Non-Gaussian Fluctuations in Relativistic Heavy-Ion Collisions

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Sakaida, Asakawa, Fujii, MK, to appear in PRC
Sakaida, Asakawa, MK, PRC90, 064911 (2014)

Nagoya Seminar, Nagoya U., 31/May/2017
Beam-Energy Scan

\begin{figure}
\centering
\begin{tikzpicture}
\begin{axis}[
    xlabel = $\mu$,
    ylabel = $T$,
    xmin=0, xmax=10,
    ymin=0, ymax=10,
    xtick={0}, ytick={0},
    axis lines=middle,
    axis line style={-latex},
    axis on top=true,
    \node at (axis cs:5,5) {Hadrons};
    \node at (axis cs:6,7) {Color SC};
\end{axis}
\end{tikzpicture}
\end{figure}
Beam-Energy Scan

Grand Canonical Ensemble

Au+Au

Cleymans

Andronic

STAR Preliminary

200 GeV
39 GeV
11.5 GeV
7.7 GeV

T_\text{ch} (GeV)

\mu_B (GeV)

0

0.1

0.2

0.3

0.4

0.5

STAR 2012

Hadrons

Color SC

beam energy

high

low

0
Beam-Energy Scan

Active experimental researches/plans for the beam-energy scan

Search for QCD phase structure / critical point
Fluctuations
Thermal Fluctuations

Observables are fluctuating even in an equilibrated medium.
Thermal Fluctuations

Observables are fluctuating even in an equilibrated medium.

- **Skewness:** \( S = \frac{\langle \delta N^3 \rangle}{\sigma^3} \)
- **Kurtosis:** \( \kappa = \frac{\langle \delta N^4 \rangle - 3\langle \delta N^2 \rangle^2}{\chi_2 \sigma^2} \)
- **Variance:** \( \langle \delta N^2 \rangle = V \chi_2 = \sigma^2 \)
- **Non-Gaussianity**

\( \delta N = N - \langle N \rangle \)
The noise is the signal.

R. Landauer
1998
A Coin Game

① Bet 250 JPY
② You get head coins of

A. 100 x 5 JPY
B. 50 x 10 JPY

Same expectation value.
A Coin Game

① Bet 250 JPY
② You get head coins of

A. 100 x 5 JPY
B. 50 x 10 JPY
C. 1 x 500 JPY

Same expectation value. But, different fluctuation.
Event-by-Event Fluctuations

Fluctuations can be measured by e-by-e analysis in experiments.

Cumulants

$\langle \delta N_p^2 \rangle$, $\langle \delta N_p^3 \rangle$, $\langle \delta N_p^4 \rangle_c$
Event-by-Event Analysis @ HIC

Fluctuations can be measured by e-by-e analysis in experiments.
Non-zero non-Gaussian cumulants have been established!

Have we measured critical fluctuations?
Fluctuations and Elemental Charge

\[ \langle \delta N_q^n \rangle_c = \langle N_q \rangle \]

\[ \langle \delta N_B^n \rangle_c = \frac{1}{3n-1} \langle N_B \rangle \]

\[ 3N_B = N_q \]

\[ \langle \delta N_B^n \rangle_c = \langle N_B \rangle \]

Free Boltzmann → Poisson

\[ \langle \delta N^n \rangle_c = \langle N \rangle \]
Fluctuations and Elemental Charge

\[ \langle \delta N^n_q \rangle_c = \langle N_q \rangle \]

\[ \langle \delta N_B^n \rangle_c = \frac{1}{3n-1} \langle N_B \rangle \]

\[ 3N_B = N_q \]

Asakawa, Heinz, Muller, 2000
Jeon, Koch, 2000
Ejiri, Karsch, Redlich, 2005

RBC-Bielefeld '09
Fluctuations and Elemental Charge

\[ \langle \delta N_q^n \rangle_c = \langle N_q \rangle \]

\[ \langle \delta N_B^n \rangle_c = \frac{1}{3n-1} \langle N_B \rangle \]

\[ 3N_B = N_q \]
Shot Noise

$S_{\text{shot}} \sim \langle \delta I^2 \rangle$

$S_{\text{shot}} = 2e^* \langle I \rangle$

charge of quasi-particles

Total charge $Q$:

$Q = e \langle N \rangle$

$\langle \delta Q^2 \rangle = e^2 \langle \delta N^2 \rangle = e^2 \langle N \rangle = eQ$

$\frac{\langle \delta Q^2 \rangle}{Q} = e$
Shot Noise

\[ S_{\text{shot}} \sim \langle \delta I^2 \rangle \]
\[ S_{\text{shot}} = 2e^* \langle I \rangle \]
charge of quasi-particles

Superconductors with Cooper Pairs

\[ e^* = 2e \]

doubled!

