \langle x(t) \rangle = N_0 e^{-k_1 t} + N_{eq}(1 - e^{-k_1 t}) \]

\[ \langle \delta x(t)^2 \rangle = N_0 (e^{-k_1 t} - e^{-2k_1 t}) + N_{eq}(1 - e^{-k_1 t}) \]

\[ \langle \delta x(t)^3 \rangle = N_0 (e^{-k_1 t} - 3e^{-2k_1 t} + 2e^{-3k_1 t}) + N_{eq}(1 - e^{-k_1 t}) \]
Chemical Reaction 2

\[
\begin{align*}
X & \xrightarrow{k_1} A \\
X & \xrightarrow{k_2} B
\end{align*}
\]

\[
\begin{align*}
N_0 &= N_{eq} \\
\langle x(t) \rangle &= N_{eq} \\
\langle \delta x(t)^2 \rangle &= N_{eq}(1 - e^{-2k_1 t}) \\
\langle \delta x(t)^3 \rangle &= N_{eq}(1 - 3e^{-2k_1 t} + 2e^{-3k_1 t})
\end{align*}
\]

Higher-order cumulants grow slower.
Time Evolution in HIC

Quark-Gluon Plasma

Hadronization

Freezeout

\[ \langle \Delta N^2 \rangle \]
\[ \Delta \eta \]

\[ \chi_{\text{HAD}} \]
\[ \chi_{\text{QGP}} \]

\[ \Delta \eta \]
Hydrodynamic Fluctuations

Landau, Lifshitz, Statistical Mechaniqs II
Kapusta, Muller, Stephanov, 2012

Diffusion equation

\[ \partial_\tau n = D \partial^2_\eta n \]

Stochastic diffusion equation

\[ \partial_\tau n = D \partial^2_\eta n + \partial_\eta \xi(\eta, \tau) \]

Stochastic Force
determined by fluctuation-dissipation relation
**Δη Dependence**

- **Initial condition:** \( \langle \delta n(\eta_1, 0) \delta n(\eta_2, 0) \rangle = \sigma_2 \delta(\eta_1 - \eta_2) \)
- **Translational invariance**

\[
Q(\tau) = \int_0^{\Delta \eta} d\eta n(\eta, \tau)
\]

\[
\langle \delta Q(\tau)^2 \rangle = \sigma_2 F_2(X) + \chi_2 (1 - F_2(X))
\]

- **Initial**
- **Equilibrium**

---

Equilibrium

\[
1 - F_2(X)
\]

---

\[
\langle \delta Q^2 \rangle \text{ at LHC}
\]
It is impossible to directly extend the theory of hydro fluctuations to treat higher orders.

- No a priori extension of FD relations to higher orders

- Theorem: Markov process + continuous variable → Gaussian random force

(cf) Gardiner, “Stochastic Methods”
Global Charge Conservation

Solve SDE or DME in a finite volume

Effect of a boundary appears only in the range of diffusion length

Effect of GCC can be read off from $\Delta \eta$ dependence.
No GCC effect in ALICE experiments!
Non-Gaussianity

fluctuations (correlations)

$\langle \delta n_1 \delta n_2 \rangle, \langle \delta n_1 \delta n_2 \delta n_3 \rangle, \langle \delta n_1 \delta n_2 \delta n_3 \delta n_4 \rangle_c, \cdots$

Non-Gaussianity

PLANCK: statistics insufficient to see non-Gaussianity…(2013)
Observables in equilibrium are fluctuating.
Fluctuations

Observables in equilibrium are fluctuating.

- Variance: $\langle \delta N^2 \rangle = V \chi_2 = \sigma^2$
- Skewness: $S = \frac{\langle \delta N^3 \rangle}{\sigma^3}$
- Kurtosis: $\kappa = \frac{\langle \delta N^4 \rangle - 3\langle \delta N^2 \rangle^2}{\chi_2 \sigma^2}$

$\delta N = N - \langle N \rangle$

Non-Gaussianity
Fluctuations can be measured by e-by-e analysis in HIC.

\[ \langle \delta N^2 \rangle, \langle \delta N^3 \rangle, \langle \delta N^4 \rangle_c, \ldots \]