QCD Thermodynamics from Gradient Flow

Masakiyo Kitazawa

for FlowQCD Collaboration

(Asakawa, Hatsuda, Iritani, Itou, MK, Suzuki)

XQCD2015, 2015/Sep./22, Wuhan, China

Yang-Mills Gradient Flow

$$\partial_t A_\mu = D_\nu G_{\mu\nu}$$

A new flow in lattice community (2010~)

Applications of Gradient Flow

1. Lattice spacing / reference scales
2. Energy-momentum tensor
   - thermodynamics
   - correlation functions
3. Topology
4. Running coupling
5. and etc…

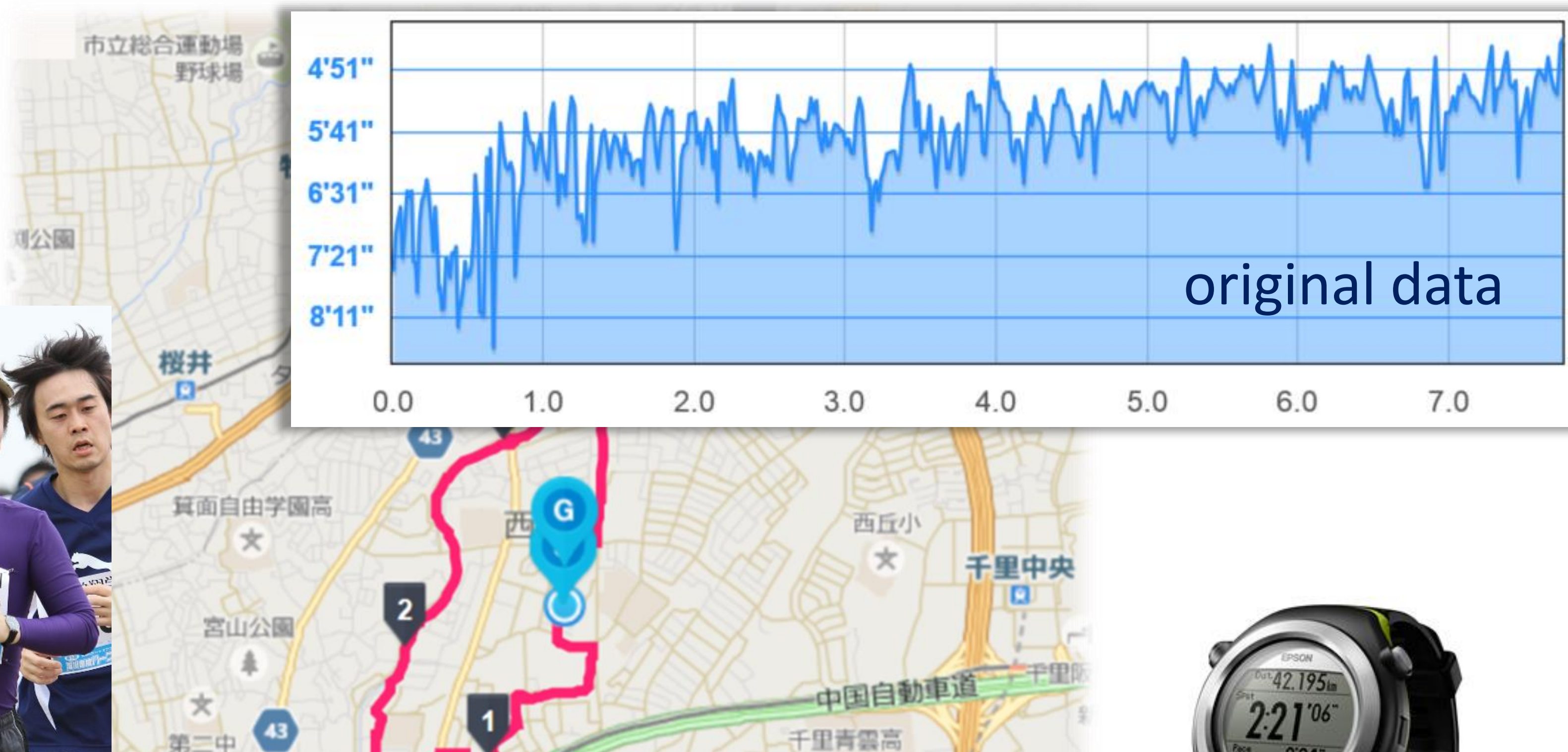
topics coverted by this talk

Yang-Mills Gradient Flow

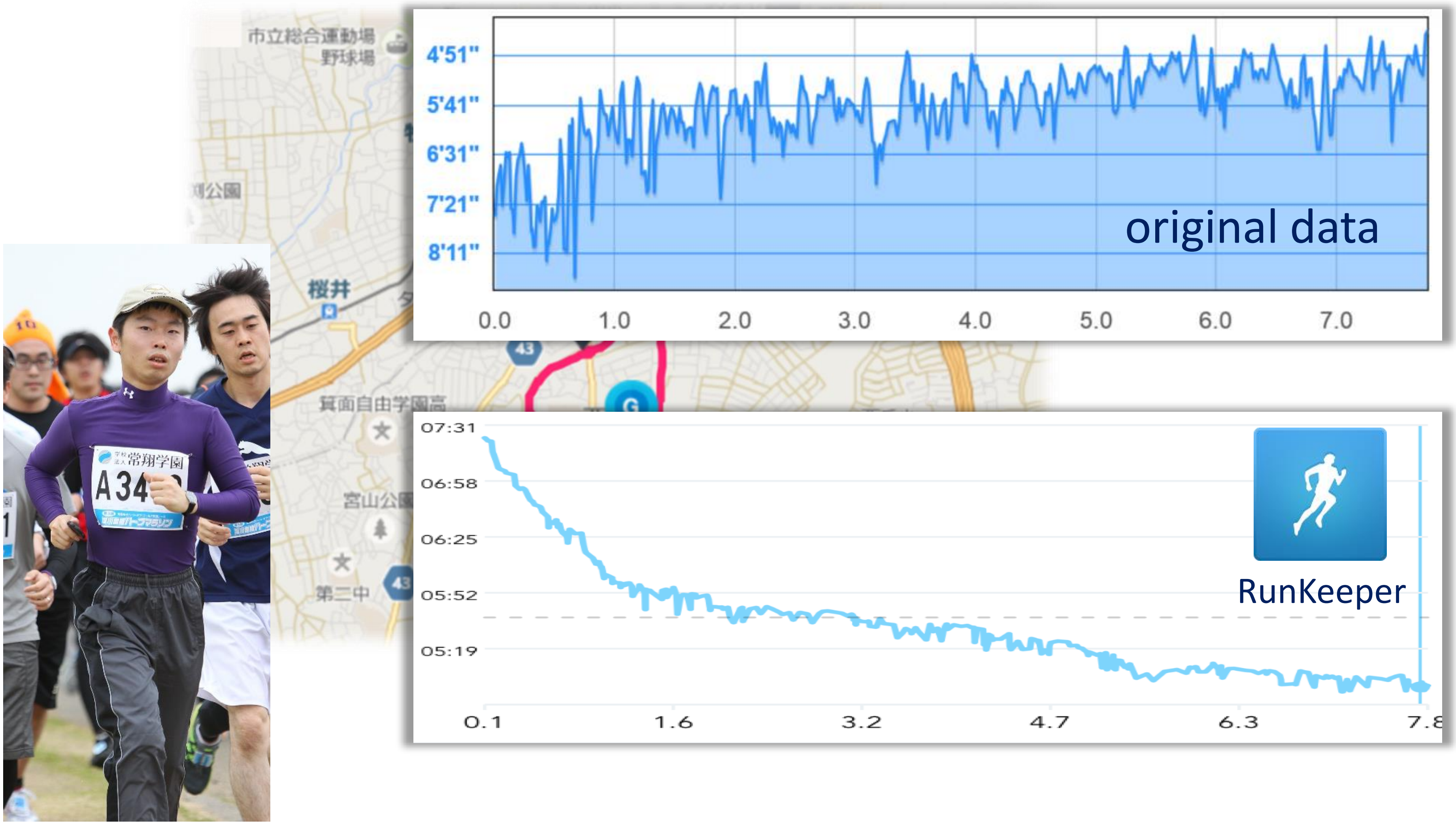
$$\partial_t A_\mu = D_\nu G_{\mu\nu}$$

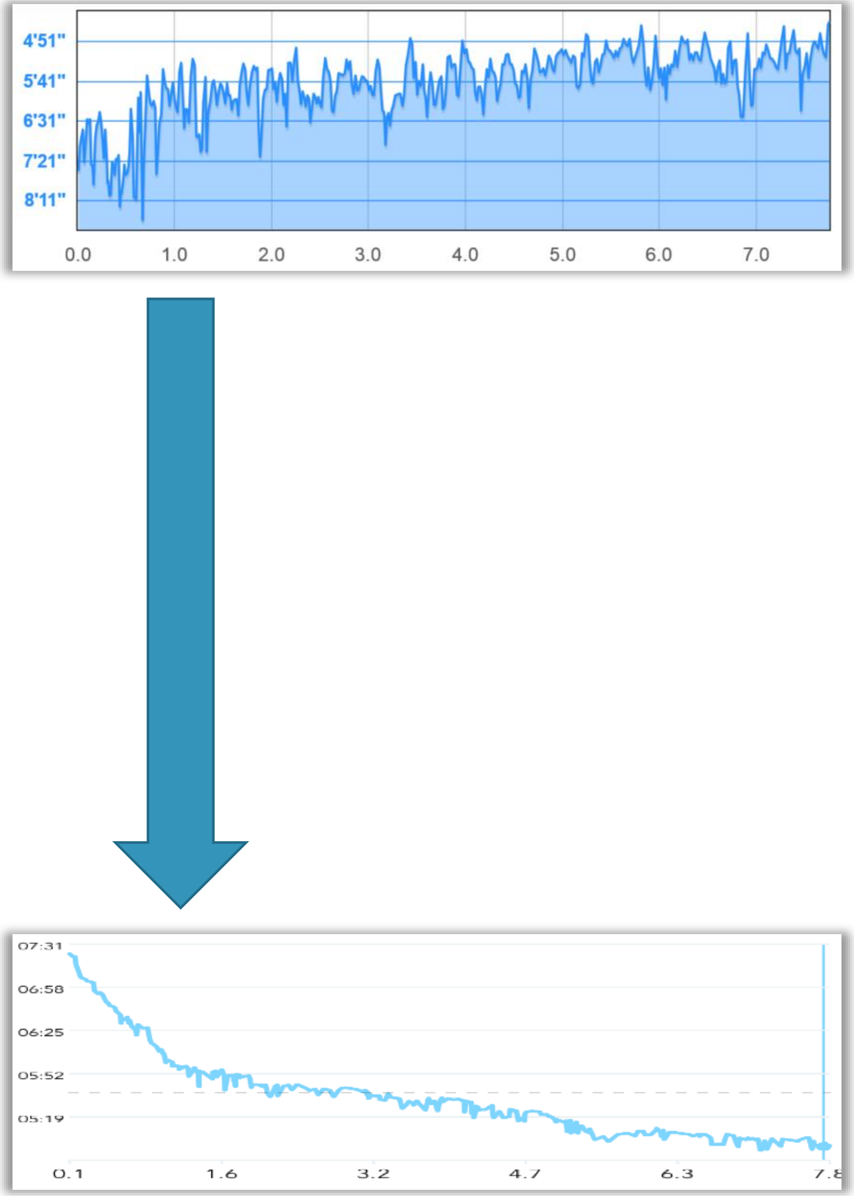
Gradient Flow and Jogging

Gradient Flow and Jogging



Gradient Flow and Jogging




4'51"
5'41"
6'31"
7'21"
8'11"
0.0
1.0
2.0
3.0
4.0
5.0
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07:31
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06:25
05:52
05:19
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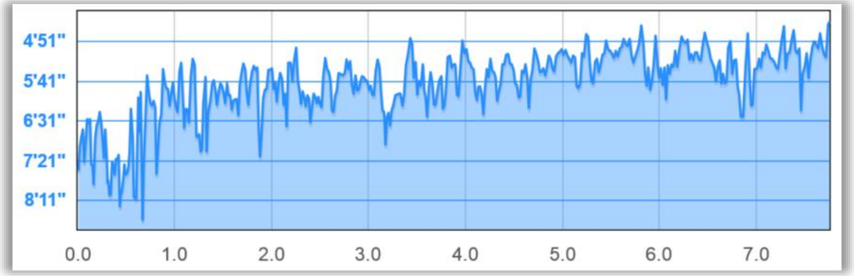
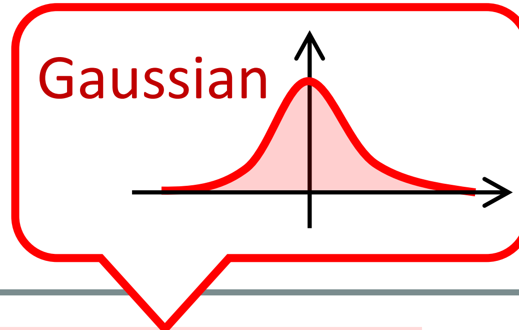
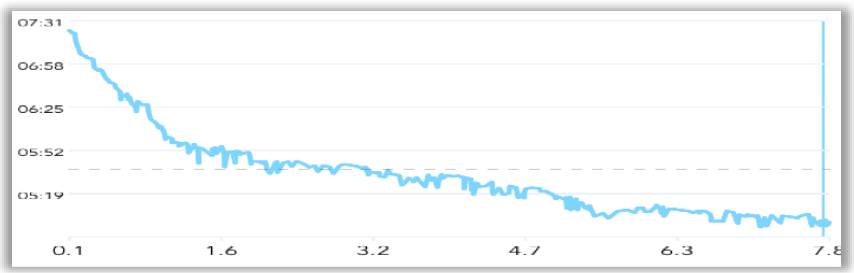
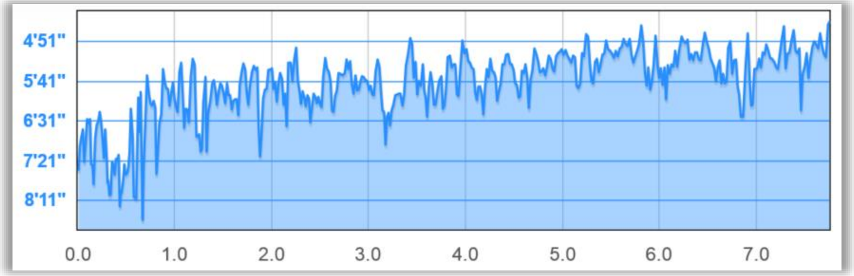
3.2
4.7
6.3