Fractional Quantum Hall Systems

\[ e^* = \frac{q}{p} e \]
Saminadayar+, PRL 79,2526 (1997)

Higher order cumulants:
Fluctuation and QCD Critical Point

Fluctuations diverge at the QCD critical point

- Geometric interpretation to signs of higher order cumulants

\[ \langle \delta N^3 \rangle = T \frac{\partial \langle \delta N^2 \rangle}{\partial \mu} \]

- More severe divergence for higher-order cumulants

Asakawa, Ejiri, MK 2009

Stephanov, 2009
Impact of Negative Third Cumulants

Once negative $<dN_B^3>$ is established, it is evidences that:

1. $\chi_B$ has a peak structure in the QCD phase diagram.
2. Hot matter beyond the peak is created in the collisions.

- No dependence on any specific models.
- Just the sign! No normalization (such as by $N_{ch}$).
Even on one blade of grass
the cool wind lives

Issa Kobayashi
1814
Physicists can feel hot early Universe 13 800 000 000 years ago in tiny fluctuations of cosmic microwave
Physicists can feel the existence of **microscopic** atoms behind random **fluctuations** of Brownian pollens

A. Einstein
1905
Physicists can feel dark matter behind fluctuations of galaxies billion light years away.
Feel hot quark wind behind fluctuations in relativistic heavy ion collisions

2010-
Clear suppression!

ex. Asakawa, Ejiri, MK, 2009

Even on one blade of grass the cool wind lives
Rapidity Window Dependences of Fluctuations
Remarks on Critical Fluctuation

Experiments cannot observe critical fluctuation in equilibrium directly.

- **Growth**
  Critical fluctuation has to be well developed. But, relaxation toward equilibration is slow around CP because of the critical slowing down.

- **Decay by diffusion**
  Fluctuations developed at CP are modified by the time evolution in later stage.
Time Evolution of Fluctuations

Diffusion modifies distribution
**Δη Dependence @ ALICE**

D-measure

\[ D = 4 \frac{\langle \delta N_Q^2 \rangle}{\langle N_Q^+ + N_Q^- \rangle} \]

- \( D \sim 3-4 \) Hadronic
- \( D \sim 1-1.5 \) Quark

**hadronic**

rapidity window
Dh Dependence @ ALICE

\[ D \sim \frac{\langle \delta N_Q \rangle^2}{\Delta \eta} \]

has to be a constant in equil. medium

Fluctuation of \( N_Q \) at ALICE is not the equilibrated one.
Time Evolution of Fluctuations

Quark-Gluon Plasma

Hadronization

Freezeout

\[ \langle \Delta N^2 \rangle \]

\[ \Delta \eta \]

\[ \chi_{\text{HAD}} \]

\[ \chi_{\text{QGP}} \]

\[ \Delta \eta \]

Variation of a conserved charge is achieved only through diffusion.

The larger \( \Delta \eta \), the slower diffusion.
How does $\langle \delta N_Q^4 \rangle_c$ behave as a function of $\Delta \eta$?

- suppression
- enhancement
How does $\langle \delta N_Q^4 \rangle_c$ behave as a function of $\Delta \eta$?

- [ ] suppression
- [ ] enhancement
Stochastic Diffusion Equation (SDE)

- **Diffusion equation**
  \[ \partial_{\tau} n = D \partial_{\eta}^2 n \]
  - Describe a relaxation of a conserved density \( n \) toward uniform state **without fluctuation**

- **Stochastic diffusion equation**
  \[ \partial_{\tau} n = D \partial_{\eta}^2 n + \partial_{\eta} \xi(\eta, \tau) \]
  \[ \langle \xi(\eta_1)\xi(\eta_2) \rangle \sim \chi \delta(\eta_1 - \eta_2) \]
  - Describe a relaxation toward **fluctuating** uniform state
  - \( \chi \): susceptibility (fluctuation in equil.)

