Thermodynamics of Gauge Theory from Gradient Flow
For FlowQCD Collaboration
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$T_{\mu\nu}$
Poincare symmetry

\[ T_{\mu\nu} \]
Poincare symmetry

\[ T_{\mu \nu} \]

Einstein Equation

\[ G_{\mu \nu} + \Lambda g_{\mu \nu} = \kappa T_{\mu \nu} \]

Hydrodynamic Eq.

\[ \partial_\mu T_{\mu \nu} = 0 \]
Definition of the operator is nontrivial because of the explicit breaking of Lorentz symmetry.

\[ T_{\mu\nu} = F_{\mu\rho} F_{\nu\rho} - \frac{1}{4} \delta_{\mu\nu} FF \]

Its measurement is extremely noisy due to high dimensionality and etc.
If we have $T_{\mu\nu}$
Thermodynamics

direct measurement of expectation values

\[ \langle T_{00} \rangle, \langle T_{ii} \rangle \]
Thermodynamics

direct measurement of expectation values

\[ \langle T_{00} \rangle, \langle T_{ii} \rangle \]

Fluctuations and Correlations

viscosity, specific heat, ...

\[ c_V \sim \langle \delta T_{00}^2 \rangle \]

\[ \eta = \langle T_{12}; T_{12} \rangle \]

If we have

\[ T_{\mu \nu} \]
Thermodynamics

direct measurement of expectation values
\[ \langle T_{00} \rangle, \langle T_{ii} \rangle \]

Fluctuations and Correlations

viscosity, specific heat, ...
\[ c_V \sim \langle \delta T_{00}^2 \rangle \]
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If we have

\[ T_{\mu\nu} \]

- confinement string
- EM distribution in hadrons

Hadron Structure

- vacuum configuration
- mixed state on 1st transition

Vacuum Structure
Measurement of Thermodynamics using gradient flow

Masakiyo Kitazawa (Osaka U.)

for FlowQCD Collaboration
Asakawa, Hatsuda, Iritani, Itou, MK, Suzuki

FlowQCD, arXiv:1312.7492[hep-lat]; to appear in PRD

LATTICE2014, 28 June 2014, New York
Flow QCD Collaboration

\[ \partial_t B_\mu = D_\nu G_{\nu\mu} \]
Flow QCD Collaboration

from Gradient flow
\[ \partial_t B_\mu = D_\nu G_{\nu\mu} \]

to Hydrodynamic flow
\[ \partial_\mu T_{\mu\nu} = 0 \]

Thermodynamics
- direct measurement of expectation values
  \[ \langle T_{00} \rangle, \langle T_{ii} \rangle \]

Fluctuations and Correlations
- viscosity, specific heat, ...
  \[ c_V \sim \langle \delta T_{00}^2 \rangle \]
  \[ \eta = \langle T_{12}; T_{12} \rangle \]
Gradient Flow
YM Gradient Flow

$$\partial_t A_\mu(t, x) = -\frac{\partial S_{\text{YM}}}{\partial A_\mu}$$

A. Ramos, plenary, Friday

$$A_\mu(0, x) = A_\mu(x)$$
YM Gradient Flow

\[ \partial_t A_\mu(t, x) = -\frac{\partial S_{YM}}{\partial A_\mu} \]

\[ A_\mu(0, x) = A_\mu(x) \]

- \boxed{\partial_t A_\mu} = D_\nu G_{\mu\nu} = \partial_\nu \partial_\nu A_\mu + \cdots

- Diffusion length \( d \sim \sqrt{8t} \)

- This is NOT the standard cooling/smearing

- All composite operators at t>0 are UV finite Luescher, Weisz, 2011
Applications of Gradient Flow

① scale setting
② running coupling
③ topology
④ operator relation
⑤ autocorrelation, etc.

