Stress Tensor Distribution around Flux Tube in SU(3) Yang-Mills Theory
Force

$m_1, q_1$

$m_2, q_2$
Force

\[ F = -G \frac{m_1 m_2}{r^2} \]

\[ F = -\frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r^2} \]

Newton
Action-at-a-distance
Faraday

Local interaction

“Action through medium”
Stress = Force per Unit Area
\[ \vec{P} = F \frac{\hat{n}}{S} \]
Stress = Force per Unit Area

Pressure

\[ \vec{P} = \frac{\vec{F}}{S} \]

In thermal medium

\[ T_{ij} = P \delta_{ij} \]

Generally, \( F \) and \( n \) are not parallel

\[ \frac{F_i}{S} = \sigma_{ij} n_j \]

Stress Tensor

\[ \sigma_{ij} = -T_{ij} \]

Landau Lifshitz
Maxwell Stress
(in Maxwell Theory)

\[
\sigma_{ij} = \varepsilon_0 E_i E_j + \frac{1}{\mu_0} B_i B_j - \frac{1}{2} \delta_{ij} \left( \varepsilon_0 E^2 + \frac{1}{\mu_0} B^2 \right)
\]

\[
\vec{E} = (E, 0, 0)
\]

\[
T_{ij} = \begin{pmatrix}
-E^2 & 0 & 0 \\
0 & E^2 & 0 \\
0 & 0 & E^2
\end{pmatrix}
\]

- Parallel to field: **Pulling**
- Vertical to field: **Pushing**
Maxwell Stress
(in Maxwell Theory)

\[ T_{ij} u_j^{(k)} = \lambda_k u_i^{(k)} \]

(k = 1, 2, 3)

length: \( \sqrt{|\lambda_k|} \)

- Distortion of field, line of the force
- Propagation of the force as local interaction
- Absolute value of the force between sources
Maxwell Stress
(in Maxwell Theory)

- Distortion of field, line of the force
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\[ T_{ij} v_j^{(k)} = \lambda_k v_i^{(k)} \]

\( (k = 1, 2, 3) \)

length: \( \sqrt{|\lambda_k|} \)
Quark—Anti-quark system

Formation of the flux tube $\rightarrow$ confinement
Quark—Anti-quark system

Formation of the flux tube $\rightarrow$ confinement

Previous Studies on Flux Tube

- Potential
- Action density
- Color-electric field

so many studies…

Cea+ (2012)
Cardoso+ (2013)
Spatial Distribution of Stress Tensor in the QQ System

- Clearly gauge invariant
- Distortion of field, line of the force
- Propagation of the force as local interaction
- Absolute value of the force between sources

Lattice simulation
SU(3) Yang-Mills
a=0.029 fm
R=0.69 fm
t/a²=2.0

\[ T_{ij} v_{j}^{(k)} = \lambda_k v_{i}^{(k)} \]

(k = 1, 2, 3)

length: \( \sqrt{|\lambda_k|} \)
Comparison: SU(3) YM vs Maxwell

SU(3) Yang-Mills (quantum)

Maxwell (classical)

Propagation of the force is clearly different in YM and Maxwell theories!
Energy-Momentum Tensor on the Lattice and Gradient Flow

\[ T_{\mu\nu} = \begin{pmatrix} T_{00} & T_{01} & T_{02} & T_{03} \\ T_{10} & T_{11} & T_{12} & T_{13} \\ T_{20} & T_{21} & T_{22} & T_{23} \\ T_{30} & T_{31} & T_{32} & T_{33} \end{pmatrix} \]

- Energy
- Momentum
- Pressure
- Stress
Definition of the operator is nontrivial because of the explicit breaking of Lorentz symmetry. Examples:

\[ T_{\mu\nu} = F_{\mu\rho}F_{\nu\rho} - \frac{1}{4} \delta_{\mu\nu} FF \]

Its measurement is extremely noisy due to high dimensionality and etc.
(Yang-Mills) Gradient Flow

\[
\frac{\partial}{\partial t} A_\mu(t, x) = - \frac{\partial S_{\text{YM}}}{\partial A_\mu}
\]

leading

\[
A_\mu(0, x) = A_\mu(x)
\]

- diffusion equation in 4-dim space
- diffusion distance \( d \sim \sqrt{8t} \)
- "continuous" cooling/smearing

Luscher 2010
Narayananan, Neuberger, 2006
Luscher, Weiss, 2011
Small Flow-Time Expansion

\[ \tilde{O}(t, x) \xrightarrow{t \to 0} \sum_i c_i(t) O_i^R(x) \]

- an operator at \( t > 0 \)
- remormalized operators of original theory

original 4-dim theory

\[ 2\sqrt{8t} \]