Review: Asakawa, MK, PPNP 90 (2016)
Hydrodynamic Fluctuations

Stochastic diffusion equation

\[ \partial_\tau n = D \partial_\eta^2 n + \partial_\eta \xi(\eta, \tau) \]

Shuryak, Stephanov, 2001

Fluctuation of \( n \) is Gaussian in equilibrium

Markov (white noise) + continuity

Gaussian noise

cf) Gardiner, “Stochastic Methods”
Baryons in Hadronic Phase

Baryons behave like Brownian pollens in water

hadronize  chem. f.o.  kinetic f.o.  

\[ p, \bar{p}, \quad n, \bar{n}, \quad \Delta(1232) \]

mesons

baryons

10~20fm
Non-Interacting Brownian Particle System

Initial condition (uniform)

Cumulants: $\langle \bar{Q}^2 \rangle_c$, $\langle \bar{Q}^3 \rangle_c$, $\langle \bar{Q}^4 \rangle_c$

Random walk

diffusion master equation: MK+, PLB(2014)
probabilistic argument: Ohnishi+, PRC(2016)
Non-Interacting Brownian Particle System

Initial condition (uniform)

cumulants: $\langle \tilde{Q}^2 \rangle_c$, $\langle \tilde{Q}^3 \rangle_c$, $\langle \tilde{Q}^4 \rangle_c$

diffusion distance $\Delta Y_{\text{drift}}$

random walk

$t \rightarrow \infty$

Poisson distribution

$\Delta Y$

Study $\Delta Y$ dependence

diffusion master equation: MK+, PLB(2014)
probabilistic argument: Ohnishi+, PRC(2016)
Diffusion Master Equation

Divide spatial coordinate into discrete cells

\[ \cdots, n_{x-1}, n_x, n_{x+1}, n_{x+2}, \cdots \]

\[ P(n) \]
Diffusion Master Equation

Divide spatial coordinate into discrete cells

Master Equation for $P(n)$

\[
\frac{\partial}{\partial t} P(n) = \gamma \sum_x [(n_x + 1) \{P(n + e_x - e_{x+1}) + P(n + e_x - e_{x-1})\} - 2n_x P(n)]
\]

Solve the DME exactly, and take $a \to 0$ limit

No approx., ex. van Kampen’s system size expansion

MK, Asakawa, Ono, 2014
MK, 2015
No initial net fluctuation:
\[ \langle Q^2 \rangle_c = \langle Q^4 \rangle_c = \langle Q^2 Q_{(tot)} \rangle_c = 0 \]
Cumulants at finite $\Delta y$ is different from initial value.

$4^{th}$ cumulant can have a sign change.

$4^{th}$ cumulant can have non-monotonic behavior.
**Δη Dependence: 4th order**

Initial Condition

\[
D_4 = \frac{\langle Q^4_{(net)} \rangle_c}{\langle Q_{(tot)} \rangle}
\]

\[
b = \frac{\langle Q^2_{(net)} Q_{(tot)} \rangle_c}{\langle Q_{(net)} \rangle}
\]

\[
c = \frac{\langle Q^2_{(tot)} \rangle_c}{\langle Q_{(tot)} \rangle}
\]

\[
D_2 = \frac{\langle Q^2_{(net)} \rangle_c}{\langle Q_{(tot)} \rangle} = 0.5
\]

Characteristic Δη dependences!
\( \Delta \eta \) Dependence: 4\(^{th}\) order

\[ D_4 = \frac{\langle Q_{(net)}^4 \rangle_c}{\langle Q_{(tot)} \rangle} \]

\[ b = \frac{\langle Q_{(net)}^2 Q_{(tot)} \rangle_c}{\langle Q_{(net)} \rangle} \]

\[ c = \frac{\langle Q_{(tot)}^2 \rangle_c}{\langle Q_{(tot)} \rangle} \]

\[ D_2 = \frac{\langle Q_{(net)}^2 \rangle_c}{\langle Q_{(tot)} \rangle} = 0.5 \]