A. Ramos, plenary, Friday
Small Flow Time Expansion of Operators and EMT
\[ \tilde{O}(t, x) \xrightarrow{t \to 0} \sum_i c_i(t) O_i^R(x) \]

an operator at \( t > 0 \)

remormalized operators of original theory
Operator Relation

\[ \tilde{O}(t, x) \rightarrow \sum_i c_i(t) \mathcal{O}_i^R(x) \]

an operator at \( t > 0 \)

remormalized operators of original theory

Luescher, Weisz, 2011
Operator Relation

\[ \tilde{O}(t, x) \xrightarrow{t \to 0} \sum_i c_i(t) \mathcal{O}_i^R(x) \]

- an operator at \( t > 0 \)
- remormalized operators of original theory

Luescher, Weisz, 2011
Constructing EMT

\[ \tilde{O}(t, x) \xrightarrow{t \to 0} \sum_i c_i(t) \mathcal{O}_i^R(x) \]

- gauge-invariant dimension 4 operators

\[
\begin{align*}
U_{\mu\nu}(t, x) &= G_{\mu\rho}(t, x)G_{\nu\rho}(t, x) - \frac{1}{4}\delta_{\mu\nu}G_{\mu\nu}(t, x)G_{\mu\nu}(t, x) \\
E(t, x) &= \frac{1}{4}\delta_{\mu\nu}G_{\mu\nu}(t, x)G_{\mu\nu}(t, x)
\end{align*}
\]
Constructing EMT 2

\[ U_{\mu\nu}(t, x) = \alpha_U(t) \left[ T_{\mu\nu}^R(x) - \frac{1}{4} \delta_{\mu\nu} T_{\rho\rho}^R(x) \right] + \mathcal{O}(t) \]

\[ E(t, x) = \langle E(t, x) \rangle + \alpha_E(t) T_{\rho\rho}^R(x) + \mathcal{O}(t) \]

Suzuki coeffs.

\[
\begin{align*}
\alpha_U(t) &= g^2 \left[ 1 + 2b_0 s_1 g^2 + O(g^4) \right] \\
\alpha_E(t) &= \frac{1}{2b_0} \left[ 1 + 2b_0 s_2 g^2 + O(g^4) \right]
\end{align*}
\]

\[ g = g\left(1/\sqrt{8t}\right) \quad s_1 = 0.03296 \ldots \quad s_2 = 0.19783 \ldots \]

See also, Patella, Parallel7E, Thu.
Constructing EMT 2

\[ U_{\mu\nu}(t, x) = \alpha_U(t) \left[ T_{\mu\nu}^R(x) - \frac{1}{4} \delta_{\mu\nu} T_{\rho\rho}^R(x) \right] + \mathcal{O}(t) \]

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\end{align*}
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\[ g = g(1/\sqrt{8t}) \]

\[ s_1 = 0.03296 \ldots \]

\[ s_2 = 0.19783 \ldots \]

Remormalized EMT

\[
T_{\mu\nu}^R(x) = \lim_{t \to 0} \left[ \frac{1}{\alpha_U(t)} U_{\mu\nu}(t, x) + \frac{\delta_{\mu\nu}}{4\alpha_E(t)} E(t, x)_{\text{subt.}} \right]
\]
Numerical Analysis on the Lattice
Gradient Flow Method

lattice regularized gauge theory

gradient flow

continuum theory (with dim. reg.)

analytic (perturbative)

$T_{\mu\nu}^R$

$U_{\mu\nu}, E$

continuum theory (with dim. reg.)
Gradient Flow Method

lattice regularized gauge theory

continuum theory (with dim. reg.)

continuum theory (with dim. reg.)

measurement on the lattice

gradient flow

analytic (perturbative)

$T_R^\mu\nu$

$U_{\mu\nu}, E$
Caveats

Perturbative relation has to be applicable!
\[ \sqrt{8t} \ll \Lambda^{-1}, T^{-1} \]

Gauge field has to be sufficiently smeared!
\[ a \ll \sqrt{8t} \]

\( T_{\mu\nu}^R \)
continuum theory
(with dim. reg.)