\( t \to 0 \) limit
Constructing EMT 1

\[ \tilde{O}(t, x) \xrightarrow{t \to 0} \sum_i c_i(t) O_i^R(x) \]

**gauge-invariant dimension 4 operators**

\[
\begin{align*}
U_{\mu\nu}(t, x) &= G_{\mu\rho}(t, x)G_{\nu\rho}(t, x) - \frac{1}{4}\delta_{\mu\nu}G_{\mu\nu}(t, x)G_{\mu\nu}(t, x) \\
E(t, x) &= \frac{1}{4}\delta_{\mu\nu}G_{\mu\nu}(t, x)G_{\mu\nu}(t, x)
\end{align*}
\]
Constructing EMT 2

\[ U_{\mu\nu}(t, x) = \alpha_U(t) \left[ T^{R}_{\mu\nu}(x) - \frac{1}{4} \delta_{\mu\nu} T^{R}_{\rho\rho}(x) \right] + \mathcal{O}(t) \]

\[ E(t, x) = \langle E(t, x) \rangle + \alpha_E(t) T^{R}_{\rho\rho}(x) + \mathcal{O}(t) \]

Suzuki coeffs.

\[
\begin{align*}
\alpha_U(t) &= g^2 \left[ 1 + 2b_0 s_1 g^2 + \mathcal{O}(g^4) \right] \\
\alpha_E(t) &= \frac{1}{2b_0} \left[ 1 + 2b_0 s_2 g^2 + \mathcal{O}(g^4) \right]
\end{align*}
\]

\[ g = g(1/\sqrt{8t}) \]

\[ s_1 = 0.03296 \ldots \]

\[ s_2 = 0.19783 \ldots \]

Remormalized EMT

\[ T^{R}_{\mu\nu}(x) = \lim_{t \to 0} \left[ \frac{1}{\alpha_U(t)} U_{\mu\nu}(t, x) + \frac{\delta_{\mu\nu}}{4\alpha_E(t)} E(t, x)_{\text{subt.}} \right] \]
Application to Thermodynamics

FlowQCD, PRD94, 114512 (2016)

Conventional Integral Method

Thermodynamic relations

\[
\frac{\partial \ln Z}{\partial a} = \frac{\partial \beta}{\partial a} \frac{\partial \ln Z}{\partial \beta} \sim \frac{\partial \beta}{\partial a} \langle S \rangle
\]

\[
T \frac{\partial (p/T^4)}{\partial T} = \frac{\varepsilon - 3p}{T^4}
\]

Our Approach Gradient Flow Method

Take expectation values

\[
\varepsilon = \langle T_{00} \rangle
\]

\[
p = \langle T_{11} \rangle
\]

Other progress: shifted boundary Giusti and Pepe (2014~)
Jarzynski’s equality Caselle+ (2018); Talk by Nada, Monday
Expectation values of $T_{\mu\nu}$

SU(3) YM theory

Wilson gauge action

Parameters:
- $N_t = 12, 16, 20-24$
- aspect ratio $5.3 < N_s/N_t < 8$
- 1500~2000 configurations

Scale from gradient flow

$\rightarrow aT_c$ and $a\Lambda_{\text{MS}}$

FlowQCD 1503.06516
t, a Dependence

\[ T/T_c = 1.67 \]

\[ T/T_c = 1.91 \]

\[ T/T_c = 1.94 \]

\[ \epsilon - 3p/T^4 \]

\[ \epsilon + p/T^4 \]

\[ \sqrt{8}t < a : \] strong discretization effect

\[ \sqrt{8}t > 1/(2T) : \] over smeared

\[ a < \sqrt{8}t < 1/(2T) : \text{Linear t dependence} \]
Double Extrapolation
\( t \rightarrow 0, \ a \rightarrow 0 \)

\[
\langle T_{\mu \nu}(t) \rangle_{\text{lat}} = \langle T_{\mu \nu}(t) \rangle_{\text{cont}} + C_{\mu \nu} t + D_{\mu \nu}(t) \frac{a^2}{t}
\]

Continuum extrapolation
\[
\langle T_{\mu \nu}(t) \rangle_{\text{cont}} = \langle T_{\mu \nu}(t) \rangle_{\text{lat}} + C(t) a^2
\]

Small \( t \) extrapolation
\[
\langle T_{\mu \nu} \rangle = \langle T_{\mu \nu}(t) \rangle + C' t
\]
Double Extrapolation

Black line: continuum extrapolated
Double Extrapolation

Fitting ranges:
- **range-1**: $0.01 < tT^2 < 0.015$
- **range-2**: $0.005 < tT^2 < 0.015$
- **range-3**: $0.01 < tT^2 < 0.02$

Systematic error from the choice of fitting range
≈ statistical error

Black line: continuum extrapolated
Error includes:

- statistical error
- choice of t range for $t \to 0$ limit
- uncertainty in $a\Lambda_{MS}$

Total error $<1.5\%$ for $T>1.1T_c$

- Excellent agreement with integral method
- High accuracy only with $\sim 2000$ confs.

