Energy-Momentum Tensor on the Lattice via Gradient Flow

Masakiyo Kitazawa (Osaka University)

FlowQCD Collab.
M. Asakawa, T. Hatsuda, T. Iritani, H. Suzuki, R. Yanagihara
PRD$_{94}$,114512(2016); PRD$_{96}$,111502(2017); arXiv:1803.05656

WHOT-QCD Collab.
S. Ejiri, K. Kanaya, H. Suzuki, Y. Taniguchi, T. Umeda, ...
PRD$_{96}$,014509(2017); arXiv:1710.10015; arXiv:1711.02262

Takaura, Suzuki, Iritani, MK, in preparation
MK, Mogliacci, et al., in preparation
Energy-Momentum Tensor

One of the most fundamental quantities in physics
Energy-Momentum Tensor

\[ T_{\mu\nu} = \begin{bmatrix}
T_{00} & T_{01} & T_{02} & T_{03} \\
T_{10} & T_{11} & T_{12} & T_{13} \\
T_{20} & T_{21} & T_{22} & T_{23} \\
T_{30} & T_{31} & T_{32} & T_{33}
\end{bmatrix} \]

All components are important physical observables!
Definition of the operator is nontrivial because of the explicit breaking of Lorentz symmetry:

\[ T_{\mu\nu} = F_{\mu\rho} F_{\nu\rho} - \frac{1}{4} \delta_{\mu\nu} F F \]

Example:

Its measurement is extremely noisy due to high dimensionality and etc.
Thermodynamics

direct measurement of expectation values
\[ \langle T_{00} \rangle, \langle T_{ii} \rangle \]

- Vacuum Structure
  - vacuum configuration
  - mixed state on 1\(^{st}\) transition

Fluctuations and Correlations

viscosity, specific heat, ...
\[ c_V \sim \langle \delta T_{00}^2 \rangle \]
\[ \eta = \langle T_{12}; T_{12} \rangle \]

If we have
\[ T_{\mu\nu} \]

- flux tube / hadrons
- stress distribution

Hadron Structure

- vacuum configuration
- mixed state on 1\(^{st}\) transition

Vacuum Structure
Contents

Constructing EMT on the lattice

Thermodynamics

- Thermodynamics
  - direct measurement of expectation values
    \( \langle T_{00} \rangle, \langle T_{ii} \rangle \)

EMT Correlation Function

- Fluctuations and Correlations
  - viscosity, specific heat, ...
  - \( \eta = \int_0^\infty dt \langle T_{12}; T_{12} \rangle \)
  - \( c_V \sim \langle \delta T_{00}^2 \rangle \)

Stress distribution in \( \bar{q}q \) system

- Hadron Structure
  - flux tube / hadrons
  - stress distribution
Yang-Mills Gradient Flow

\[
\frac{\partial}{\partial t} A_\mu(t, x) = - \frac{\partial S_{YM}}{\partial A_\mu}
\]

\[
A_\mu(0, x) = A_\mu(x)
\]

- diffusion equation in 4-dim space
- diffusion distance \( d \sim \sqrt{8t} \)
- “continuous” cooling/smearing
- No UV divergence at \( t > 0 \)

Luscher 2010
Narayanan, Neuberger, 2006
Luscher, Weiss, 2011
Yang-Mills Gradient Flow

\[ \frac{\partial}{\partial t} A_\mu(t, x) = - \frac{\partial S_{\text{YM}}}{\partial A_\mu} \]

Luscher 2010
Narayanan, Neuberger, 2006
Luscher, Weiss, 2011

\[ A_\mu(0, x) = A_\mu(x) \]

Applications
scale setting / topological charge / running coupling
noise reduction / defining operators / ...
Small Flow-Time Expansion

