近接作用として理解する
閉じ込め相互作用

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(大阪大学)

FlowQCD Collab.: Asakawa, Hatsuda, Iritani, Suzuki, Yanagihara
PLB, in press [arXiv:1803.05656]
Force

Action-at-a-distance

Newton
1687

\[ F = -G \frac{m_1 m_2}{r^2} \]

Local interaction

Faraday
1839

\[ F = -\frac{1}{4\pi\varepsilon_0} \frac{q_1 q_2}{r^2} \]
Stress = Force per Unit Area
Stress = Force per Unit Area

Pressure

\[
\vec{P} = \frac{\vec{F}}{S}
\]

\[
\vec{P} = P \hat{n}
\]
**Stress = Force per Unit Area**

**Pressure**

\[ \vec{P} = \frac{\vec{F}}{S} \]

In thermal medium

\[ T_{ij} = P \delta_{ij} \]

**Generally, F and n are not parallel**

\[ \frac{F_i}{S} = \sigma_{ij} n_j \]

**Stress Tensor**

\[ \sigma_{ij} = -T_{ij} \]

Landau Lifshitz
Maxwell Stress
(in Maxwell Theory)

$$\sigma_{ij} = \varepsilon_0 E_i E_j + \frac{1}{\mu_0} B_i B_j - \frac{1}{2} \delta_{ij} \left( \varepsilon_0 E^2 + \frac{1}{\mu_0} B^2 \right)$$

$$\vec{E} = (E, 0, 0)$$

$$T_{ij} = \begin{pmatrix} -E^2 & 0 & 0 \\ 0 & E^2 & 0 \\ 0 & 0 & E^2 \end{pmatrix}$$

- Parallel to field: **Pulling**
- Vertical to field: **Pushing**
Maxwell Stress
(in Maxwell Theory)

\[ T_{ij} v_j^{(k)} = \lambda_k v_i^{(k)} \]

(length: \( \sqrt{|\lambda_k|} \))

Definite physical meaning
- Distortion of field, line of the field
- Propagation of the force as local interaction
Quark-Anti-quark system

Formation of the flux tube $\rightarrow$ confinement

Previous Studies on Flux Tube

- Potential
- Action density
- Color-electric field

so many studies...  
  Cea+ (2012)  
  Cardoso+ (2013)
Stress Tensor in $Q\bar{Q}$ System

Yanagihara+, 1803.05656 PLB, in press

Lattice simulation
SU(3) Yang-Mills
a=0.029 fm
R=0.69 fm
t/a²=2.0

Definite physical meaning
- Distortion of field, line of the field
- Propagation of the force as local interaction
- Manifestly gauge invariant
SU(3) YM vs Maxwell

SU(3) Yang-Mills (quantum)

Maxwell (classical)

Propagation of the force is clearly different in YM and Maxwell theories!
Energy-Momentum Tensor

\[ T_{\mu\nu} = \begin{bmatrix} T_{00} & T_{01} & T_{02} & T_{03} \\ T_{10} & T_{11} & T_{12} & T_{13} \\ T_{20} & T_{21} & T_{22} & T_{23} \\ T_{30} & T_{31} & T_{32} & T_{33} \end{bmatrix} \]

All components are important physical observables!
Definition of the operator is nontrivial because of the explicit breaking of Lorentz symmetry.

\[ T_{\mu\nu} = F_{\mu\rho} F_{\nu\rho} - \frac{1}{4} \delta_{\mu\nu} F F \]

Its measurement is extremely noisy due to high dimensionality and etc.
Thermodynamics

direct measurement of expectation values

\[ \langle T_{00} \rangle, \langle T_{ii} \rangle \]

Fluctuations and Correlations

viscosity, specific heat, ...

\[ c_V \sim \langle \delta T_{00}^2 \rangle \]

\[ \eta = \langle T_{12}; T_{12} \rangle \]

If we have

\[ T_{\mu\nu} \]

- flux tube / hadrons
- EM form factors

Hadron Structure

- vacuum configuration
- mixed state on 1st transition

Vacuum Structure
Contents

Constructing EMT on the lattice with gradient flow

Thermodynamics

EMT Correlation Function

Stress distribution in $\bar{q}q$ system
Yang-Mills Gradient Flow

\[
\frac{\partial}{\partial t} A_\mu(t, x) = - \frac{\partial S_{YM}}{\partial A_\mu}
\]

\[
A_\mu(0, x) = A_\mu(x)
\]