(perturbative)

\( U_{\mu\nu}, E \)
continuum theory
(with dim. reg.)

measurement on the lattice

analytic
gradient flow
Caveats

Gauge field has to be sufficiently smeared!

\[ a \ll \sqrt{8t} \]

Perturbative relation has to be applicable!

\[ \sqrt{8t} \ll \Lambda^{-1}, T^{-1} \]

Continuum theory (with dim. reg.)

\[ T^{R}_{\mu\nu} \]

Gradient flow

\[ U_{\mu\nu}, E \]

Analytic (perturbative)

Continuum theory (with dim. reg.)

\[ a \ll \sqrt{8t} \ll \Lambda^{-1} \]
\[ \tilde{T}_{\mu\nu}(t) = \frac{1}{\alpha_U(t)} U_{\mu\nu}(t) + \frac{\delta_{\mu\nu}}{4\alpha_E(t)} E(t)_{\text{subt.}} \]

\[ T^R_{\mu\nu} = \lim_{t \to 0} \tilde{T}_{\mu\nu}(t) \]

\[ \langle \tilde{T}_{\mu\nu}(t) \rangle \quad \text{in continuum} \]

\[ \langle T^R_{\mu\nu} \rangle \quad \text{non-perturbative region} \]

\[ \langle T^R_{\mu\nu} \rangle \quad \text{O}(t) \text{ effect} \]

\[ t \quad \text{classical} \]
\[ \tilde{T}_{\mu\nu}(t) = \frac{1}{\alpha_U(t)} U_{\mu\nu}(t) + \frac{\delta_{\mu\nu}}{4\alpha_E(t)} E(t)_{\text{subt.}} \]

\[ T^R_{\mu\nu} = \lim_{t \to 0} \tilde{T}_{\mu\nu}(t) \]

\[ \langle \tilde{T}_{\mu\nu}(t) \rangle \text{ in continuum} \]

\[ \langle T^R_{\mu\nu} \rangle \text{ non-perturbative region} \]

\[ \text{O(t) effect} \]

\[ \text{on the lattice} \]

\[ \Box t \to 0 \text{ limit with keeping } t \gg a^2 \]
Numerical Simulation

- SU(3) YM theory
- Wilson gauge action

Simulation 1
(arXiv:1312.7492)
- lattice size: $32^3 \times N_t$
- $N_t = 6, 8, 10$
- $\beta = 5.89 - 6.56$
- ~300 configurations

using SX8 @ RCNP
SR16000 @ KEK

Simulation 2
(new, preliminary)
- lattice size: $64^3 \times N_t$
- $N_t = 10, 12, 14, 16$
- $\beta = 6.4 - 7.4$
- ~2000 configurations

using BlueGeneQ @ KEK
efficiency ~40%

twice finer lattice!
Numerical Simulation

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efficiency ~40%
ε-3p at $T=1.65T_c$

\[ \tilde{T}_{\mu\nu}(t) = \frac{1}{\alpha_U(t)} U_{\mu\nu}(t) + \frac{\delta_{\mu\nu}}{4\alpha_E(t)} E(t)_{\text{subt.}} \]

\[ T_{\mu\nu}^R = \lim_{t \to 0} \tilde{T}_{\mu\nu}(t) \]

\[ \sqrt{8t} = 2a \]

Emergent plateau!

\[ 2a \lesssim \sqrt{8t} \lesssim 0.4T^{-1} \]

$Nt=6,8,10$

$\sim 300$ confs.

the range of $t$ where the EMT formula is successfully used!
$\varepsilon$-3p at $T = 1.65T_c$

$$\tilde{T}_{\mu\nu}(t) = \frac{1}{\alpha_U(t)} U_{\mu\nu}(t) + \frac{\delta_{\mu\nu}}{4\alpha_E(t)} E(t)_{\text{subt.}}$$

$T^R_{\mu\nu} = \lim_{t \to 0} \tilde{T}_{\mu\nu}(t)$

Emergent plateau!

$$2a \lesssim \sqrt{8t} \lesssim 0.4T^{-1}$$

the range of $t$ where the EMT formula is successfully used!

$Nt = 6, 8, 10$

$\sim 300$ confs.
Entropy Density at $T=1.65T_c$

Emergent plateau!

$2a \lesssim \sqrt{8t} \lesssim 0.4T^{-1}$

$\sqrt{8t} = 2a$

Nt=6,8,10
~300 confs.

Direct measurement of e+p on a given T!