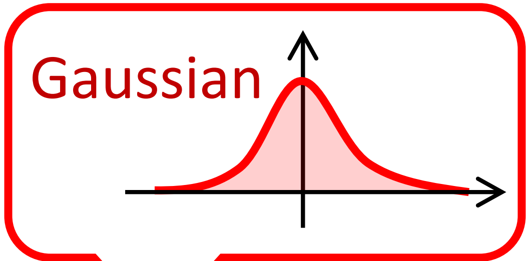




① $x(t) \to x'(t) \sim \int dt' \exp\left[-\frac{(t-t')^2}{2\sigma^2}\right] x(t')$



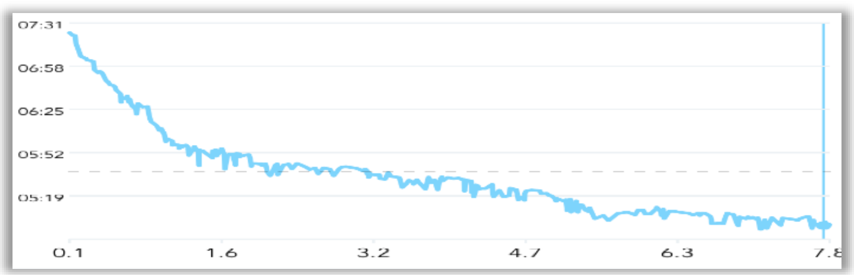


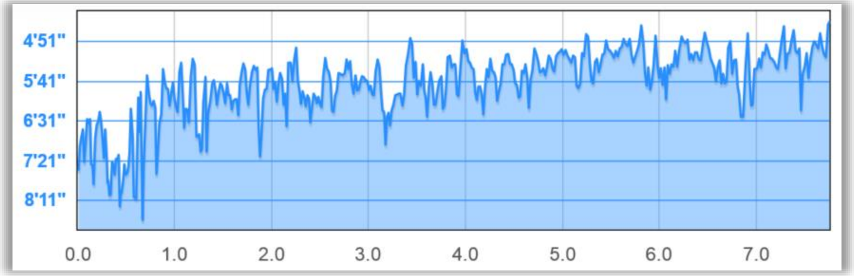


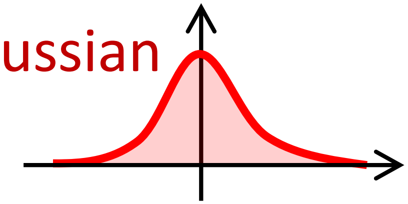
① $$x(t) \to x'(t) \sim \int dt' \exp\left[-\frac{(t-t')^2}{2\sigma^2}\right] x(t')$$

$$\sigma = \sqrt{2s}$$

② $$\frac{d}{ds} x(t;s) = \frac{d^2}{dt^2} x(t,s) \qquad x(t;0) = x(t)$$







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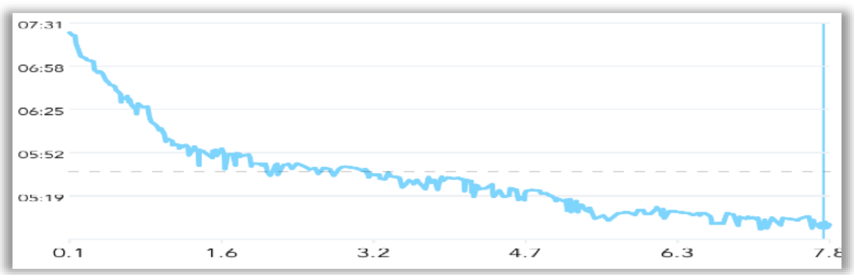

YM Gradient Flow

$$\partial_t A_\mu = D_\nu G_{\mu\nu}$$
$$= \partial_\nu \partial_\nu A_\mu + \cdots$$

Gauge invariant version of
4-dim. diffusion equation

YM Gradient Flow

$$\partial_t A_\mu = D_\nu G_{\mu\nu}$$

= Continuous smoothing transformation of gauge field

- t: "flow time" dim: $[E^{-2}]$
- smearing length $r = \sqrt{8t}$

Remarks:

- All observables are UV finite at t>0. Luscher, Weiss, 2011
- Smoothed field is no longer the original gauge field!

Applications of Gradient Flow

1. Lattice spacing / reference scales

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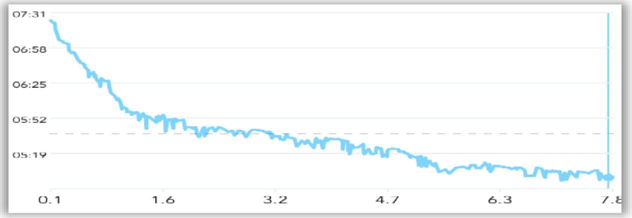
Observables at Nonzero Flow Time

$\langle \mathcal{O} \rangle_t$ →



- function of *t*
- independent of renormalization

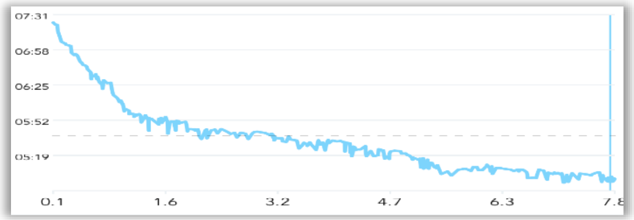
Luescher, Weisz, 2011

independent of lattice spacing
(up to $O(a^2)$ corrections)

The *t* dep. can be used for the **scale setting** of the lattice.

Observables at Nonzero Flow Time

$$\langle \mathcal{O} \rangle_t$$



- function of *t*
- independent of renormalization

Choice 1: $\mathcal{O} = t^2 E(t)$

Luescher, 2010

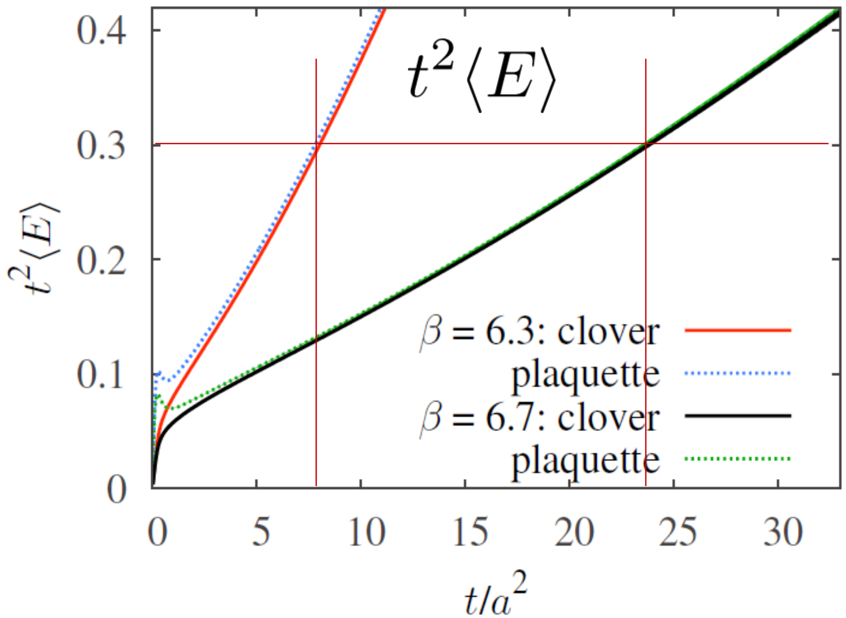
Choice 2: $\mathcal{O} = t\frac{d}{dt}t^2 E(t)$

Budapest-Wuppertal, 2012

discretization effect suppressed

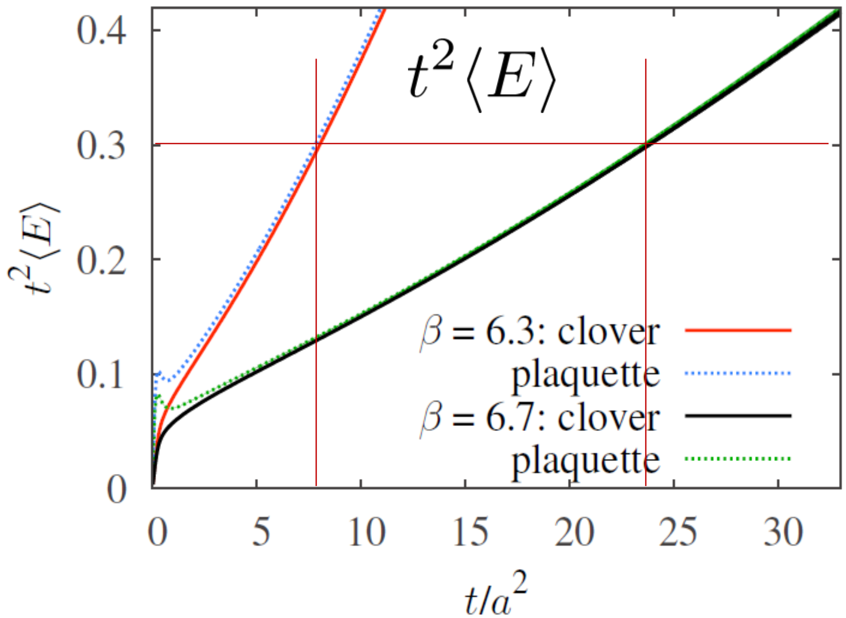
$$E = \frac{1}{4} G^a_{\mu\nu} G^a_{\mu\nu}$$

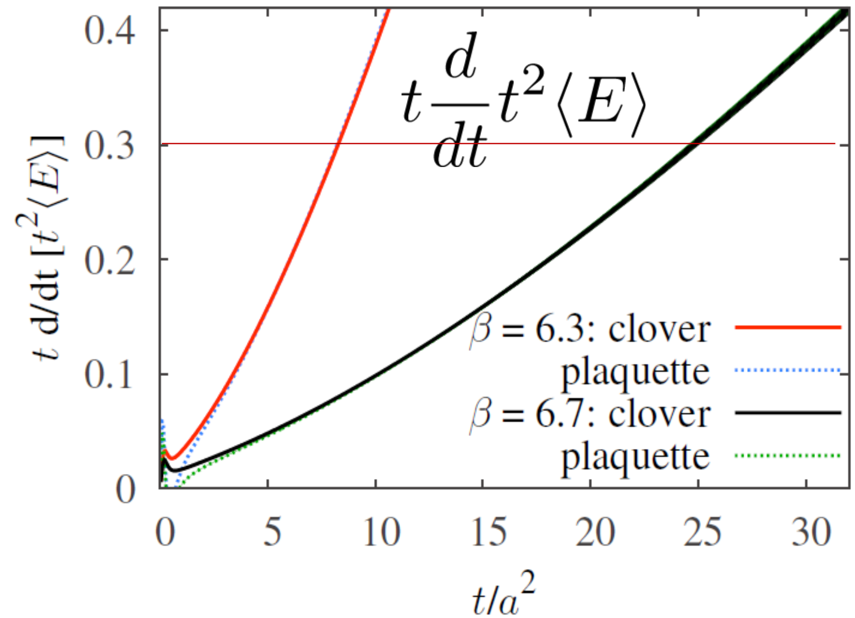
Behavior of $t^2\langle E \rangle$



$$t_0 \;:\; t^2\langle E\rangle|_{t=t_0} = 0.3$$

Behavior of t^2<E>





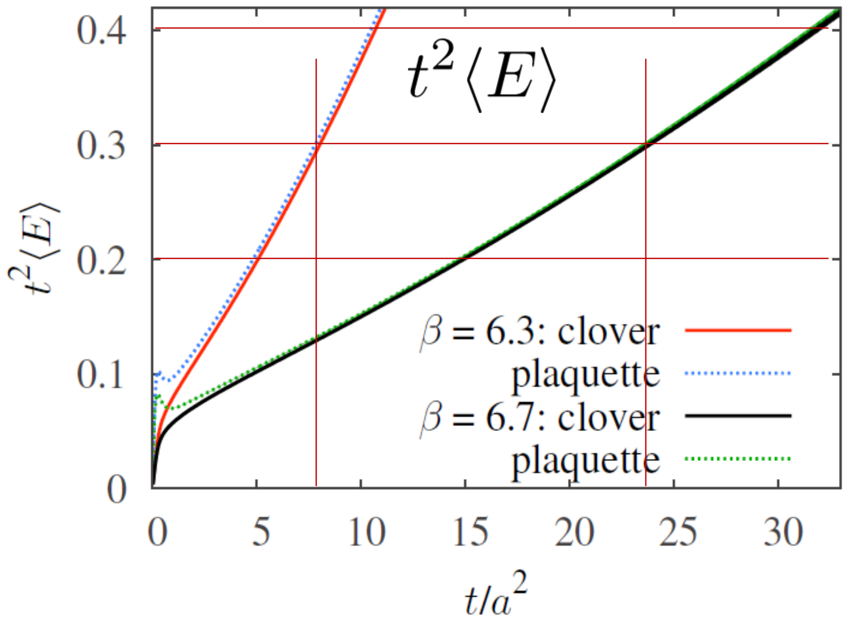
$$t_0 \; : \; t^2\langle E\rangle|_{t=t_0} = 0.3$$

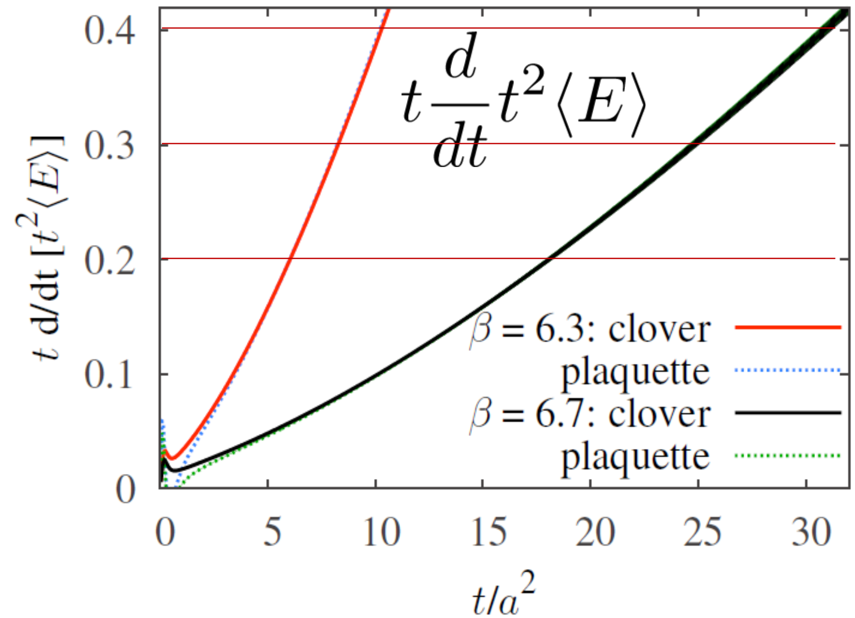
$$w_0 \; : \; t\frac{d}{dt}t^2\langle E\rangle|_{t=w_0^2} = 0.3$$

Advangates:

- Less statistical errors
- No fit analyses
- No measurement of non-local obs.