- \(t\): “flow time”
- \(\text{dim: [length}^2\text{]}\)
- Leading

- Diffusion equation in 4-dim space
- Diffusion distance \(d \sim \sqrt{8t}\)
- “Continuous” cooling/smearing
- No UV divergence at \(t > 0\)

Luscher 2010
Narayanan, Neuberger, 2006
Luscher, Weiss, 2011
Yang-Mills Gradient Flow

\[ \frac{\partial}{\partial t} A_\mu(t, x) = - \frac{\partial S_{\text{YM}}}{\partial A_\mu} \]

Luscher 2010
Narayanan, Neuberger, 2006
Luscher, Weiss, 2011

\[ A_\mu(0, x) = A_\mu(x) \]

leading

\[ \partial_t A_\mu = D_\nu G_{\mu\nu} = \partial_\nu \partial_\nu A_\mu + \cdots \]

Applications
scale setting / topological charge / running coupling
noise reduction / defining operators / ...

t: “flow time”
dim: [length²]
Small Flow-Time Expansion

\[ \tilde{O}(t, x) \xrightarrow{t \to 0} \sum_{i} c_i(t) \mathcal{O}_i^R (x) \]

an operator at \( t > 0 \)

remormalized operators of original theory

original 4-dim theory

\( t \to 0 \) limit
Constructing EMT 1

\[ \tilde{O}(t, x) \xrightarrow{t \to 0} \sum_i c_i(t) O_i^R(x) \]

Gauge-invariant dimension 4 operators

\[
\begin{align*}
U_{\mu\nu}(t, x) &= G_{\mu\rho}(t, x)G_{\nu\rho}(t, x) - \frac{1}{4}\delta_{\mu\nu}G_{\mu\nu}(t, x)G_{\mu\nu}(t, x) \\
E(t, x) &= \frac{1}{4}\delta_{\mu\nu}G_{\mu\nu}(t, x)G_{\mu\nu}(t, x)
\end{align*}
\]
Constructing EMT 2

\[ U_{\mu\nu}(t, x) = C_U(t) \left[ T^R_{\mu\nu}(x) - \frac{1}{4} \delta_{\mu\nu} T^R_{\rho\rho}(x) \right] + \mathcal{O}(t) \]

\[ E(t, x) = \langle E(t, x) \rangle + \alpha_E(t) T^R_{\rho\rho}(x) + \mathcal{O}(t) \]

\[ T^R_{\mu\nu}(x) = \lim_{t \to 0} \left[ c_1(t) U_{\mu\nu}(t, x) + \delta_{\mu\nu} c_2(t) E(t, x)_{\text{subt.}} \right] \]
Perturbative Coefficients

\[ T_{\mu\nu}(t) = c_1(t) U_{\mu\nu}(t) + \delta_{\mu\nu} c_2(t) E(t) \]

<table>
<thead>
<tr>
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<td>( c_1(t) )</td>
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Suzuki (2013)

- **Choice of the scale of** \( g^2 \)

\[ c_1(t) = c_1 \left( g^2 (\mu(t)) \right) \]

Previous: \( \mu_d(t) = 1/\sqrt{8t} \)

Improved: \( \mu_0(t) = 1/\sqrt{2e^\gamma E t} \)

Harlander+ (2018)
**Perturbative Coefficients**

\[ T_{\mu\nu}(t) = c_1(t)U_{\mu\nu}(t) + \delta_{\mu\nu}c_2(t)E(t) \]

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- **Suzuki (2013)**
- **Harlander+ (2018)**

**Choice of the scale of \( g^2 \)**

\[ c_1(t) = c_1 \left( g^2(\mu(t)) \right) \]

**Previous:** \( \mu_d(t) = 1/\sqrt{8t} \)

**Improved:** \( \mu_0(t) = 1/\sqrt{2e^{\gamma E} t} \)

- **Harlander+ (2018)**
Constructing EMT on the lattice

Thermodynamics

EMT Correlation Function

Stress distribution in $\bar{q}q$ system
Thermodynamics of SU(3) YM

- **Integral method**
  - Most conventional / established
  - Use thermodynamic relations
    Boyd+ 1995; Borsanyi, 2012

- **Moving-frame method**
  - Giusti, Pepe, 2014~

- **Non-equilibrium method**
  - Use Jarzynski’s equality Caselle+, 2016; 2018

- **Gradient-flow method**
  - Take expectation values of EMT

  \[ p = \frac{T}{V} \ln Z \]

  \[ T \frac{\partial (p/T^4)}{\partial T} = \frac{\varepsilon - 3p}{T^4} \]