Luescher, Weisz, 2011
Suzuki, 2013

\[ \tilde{O}(t, x) \xrightarrow{t \rightarrow 0} \sum_i c_i(t) O_i^R(x) \]

an operator at \( t > 0 \)
remormalized operators of original theory

original 4-dim theory

\[ 2\sqrt{2t} \]
t\( \rightarrow 0 \) limit
Constructing EMT 1

Gauge-invariant dimension 4 operators

\[
\tilde{O}(t, x) \xrightarrow{t \to 0} \sum_i c_i(t) \mathcal{O}_i^R(x)
\]

\[
\begin{align*}
U_{\mu\nu}(t, x) &= G_{\mu\rho}(t, x)G_{\nu\rho}(t, x) - \frac{1}{4}\delta_{\mu\nu}G_{\mu\nu}(t, x)G_{\mu\nu}(t, x) \\
E(t, x) &= \frac{1}{4}\delta_{\mu\nu}G_{\mu\nu}(t, x)G_{\mu\nu}(t, x)
\end{align*}
\]

Suzuki, 2013
Constructing EMT 2

\begin{align*}
U_{\mu\nu}(t, x) &= \alpha_U(t) \left( T_{\mu\nu}^R(x) - \frac{1}{4} \delta_{\mu\nu} T_{\rho\rho}^R(x) \right) + \mathcal{O}(t) \\
E(t, x) &= \langle E(t, x) \rangle + \alpha_E(t) T_{\rho\rho}^R(x) + \mathcal{O}(t)
\end{align*}

Remormalized EMT

\[ T_{\mu\nu}^R(x) = \lim_{t \to 0} \left[ \frac{1}{\alpha_U(t)} U_{\mu\nu}(t, x) + \frac{\delta_{\mu\nu}}{4\alpha_E(t)} E(t, x)_{\text{subt.}} \right] \]

\text{Suzuki coeffs.}

\[ \begin{aligned}
\alpha_U(t) &= g^2 \left[ 1 + 2b_0 s_1 g^2 + \mathcal{O}(g^4) \right] \\
\alpha_E(t) &= \frac{1}{2b_0} \left[ 1 + 2b_0 s_2 g^2 + \mathcal{O}(g^4) \right]
\end{aligned} \]

\[ g = g(1/\sqrt{8}t), \quad s_1 = 0.03296 \ldots, \quad s_2 = 0.19783 \ldots \]
Gradient Flow for Fermions

\[ \partial_t \psi(t, x) = D_\mu D_\mu \psi(t, x) \]
\[ \partial_t \overline{\psi}(t, x) = \psi(t, x) \overleftarrow{D_\mu} \overleftarrow{D_\mu} \]
\[ D_\mu = \partial_\mu + A_\mu(t, x) \]

- Not “gradient” flow but a “diffusion” equation.
- Divergence in field renormalization of fermions.
- All observables are finite at \( t>0 \) once \( Z(t) \) is fixed.
- Energy-momentum tensor from SFTE Makino, Suzuki, 2014

Luscher, 2013
Makino, Suzuki, 2014
Taniguchi+ (WHOT) 2016; 2017
Higher Order Analysis

The 2-loop analysis of $\alpha_U, \alpha_E$ is now available! (full QCD / pure gauge)

Harlander+, 1808.09837

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<thead>
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<th>LO</th>
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Suzuki (2013)

Effect on thermodynamics (pure gauge)

- $e-3p$: negligible (<0.5%)
- $e+p$: 2-3% increase
## Higher Order Analysis

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(full QCD / pure gauge)

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Harlander+, 1808.09837

**Effect on thermodynamics**  
(pure gauge)

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Higher Order Analysis

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Harlander+, 1808.09837
Constructing EMT on the lattice

Thermodynamics
- direct measurement of expectation values
  \( \langle T_{00}, T_{ii} \rangle \)

