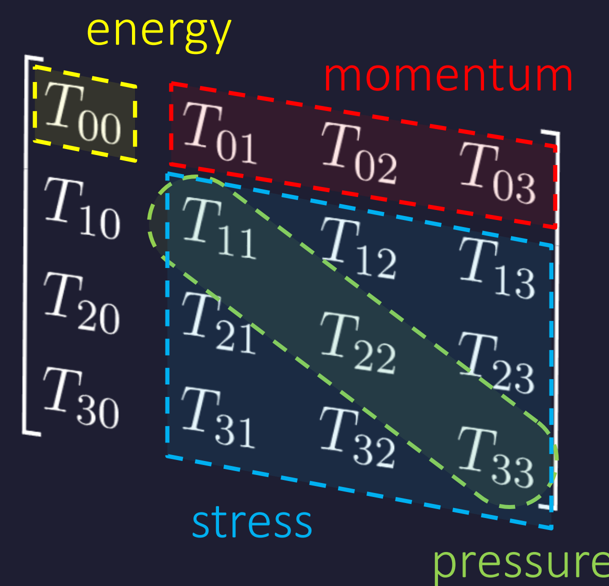
Thermodynamics of SU(3) gauge theory from gradient flow

Masakiyo Kitazawa

(Osaka U.)

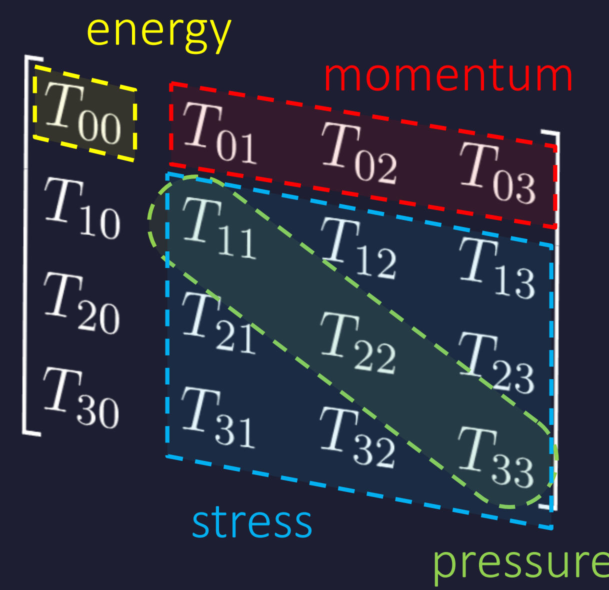
Asakawa, Hatsuda, Itou, MK, Suzuki (FlowQCD Collab.),
arXiv:1312.7492[hep-lat]

$$T_{\mu\nu}$$



Noether current / generator of space-time translation

$$T_{\mu\nu}$$



Noether current / generator of space-time translation

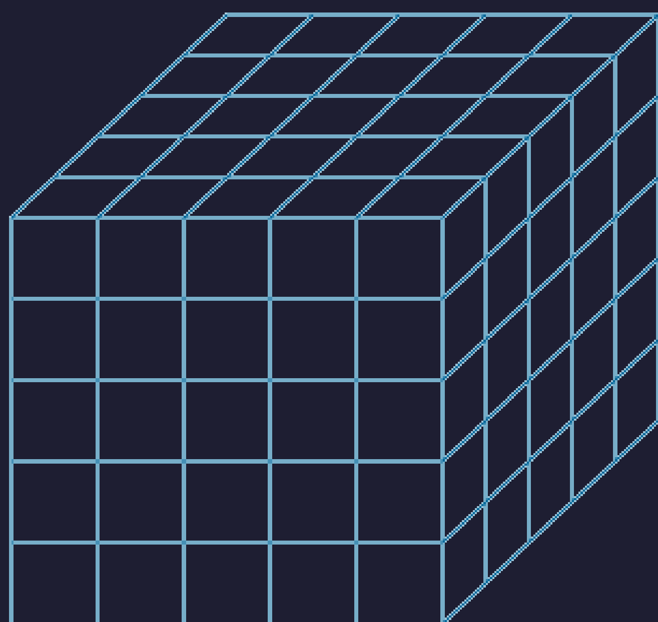
Einstein Equation

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}$$

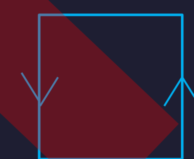
Hydrodynamic Eq.

$$\partial_\mu T_{\mu\nu} = 0$$

The definition of $T_{\mu\nu}$ on the lattice is nontrivial...



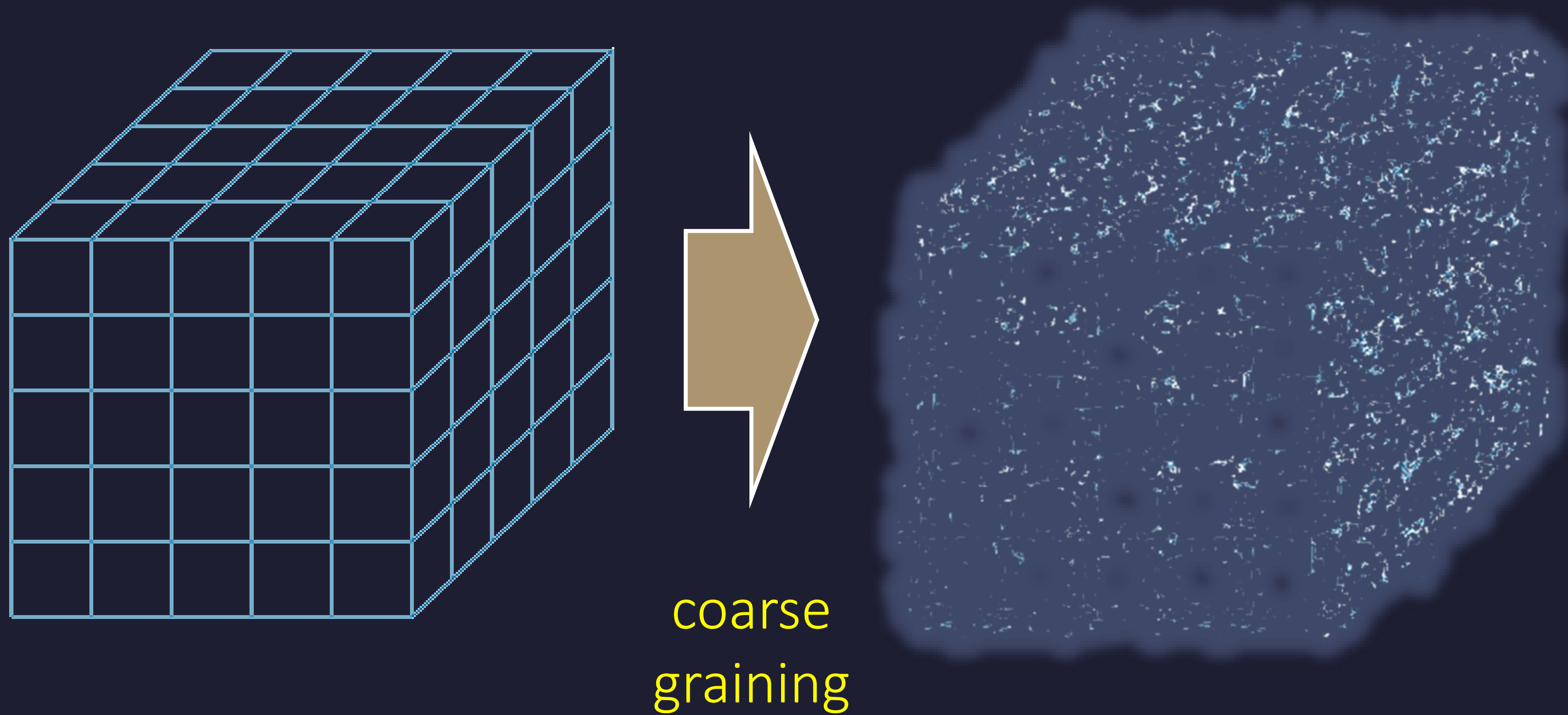
$$T_{\mu\nu} = F_{\mu\rho}F_{\nu\rho} - \frac{1}{4}\delta_{\mu\nu}FF$$

$$F_{\mu\nu} =$$ 

... because of the lack of translational symmetry

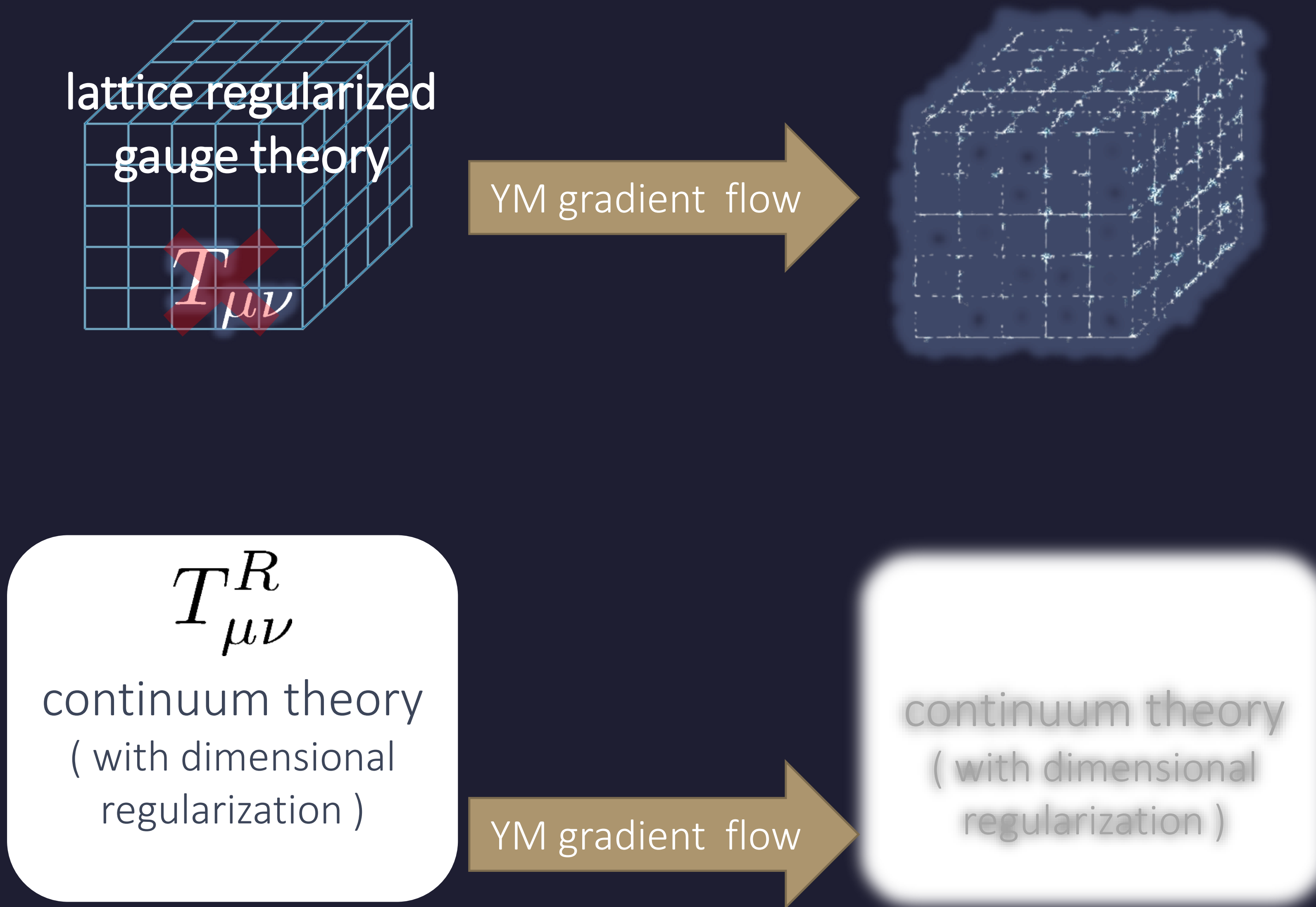
cf) Caracciolo+ (1989)