NO integral / NO vacuum subtraction
Continuum Limit

$32^3 \times N_t$

$N_t = 6, 8, 10$

$T/T_c = 0.99, 1.24, 1.65$

Boyd+1996
Continuum Limit

Comparison with previous studies

32³xNt
Nt = 6, 8, 10
T/Tc=0.99, 1.24, 1.65
Numerical Simulation

- SU(3) YM theory
- Wilson gauge action

**Simulation 1**
(arXiv:1312.7492)
- lattice size: $32^3 \times N_t$
- $N_t = 6, 8, 10$
- $\beta = 5.89 - 6.56$
- ~300 configurations

**Simulation 2**
(new, preliminary)
- lattice size: $64^3 \times N_t$
- $N_t = 10, 12, 14, 16$
- $\beta = 6.4 - 7.4$
- ~2000 configurations

using SX8 @ RCNP
SR16000 @ KEK

using BlueGeneQ @ KEK
efficiency ~40%
Entropy Density on Finer Lattices

$T = 2.31T_c$

$64^3 \times N_t$

$N_t = 10, 12, 14, 16$

2000 confs.

The wider plateau on the finer lattices

Plateau may have a nonzero slope

FlowQCD, 2013

$T = 1.65T_c$
Continuum Extrapolation

- $T = 2.31T_c$
- 2000 confs
- $N_t = 10 \sim 16$

Continuum extrapolation is stable

$a \to 0$ limit with fixed $t/a^2$

$\sqrt{8tT} = 1312.7492$

$t/a^2 = 1.3$

$t/a^2 = 1.0$
Summary

\[ T^R_{\mu\nu}(x) \]
EMT formula from gradient flow

\[ T^R_{\mu\nu}(x) = \lim_{t \to 0} \left[ \frac{1}{\alpha_U(t)} U_{\mu\nu}(t, x) + \frac{\delta_{\mu\nu}}{4\alpha_E(t)} E(t, x)_{\text{subt.}} \right] \]

This formula can successfully define and calculate the EMT on the lattice.

This operator provides us with novel approaches to measure various observables on the lattice!

This method is direct, intuitive and less noisy.
Other observables
full QCD Makino,Suzuki,2014
non-pert. improvement Patella 7E(Thu)

Thermodynamics
direct measurement of expectation values
\(\langle T_{00}\rangle, \langle T_{ii}\rangle\)

\(\eta = \langle T_{12}; T_{12}\rangle\)

\(c_V \sim \langle \delta T_{00}^2 \rangle\)

\(\rightarrow \) confinement string
\(\rightarrow \) EM distribution in hadrons

Hadron Structure

\(\rightarrow \) vacuum configuration
\(\rightarrow \) mixed state on 1st transition

Vacuum Structure

Many Future Studies!!

Fluctuations and Correlations
viscosity, specific heat,

O(a) improvement
Nogradi, 7E(Thu); Sint, 7E(Thu)
Monahan, 7E(Thu)

and etc.
One More Thing...
One More Thing...

Fluctuations and Correlations

viscosity, specific heat, ...

\[ c_V \sim \langle \delta T_{00}^2 \rangle \]

\[ \eta = \langle T_{12}; T_{12} \rangle \]
Energy Correlation Function

\[ \langle T_{00}(\tau)T_{00}(0) \rangle \]

\( T = 2.31T_c \)
\( b = 7.2, \ N_t = 16 \)
2000 confs
\( p = 0 \) correlator

\( \tau < 2\sqrt{2t} \)
smeared

\( \tau/a \)
Energy Correlation Function

\[ \langle T_{00}(\tau)T_{00}(0) \rangle \]

\( T = 2.31T_c \)
\( b = 7.2, \ N_t = 16 \)
\( 2000 \) confs
\( p = 0 \) correlator

\( \tau \) independent const.
\( \rightarrow \) energy conservation

\( \tau < 2\sqrt{2t} \)
smeared
Energy Correlation Function

\[ \langle T_{00}(\tau)T_{00}(0) \rangle \]

- \( T = 2.31T_c \)
- \( b = 7.2, N_t = 16 \)
- 2000 confs
- \( p = 0 \) correlator

- \( \tau \) independent const.
- \( \rightarrow \) energy conservation

- specific heat
  \[ c_V = \frac{\langle \delta E^2 \rangle}{VT^2} \]
- \( \rightarrow \) Novel approach to measure specific heat!
Keep your attention to this new flow just like...
Keep your attention to this new flow

just like...

Thank you for your attention!