EMT Correlation Function

Stress distribution in \( \bar{q}q \) system
t, a Dependence

\[ \frac{T}{T_c} = 1.67. \]

\[ \frac{(\varepsilon - 3P)/T^4}{T^2} \]

\[ T^2 \]

\[ \frac{(\varepsilon + P)/T^4}{T^2} \]

\[ T^2 \]

\[ \begin{cases} \sqrt{8t} < a : \text{strong discretization effect} \\ \sqrt{8t} > 1/(2T) : \text{over smeared} \end{cases} \]

\[ a < \sqrt{8t} < 1/(2T) : \text{Linear t dependence} \]

\[ \beta = 6.719, 64^3 \times 12 \]

\[ \beta = 6.941, 96^3 \times 16 \]

\[ \beta = 7.117, 128^3 \times 20 \]
Double Extrapolation
$t \to 0$, $a \to 0$

\[
\langle T_{\mu\nu}(t) \rangle_{\text{latt}} = \langle T_{\mu\nu}(t) \rangle_{\text{phys}} + C_{\mu\nu} t + D_{\mu\nu}(t) \frac{a^2}{t}
\]

O(t) terms in SFTE lattice discretization

Continuum extrapolation
\[
\langle T_{\mu\nu}(t) \rangle_{\text{cont}} = \langle T_{\mu\nu}(t) \rangle_{\text{latt}} + C(t) a^2
\]

Small t extrapolation
\[
\langle T_{\mu\nu} \rangle = \langle T_{\mu\nu}(t) \rangle + C't
\]
Double Extrapolation

Black line: continuum extrapolated
Double Extrapolation

Fitting ranges:
- **range-1**: $0.01 < tT^2 < 0.015$
- **range-2**: $0.005 < tT^2 < 0.015$
- **range-3**: $0.01 < tT^2 < 0.02$

Systematic error from the choice of fitting range
\[ \approx \text{statistical error} \]
Error includes:
- statistical error
- choice of t range for \( t \to 0 \) limit
- uncertainty in \( a\Lambda_{\text{MS}} \)

Total error <1.5% for \( T > 1.1T_c \)

- Excellent agreement with integral method
- High accuracy only with ~2000 confs.
Thermodynamics of SU(3) YM

- **Integral method**
  - Most conventional / established
  - Use thermodynamic relations
    - Boyd+ 1995; Borsanyi, 2012

- **Gradient-flow method**
  - Take expectation values of EMT

- **Moving-frame method**
  - Giusti, Pepe, 2014-

- **Non-equilibrium method**
  - Use Jarzynski’s equality
    - Caselle+, 2016; 2018

- **Differential method**
  - Shirogane+(WHOT-QCD), 2016~

\[
p = \frac{T}{V} \ln Z
\]

\[
T \frac{\partial (p/T^4)}{\partial T} = \frac{\varepsilon - 3p}{T^4}
\]

\[
\begin{align*}
\varepsilon &= \langle T_{00} \rangle \\
p &= \langle T_{11} \rangle
\end{align*}
\]
SU(3) YM EoS: Comparison

- Measurement of thermodynamics with various methods.
- All results are in good agreement.
- But, non-negligible discrepancy at $T/T_c \approx 1.3$?
Effect on thermodynamics (pure gauge)

- e-3p: negligible (<0.5%)
- e+p: 2-3% increase

Takaura, Suzuki, Iritani, MK, in prep.

Efect of Higher-Order Coeffs.
Agreement with integral method except for $N_t=4, 6$
No stable extrapolation for $N_t=4, 6$
Statistical error is substantially suppressed!

Physical mass: Kanaya+ (WHOT-QCD), 1710.10015
Pressure anisotropy in finite system

Casimir effect

Finite system at nonzero $T$

MK, Mogliacci, et al. in preparation

$V = L_x \times L_y \times L_z$

$L_x \ll L_y = L_z$

pressure anisotropy

$T_{11} \neq T_{22} = T_{33}$
Pressure Anisotropy

MK, Mogliacci, et al. in prep.