Rough Idea

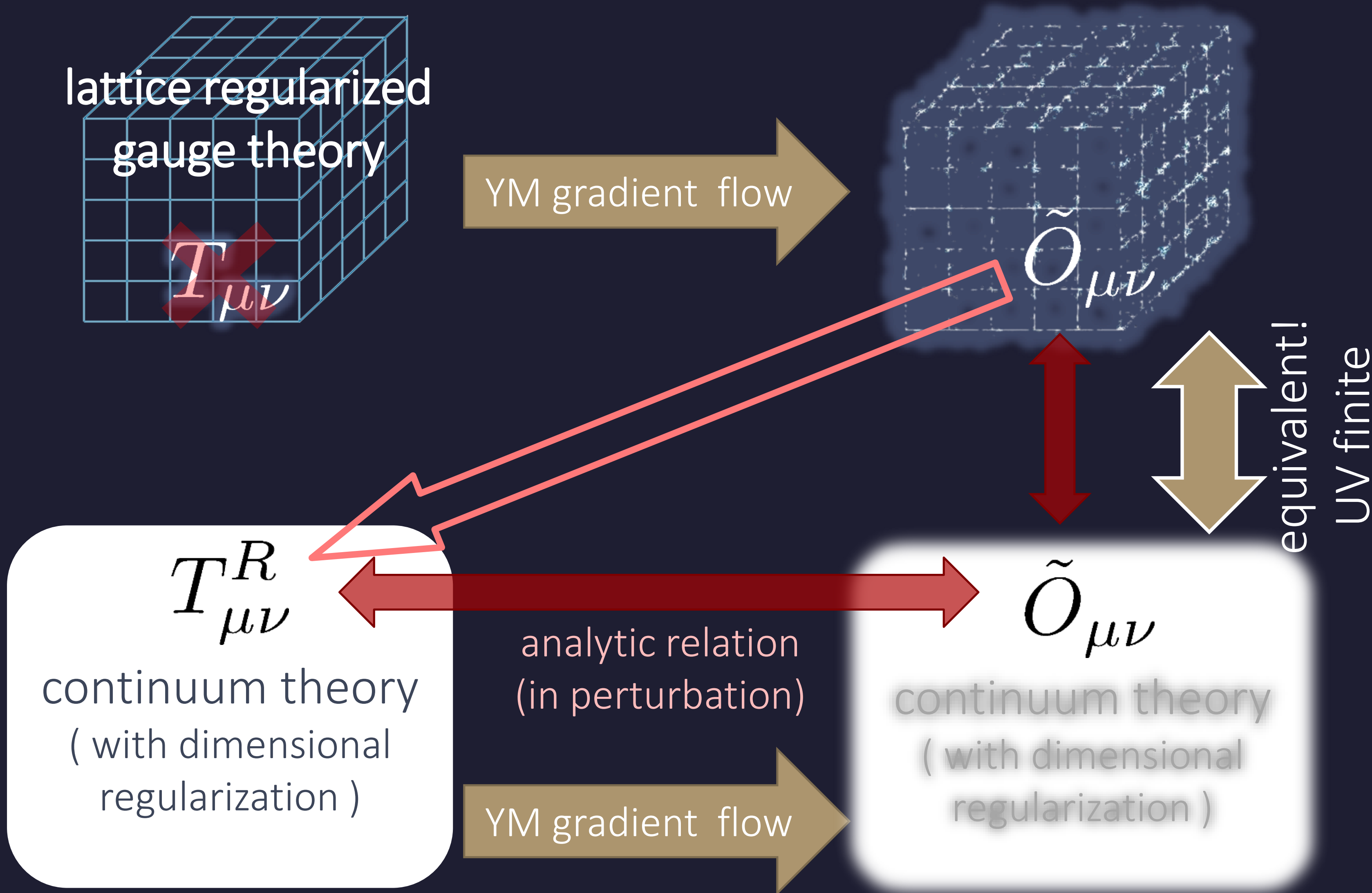


no translational invariance

translational symmetry is recovered!



Luescher, Weiss (2012)
Suzuki (2013)



Luescher, Weiss (2012)
Suzuki (2013)

What we can measure with $T_{\mu\nu}$

$\langle T_{\mu\nu} \rangle$ ⇒
- bulk thermodynamics (energy density, pressure)

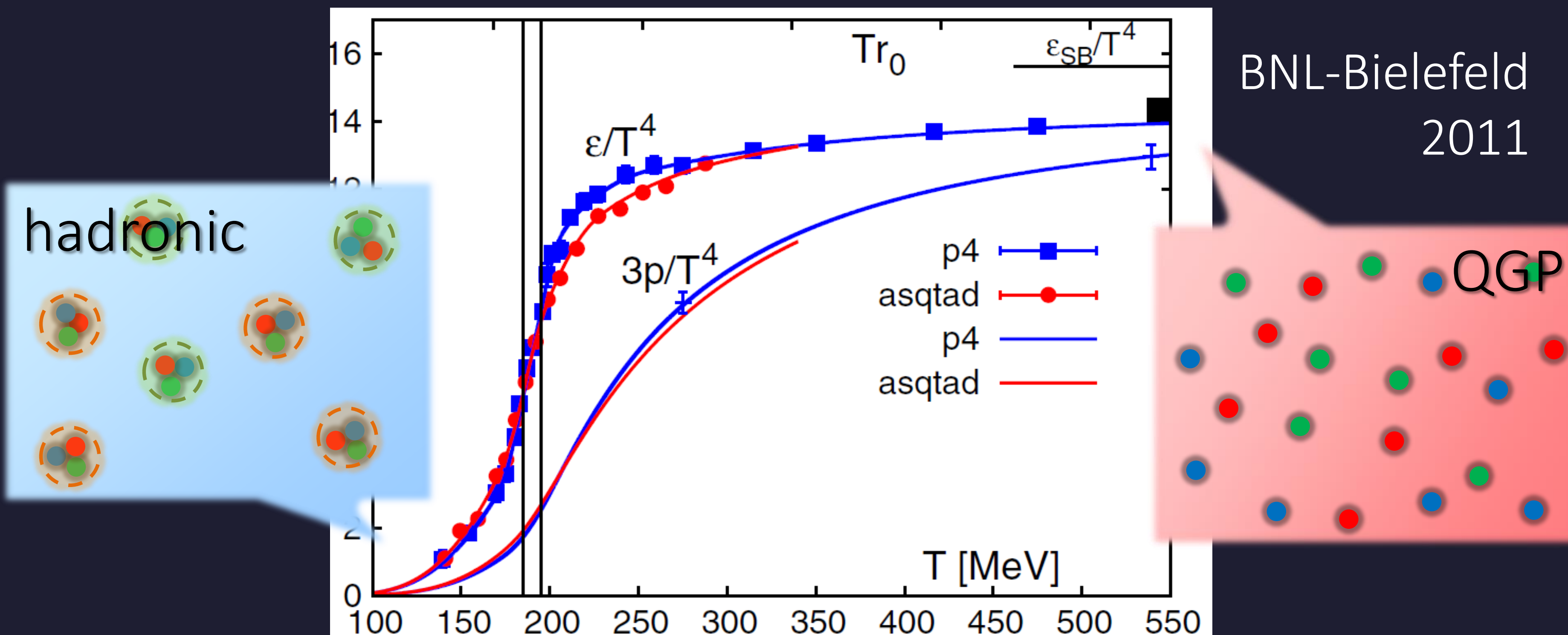
$\langle T_{\mu\nu}(x) T_{\mu\nu}(0) \rangle$ ⇒
- correlation functions
- viscosity, thermal excitation
- vacuum structure?

$\langle (\delta T_{\mu\nu})^n \rangle$ ⇒
- fluctuations, specific heat
- non-Gaussian fluctuations, etc.