Behavior of t^2<E>





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$$w_0 \ : \ t\frac{d}{dt}t^2\langle E\rangle|_{t=w_0^2} = 0.3$$

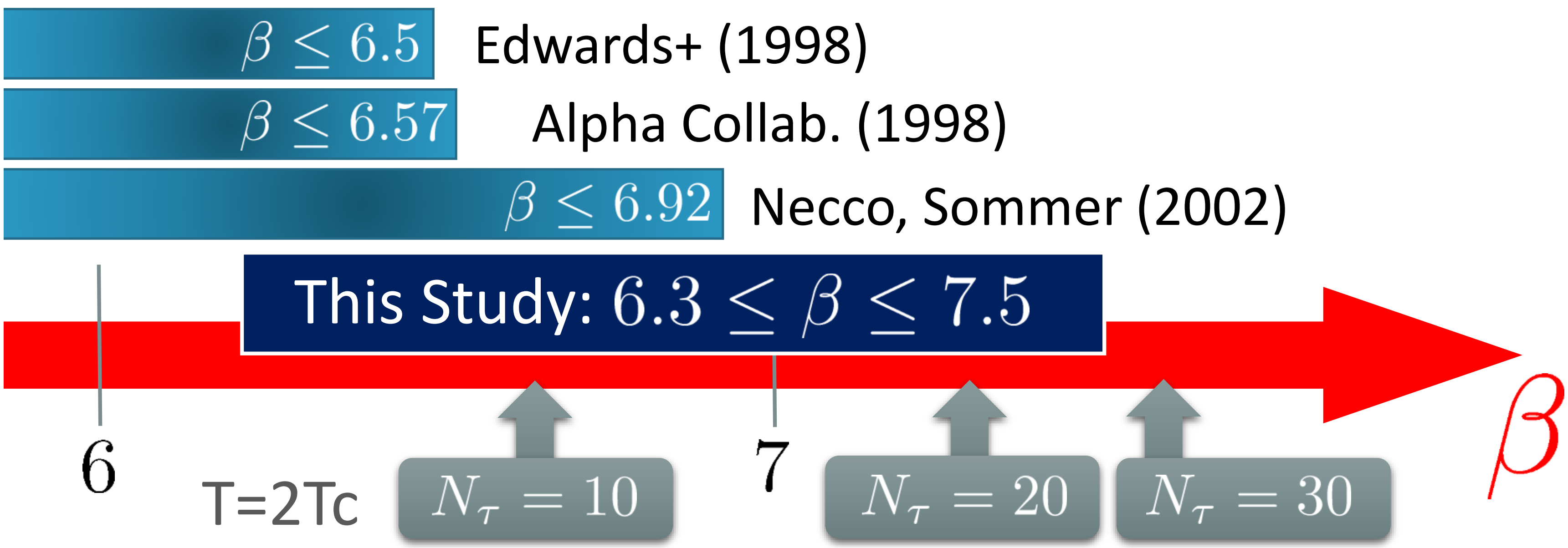
Our Choice

value in rhs is arbitrary

$$t_{0.2},\ t_{0.3},\ t_{0.4}$$

$$w_{0.2},\ w_{0.3},\ w_{0.4}$$

Lattice Spacing of SU(3) Wilson Action

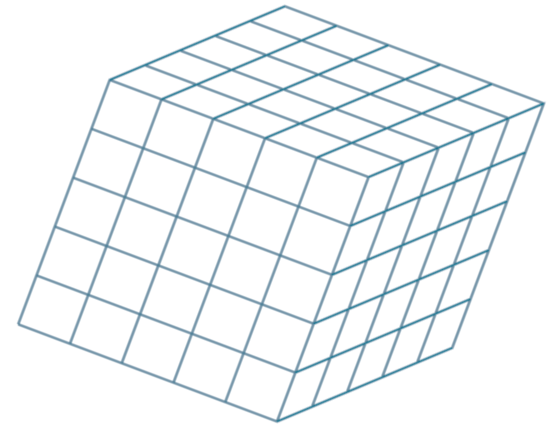


our parametrization for *a*

$$\log\left(\frac{w_{0.4}}{a}\right)(\beta) = \frac{4\pi^2}{33}\beta - 8.6853 + \frac{37.422}{\beta} - \frac{143.84}{\beta^2}$$

stat. err. < 0.4% / sys. err. < 0.7%

Numerical Setting

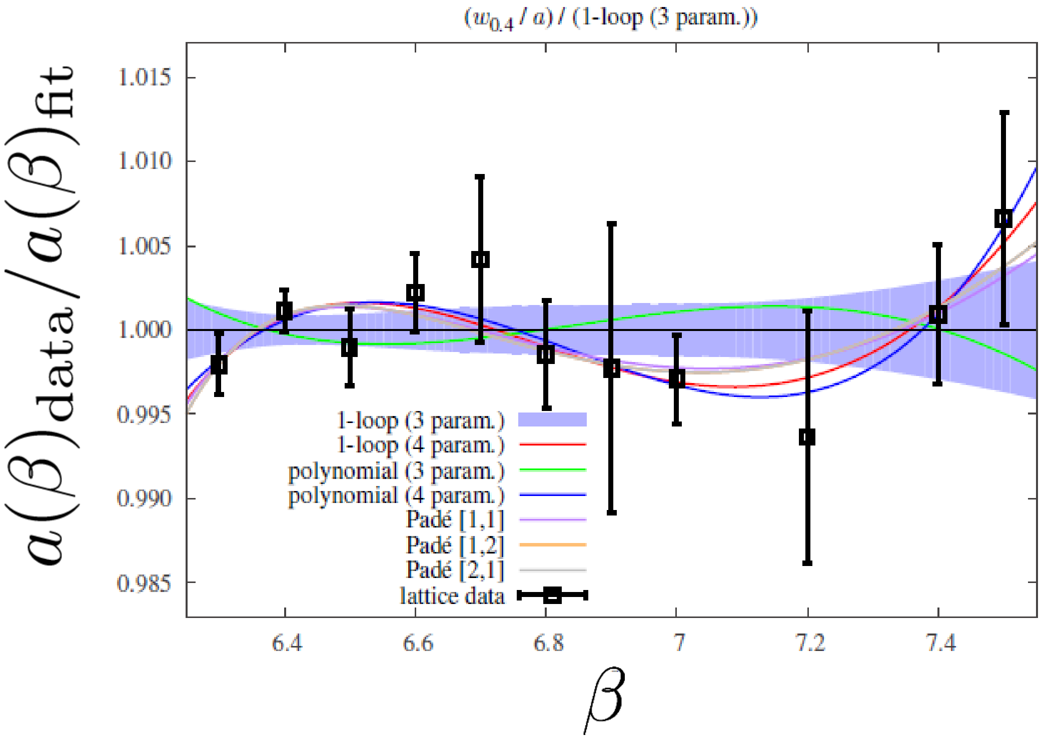


- SU(3) YM theory
- Wilson gauge action
- $w_{0.4}$ / $w_{0.2}$ scaling

b	size	N_{conf}	b	size	N_{conf}
6.3	64^4	30	6.9	64^4	30
6.4	64^4	100	7.0	96^4	60
6.5	64^4	49	7.2	96^4	53
6.6	64^4	100	7.4	128^4	40
6.7	64^4	30	7.5	128^4	60
6.8	64^4	100			

Each configuration is separated by 1000 updates (HB+OR^5)
BlueGene/Q @ KEK

Parametrization with $w_{0.4}$



- Use various fitting funcs
  → estimate sys. err.
- For β=7.4 and 7.5, $w_{0.4}$ is obtained from $w_{0.2}$.

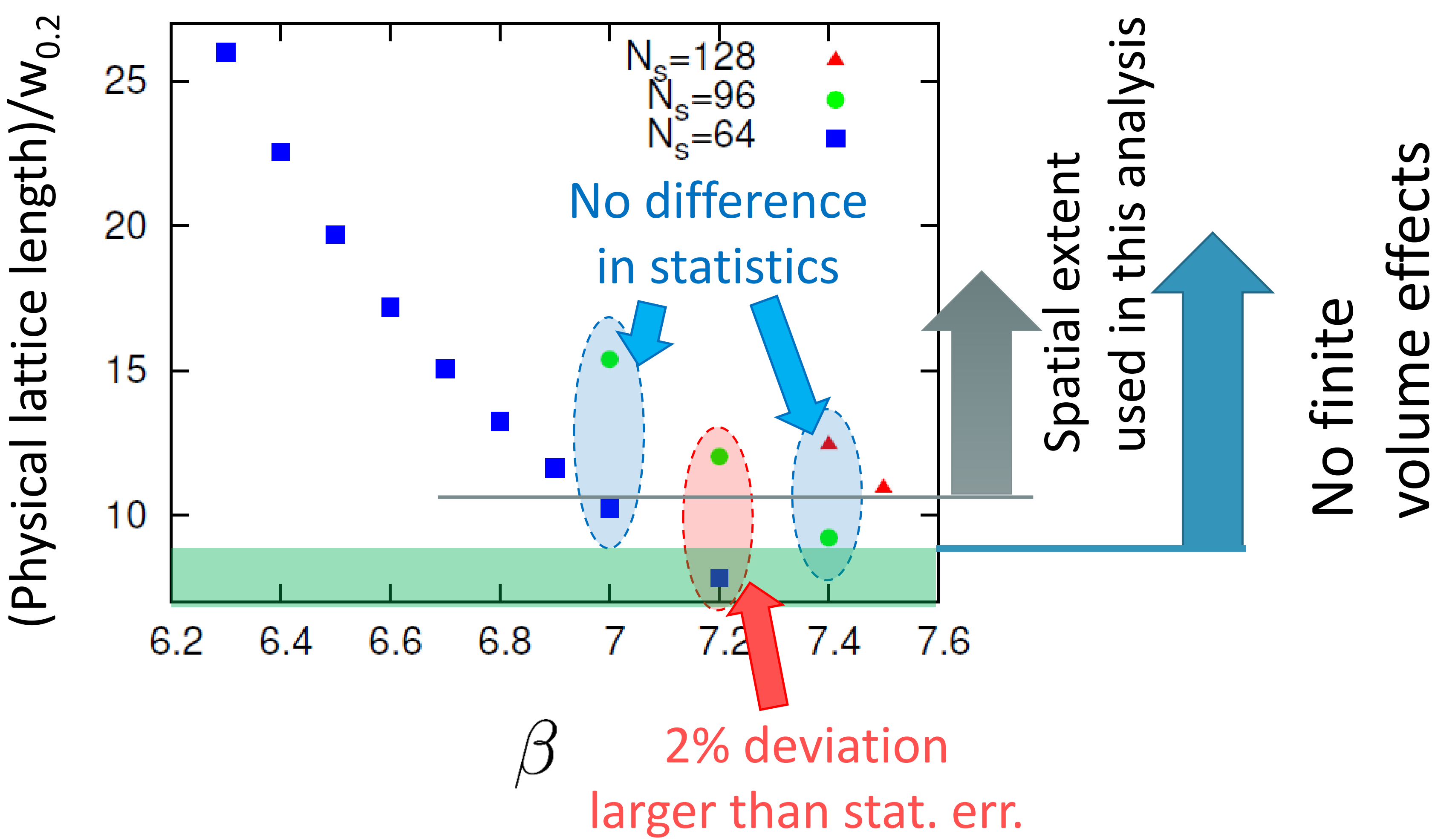
$$\log\left(\frac{w_{0.4}}{a}\right)(\beta) = \frac{4\pi^2}{33}\beta - 8.6853 + \frac{37.422}{\beta} - \frac{143.84}{\beta^2}$$

stat. err. < 0.4% / sys. err. < 0.7%

3 parameter fit

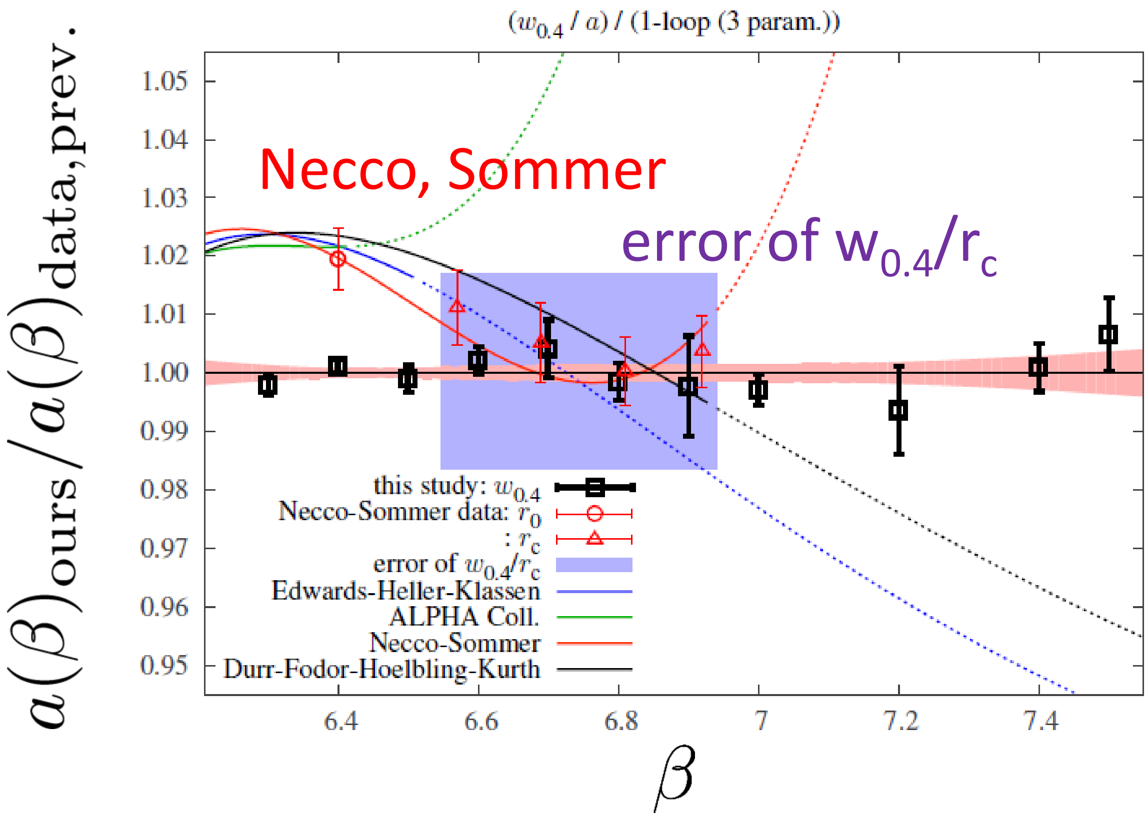
$$\chi^2/\mathrm{dof} = 0.92$$

Finite Volume Effects



No finite volume effects within statistics

Comparison with Previous Studies



Edwards, Heller, Klassen, 1998
Alpha-Collaboration, 1998
Necco, Sommer, 2002
Durr, Fodor, Hoelbling, 2007

our parametrization by $w_{0.4}$

Recent study:
Francis+, arXiv:1503.05652

- Agreements in available ranges.
- Need the analysis of the topological freezing effect.