\( \Delta \eta \) = 1.0 at ALICE

\( \Delta \eta = 1.6 \) baryon #

\[ D \sim M^{-1} \]
$4^{\text{th}}$ order: w/ Critical Fluctuation

Initial Condition

\[ D_4 = \frac{\langle Q_{(\text{net})}^4 \rangle_c}{\langle Q_{(\text{tot})} \rangle} = 4 \]

\[ b = \frac{\langle Q_{(\text{net})}^2 Q_{(\text{tot})} \rangle_c}{\langle Q_{(\text{net})} \rangle} \]

\[ c = \frac{\langle Q_{(\text{tot})}^2 \rangle_c}{\langle Q_{(\text{tot})} \rangle} \]

\[ D_2 = \frac{\langle Q_{(\text{net})}^2 \rangle_c}{\langle Q_{(\text{tot})} \rangle} = 1 \]

$D \sim M^{-1}$
Non monotonic behavior of cumulants.

Approach initial value as $\Delta y \rightarrow \text{large}$

finite volume effect: Sakaida+, PRC064911(2014)
More sophisticated analysis with factorial cumulants, MK, Luo (2017)
Non-Interacting Brownian Particle System

Initial condition (uniform)

cumulants: $\langle Q^2 \rangle_c, \langle Q^3 \rangle_c, \langle Q^4 \rangle_c$

trace back

time evolution

$\Delta Y$

diffusion distance $\leftrightarrow$ diffusion coefficient
$\Delta \eta$ Dependence: 3$^{\text{rd}}$ order

Initial Condition

\[
D_3 = \frac{\langle Q_{(\text{net})}^3 \rangle_c}{\langle Q_{(\text{net})} \rangle} \\
a = \frac{\langle Q_{(\text{net})} Q_{(\text{tot})} \rangle_c}{\langle Q_{(\text{net})} \rangle} \\
D_2 = \frac{\langle Q_{(\text{net})}^2 \rangle_c}{\langle Q_{(\text{tot})} \rangle} = 0.5
\]

$D \sim M^{-1}$

$\langle \delta N^3 \rangle(\eta)/\text{Skellam}$

$\Delta \eta = 1.0$ at ALICE

$\Delta \eta = 1.6$ at ALICE

$\Delta \eta = 1.0$ baryon #
Non-Gaussian fluctuations are one of the most interesting topics in relativistic heavy ion collisions.

Using fluctuation observables, we can explore early thermodynamics and QCD phase structure.

Rapidity window dependences of higher-order cumulants encode various information on fluctuation.

More information in future experiments. More theoretical studies are required!
Search for QCD Critical Point

Sakaida, Asakawa, Fujii, MK, to appear in PRC
arXiv:1703.08008
Remarks on Critical Fluctuation 1

Experiments cannot observe critical fluctuation in equilibrium directly.

- **Growth**
  Critical fluctuation has to be well developed. But, relaxation toward equilibration is slow around CP because of the critical slowing down.

- **Decay by diffusion**
  Fluctuations developed at CP are modified by the time evolution in later stage.
Remarks on Critical Fluctuation 2

Critical fluctuation is a conserved mode!

Fujii 2003; Fujii, Ohtani, 2004; Son, Stephanov, 2004

Fluctuations of $\sigma$ and $n_B$ are coupled around the CP!

$$\delta \sigma \simeq \delta n_B$$

$F(\sigma, n) = A\sigma^2 + B\sigma n + Cn^2 + \cdots$
Dynamical Evolution of Critical Fluctuations

- Evolution of correlation length
  - Berdnikov, Rajagopal (2000)
  - Asakawa, Nonaka (2002)

- Higher orders (spatially uniform “σ” mode)
  - Mukherjee, Venugopalan, Yin (2015)

- Dynamical evolution in chiral fluid model
  - Nahrgang, Herold, ... (2014~)

- Correlation functions
  - Kapusta, Torres-Rincon (2012)
Aim of This Study

- Describe conserved nature of critical fluctuation.
- We want to study experimental observables.
  - Focus on a conserved charge (baryon number)
  - Study evolution of conserved-charge fluctuation
- Concentrate on 2nd order fluctuation. (not higher)
- We study
  - Rapidity window dependence of the cumulant
  - 2-particle correlation function