Asakawa, Ejiri, MK (2009)

pink chars:
T>0 physics

QCD EoS
(Energy Density, Pressure)



- Rapid increase of ε/T^4 around T=150-200 MeV
- Crossover transition
- Rapid but smooth change of medium from hadronic to QGP-like

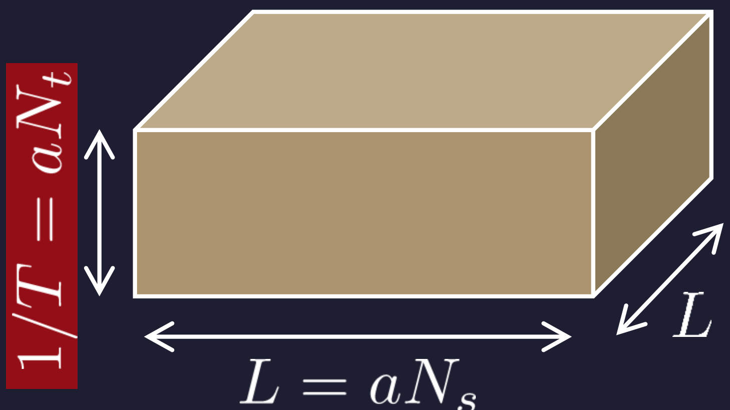
QCD Thermodynamics

$$Z(T) = \mathrm{Tr}\left[e^{-H/T}\right]$$

$$= \int \mathcal{D}A \exp\left[-\int_0^{1/T} d\tau \int_V d^3x \mathcal{L}_E\right]$$

Thermodynamic relations

$$\varepsilon = \frac{T^2}{V}\frac{\partial \ln Z}{\partial T} \qquad p = T\frac{\partial \ln Z}{\partial V}$$



How do we take T and V derivatives?

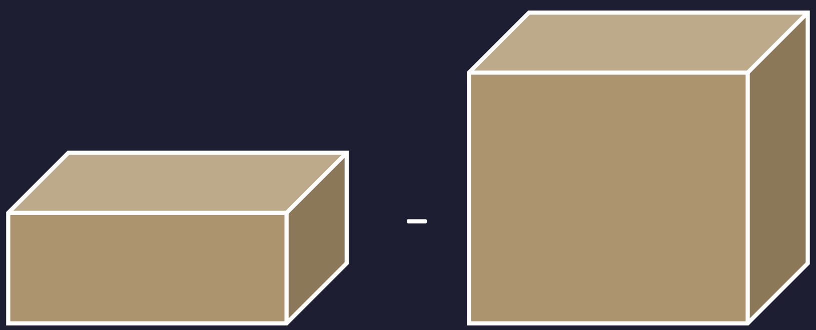
Lattice Spacing Derivative

Changing lattice spacing a ⇨ 1/T and V change

$$\frac{\partial \ln Z}{\partial a} \sim \varepsilon - 3p$$

$$\frac{\partial \ln Z}{\partial a} = \frac{\partial \beta}{\partial a}\frac{\partial \ln Z}{\partial \beta} \sim \frac{\partial \beta}{\partial a}\langle S \rangle$$

$$\beta = 2N_c/g^2$$

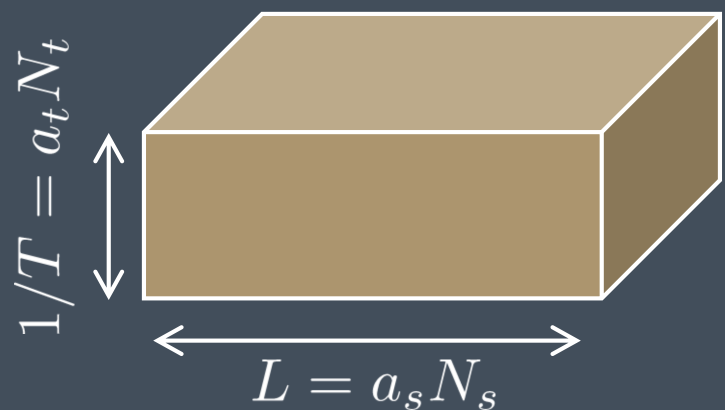


$\frac{\partial \beta}{\partial a}$, $\langle S \rangle$ ⇨ $[\varepsilon - 3p]_{\mathrm{thermodyn.}} = [\varepsilon - 3p]_T - [\varepsilon - 3p]_{\mathrm{vac}}$

Differential Method

anisotropic lattice with

$$a_s, a_t$$



$$\varepsilon \sim \frac{\partial \ln Z}{\partial a_t}$$

$$p \sim \frac{\partial \ln Z}{\partial a_s}$$

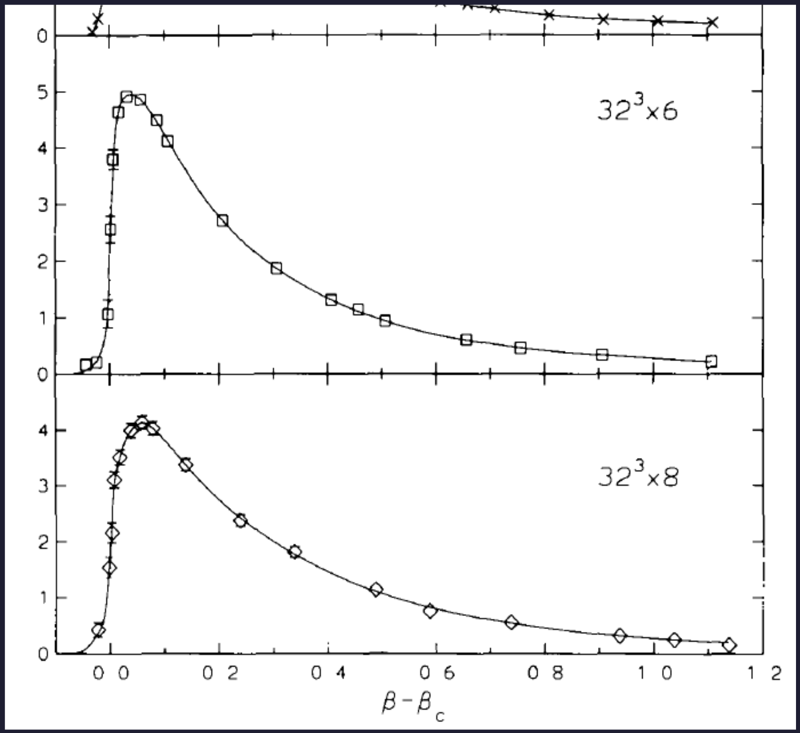
- 2 independent “beta functions”
- perturbative result Karsch(1982)
- negative pressure with Karsch coeffs.
- vacuum subtraction

Integral Method

$$T\frac{\partial(p/T^4)}{\partial T} = \frac{\varepsilon - 3p}{T^4}$$

$$\frac{p}{T^4} = \int_{T_0}^{T} dT \frac{\varepsilon - 3p}{T^5}$$

Boyd+ 1996



- measurements of e-3p for many T
- vacuum subtraction for each T
- information on beta function

Gradient Flow Method

$$\langle T_{\mu\nu} \rangle$$

Gradient Flow

Luescher, 2010

$$\partial_t B_\mu(t,x) = D_\nu G_{\mu\nu}$$

$$B_\mu(0,x) = A_\mu(x)$$

$$G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + [B_\mu, B_\nu]$$

t: “flow time”
dim:[length^2]

steepest descent direction of the action

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$$B_\mu(0,x) = A_\mu(x)$$

$$G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + [B_\mu, B_\nu]$$

- modify gauge field toward the stationary point of the action
- smoothing similarly to diffusion equation

$$\partial_t B_\mu = D_\nu G_{\mu\nu} = \partial_\nu \partial_\nu B_\mu + \cdots$$

- diffusion length $d \sim \sqrt{8t}$
- All composite operators at t>0 are UV finite Luescher,Weisz,2011

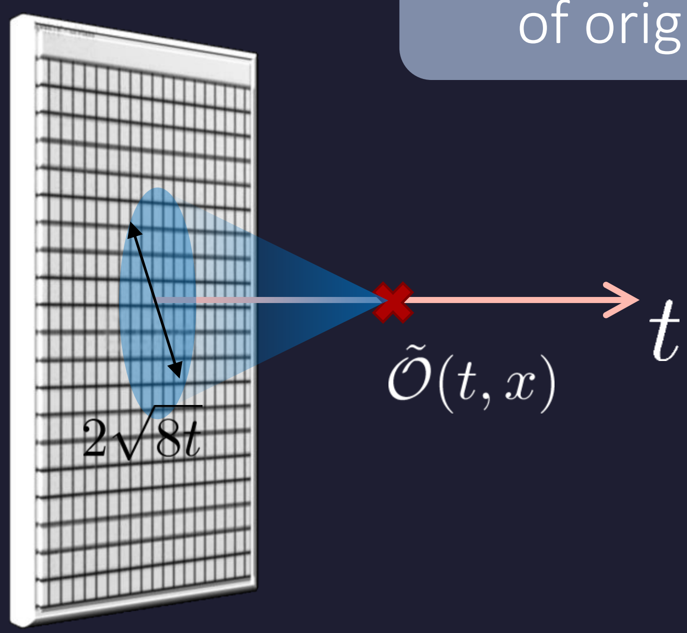
Operator Relation

Luescher, Weisz, 2011

$$\tilde{\mathcal{O}}(t,x) \xrightarrow[t \to 0]{} \sum_i c_i(t)\mathcal{O}_i^R(x)$$

an operator at t>0

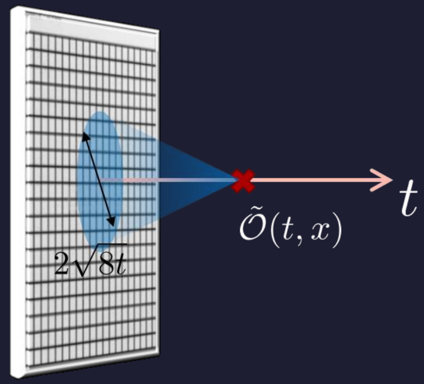
remormalized operators of original theory



Constructing EMT

Suzuki, 2013

$$\tilde{\mathcal{O}}(t,x) \underset{t\to 0}{\longrightarrow} \sum_i c_i(t)\mathcal{O}_i^R(x)$$



- **gauge-invariant dimension 4 operators**

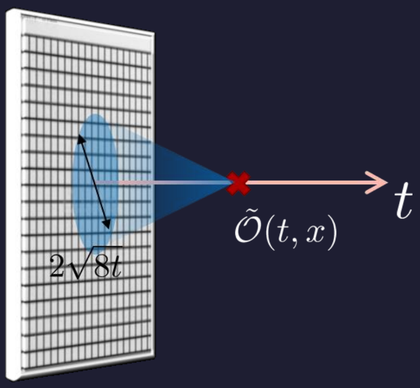
$$\begin{cases} U_{\mu\nu}(t,x) = G_{\mu\rho}(t,x)G_{\nu\rho}(t,x) - \frac{1}{4}\delta_{\mu\nu}G_{\mu\nu}(t,x)G_{\mu\nu}(t,x) \\ E(t,x) = \frac{1}{4}\delta_{\mu\nu}G_{\mu\nu}(t,x)G_{\mu\nu}(t,x) \end{cases}$$