Free scalar field

\[ L_2 = L_3 = \infty \]

Mogliacci+, 1807.07871

\[ L_1 T = \frac{N_x}{N_t} \]
Pressure Anisotropy

Free scalar field
- \( L_2 = L_3 = \infty \)
  - Mogliacci+, 1807.07871

Lattice results
- Periodic BC
- \( N_s^2 \times N_x \times N_t = 72^2 \times N_x \times 12 \)
- \( N_x = 12, 14, 16, 18 \)
- Only \( t \to 0 \) limit (fixed \( a \))

Medium near \( T_c \) is remarkably insensitive to finite size!
How do we understand??
Contents

Constructing EMT on the lattice

Thermodynamics

EMT Correlation Function

Stress distribution in \( \bar{q}q \) system
EMT Correlator: Motivation

- **Transport Coefficient**
  
  Kubo formula $\rightarrow$ viscosity
  \[
  \eta = \int_0^\infty dt \int_0^{1/T} d\tau \int d^3x \langle T_{12}(x, -i\tau)T_{12}(0, t) \rangle
  \]
  
  Karsch, Wyld, 1987
  Nakamura, Sakai, 2005
  Meyer; 2007, 2008
  ...
  Borsanyi+, 2018
  Astrakhantsev+, 2018

- **Energy/Momentum Conservation**
  \[
  \langle \bar{T}_{0\mu}(\tau)\bar{T}_{\rho\sigma}(0) \rangle : \tau\text{-independent constant}
  \]

- **Fluctuation-Response Relations**
  \[
  c_V = \frac{\langle \delta E^2 \rangle}{VT^2}
  \quad E + p = \frac{\langle \bar{T}_{01}^2 \rangle}{VT} = \frac{\langle \bar{T}_{11}\bar{T}_{00} \rangle}{VT}
  \]
EMT Euclidean Correlator

Confirmation of fluctuation-response relations

New method to measure $c_V$

Similar result for (41;41) channel: Borsanyi+, 2018

Perturbative analysis: Eller, Moore, 2018
New measurement of $c_V$:

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Fluctuation-Response Relations

\[
\langle T_{44}(\tau)T_{44}(0) \rangle \quad \langle T_{44}(\tau)T_{11}(0) \rangle \quad \langle T_{41}(\tau)T_{41}(0) \rangle
\]

$T/T_c = 2.24$

Confirmation of FRR:

\[
E + p = \frac{\langle \tilde{T}_{01}^2 \rangle}{VT} = \frac{\langle \tilde{T}_{11} \tilde{T}_{00} \rangle}{VT}
\]

2+1 QCD:
Taniguchi+ (WHOT-QCD), 1711.02262
Constructing EMT on the lattice

Thermodynamics

EMT Correlation Function

Stress distribution in $\bar{q}q$ system
Stress = Force per Unit Area
Stress = Force per Unit Area

Pressure

\[ \vec{P} = \frac{\vec{F}}{S} \]

\[ \vec{P} = P\vec{n} \]
Pressure

\[ \vec{P} = \frac{\vec{F}}{S} \]

In thermal medium

\[ T_{ij} = P \delta_{ij} \]

Generally, F and n are not parallel

\[ \frac{F_i}{S} = \sigma_{ij} n_j \]

Stress Tensor

\[ \sigma_{ij} = -T_{ij} \]

Landau Lifshitz
Force

Action-at-a-distance

Newton 1687

\[ F = -G \frac{m_1 m_2}{r^2} \]

Local interaction

Faraday 1839

\[ F = -\frac{1}{4\pi \epsilon_0} \frac{q_1 q_2}{r^2} \]
Maxwell Stress
(in Maxwell Theory)

\[ \sigma_{ij} = \varepsilon_0 E_i E_j + \frac{1}{\mu_0} B_i B_j - \frac{1}{2} \delta_{ij} \left( \varepsilon_0 E^2 + \frac{1}{\mu_0} B^2 \right) \]

\[ \vec{E} = (E, 0, 0) \]

\[ T_{ij} = \begin{pmatrix} -E^2 & 0 & 0 \\ 0 & E^2 & 0 \\ 0 & 0 & E^2 \end{pmatrix} \]