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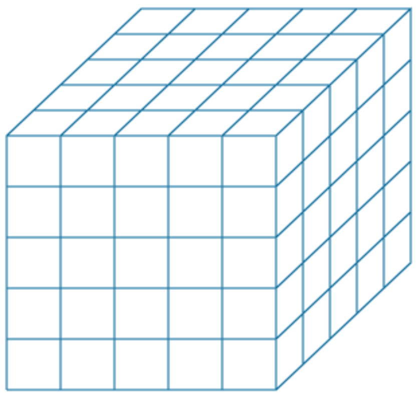
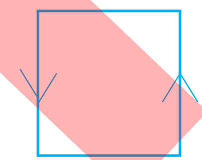
$T_{\mu\nu}$: nontrivial observable on the lattice

① Definition of the operator is nontrivial

because of the explicit breaking of Lorentz symmetry



ex: $T_{\mu\nu} = F_{\mu\rho}F_{\nu\rho} - \frac{1}{4}\delta_{\mu\nu}FF$

$F_{\mu\nu} =$ 

② Its measurement is extremely noisy

due to high dimensionality and etc.

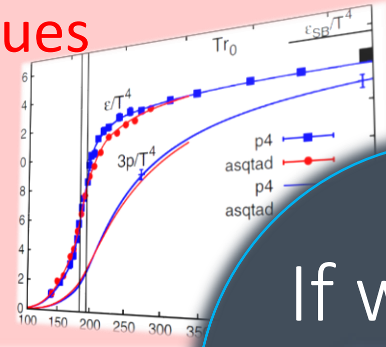
If we have

$$T_{\mu\nu}$$

Thermodynamics

direct measurement of expectation values

$$\langle T_{00}\rangle, \langle T_{ii}\rangle$$



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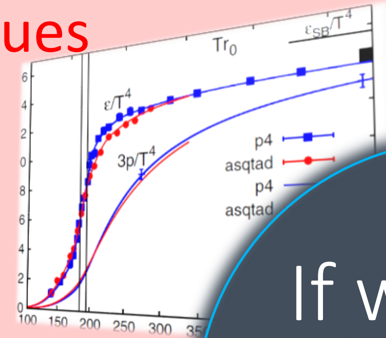

Fluctuations and Correlations

viscosity, specific heat, ...

$$\eta = \int_0^\infty dt \langle T_{12}; T_{12} \rangle$$

$$c_V \sim \langle \delta T_{00}^2 \rangle$$

If we have

$T_{\mu\nu}$

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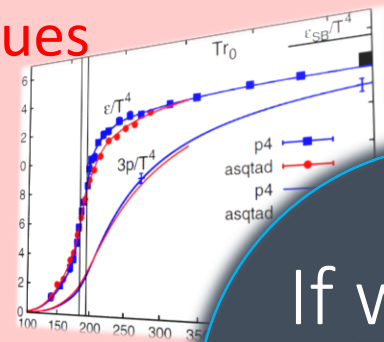

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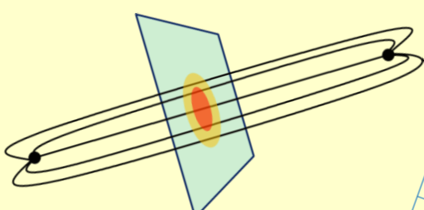
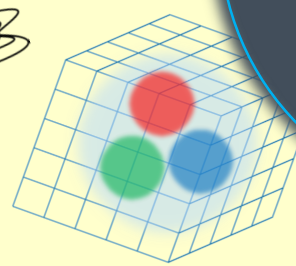
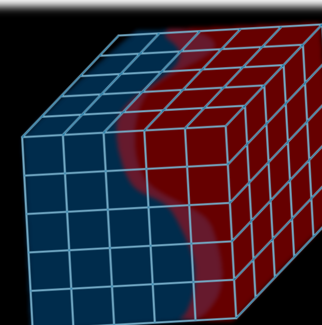




- confinement flux tube
- EM distribution in hadrons

Hadron Structure



- vacuum configuration
- mixed state on 1st transition

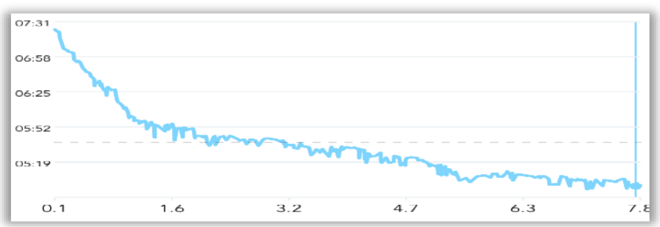
Vacuum Structure

Small Flow-time Expansion

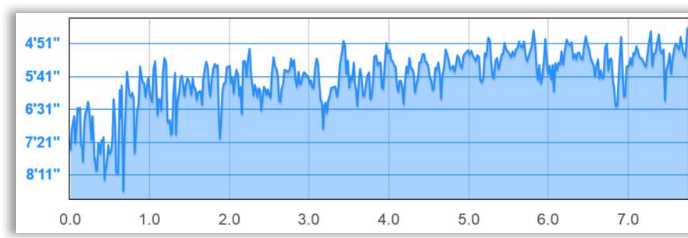
Luescher, Weisz, 2011
Suzuki, 2013

$$\tilde{\mathcal{O}}(t,x) \xrightarrow[t \to 0]{} \sum_i c_i(t)\mathcal{O}_i^R(x)$$

An operator
in blurred world



Renormalized operators
in original theory



less noisy, continuous

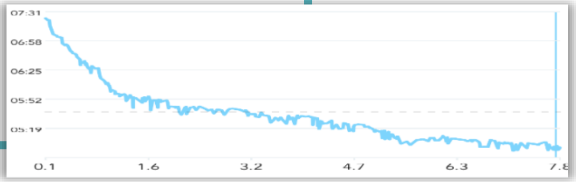
noisy, lattice discretized

SFTE of Energy-Momentum Tensor

Suzuki, 2013

- gauge-invariant dimension 4 operators

$$\begin{cases} U_{\mu\nu}(t,x) = G_{\mu\rho}(t,x)G_{\nu\rho}(t,x) - \frac{1}{4}\delta_{\mu\nu}G_{\mu\nu}(t,x)G_{\mu\nu}(t,x) \\ E(t,x) = \frac{1}{4}\delta_{\mu\nu}G_{\mu\nu}(t,x)G_{\mu\nu}(t,x) \end{cases}$$



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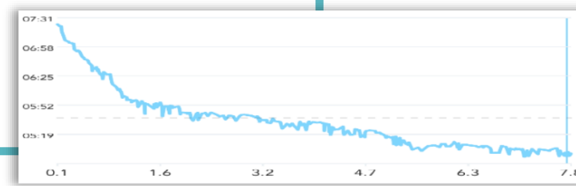
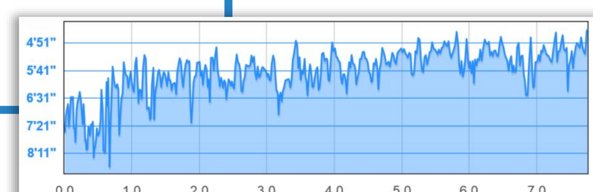


$$U_{\mu\nu}(t,x) = \alpha_U(t)\left[T^R_{\mu\nu}(x) - \frac{1}{4}\delta_{\mu\nu}T^R_{\rho\rho}(x)\right] + \mathcal{O}(t)$$

$$E(t,x) = \langle E(t,x)\rangle + \alpha_E(t)T^R_{\rho\rho}(x) + \mathcal{O}(t)$$



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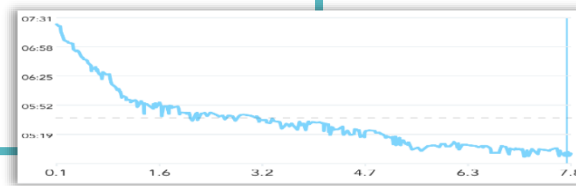


Remormalized EMT

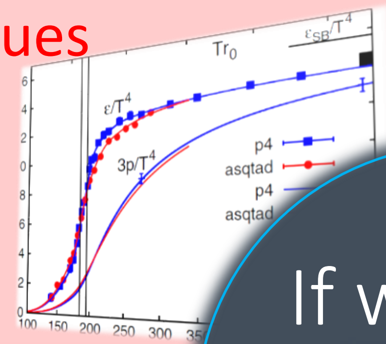
$$T^R_{\mu\nu}(x) = \lim_{t\to 0}\left[\frac{1}{\alpha_U(t)}U_{\mu\nu}(t,x) + \frac{\delta_{\mu\nu}}{4\alpha_E(t)}E(t,x)_{\text{subt.}}\right]$$

Suzuki coeffs. $\begin{cases} \alpha_U(t) = g^2\left[1 + 2b_0 s_1 g^2 + O(g^4)\right] \\ \alpha_E(t) = \frac{1}{2b_0}\left[1 + 2b_0 s_2 g^2 + O(g^4)\right] \end{cases}$ $g = g(\sqrt{8t})$

Thermodynamics

direct measurement of expectation values

$\langle T_{00} \rangle, \langle T_{ii} \rangle$



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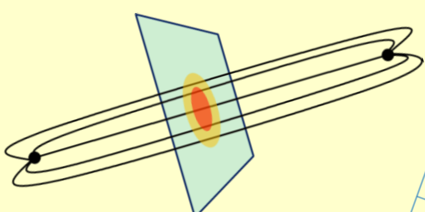

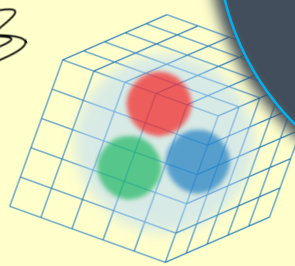

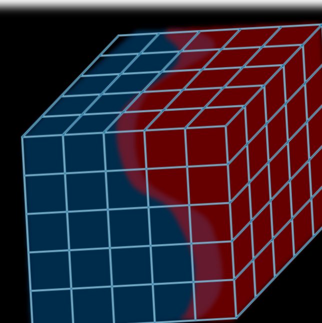

- confinement string
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Hadron Structure

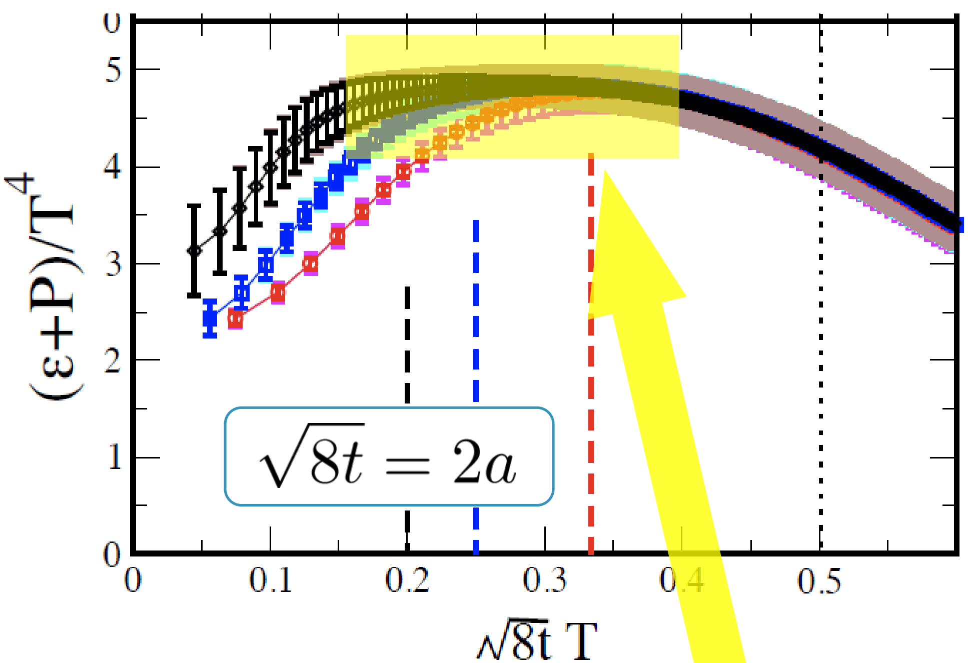
- vacuum configuration
- mixed state on 1st transition