Constructing EMT 2

Suzuki, 2013

$$U_{\mu\nu}(t,x) = \alpha_U(t)\left[T^R_{\mu\nu}(x) - \frac{1}{4}\delta_{\mu\nu}T^R_{\rho\rho}(x)\right] + \mathcal{O}(t)$$

$$E(t,x) = \langle E(t,x)\rangle + \alpha_E(t)T^R_{\rho\rho}(x) + \mathcal{O}(t)$$



Suzuki coefficients

$$\begin{cases} \alpha_U(t) = g^2\left[1 + 2b_0 s_1 g^2 + O(g^4)\right] \\ \alpha_E(t) = \dfrac{1}{2b_0}\left[1 + 2b_0 s_2 g^2 + O(g^4)\right] \end{cases}$$

$g = g(1/\sqrt{8t})$

$s_1 = 0.03296\ldots$

$s_2 = 0.19783\ldots$

Constructing EMT 2

$$U_{\mu\nu}(t,x) = \alpha_U(t)\left[T^R_{\mu\nu}(x) - \frac{1}{4}\delta_{\mu\nu}T^R_{\rho\rho}(x)\right] + \mathcal{O}(t)$$

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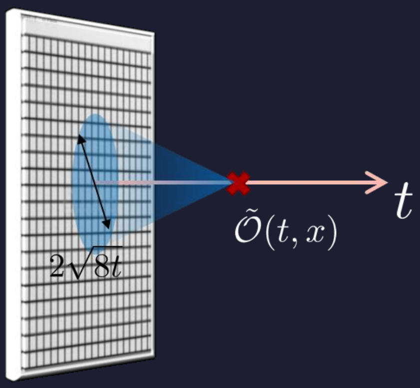

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$g = g(1/\sqrt{8t})$

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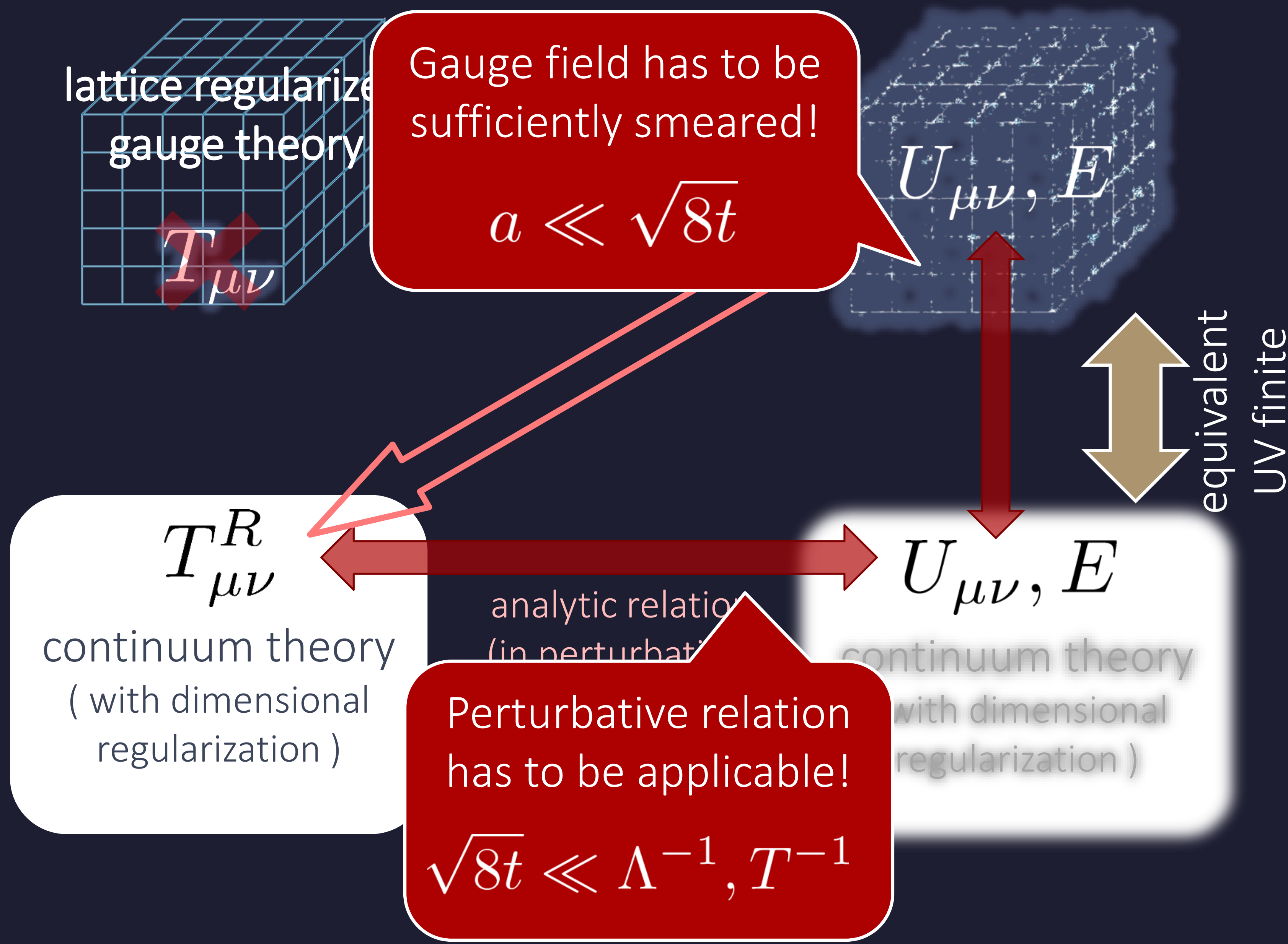
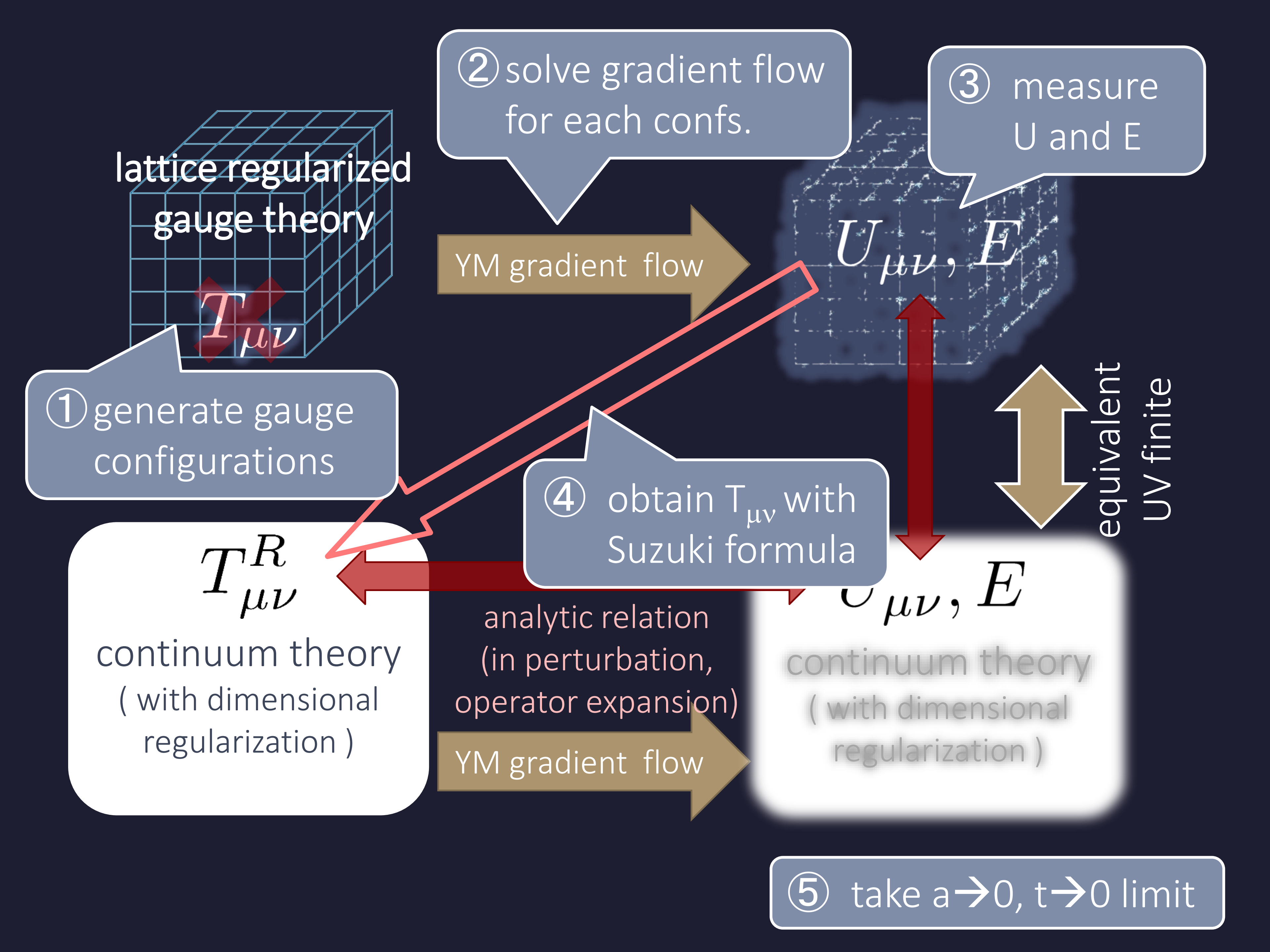
Remormalized EMT

$$T^R_{\mu\nu}(x) = \lim_{t\to 0}\left[\frac{1}{\alpha_U(t)}U_{\mu\nu}(t,x) + \frac{1}{4\alpha_E(t)}\delta_{\mu\nu}E(t,x)_{\mathrm{subt.}}\right]$$