Our Main Conclusion

Non-monotonicity in cumulants or correlation func. = Signal of QCD-CP
Cumulants and Correlation Function

\[ Q = \int_V dx n(x) \]  

**total charge**  

\[ \langle \delta Q^2 \rangle = \int_V dx dy \langle \delta n(x) \delta n(y) \rangle \]  

**2\textsuperscript{nd} order cumulant (fluctuation)**  

\[ \langle \delta Q^2 \rangle_{\Delta y} = \int_{\Delta y} dy (\Delta y - |y|) \langle \delta n(y) \delta n(0) \rangle \]  

**correlation function**  

**1-to-1 correspondence**  

**1-dim case**
Stochastic Diffusion Equation (SDE)

- **Diffusion equation**
  \[ \partial_\tau n = D \partial_\eta^2 n \]
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- **Stochastic diffusion equation**
  \[ \partial_\tau n = D \partial_\eta^2 n + \partial_\eta \xi(\eta, \tau) \]
  \[ \langle \xi(\eta_1)\xi(\eta_2) \rangle \sim \chi \delta(\eta_1 - \eta_2) \]
  - Describe a relaxation toward *fluctuating* uniform state
  - \( \chi \): susceptibility (fluctuation in equil.)

Review: Asakawa, MK, PPNP 90 (2016)
Soft Mode of QCD Critical Point

Fujii 2003; Fujii, Ohtani, 2004; Son, Stephanov, 2004

- Effective potential

\[ F(\sigma, n) = A\sigma^2 + B\sigma n + Cn^2 + \cdots \]

- Time dependent Ginzburg-Landau

\[
\begin{pmatrix}
\dot{\sigma} \\
\dot{n}
\end{pmatrix} =
\begin{pmatrix}
\Gamma_{\sigma\sigma} & \Gamma_{\sigma n} \\
\Gamma_{n\sigma} & \Gamma_{nn}
\end{pmatrix}
\begin{pmatrix}
\sigma \\
n
\end{pmatrix}
\sim k^2
\]

For slow and long wavelength,

\[
\partial_\tau n = D(\tau)\partial^2_\eta n + \partial_\eta \xi
\]

\[\langle \xi(\eta_1)\xi(\eta_2) \rangle \sim \chi(\tau)\delta(\eta_1 - \eta_2)\]

SDE

singularities in \( D(\tau) \) and \( \chi(\tau) \)
Critical behavior

- 3D Ising (r,H)
- model H

Temperature dep.
\[ K(\Delta y) = \frac{\langle \delta Q^2 \rangle}{\langle \delta Q^2 \rangle_{\text{eq.}}} \]

- monotonically decreasing

\[ \chi(\tau) \]
monotonically increasing

\[ K(\Delta y) \]
monotonically decreasing

Analytic property
Δη Dependence @ ALICE

D-measure

\[ D = 4 \frac{\langle \delta N_Q^2 \rangle}{\langle N_Q^+ + N_Q^- \rangle} \]

- \( D \sim 3-4 \) Hadronic
- \( D \sim 1-1.5 \) Quark

hadronic

rapidity window
Crossover / Correlation Func.

\[ C(\bar{y}) = \frac{\langle \delta n(\bar{y})\delta n(0) \rangle}{\chi_{\text{hadron}}} \]

- monotonically decreasing

Analytic property:
- \( \chi(\tau) \) monotonically increasing
- \( C(\bar{y}) \) monotonically decreasing
Critical Point / Cumulant

\[ K(\Delta y) = \frac{\langle \delta Q^2 \rangle}{\langle \delta Q^2 \rangle_{\text{eq.}}} \]

- non-monotonic \( \Delta y \) dep.

Analytic property

- \( K(\Delta y) \) non-monotonic
- \( \chi(\tau) \) non-monotonic
Criticap Point / Correlation Func.

\[ C(\bar{y}) = \langle \delta n(\bar{y})\delta n(0) \rangle / \chi_{\text{hadron}} \]

- non-monotonic \(\Delta y\) dep.