- Parallel to field: **Pulling**
- Vertical to field: **Pushing**
Maxwell Stress
(in Maxwell Theory)

\[ T_{ij} v_j^{(k)} = \lambda_k v_i^{(k)} \]

\((k = 1, 2, 3)\)

length: \(\sqrt{|\lambda_k|}\)

Definite physical meaning:
- Distortion of field, line of the field
- Propagation of the force as local interaction
Quark—Anti-quark system

Formation of the flux tube $\rightarrow$ confinement

Previous Studies on Flux Tube

- Potential
- Action density
- Color-electric field

so many studies...  

Cea+ (2012)  

Cardoso+ (2013)
Lattice simulation
SU(3) Yang-Mills
$a=0.029\ \text{fm}$
$R=0.69\ \text{fm}$
t/$a^2=2.0$

Yanagihara+, 1803.05656

Lattice simulation
SU(3) Yang-Mills
$a=0.029\ \text{fm}$
$R=0.69\ \text{fm}$
t/$a^2=2.0$

- Distortion of field, line of the field
- Propagation of the force as local interaction
- Manifestly gauge invariant

Definite physical meaning
Lattice Setup

- SU(3) Yang-Mills (Quenched)
- Wilson gauge action
- Clover operator
- APE smearing / multi-hit
- Fine lattices (a=0.029-0.06 fm)
- Continuum extrapolation

Simulation: bluegene/Q@KEK

Yanagihara+, 1803.05656
Continuum Extrapolation at mid-point

- $a \rightarrow 0$ extrapolation with fixed $t$
- $t \to 0$ Extrapolation at mid-point

- a $\to 0$ extrapolation with fixed $t$
- Then, $t \to 0$ with three ranges
Stress Distribution on Mid-Plane

From rotational symm. & parity

EMT is diagonalized in Cylindrical Coordinates

\[
T_{cc'}(r) = \begin{pmatrix}
T_{rr} & T_{\theta\theta} \\
T_{\theta\theta} & T_{zz} \\
T_{zz} & T_{44}
\end{pmatrix}
\]

\[
T_{rr} = \vec{e}_r^T T \vec{e}_r,
\]

\[
T_{\theta\theta} = \vec{e}_\theta^T T \vec{e}_\theta
\]

Degeneracy in Maxwell theory

\[
T_{rr} = T_{\theta\theta} = -T_{zz} = -T_{44}
\]
Mid-Plane

Degeneracy: $T_{44} \simeq T_{zz}$, $T_{rr} \simeq T_{\theta\theta}$

Separation: $T_{zz} \neq T_{rr}$

Nonzero trace anomaly $\sum T_{cc} \neq 0$

Continuum Extrapolated!

In Maxwell theory

$T_{rr} = T_{\theta\theta} = -T_{zz} = -T_{44}$
Degeneracy: \( T_{44} \approx T_{zz} \), \( T_{rr} \approx T_{\theta\theta} \)

Separation: \( T_{zz} \neq T_{rr} \)

Nonzero trace anomaly: \( \sum T_{cc} \neq 0 \)
Force

**Force from Potential**

\[ F_{\text{pot}} = - \frac{dV}{dR} \]

**Force from Stress**

\[ F_{\text{stress}} = \int_{\text{mid.}} d^2x T_{zz}(x) \]
Force

**Force from Potential**

\[ F_{\text{pot}} = -\frac{dV}{dR} \]

**Force from Stress**

\[ F_{\text{stress}} = \int_{\text{mid.}} d^2 x T_{zz}(x) \]

*Newton 1687*

*Faraday 1839*
**Force**

- Force from Potential
  \[ F_{\text{pot}} = -\frac{dV}{dR} \]