Numerical Simulation on the Lattice


gauge theory
$T_{\mu\nu}$
Gauge field has to be sufficiently smeared!
$a \ll \sqrt{8t}$
$U_{\mu\nu}, E$
equivalent
UV finite
$T^R_{\mu\nu}$
continuum theory
(with dimensional regularization)
$U_{\mu\nu}, E$
continuum theory
with dimensional regularization)
Perturbative relation has to be applicable!
$\sqrt{8t} \ll \Lambda^{-1}, T^{-1}$



lattice regularized gauge theory
$T_{\mu\nu}$
① generate gauge configurations
② solve gradient flow for each confs.
YM gradient flow
③ measure U and E
$U_{\mu\nu}, E$
equivalent
UV finite
④ obtain $T_{\mu\nu}$ with Suzuki formula
$T^R_{\mu\nu}$
continuum theory
(with dimensional regularization)
analytic relation
(in perturbation, operator expansion)
YM gradient flow
$U_{\mu\nu}, E$
continuum theory
(with dimensional regularization)
⑤ take a→0, t→0 limit

Numerical Simulation 1

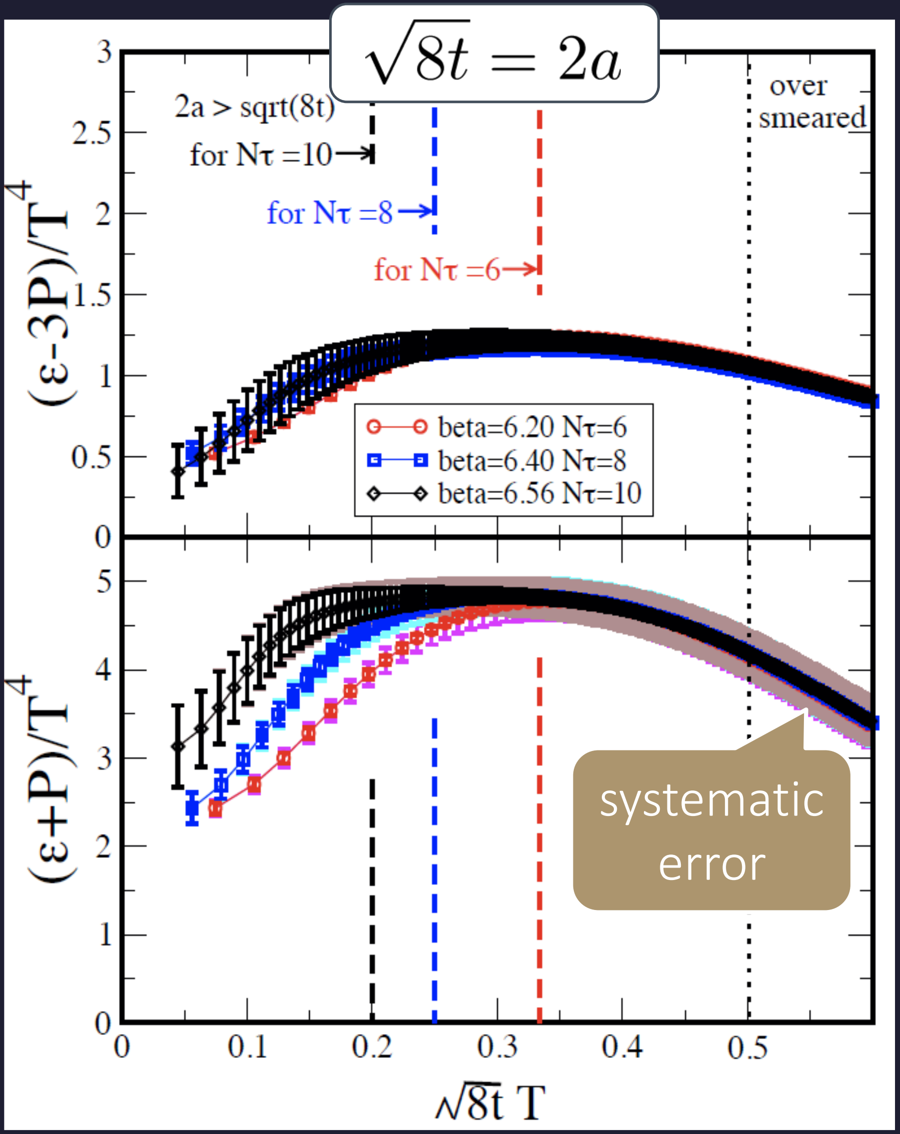
FlowQCD, 1312.7492

- SU(3) YM theory
- Wilson gauge action
- lattice size: $32^3 x N_t$
- N_t=6, 8, 10
- configurations: 100-300

N_τ	6	8	10	T/T_c
β	6.20	6.40	6.56	1.65
	6.02	6.20	6.36	1.24
	5.89	6.06	6.20	0.99

Scale setting:
alpha Collab., NPB538,669(1999)

YM Thermodynamics

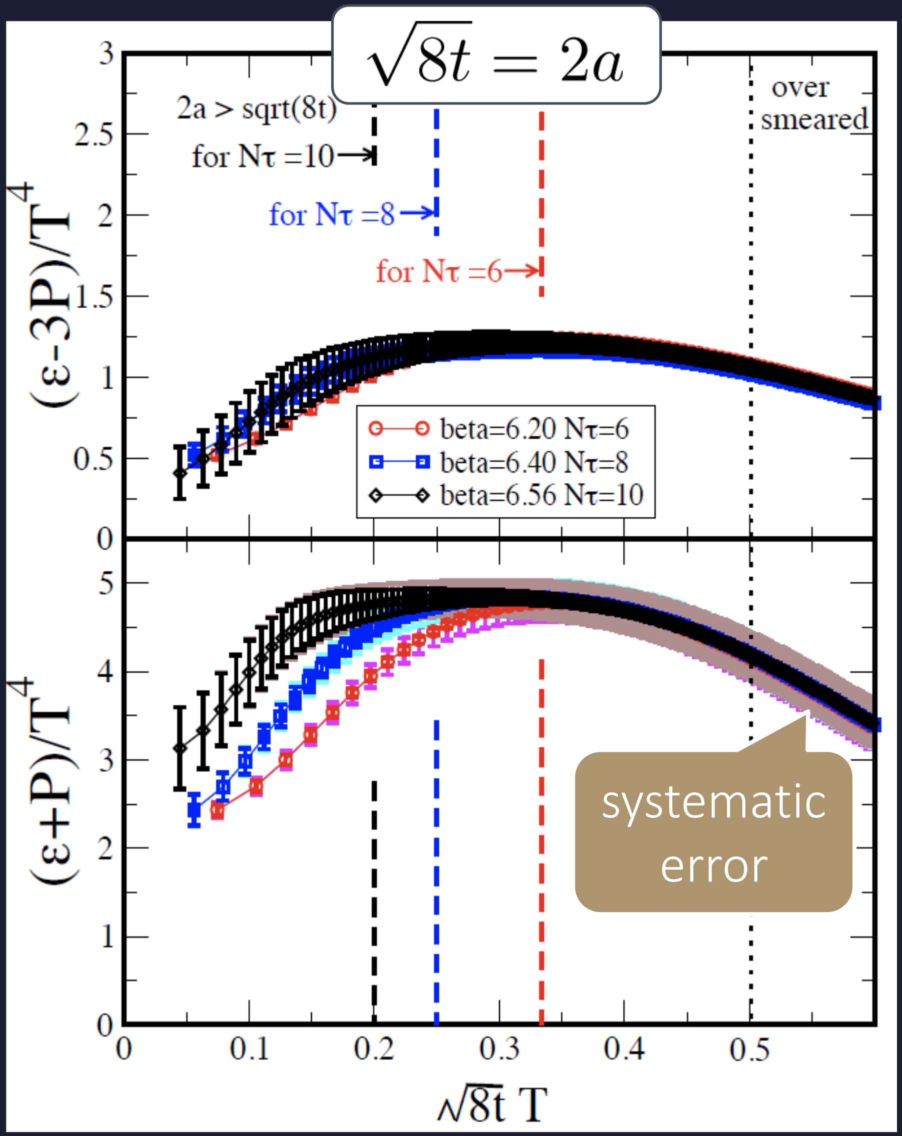


Emergent plateau!

$$2a \lesssim \sqrt{8t} \lesssim 0.4T^{-1}$$

There exists a wide range of t at which the Suzuki formula is safely used with Nt=10.