Analytic property: \(C(\Delta y)\) non-monotonic \(\rightarrow\) \(\chi(\tau)\) non-monotonic
Weaker Critical Enhancement

\[ K(\Delta y) = \frac{\langle \delta Q^2 \rangle}{\langle \delta Q^2 \rangle_{eq.}} \quad C(\bar{y}) = \frac{\langle \delta n(\bar{y})\delta n(0) \rangle}{\chi_{\text{hadron}}} \]

- Non-monotonicity in \( K(\Delta y) \) disappears.
- But \( C(y) \) is still non-monotonic.

Analytic property \( K(\Delta y), C(\bar{y}) \) monotonically.

\( C(y) \) is better to see non-monotonicity.
Away from the CP

\[ K(\Delta y) = \frac{\langle \delta Q^2 \rangle}{\langle \delta Q^2 \rangle_{eq.}} \]

\[ C(\bar{y}) = \frac{\langle \delta n(\bar{y}) \delta n(0) \rangle}{\chi_{\text{hadron}}} \]

- Signal of the critical enhancement can be clearer on a path away from the CP.

Away from the CP \(\rightarrow\) Weaker critical slowing down
Summary

- Soft mode of the QCD critical point is a conserved mode. Its time evolution depends on the size defining the charge.

- Time evolution of conserved charges (especially baryon number) is well described by the stochastic diffusion equation.

- A non-monotonic behavior of cumulant or correlation function is the signal of the critical enhancement!

Suggestion to experimentalists

- To find the CP, measure
  - $\Delta y$ dep. of $2^{nd}$ order cumulant
  - $y$ dep. of correlation function

- Study lower-order fluctuation in more detail
Future Studies

- **Experimental side:**
  -rapidity window dependences
  -baryon number cumulants
  -BES for SPS- to LHC-energies

- **Theoretical side:**
  -rapidity window dependences in dynamical models
  -description of non-equilibrium non-Gaussianity
  -accurate measurements on the lattice

- **Both sides:**
  -Compare theory and experiment carefully
  -Let’s accelerate our understanding on fluctuations!
Themal Blurring

Ohnishi, MK, Asakawa, PRC, in press
Fluctuations: Theory vs Experiment

Theoretical analyses based on statistical mechanics
- lattice, critical point, effective models, ...

Experiments
- Fluctuations in a momentum space
- detector

Discrepancy in phase spaces
- Asakawa, Heinz, Muller, 2000;
- Jeon, Koch, 2000;
- Shuryak, Stephanov, 2001
 Connecting Phase Spaces

\[ \frac{p_z}{E} = v_z = \frac{\dot{z}}{t} \]

Under Bjorken picture,

coordinate-space rapidity \( Y \)

\[ \parallel \]

momentum-space rapidity \( y \)

of medium

\[ \parallel \]

momentum-space rapidity \( y \)

of individual particles

\[ \Delta y \simeq \Delta Y \]
Blast wave squeezes the distribution in rapidity space

\[
w = \frac{m}{T}
\]

\{  
  \cdot \text{ pions} \quad w \approx 1.5 
  \cdot \text{ nucleons} \quad w \approx 9 
\}

- blast wave
- flat freezeout surface
Thermal distribution in $y$ space

\[ w = \frac{m}{T} \]

- pions \( w \approx 1.5 \)
- nucleons \( w \approx 9 \)

Rapidity distribution can be well approximated by Gaussian.

- blast wave
- flat freezeout surface
$\Delta \eta$ Dependence

Initial condition (before blurring)
no e-v-e fluctuations

Cumulants after blurring
can take nonzero values

With $\Delta y=1$, the effect is not well suppressed

\[
 w = \frac{m}{T}
\]
\[
 \begin{cases} 
 \text{pions} & w \approx 1.5 \\
 \text{nucleons} & w \approx 9
\end{cases}
\]
Diffusion + Thermal Blurring

Thermal blurring can be regarded as a part of diffusion

Chemical f.o. (coordinate space)

Kinetic f.o. (coordinate space)

Kinetic f.o. (momentum space)

Total diffusion: 

\[ P(x - x'') = \int dx' P_1(x - x') P_2(x' - x'') \]
Is the centrality dependence understood solely by the thermal blurring at kinetic f.o.?
Assumptions:

- Centrality independent cumulant at kinetic f.o.
- Thermal blurring at kinetic f.o.

Centrality dep. of fluctuation can be described by a simple thermal blurring picture.