Vacuum Structure

SU(3) Thermodynamics with Nt=6, 8, 10

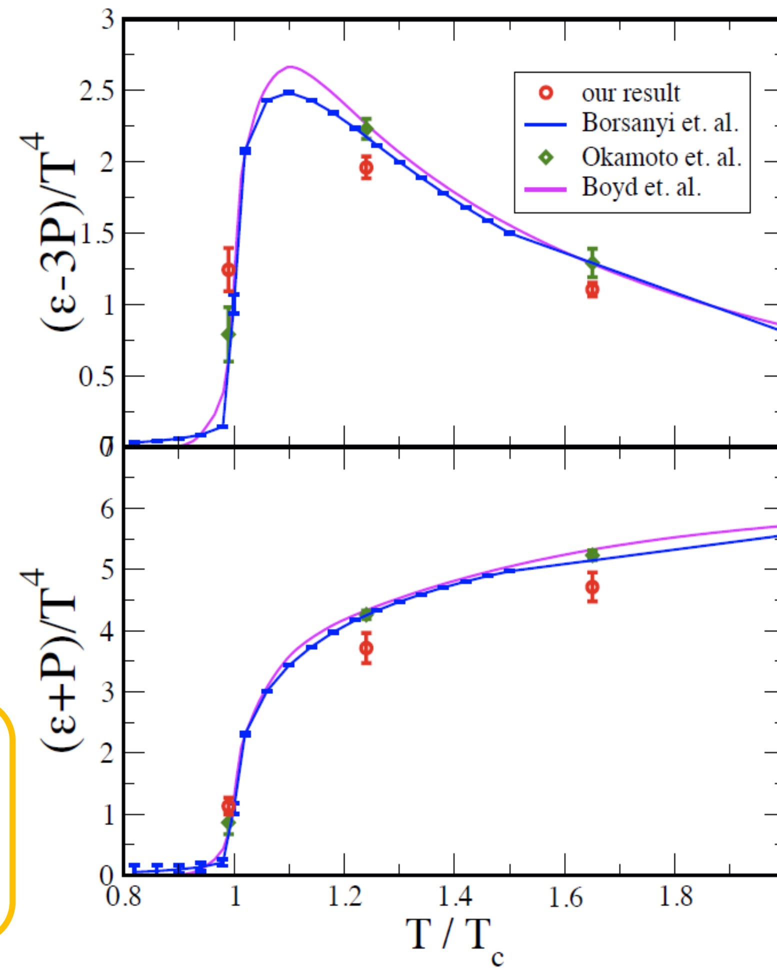
FlowQCD, PRD90,011501 (2014)

Entropy density T=1.65Tc



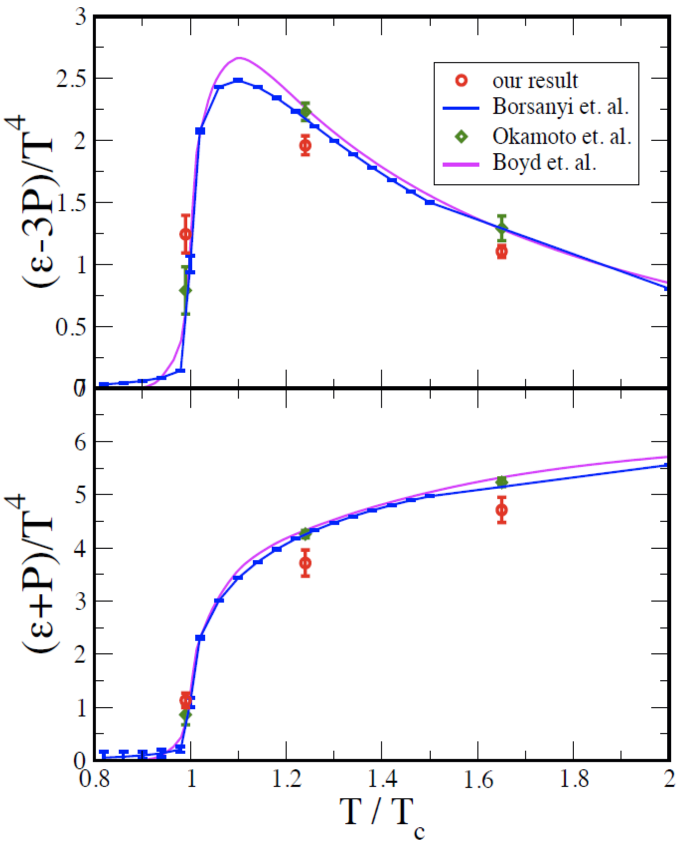
Nt=6,8,10
~300confs.

Stable region for
$2a < \sqrt{8t} < 0.4T^{-1}$

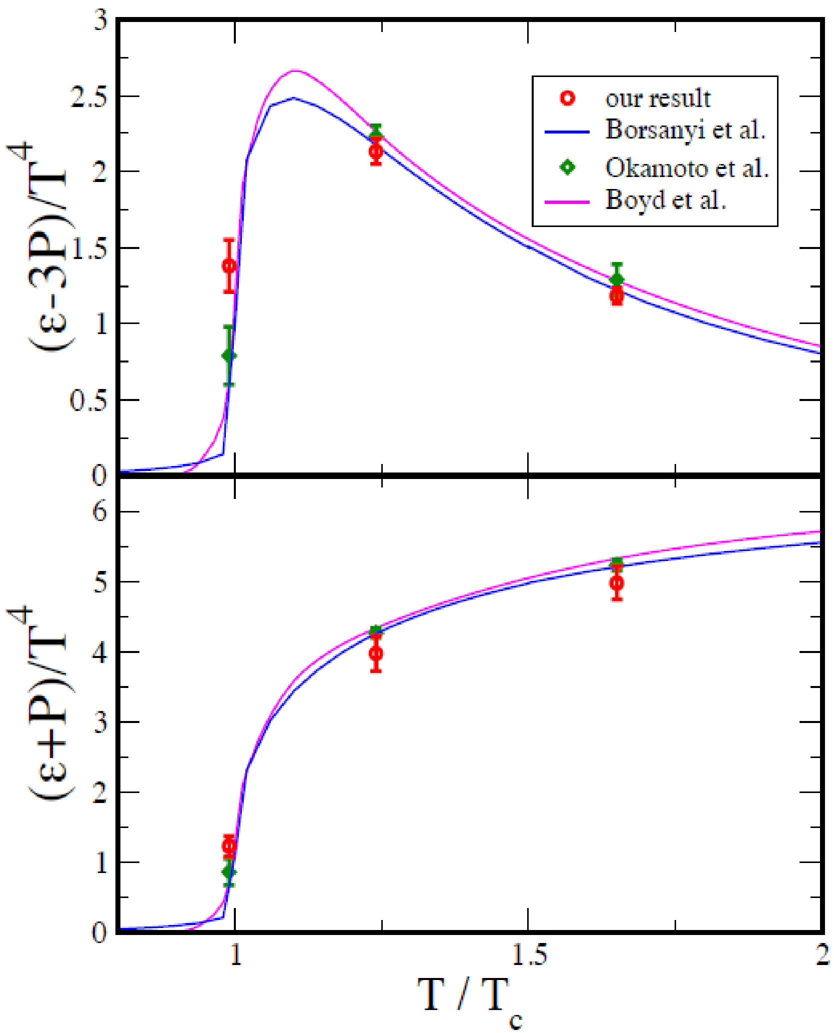


An Erratum: Error in Suzuki Coefficients

FlowQCD, PRD90,011501 (2014)



Modified (in erratum)



$$\begin{cases} \alpha_U(t) = g^2 \left[1 + 2b_0 s_1 g^2 + O(g^4)\right] \\ \alpha_E(t) = \dfrac{1}{2b_0} \left[1 + 2b_0 s_2 g^2 + O(g^4)\right] \end{cases}$$

Simulation on Fine Lattices: Nt = 12 - 32

Simulation for T=1.66Tc

for $(e-3p)/T^4$

Nt	beta	N_{conf}(T>0/vac)
12	6.719	2000/700
16	6.941	1680/830
20	7.117	2000/1020

- 5.33<Ns/Nt<6.4
- need vacuum simulation

for $(e+p)/T^4$

Nt	Ns/Nt	beta	Nconf
12	5.33	6.719	2k
16	16	6.941	20k
20	9.6	7.117	22k
24	8	7.265	20k
32	6	7.500	18k

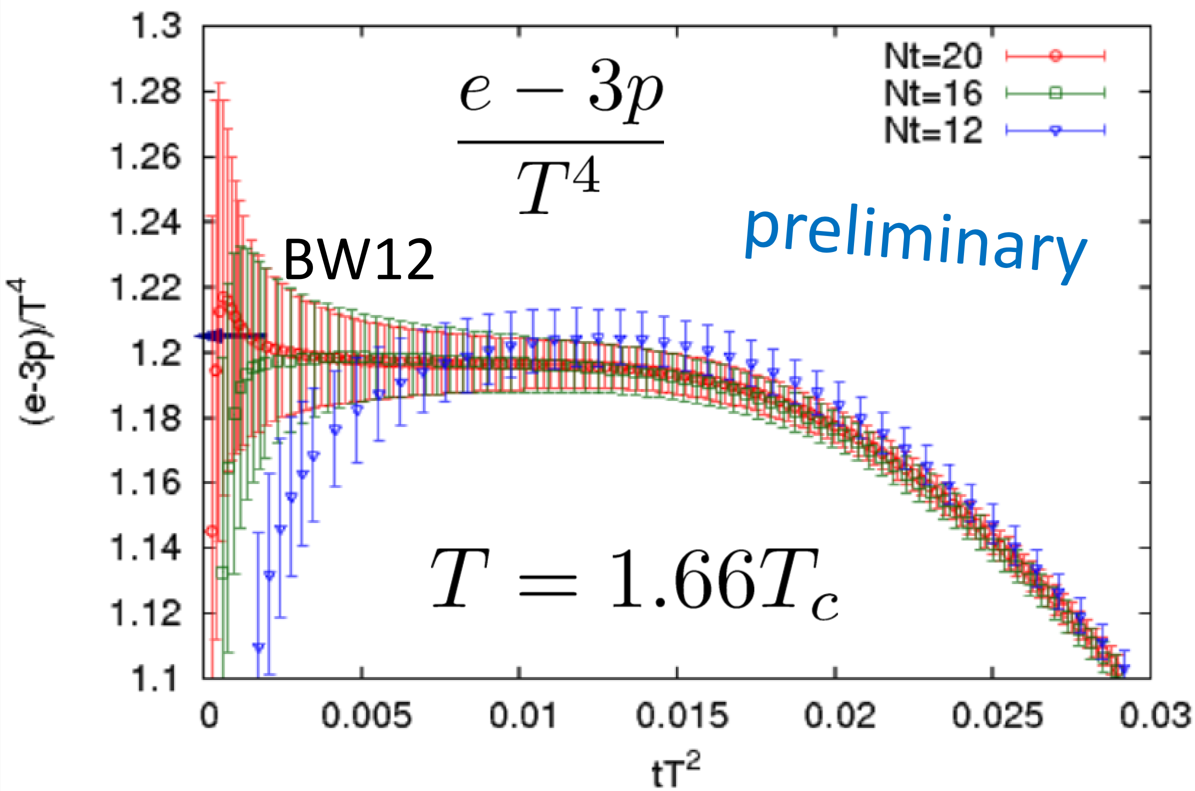
- no vacuum simulation required

FlowQCD, in prep.

New Results: Thermodynamics (e-3p)

$$\tilde{T}_{\mu\nu}(t) = \frac{1}{\alpha_U(t)} U_{\mu\nu}(t) + \frac{\delta_{\mu\nu}}{4\alpha_E(t)} E(t)_{\rm subt.}$$

FlowQCD, in prep.

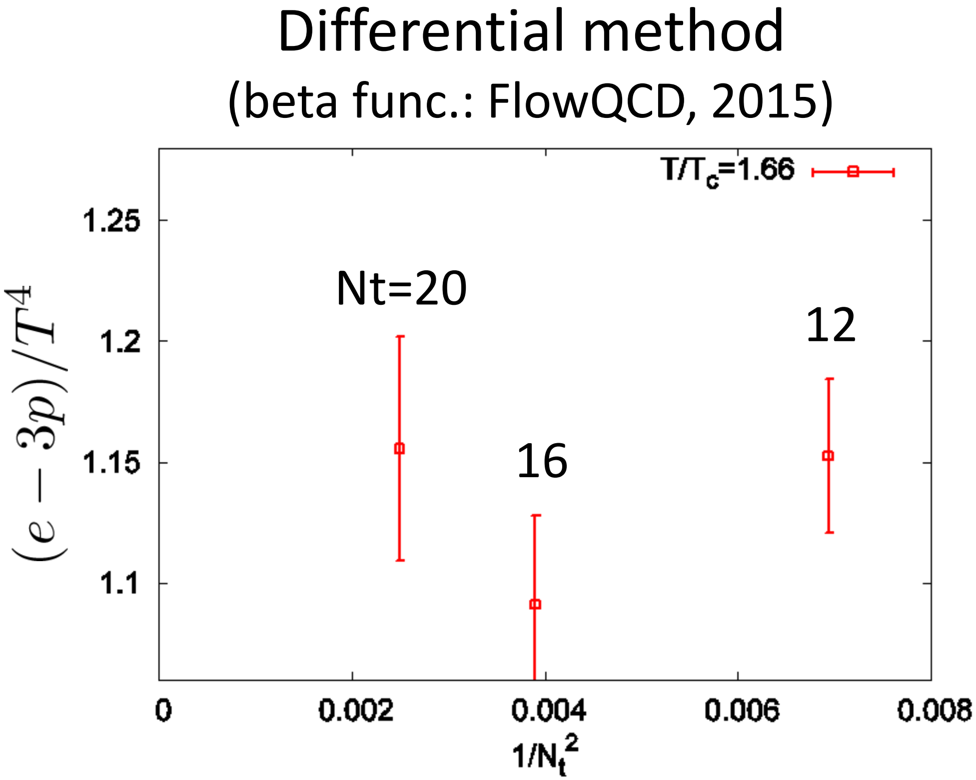


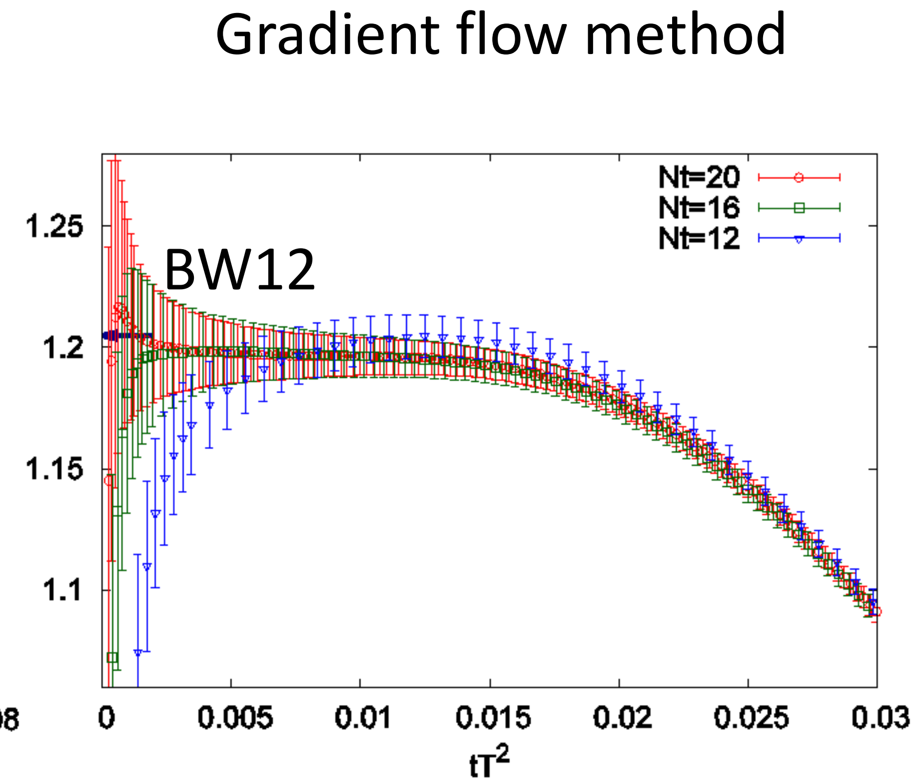
- Good agreement with previous study
- Stable plateau for