YM Thermodynamics



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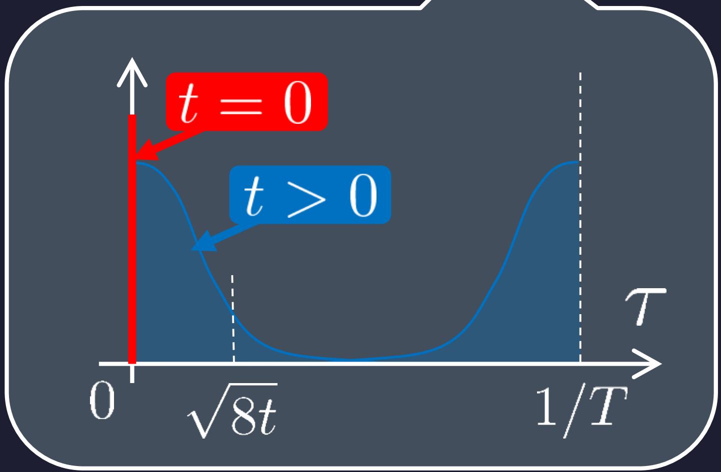
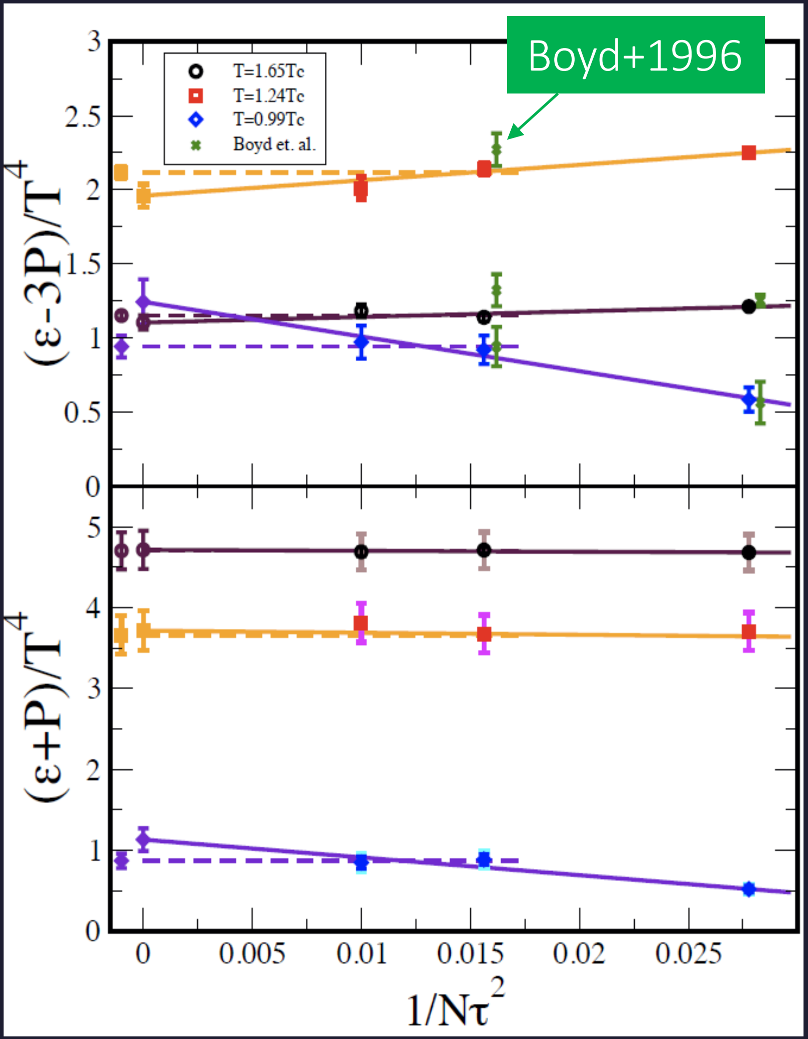


There exists a wide range of t at which the Suzuki formula is safely used with Nt=10.

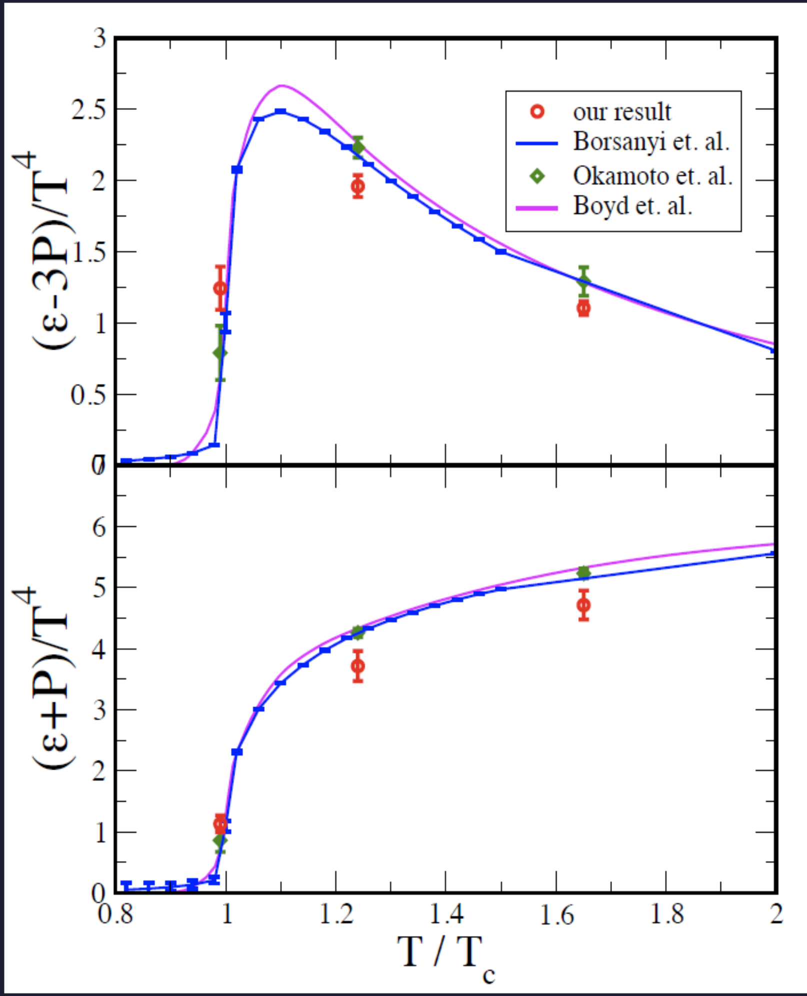
Continuum Limit



$a \to 0$ limit with fixed $\sqrt{8t}T = 0.4$

- Statistical error of e-3p is significantly smaller than Boyd+1996 which used ~10000 confs.
- No integral! Direct measurement of e and p at a given T
- no vacuum subtraction for e+p

Comparison with Integral Method



Numerical Simulation 2

- SU(3) YM theory
- Wilson gauge action
- lattice size: $32^3 \mathrm{x} N_t$
- N_t = 6, 8, 10, 32
- configurations: 100-300
- β = 5.89 – 6.56

twice finer lattice spacing!

- lattice size: $64^3 \mathrm{x} N_t$
- N_t= 8, 10, ... ,16, 18, 64
- configurations: ~2000
- β = 6.4 – 7.4

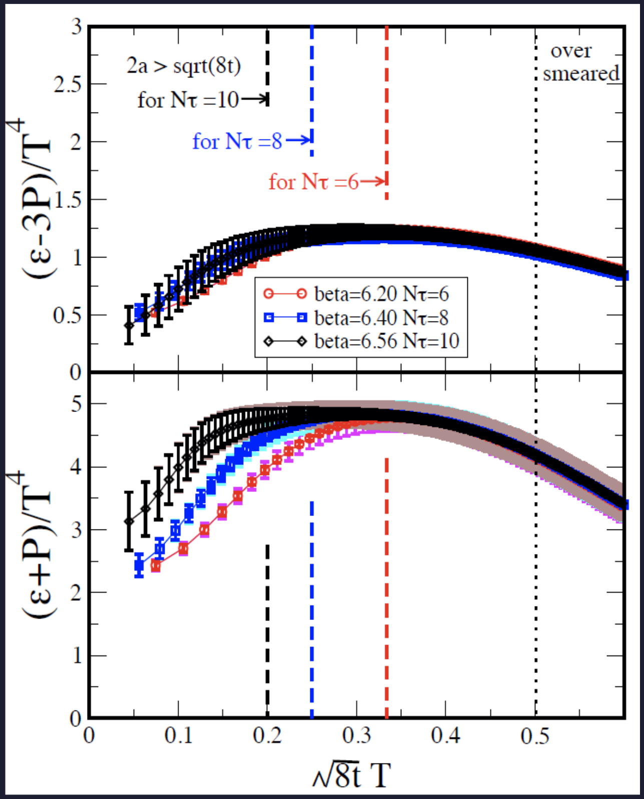
On BlueGene/Q @ KEK

Efficiency of our code:

- Gauge update (HB+OR) : ~25%
- Gradient flow (RK^4) : ~40%

t Dependence of e+p

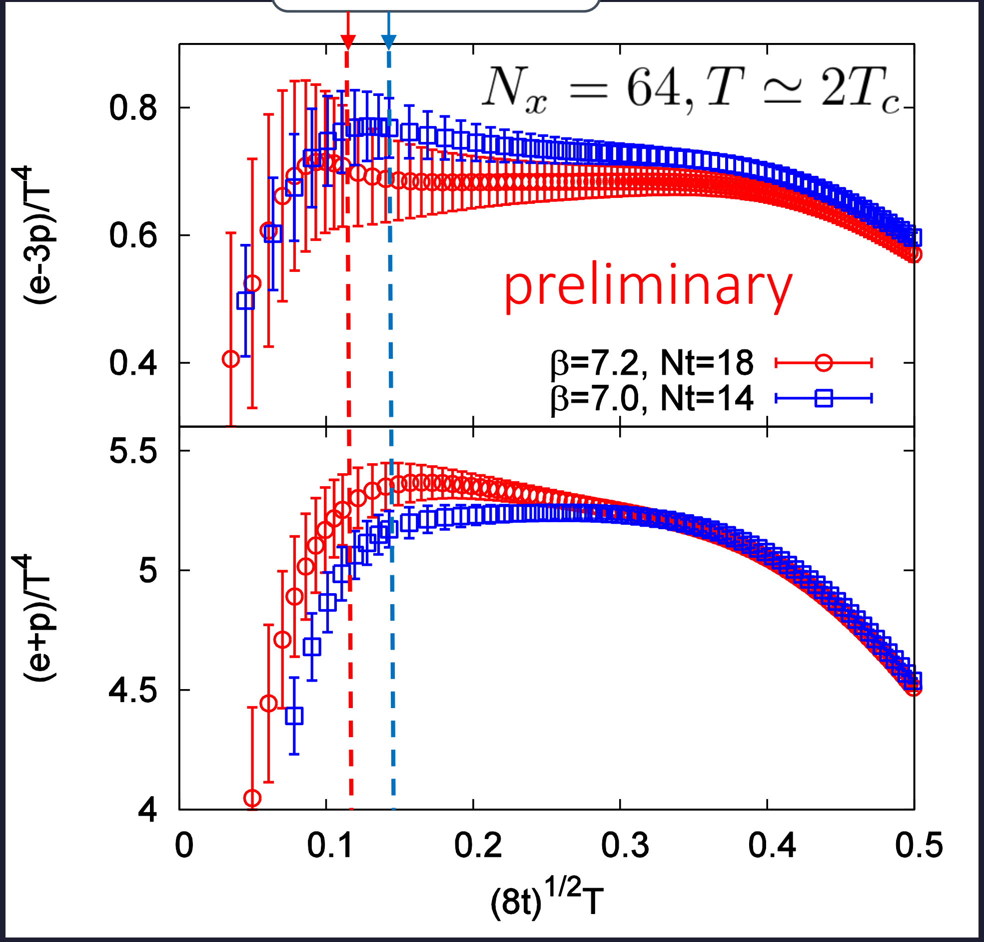
$N_x = 32, T \simeq 1.65T_c$



FlowQCD,1312.7492

$\sqrt{8t} = 2a$

NEW!!