See also, talk by Nada, Monday
Agreement with integral method except for $N_t=4, 6$
No stable extrapolation for $N_t=4, 6$
Suppression of statistical error

Physical mass: Kanaya+ (WHOT-QCD), 1710.10015
- Independent plateau in all channels $\Rightarrow$ conservation law
- Confirmation of linear-response relations
- New analysis of specific heat

$$\frac{s}{T^3} = \frac{\langle \bar{T}_{44}(\tau) \bar{T}_{11}(0) \rangle}{VT^5} = \frac{\langle \bar{T}_{41}(\tau) \bar{T}_{41}(0) \rangle}{VT^5} \quad c_V = \frac{\langle \bar{T}_{00}^2 \rangle}{VT^2}$$
Analysis of Stress Tensor in Q\(\bar{Q}\) System
Preparing Static $Q\bar{Q}$

Wilson loop $W(R, T)$

- APE smearing for spatial links
- Multi-hit for temporal links
- No gradient flow for $W(R, T)$

Potential at $b=6.6$
(a=0.038 fm)

$$V(R) = - \lim_{T \to \infty} \log \langle W(R, T) \rangle$$

$$\langle O(x) \rangle_{Q\bar{Q}} = \lim_{T \to \infty} \frac{\langle \delta O(x) \delta W(R, T) \rangle}{\langle W(R, T) \rangle}$$
Lattice Setup

- SU(3) Yang-Mills (Quenched)
- Wilson gauge action
- Clover operator
- APE smearing / multi-hit
- Fine lattices (a=0.029-0.06 fm)
- Continuum extrapolation
- Simulation: bluegene/Q@KEK
Grand state saturation under control

Appearance of plateau for $t/a^2<4$, $T/a>15$

$\beta=6.819 \ (a=0.029 \ \text{fm}), \ R=0.46 \ \text{fm}$

Ground State Saturation
Ground State Saturation

\[ W(R, T) \]

\[ \langle T_{zzz}(t, 0) \rangle_{Q\bar{Q}} \text{ [GeV/fm}^3] \]

\[ -\langle T_{zzz}(t, 0) \rangle_{Q\bar{Q}} \text{ [GeV/fm}^3] \]

\[ T/\bar{a} \]

\[ t/\bar{a}^2 = 1.0 \]

\[ t/\bar{a}^2 = 3.0 \]

\[ t/\bar{a}^2 = 4.0 \]

\[ t/\bar{a}^2 = 1.7 \]

\[ t/\bar{a}^2 = 2.0 \]

\[ t/\bar{a}^2 = 6.0 \]

\[ \beta = 6.819 (a = 0.029 \text{ fm}), R = 0.46 \text{ fm} \]

Appearance of plateau for \( t/\bar{a}^2 < 4, T/\bar{a} > 15 \)

Grand state saturation under control
Continuum Extrapolation

- $a \to 0$ extrapolation with fixed $t$

Diagram showing the continuum limit extrapolation with different lattice spacings and discretization effects.
- $\langle T_{zz}(t,0) \rangle_{Q\bar{Q}}$ [GeV/fm$^3$]

- $t \to 0$ Extrapolation

- $a \to 0$ extrapolation with fixed $t$

- Then, $t \to 0$ with three ranges
Symmetry on Mid-Plane

From rotational symm. & parity

EMT is diagonalized in Cylindrical Coordinates

\[ T_{cc'}(r) = \begin{pmatrix} T_{rr} & T_{\theta\theta} \\ T_{\theta\theta} & T_{zz} \end{pmatrix} \]

\[ T_{\theta\theta} = \overrightarrow{e_\theta}^T T \overrightarrow{e_\theta} \]

Degeneracy in Maxwell theory

\[ T_{rr} = T_{\theta\theta} = -T_{zz} = -T_{44} \]
Degeneracy: $T_{44} \simeq T_{zz}$, $T_{rr} \simeq T_{\theta\theta}$

Separation: $T_{zz} \neq T_{rr}$

Nonzero trace anomaly $\sum T_{cc} \neq 0$

Continuum Extrapolated!
Degeneracy: \( T_{44} \simeq T_{zz}, \quad T_{rr} \simeq T_{\theta\theta} \)