$$tT^2 < 0.015$$
$$(\sqrt{8t} < 0.35T^{-1})$$

BW12: Budapest-Wuppertal, 2012

v.s. Conventional Methods (e-3p, 1.66Tc)





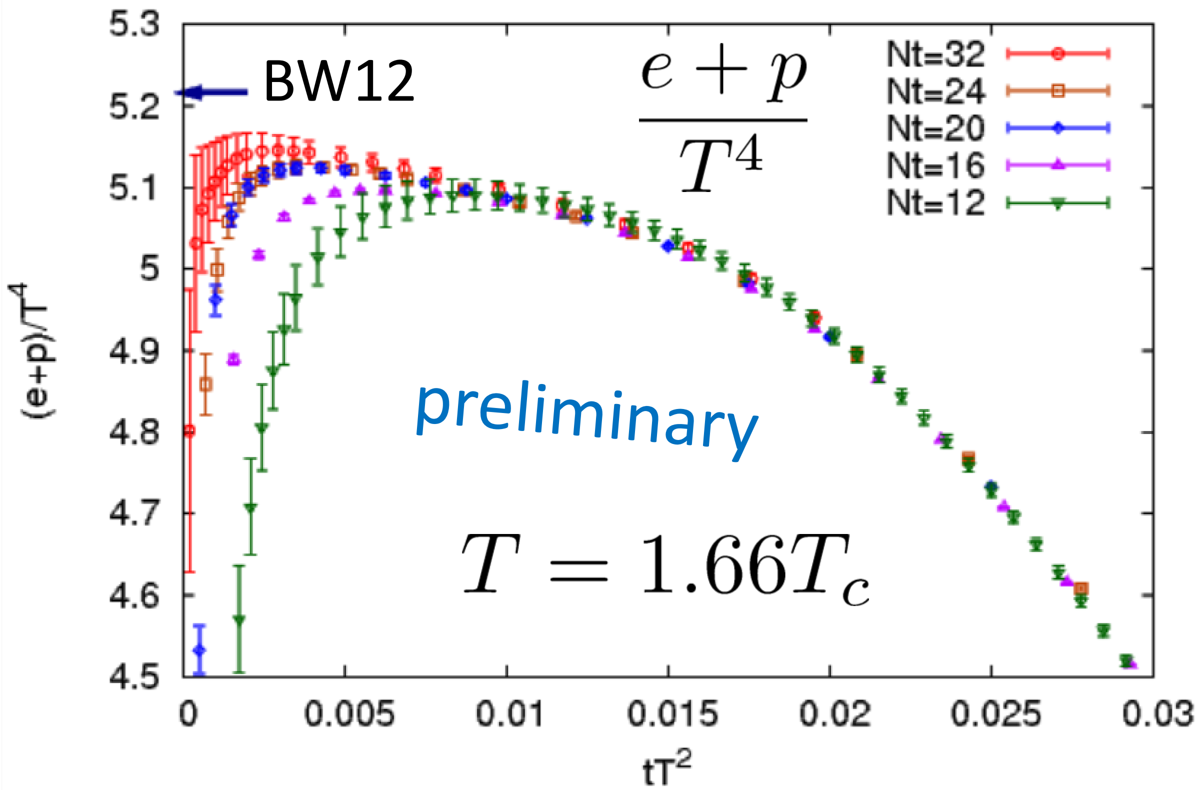
- A consistent result for two methods
- Smaller error in gradient flow method

New Results: Thermodynamics (e+p)

$$\tilde{T}_{\mu\nu}(t) = \frac{1}{\alpha_U(t)} U_{\mu\nu}(t) + \frac{\delta_{\mu\nu}}{4\alpha_E(t)} E(t)_{\rm subt.}$$

FlowQCD, in prep.

$$T^R_{\mu\nu} = \tilde{T}_{\mu\nu}(t) + O(t)$$



- ☐ Existence of O(t) effect
- ☐ Linear behavior for $tT^2 < 0.015$ $(\sqrt{8t} < 0.35T^{-1})$

- Double limit (a→0, t→0) has to be taken.

BW12: Budapest-Wuppertal, 2012

New Results: Thermodynamics (e+p)

$$\tilde{T}_{\mu\nu}(t) = \frac{1}{\alpha_U(t)} U_{\mu\nu}(t) + \frac{\delta_{\mu\nu}}{4\alpha_E(t)} E(t)_{\text{subt.}}$$

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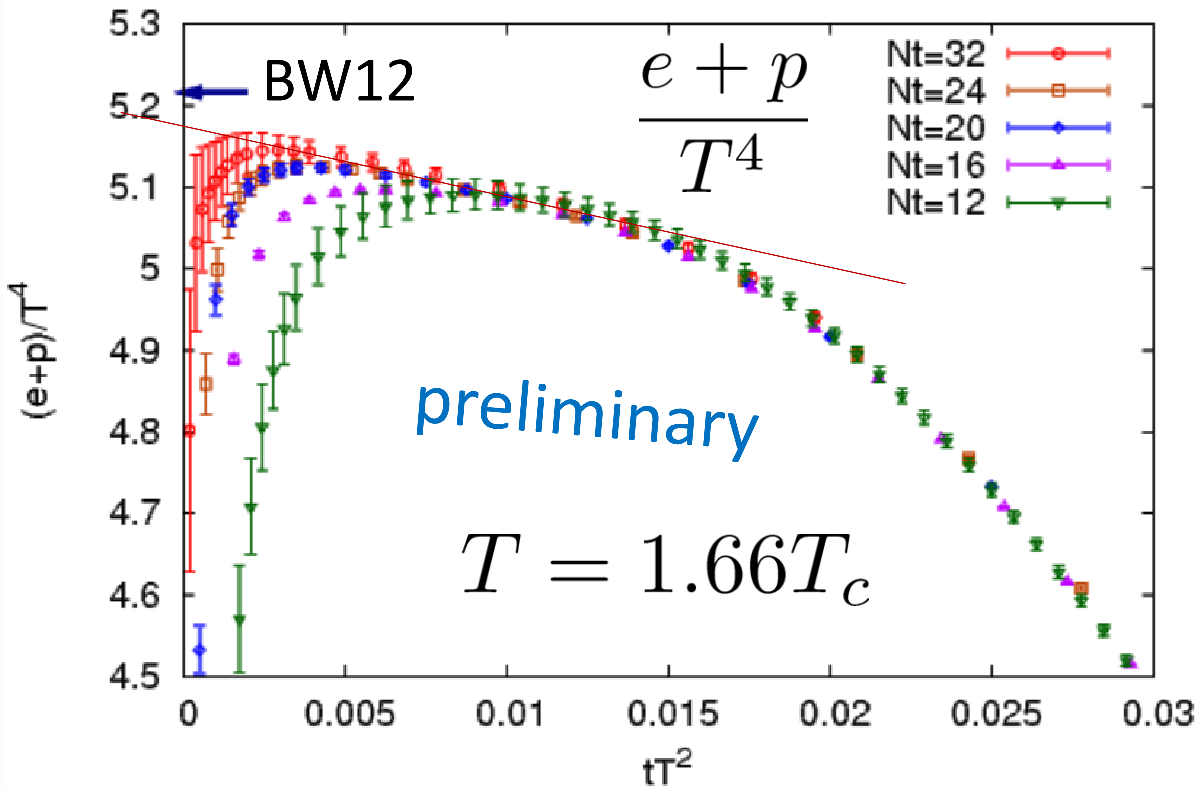

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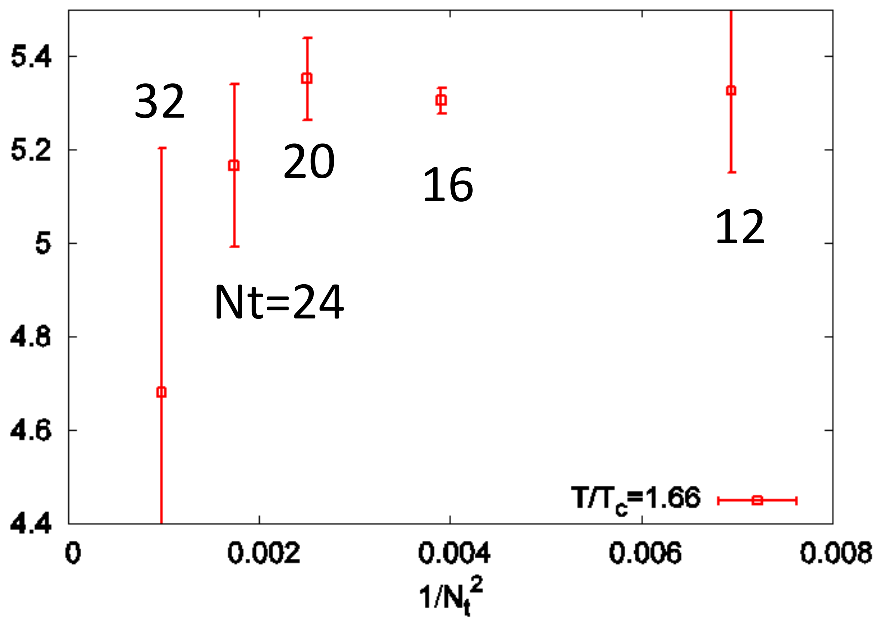
- Double limit (a→0, t→0) has to be taken.

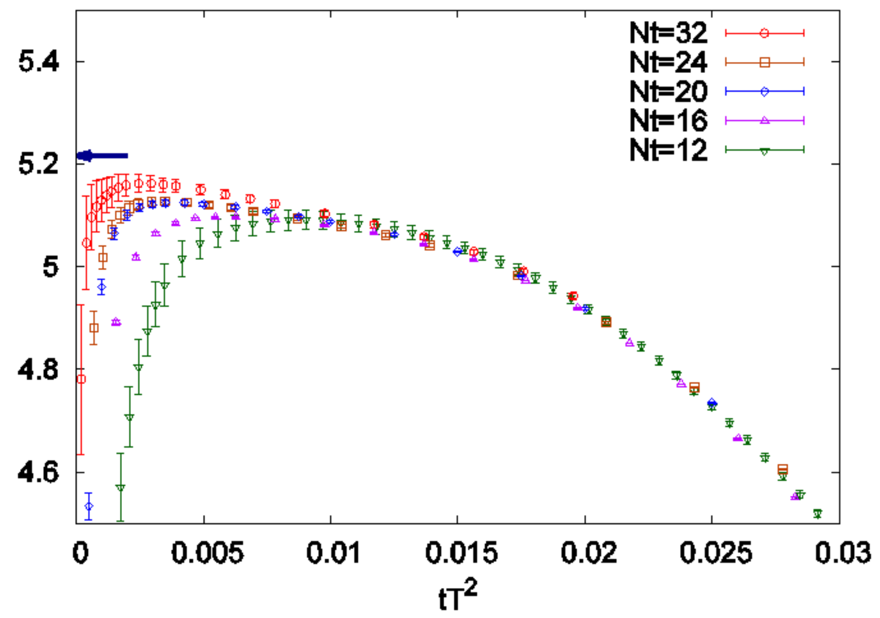
BW12: Budapest-Wuppertal, 2012

v.s. Differential Method (e+p)

Differential method
(Karsch coeffs.: Karsch+, 2000)



Gradient flow method

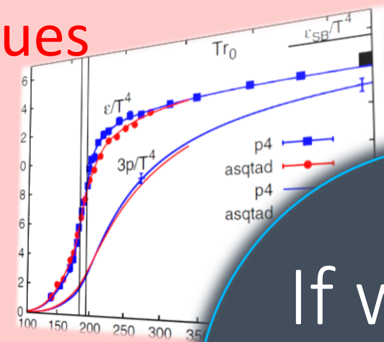


- A consistent result for two methods
  - deviation may be attributed to c_σ
- Smaller error in gradient flow method
  - the advantage become more prominent on finer lattices

Thermodynamics

direct measurement of expectation values

$\langle T_{00} \rangle, \langle T_{ii} \rangle$



Fluctuations and Correlations

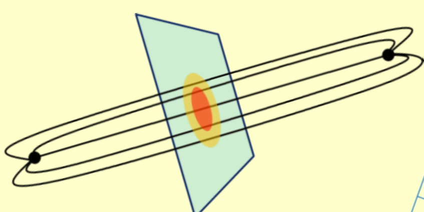
viscosity, specific heat, ...