Plateau region extends toward small t!

Summary

$$T^{R}_{\mu\nu}(x)$$

Summary

EMT formula from gradient flow

$$T_{\mu\nu}^{R}(x) = \lim_{t\to 0}\left[\frac{1}{\alpha_U(t)}U_{\mu\nu}(t,x) + \frac{1}{4\alpha_E(t)}\delta_{\mu\nu}E(t,x)_{\mathrm{subt.}}\right]$$

Our strategy can successfully define the EMT on the lattice in practical simulations

This operator provides us with novel approaches to measure observables on the lattice!

They are direct, intuitive and less noisy

Many Future Works

- ☐ precision measurement of YM thermodynamics
- ☐ EMT correlation functions → measurement of viscosity
- ☐ specific heat, non-Gaussian fluctuations, etc.
- ☐ scale setting

- ☐ taking double limit a→0, t→0
- ☐ full QCD Makino,Suzuki,1403.4772

from **Gradient** flow → to **Hydrodynamic** flow

Two Point Functions

$$\langle T_{\mu\nu}(x,t)T_{\mu\nu}(0,0)\rangle$$

EMT Correlator

- Kubo Formula: T_{12} correlator ←→ shear viscosity

$$\eta = \int_0^\infty dt \int_0^{1/T} d\tau \int d^3x \langle T_{12}(x, -i\tau) T_{12}(0, t) \rangle$$

  - Hydrodynamics describes long range behavior of $T_{\mu\nu}$

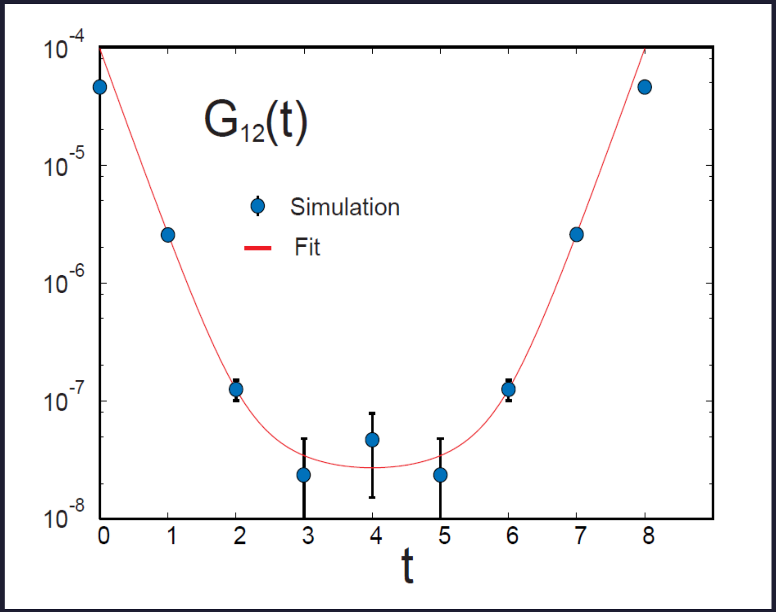
- Energy fluctuation ←→ specific heat

$$c_V = \frac{\langle \delta E^2 \rangle}{V T^2}$$

EMT Correlator : Noisy...

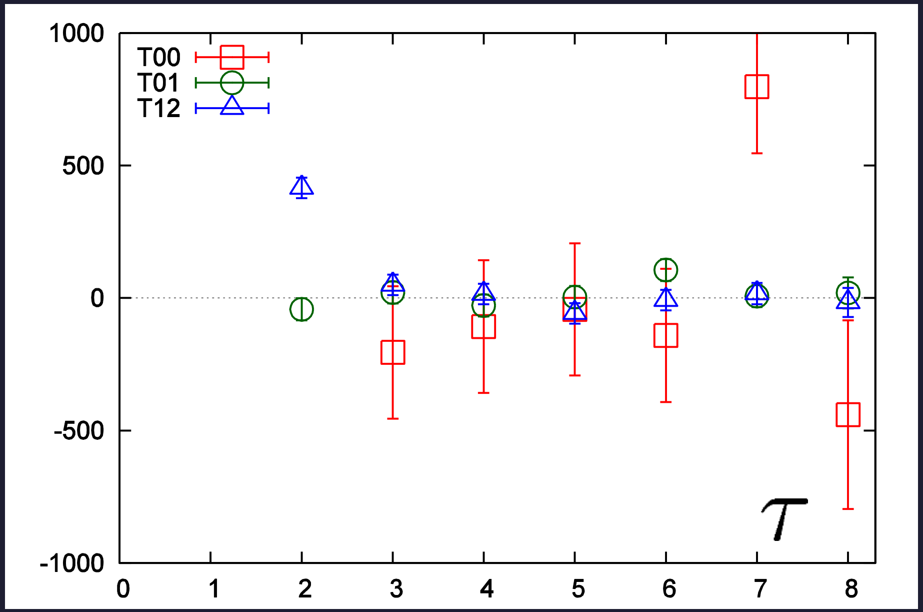
With naïve EMT operators

$$\langle T_{12}(\tau) T_{12}(0) \rangle$$



Nakamura, Sakai, PRL,2005
N_t=8
improved action
~10^6 configurations

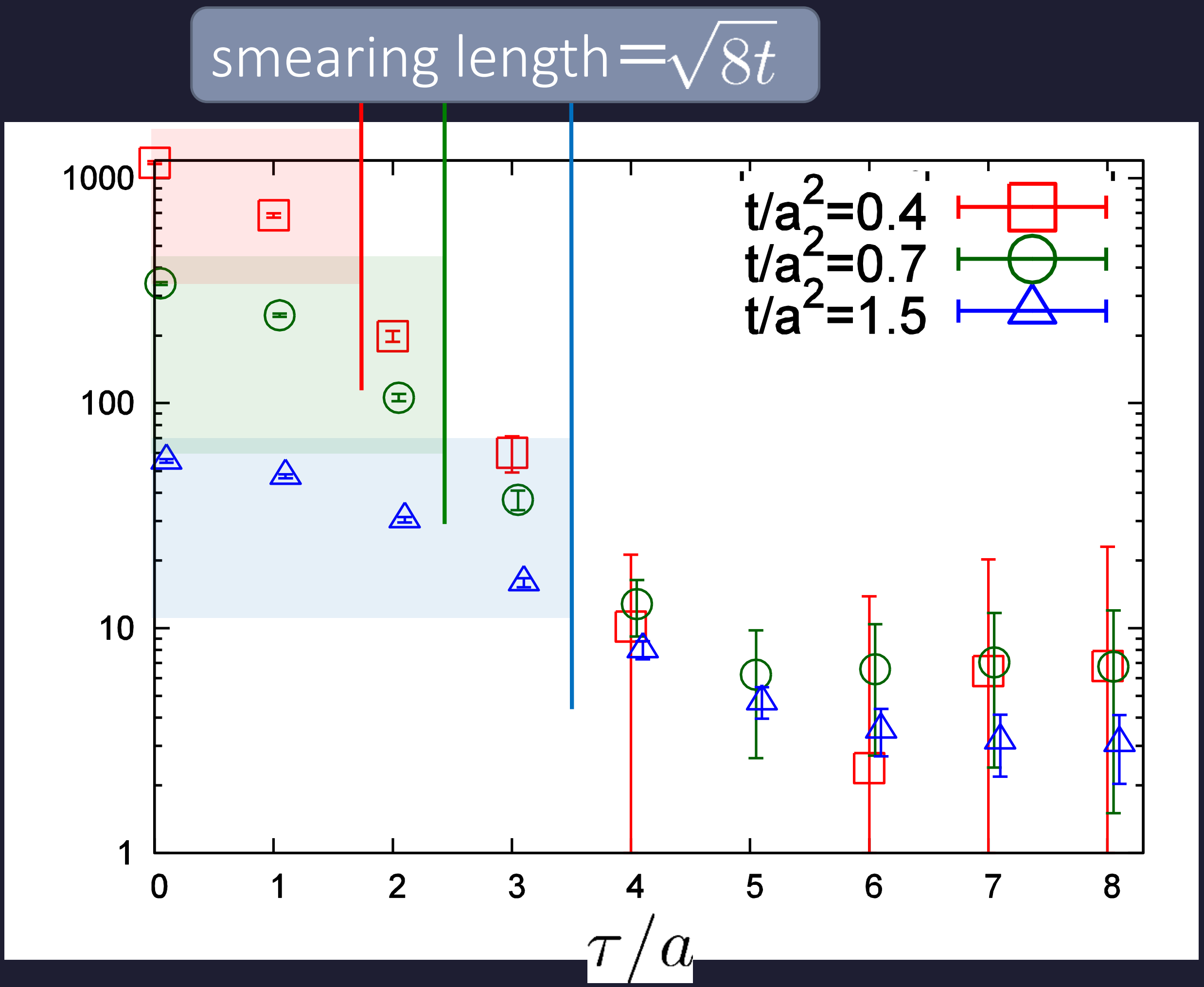
$$\langle T_{\mu\nu}(\tau) T_{\mu\nu}(0) \rangle$$