Separation: \( T_{zz} \neq T_{rr} \)

Nonzero trace anomaly \( \sum T_{cc} \neq 0 \)
Force

**Force from Potential**

\[ F_{\text{pot}} = -\frac{dV}{dR} \]

**Force from Stress**

\[ F_{\text{stress}} = \int_{\text{mid.}} d^2 x T_{zz}(x) \]
**Forces**

**Force from Potential**

\[ F_{\text{pot}} = -\frac{dV}{dR} \]

**Force from Stress**

\[ F_{\text{stress}} = \int_{\text{mid.}} d^2 x T_{zz}(x) \]
Force from Potential

\[ F_{\text{pot}} = -\frac{dV}{dR} \]

Force from Stress

\[ F_{\text{stress}} = \int_{\text{mid.}} d^2x T_{zz}(x) \]

Agreement!
Stress Tensor Distribution
in Dual Abelian-Higgs Model

Yanagihara+, in prep.
Abelian-Higgs Model

\[ \mathcal{L}_{\text{AH}} = -\frac{1}{4} F_{\mu\nu}^2 + \left| (\partial_\mu + igA_\mu) \phi \right|^2 - \lambda \left( \phi^2 - v^2 \right)^2 \]

- flux-tube solution w/ monopoles Nielsen, Olesen (1973)
- model for QCD vacuum (dual-Ginzburg-Landau)
- describe symmetry breaking/restoration
- nonzero trace anomaly
Abelian-Higgs Model

\[ \mathcal{L}_{AH} = -\frac{1}{4} F_{\mu\nu}^2 + |(\partial_{\mu} + igA_{\mu})\phi|^2 - \lambda(\phi^2 - \nu^2)^2 \]

GL parameter: \( \kappa = \sqrt{\lambda/g} \)

- type-I: \( \kappa < 1/\sqrt{2} \)
- type-II: \( \kappa > 1/\sqrt{2} \)
- Bogomol'nyi bound: \( \kappa = 1/\sqrt{2} \)

Infinitely long tube

- degeneracy
  \[ T_{zz}(r) = T_{44}(r) \quad \text{Luscher, 1981} \]
- conservation law
  \[ \frac{d}{dr}(rT_{rr}) = T_{\theta\theta} \]
Stress Tensor in AH Model

Bogomol’nyi bound: \( \kappa = 1/\sqrt{2} \)

\[
T_{rr} = T_{\theta\theta} = 0
\]

Stress Tensor in AH Model

Type-I \( \kappa = 0.1 \)

\[ T_{rr} > 0 \]

\[ T_{\theta\theta} \]

Gauge dominant

Type-II \( \kappa = 3.0 \)

\[ T_{rr} < 0 \]

Higgs dominant

- No degeneracy bw \( T_{rr} \) & \( T_{\theta\theta} \)

conservation law

\[ \frac{d}{dr} (rT_{rr}) = T_{\theta\theta} \]
Comparison

Abelian-Higgs

Lattice

(c) $R = 0.92 \text{ fm}$

- $\langle T_{44}^R(r) \rangle_{QQ} \text{ [GeV/fm}^3\text{]}$
- $\langle T_{zz}^R(r) \rangle_{QQ} \text{ [GeV/fm}^3\text{]}$
- $\langle T_{rr}^R(r) \rangle_{QQ} \text{ [GeV/fm}^3\text{]}$
- $\langle T_{\theta\theta}^R(r) \rangle_{QQ} \text{ [GeV/fm}^3\text{]}$

Type-I; infinitely long

- $T_{rr} > 0$
  → Suggest type-I (if dual-SC picture is correct)?
- $T_{rr} \simeq T_{00}$
  → Translationally-invariant flux tube is not formed.
Summary

- First non-perturbative analysis of the stress tensor distribution in quark-anti-quark systems
- New insights into the nature of the flux tube
- “Action-at-a-distance” force is given by the sum of local interaction.
- Non-trivial degeneracy in mid-plane
- $T_{rr} > 0 \rightarrow$ type-I vacuum? / $T_{rr} \sim T_{\theta \theta} \rightarrow$ Is $R$ still small?
First non-perturbative analysis of the stress tensor distribution in quark-anti-quark systems

New insights into the nature of the flux tube

“Action-at-a-distance” force is given by the sum of local interaction

Non-trivial degeneracy in mid-plane

$T_{rr} > 0 \rightarrow$ type-I vacuum? / $T_{rr} \sim T_{\theta\theta} \rightarrow$ Is R still small?

So many future studies

- Nonzero temperature / excited states
- EMT distribution inside hadrons
- Model study of the flux tube with finite length