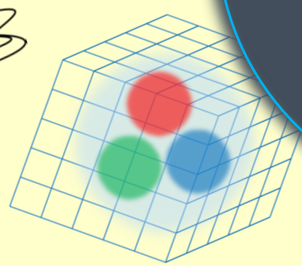
$$\eta = \int_0^\infty dt \langle T_{12}; T_{12} \rangle$$

$$c_V \sim \langle \delta T_{00}^2 \rangle$$

If we have

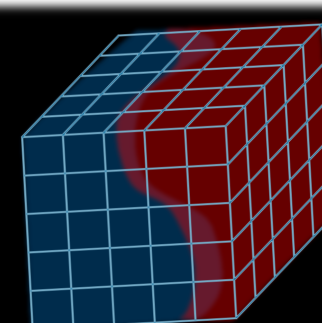
$$T_{\mu\nu}$$





- ➢ confinement string
- ➢ EM distribution in hadrons

Hadron Structure



- ➢ vacuum configuration
- ➢ mixed state on 1st transition

Vacuum Structure

EMT Correlator

- Kubo Formula: T_{12} correlator ←→ shear viscosity

$$\eta = \int_0^\infty dt \int_0^{1/T} d\tau \int d^3x \langle T_{12}(x, -i\tau) T_{12}(0,t) \rangle$$