Nt=16
standard action
5×10^4 configuration

... no signal

$$\int d^3x \langle T_{12}(x,\tau) T_{12}(0,0) \rangle$$



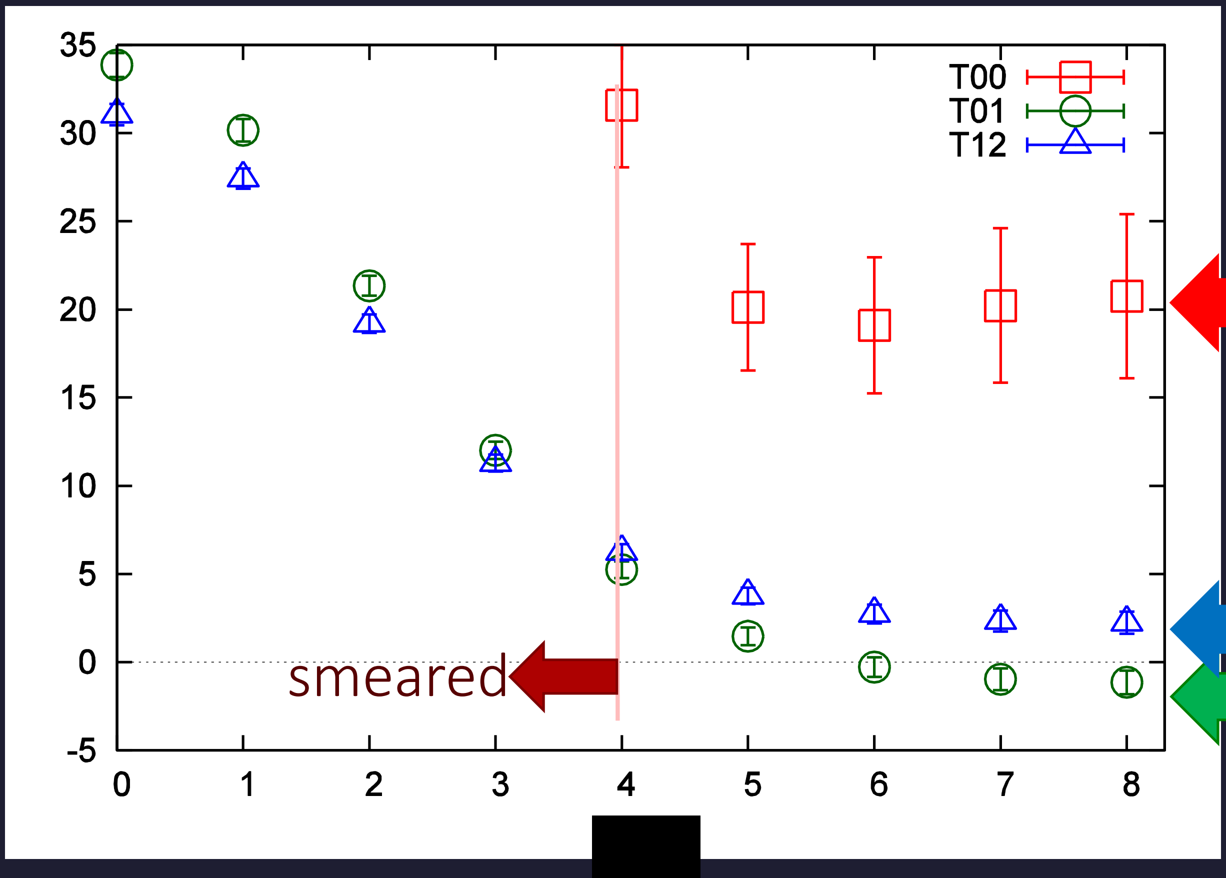
64^3x16
β=7.2 (T~2.2Tc)
1200 confs

- converge at $\tau > \sqrt{8t}$
- improvement of the statistics at large t

Correlation Function

$$C_{\mu\nu}(\tau) = \int d^3x \langle T_{\mu\nu}(x,\tau) T_{\mu\nu}(0,0) \rangle$$

64^3x16
β=7.2 (T~2.2Tc)
1200 confs
t/a^2=1.9



$C_{44}(\tau)$:constant
← conservation law!

$$\partial_\tau \langle \delta E(\tau) \delta E(0) \rangle = 0$$
$$(\text{for } \tau \neq 0)$$

$C_{12}(\tau)$

$C_{41}(\tau)$
negative ← $i^2=-1$

Energy Fluctuation and Specific Heat

Specific Heat

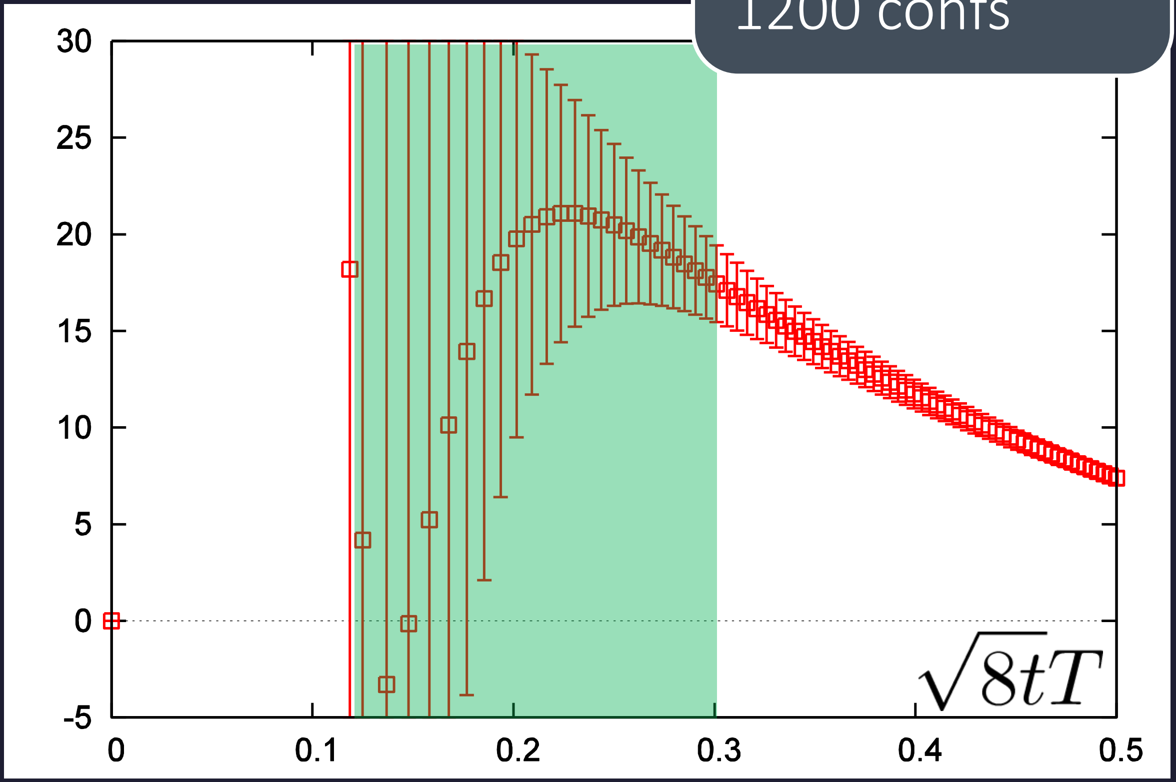
$$c_V = \frac{1}{V}\left.\frac{\partial E}{\partial T}\right|_V$$

$$= \frac{\langle \delta E^2 \rangle}{VT^2}$$

$$= \frac{\langle \delta E(\tau)\delta E(0) \rangle}{VT^2}$$

$$\frac{c_V}{T^3} = \frac{\langle \delta E(\beta/2)\delta E(0) \rangle}{VT^5}$$

64^3x16
β=7.2 (T~2.2Tc)
1200 confs



Energy Fluctuation and Specific Heat

Specific Heat

$$c_V = \frac{1}{V}\frac{\partial E}{\partial T}\bigg|_V$$

$$= \frac{\langle \delta E^2 \rangle}{VT^2}$$

$$= \frac{\langle \delta E(\tau)\delta E(0) \rangle}{VT^2}$$

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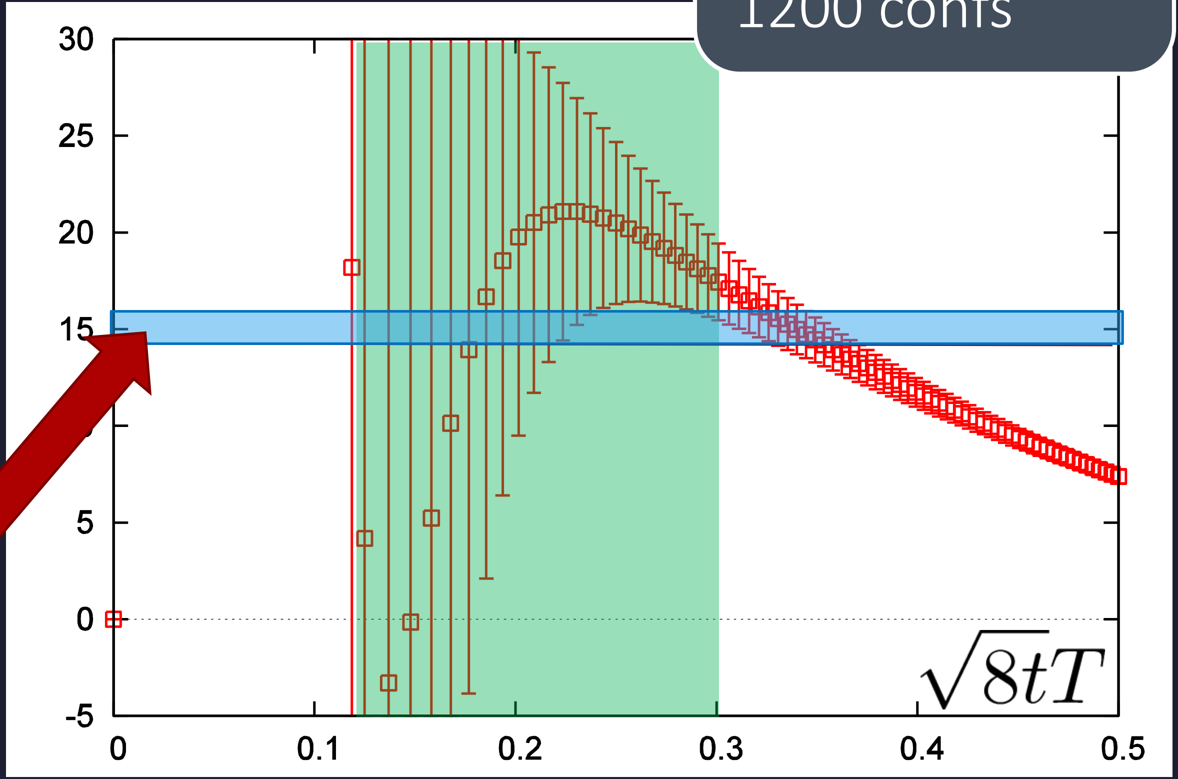

Gavai, et al., 2005
differential method
for T=2Tc

Novel approach to measure c_V