- Force from Stress
  \[ F_{\text{stress}} = \int_{\text{mid.}} d^2x T_{zz}(x) \]

---

**Good Agreement!**

- Newton 1687
- Faraday 1839
Abelian-Higgs Model

\[ \mathcal{L}_{\text{AH}} = -\frac{1}{4} F_{\mu\nu}^2 + |(\partial_{\mu} + igA_{\mu})\phi|^2 - \lambda(\phi^2 - v^2)^2 \]

**GL parameter:** \( \kappa = \sqrt{\lambda/g} \)

- **type-I:** \( \kappa < 1/\sqrt{2} \)
- **type-II:** \( \kappa > 1/\sqrt{2} \)
- **Bogomol’nyi bound:** \( \kappa = 1/\sqrt{2} \)

**Infinitely long tube**

- degeneracy
  \[ T_{zz}(r) = T_{44}(r) \quad \text{Luscher, 1981} \]
- momentum conservation
  \[ \frac{d}{dr} (rT_{rr}) = T_{\theta\theta} \]
Stress Tensor in AH Model

ininitely-long flux tube

Bogomol’nyi bound: \( \kappa = 1/\sqrt{2} \)

\[
T_{rr} = T_{\theta\theta} = 0
\]

Stress Tensor in AH Model
indefinitely-long flux tube

**Type-I**

$$\kappa = 0.1$$

- No degeneracy between $T_{rr}$ and $T_{\theta\theta}$
- $T_{\theta\theta}$ changes sign

**Type-II**

$$\kappa = 3.0$$

- conservation law
  $$\frac{d}{dr} (r T_{rr}) = T_{\theta\theta}$$
Stress Tensor in AH Model
infinitely-long flux tube

Type-I

\[ \kappa = 0.1 \]

- No degeneracy bw \( T_{rr} \) & \( T_{\theta\theta} \)
- \( T_{\theta\theta} \) changes sign

Inconsistent with lattice result

\[ T_{rr} \simeq T_{\theta\theta} \]
Flux Tube with Finite Length

R=0.92 fm

Left: $T_{zz}(0), T_{rr}(0)$ reproduce lattice result

Right: A parameter satisfying $T_{rr} \approx T_{\theta\theta}$

No parameter can reproduce lattice data at R=0.92 fm.
The analysis of energy-momentum tensor on the lattice is now available, and various studies are ongoing!

- gradient flow method
- higher-order perturbative coefficients
The analysis of energy-momentum tensor on the lattice is now available, and various studies are ongoing!
- gradient flow method
- higher-order perturbative coefficients

So many future studies
- Flux tube at nonzero temperature
- EMT distribution inside hadrons
- viscosity / other operators / instantons / full QCD
A Naïve Question

Put a single quark in QCD vacuum

How does energy density and stress behave in this system?
backup
EMT on the Lattice: Conventional

Lattice EMT Operator Caracciolo+, 1990

\[ T_{\mu \nu} = Z_6 T_{\mu \nu}^{[6]} + Z_3 T_{\mu \nu}^{[3]} + Z_1 \left( T_{\mu \nu}^{[1]} - \langle T_{\mu \nu}^{[1]} \rangle \right) \]

\[ T_{\mu \nu}^{[6]} = (1 - \delta_{\mu \nu}) F_{\mu \rho}^a F_{\nu \rho}^a, \quad T_{\mu \nu}^{[3]} = \delta_{\mu \nu} \left( F_{\mu \rho}^a F_{\nu \rho}^a - \frac{1}{4} F_{\rho \sigma}^a F_{\rho \sigma}^a \right), \quad T_{\mu \nu}^{[1]} = \delta_{\mu \nu} F_{\rho \sigma}^a F_{\rho \sigma}^a \]