  - Hydrodynamics describes long range behavior of $T_{\mu\nu}$

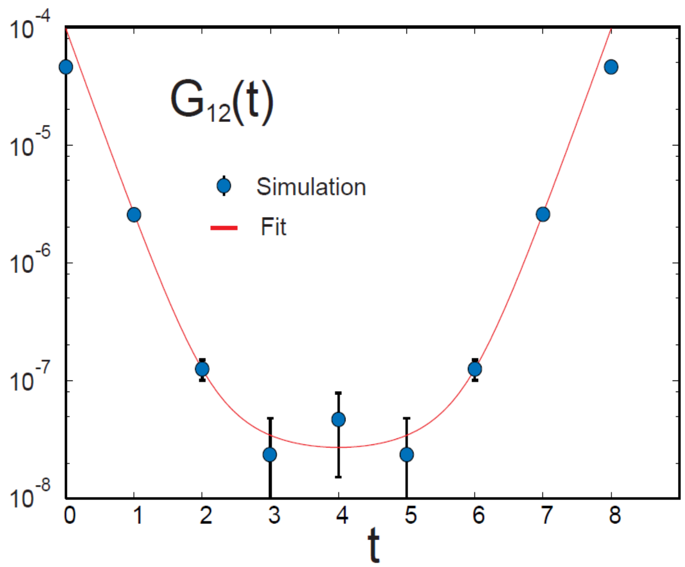
- Energy fluctuation ←→ specific heat

$$c_V = \frac{\langle \delta E^2 \rangle}{VT^2}$$

EMT Correlator : Extremely Noisy…

With naïve EMT operators

$$\langle T_{12}(\tau) T_{12}(0) \rangle$$



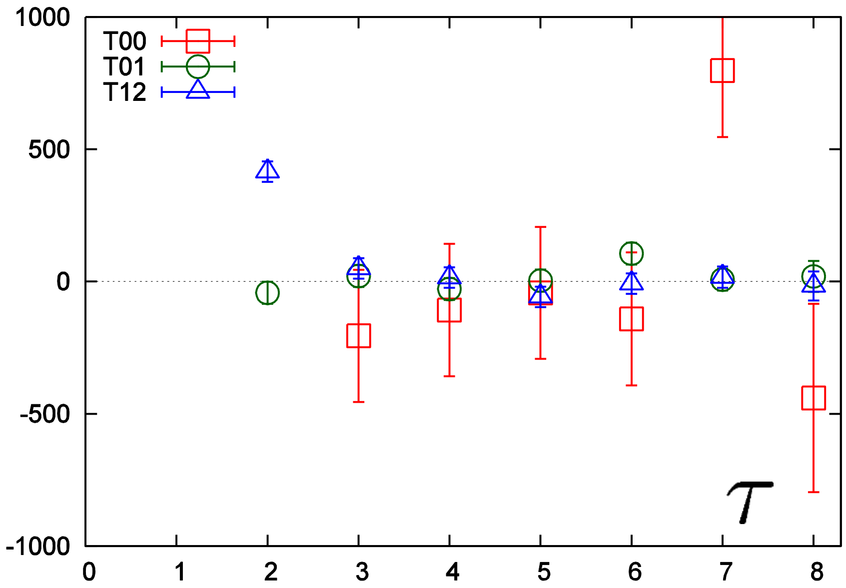
Nakamura, Sakai, PRL,2005

N_t=8

improved action

~10^6 configurations

$$\langle \delta T_{\mu\nu}(\tau) \delta T_{\mu\nu}(0) \rangle$$



Nt=16

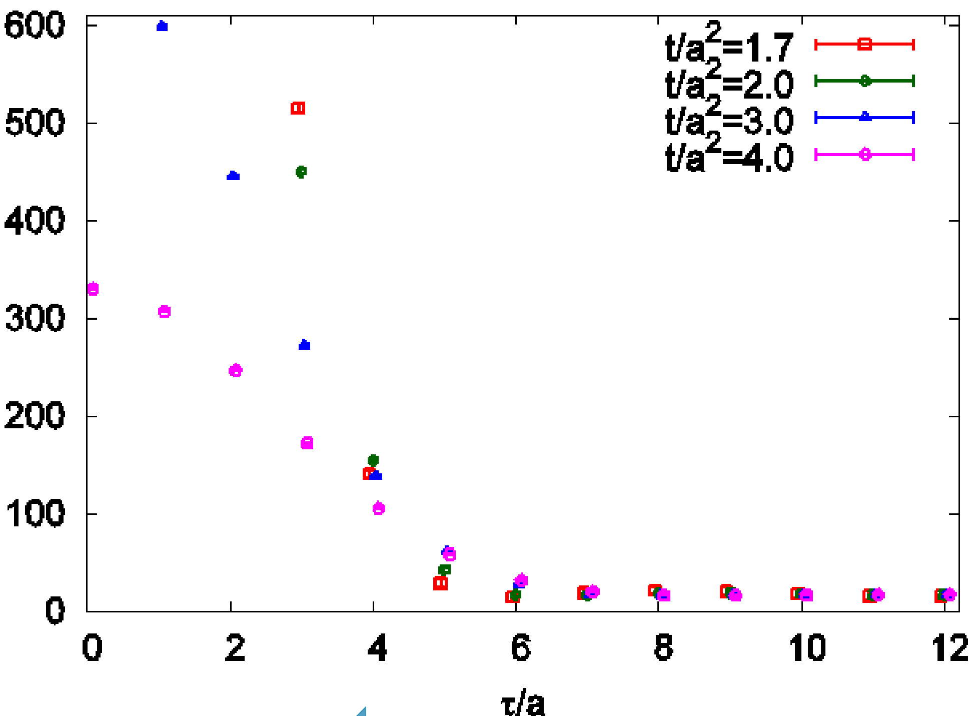
standard action

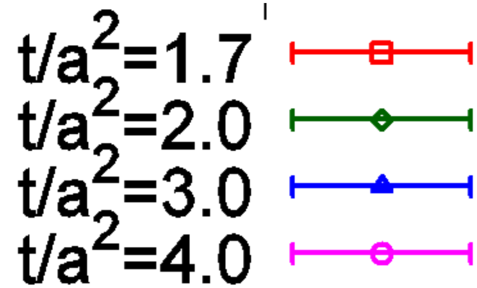
50k configurations

… no signal

Correlation Functions

$$\langle \delta T_{00}(\tau) \delta T_{00}(0) \rangle / T^5$$





Smeared

$\tau < 2\sqrt{2t}$

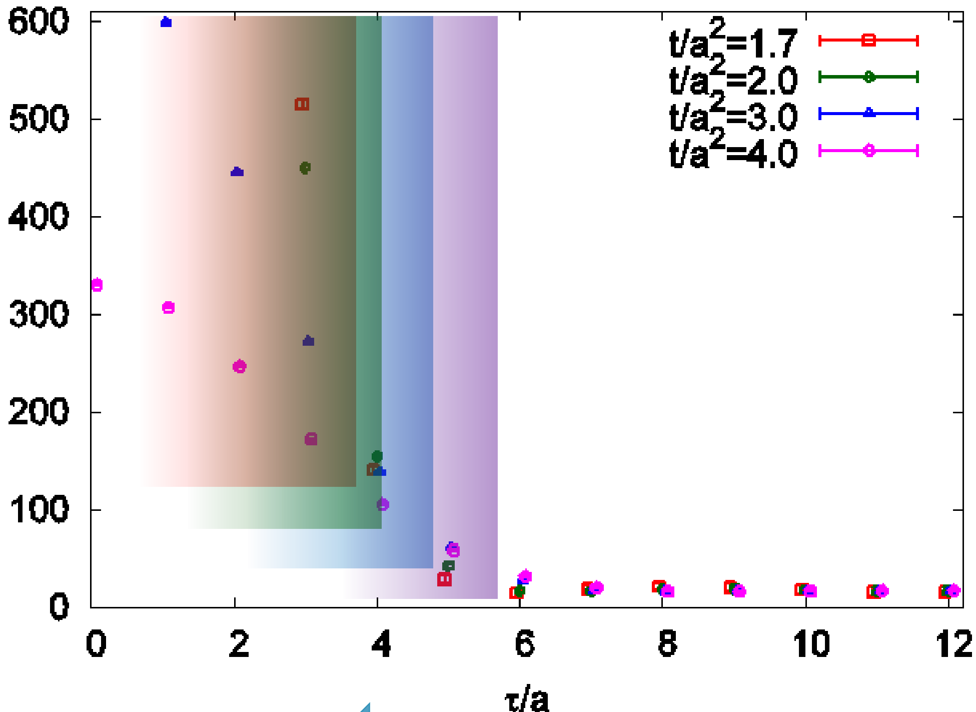
T=1.66Tc

96^3x24

50k confs

Correlation Functions

$$\langle \delta T_{00}(\tau) \delta T_{00}(0) \rangle / T^5$$



$t/a^2=1.7$
$t/a^2=2.0$
$t/a^2=3.0$
$t/a^2=4.0$

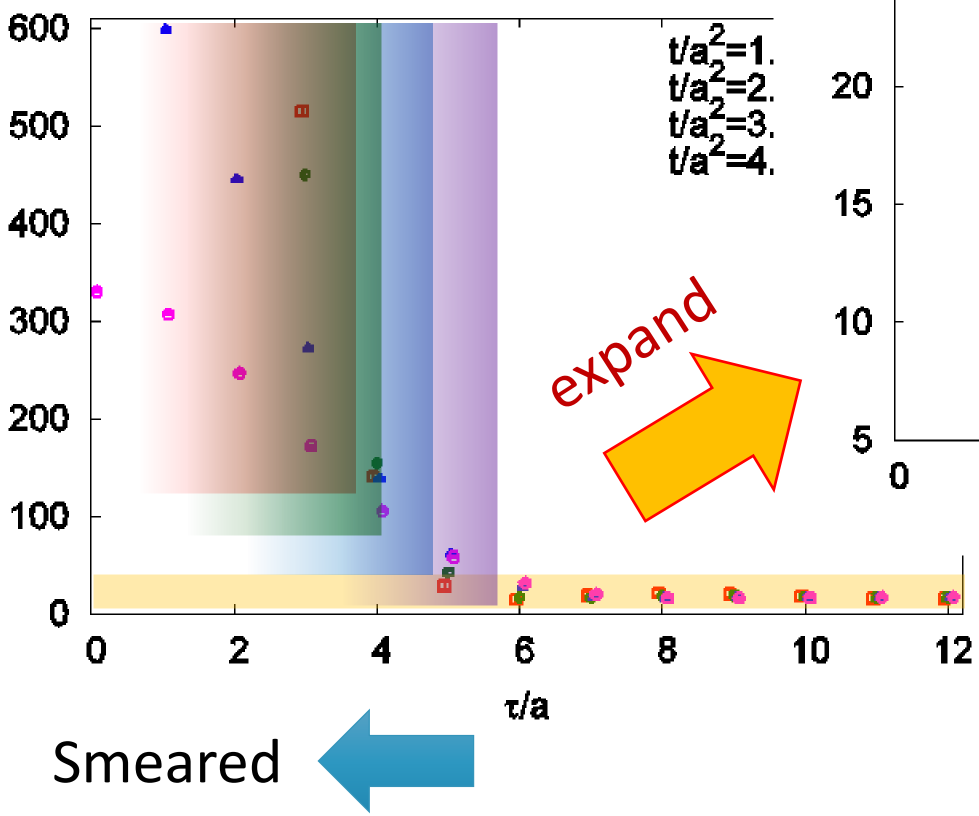
Smeared

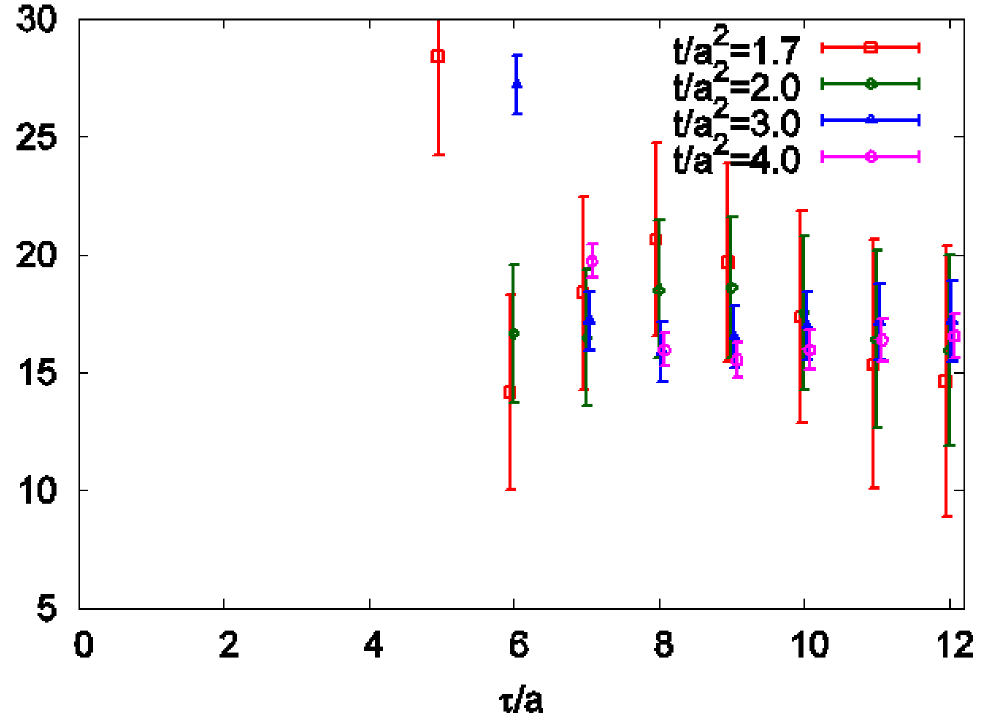
$\tau < 2\sqrt{2t}$

T=1.66Tc
96^3x24
50k confs

Correlation Functions

$$\langle \delta T_{00}(\tau)\delta T_{00}(0)\rangle / T^5$$





Smeared

$\tau < 2\sqrt{2t}$

T=1.66Tc

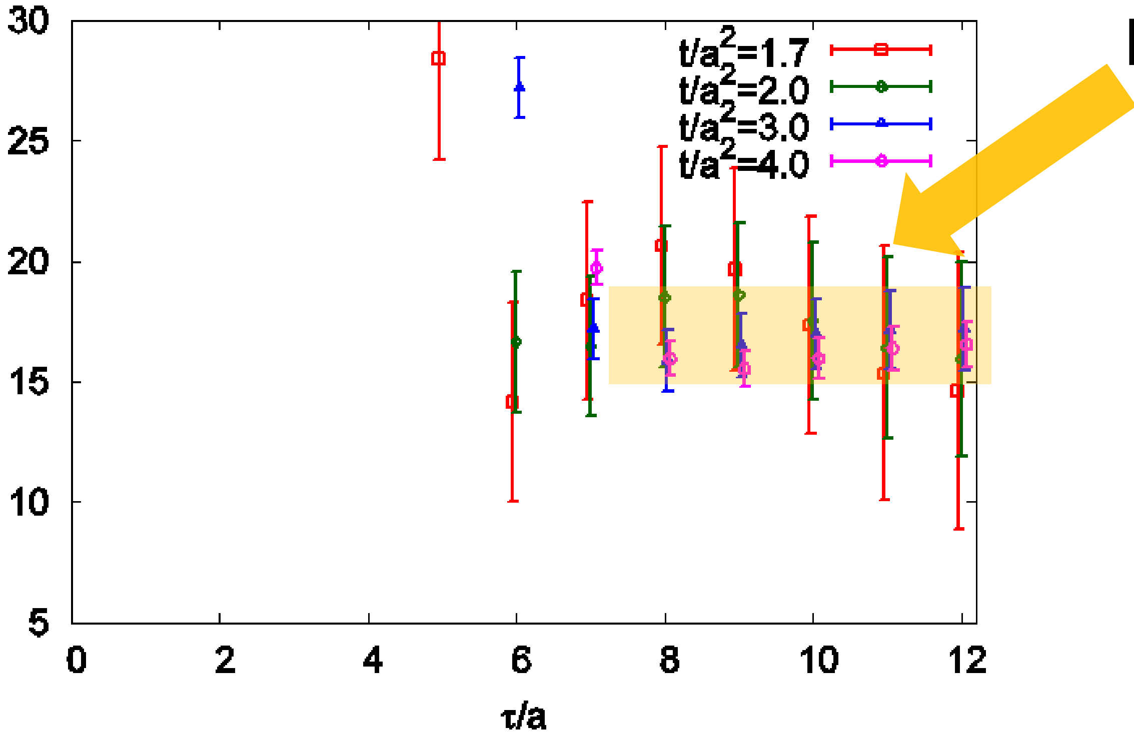
96^3x24

50k confs

Energy Correlation Function

T=1.66Tc
96^3x24
50k confs

$$\langle \delta T_{00}(\tau) \delta T_{00}(0) \rangle / T^5$$

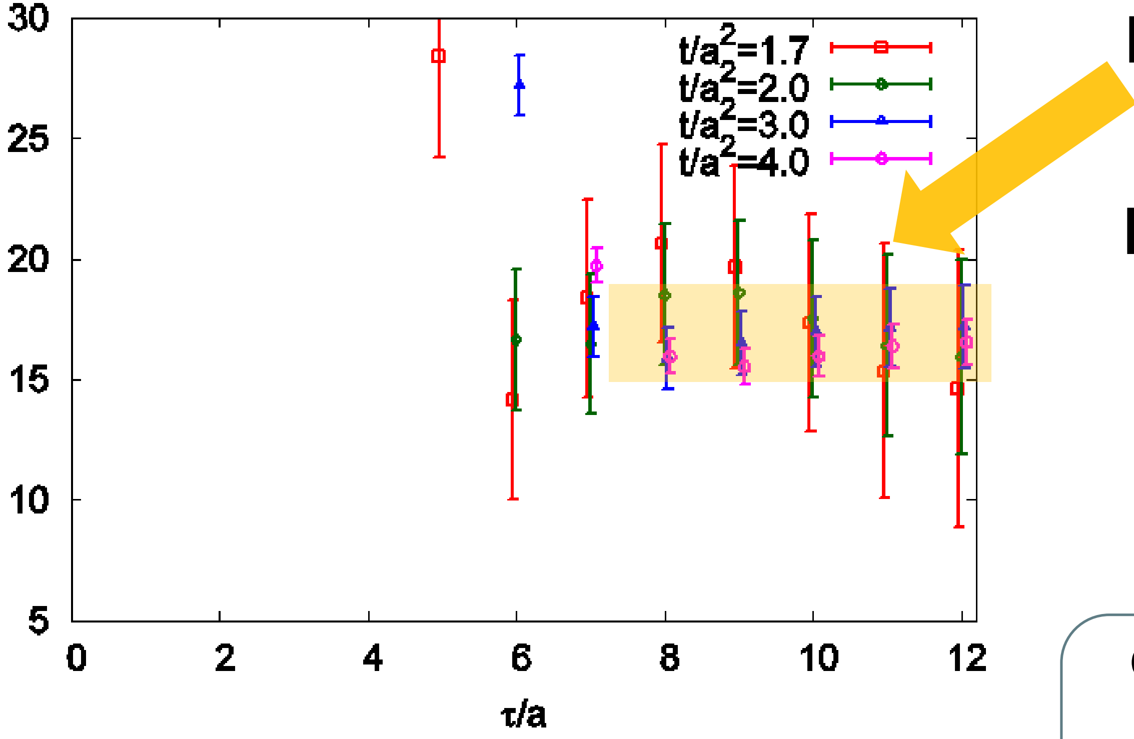


□ τ independent const.
→ energy conservation

Energy Correlation Function

T=1.66Tc
96^3x24
50k confs

$$\langle \delta T_{00}(\tau) \delta T_{00}(0) \rangle / T^5$$



- τ independent const.
  → energy conservation
- specific heat

$$c_V = \frac{\langle \delta E^2 \rangle}{VT^2}$$

→ Novel approach to measure specific heat!

Gavai, Gupta, Mukherjee, 2005

$$c_V/T^3 = 15(1) \quad T/T_c = 2$$

$$= 18(2) \quad T/T_c = 3$$

differential method / cont lim.

Gradient Flow for **Full QCD**

1. Lattice spacing / reference scales

2. Energy-momentum tensor
   - thermodynamics
   - correlation functions

3. Topology

4. Running coupling

Gradient Flow for **Full QCD**

only with gradient flow for gauge field

1. Lattice spacing / reference scales Possible (BMW,2012)

2. Energy-momentum tensor
   - thermodynamics
   - correlation functions

3. Topology Possible

4. Running coupling Possible

Gradient Flow for **Full QCD**

only with gradient flow for gauge field

1. Lattice spacing / reference scales Possible (BMW,2012)
2. Energy-momentum tensor
   - thermodynamics
   - correlation functions

   Gradient flow for **fermion field** is needed as well as SFTE
3. Topology Possible
4. Running coupling Possible

Gradient Flow for Fermion Field

A choice

$$\begin{cases} \partial_t \psi(t) = D_\mu D_\mu \psi(t) \\ \partial_t \bar{\psi}(t) = \bar{\psi}(t) \overleftarrow{D}_\mu \overleftarrow{D}_\mu \end{cases}$$

Luscher, 2013

Gradient Flow for Fermion Field

A choice
$$\begin{cases} \partial_t \psi(t) = D_\mu D_\mu \psi(t) \\ \partial_t \bar{\psi}(t) = \bar{\psi}(t) \overleftarrow{D}_\mu \overleftarrow{D}_\mu \end{cases}$$

Luscher, 2013

Fermion propagator

$$\langle \psi(t_1, x) \bar{\psi}(t_2, y) \rangle = \int dx' dy' K(t_1, x; 0, x') S(x', y') K(0, y'; t_2, y)$$

- K: "fundamental solution" $(\partial_t - D_\mu D_\mu) K = 0$
- S: propagator at t=0

- Study of chiral condensate Luscher, 2013
- Application to QCD thermodynamics: just started

by FlowQCD + WHOT QCD = FloWHOT Collaboration

Summary

YM Gradient Flow

$$\partial_t A_\mu = D_\nu G_{\mu\nu}$$

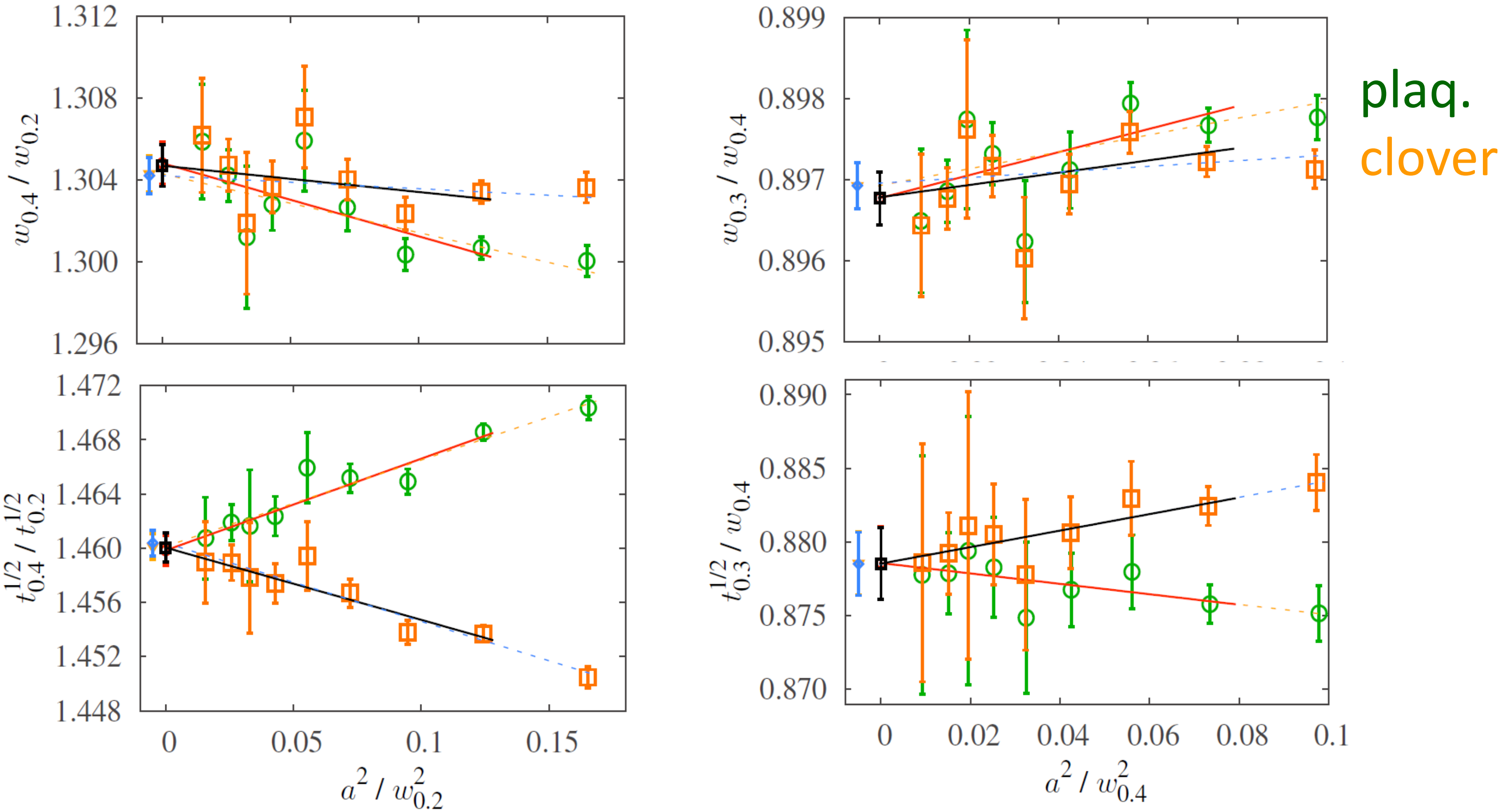
- A smoothing transformation of gauge field
- Many applications: scale setting, thermodynamics, …

Many future studies

- EMT correlation functions → transport coefficients, etc.
- Topological property of gauge theory
- Flow for fermion field → Full QCD thermodynamics

Backup

Various Reference Scales



$\sqrt{t_{0.4}}/w_{0.4}$	$\sqrt{t_{0.3}}/w_{0.4}$	$\sqrt{t_{0.2}}/w_{0.4}$	$w_{0.3}/w_{0.4}$	$w_{0.2}/w_{0.4}$
1.0164(32)(3)	0.8785(24)(0)	0.6952(18)(2)	0.8968(3)(2)	0.7665(6)(2)

$r_c/w_{0.4}$	$r_0/w_{0.4}$	$\sqrt{\sigma}w_{0.4}$	$T_c w_{0.4}$	$w_{0.4}\Lambda_{\overline{\mathrm{MS}}}$
1.328(21)(7)	2.587(45)	0.455(8)	0.285(5)	0.233(19)