- Fit to thermodynamics: \( Z_3, Z_1 \)
- Shifted-boundary method: \( Z_6, Z_3 \) Giusti, Meyer, 2011; 2013; Giusti, Pepe, 2014~; Borsanyi+, 2018

Multi-level algorithm

- effective in reducing statistical error of correlator Meyer, 2007; Borsanyi, 2018; Astrakhantsev+, 2018
Gradient Flow Method

Smeared world by gradient flow

Measure on the lattice

Take Extrapolation \((t,a) \to (0,0)\)

\[
\langle T_{\mu\nu}(t) \rangle_{\text{lat}} = \langle T_{\mu\nu}(t) \rangle_{\text{phys}} + C_{\mu\nu} t + D_{\mu\nu} \frac{a^2}{t} + \ldots
\]

O(t) terms in SFTE lattice discretization
Numerical Simulation

- Expectation values of $T_{\mu\nu}$
- SU(3) YM theory
- Wilson gauge action
- Parameters:
  - $N_t = 12, 16, 20-24$
  - aspect ratio $5.3 < N_s/N_t < 8$
  - 1500~2000 configurations

- Scale from gradient flow
  \[ \rightarrow aT_c \text{ and } a\Lambda_{\text{MS}} \]

FlowQCD, 1503.06516
Mid-Point Correlator

\[ \langle T_{44}(\tau_{\text{mid}})T_{44}(0) \rangle \quad \langle T_{44}(\tau_{\text{mid}})T_{11}(0) \rangle \quad \langle T_{41}(\tau_{\text{mid}})T_{41}(0) \rangle \]

- \((44;11), (41;41)\) channels: confirmation of FRR
- \((44;44)\) channel: \textit{new} measurement of \(c_V\)

\[
C_V = \frac{\langle \delta E^2 \rangle}{VT^2}
\]

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<th>(T/T_c)</th>
<th>(C_{44;44}(\tau_m) )</th>
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2+1 QCD: Taniguchi+ (WHOT-QCD), 1711.02262
SU(3) YM vs Maxwell

SU(3) Yang-Mills (quantum)

Maxwell (classical)

Propagation of the force is clearly different in YM and Maxwell theories!
Preparing Static $Q\bar{Q}$

Wilson loop

$W(R, T)$

$V(R) = - \lim_{T \to \infty} \log \langle W(R, T) \rangle$

$\langle O(x) \rangle_{Q\bar{Q}} = \lim_{T \to \infty} \frac{\langle \delta O(x) \delta W(R, T) \rangle}{\langle W(R, T) \rangle}$

- APE smearing for spatial links
- Multi-hit for temporal links
- No gradient flow for $W(R, T)$

Potential at $b=6.6$
(a=0.038 fm)
Grand state saturation under control

Appearance of plateau for $t/a^2 < 4$, $T/a > 15$

$\beta = 6.819$ ($a = 0.029$ fm), $R = 0.46$ fm
Ground State Saturation

\( \beta = 6.819 \ (a = 0.029 \text{ fm}), \ R = 0.46 \text{ fm} \)

Appearance of plateau for \( t/a^2 < 4, \ T/a > 15 \)

Grand state saturation under control
Abelian-Higgs Model

\[ \mathcal{L}_{AH} = -\frac{1}{4} F_{\mu\nu}^2 + \left| (\partial_\mu + ig A_\mu) \phi \right|^2 - \lambda (\phi^2 - \nu^2)^2 \]

**GL parameter:** \( \kappa = \sqrt{\lambda/g} \)

- type-I: \( \kappa < 1/\sqrt{2} \)
- type-II: \( \kappa > 1/\sqrt{2} \)
- Bogomol'nyi bound: \( \kappa = 1/\sqrt{2} \)

**Infinitely long tube**

- degeneracy: \( T_{zz}(r) = T_{44}(r) \) Luscher, 1981
- conservation law: \( \frac{d}{dr} (r T_{rr}) = T_{\theta\theta} \)