- **Differential method**
  - Shirogane+(WHOT-QCD), 2016~
\[ T_{\mu\nu}(t) = c_1(t) U_{\mu\nu}(t) + \delta_{\mu\nu} c_2(t) E(t) \]

FlowQCD2016 (c1: 1-loop / c2: 2-loop)

\[ \sqrt{8t} < a : \text{strong discretization} \]
\[ \sqrt{8t} > 1/(2T) : \text{over smeared} \]

\[ a < \sqrt{8t} < 1/(2T) \] Stable t dependence
Double Extrapolation
\( t \to 0, \ a \to 0 \)

\[
\langle T_{\mu\nu}(t) \rangle_{\text{latt}} = \langle T_{\mu\nu}(t) \rangle_{\text{phys}} + C_{\mu\nu} t + \left( D_{\mu\nu}(t) \frac{a^2}{t} \right) 
\]

O(t) terms in SFTE lattice discretization

Continuum extrapolation
\[
\langle T_{\mu\nu}(t) \rangle_{\text{cont}} = \langle T_{\mu\nu}(t) \rangle_{\text{lat}} + C(t) a^2 
\]

Small t extrapolation
\[
\langle T_{\mu\nu} \rangle = \langle T_{\mu\nu}(t) \rangle + C't 
\]
Double Extrapolation

\[ \frac{T}{T_c} = 1.67 \]

Black line: continuum extrapolated
Fitting ranges:

- range-1: $0.01 < tT^2 < 0.015$
- range-2: $0.005 < tT^2 < 0.015$
- range-3: $0.01 < tT^2 < 0.02$

Systematic error from the choice of fitting range ≈ statistical error

Black line: continuum extrapolated
Temperature Dependence

Error includes
- statistical error
- choice of t range for t→0 limit
- uncertainty in aΛ_{MS}

total error <1.5% for T>1.1T_c

- Excellent agreement with integral method
- High accuracy only with ~2000 confs.
Higher Order Coefficient: $\varepsilon + p$

- **NLO (1-loop)**
  - $T/T_C = 1.68$ (NLO) $\mu = \mu_0$
  - $64^3 \times 12$ Range-1
  - $96^3 \times 16$ Range-2
  - $128^3 \times 20$

- **N^2LO (2-loop)**
  - $T/T_C = 1.68$ (N^2LO) $\mu = \mu_0$
  - $64^3 \times 12$ Range-1
  - $96^3 \times 16$ Range-2
  - $128^3 \times 20$

- t dependence becomes milder with higher order coeff.
- 1-loop $\rightarrow$ 2-loop: about 2% increase
- Systematic analysis: $\mu_o$ or $\mu_d$, uncertainty of $\Lambda$, fit range
- Extrapolation func: linear, higher order term in $c_1$ ($\sim g^6$)

Iritani, MK, Suzuki, Takaura, in prep.
Higher Order Coefficient: $\varepsilon$-3p

No difference b/w 2- & 3-loops: 2-loop is already good!

Systematic analysis: $\mu_0$ or $\mu_d$, uncertainty of $\Lambda$, fit range

Extrapolation func: linear, higher order term in $c_2$ ($\sim g^8$)
Effect of Higher-Order Coeffs.

Iritani, MK, Suzuki, Takaura, in prep.

Effect of higher order $c_1$ & $c_2$ (pure gauge)

- e-3p: negligible (<0.5%)
- e+p: ~2% increase

Systematic error: $\mu_0$ or $\mu_d$, $\Lambda$, $t \to 0$ function, fit range
Gradient Flow for Fermions

\[ \partial_t \psi(t, x) = D_\mu D_\mu \psi(t, x) \]
\[ \partial_t \bar{\psi}(t, x) = \psi(t, x) \bar{D}_\mu \bar{D}_\mu \]
\[ D_\mu = \partial_\mu + A_\mu(t, x) \]

- Not “gradient” flow but a “diffusion” equation.
- Divergence in field renormalization of fermions.
- All observables are finite at \( t > 0 \) once \( Z(t) \) is fixed.

\[ \tilde{\psi}(t, x) = Z(t) \psi(t, x) \]

- Energy-momentum tensor from SFTE Makino, Suzuki, 2014

Luscher, 2013
Makino, Suzuki, 2014
Taniguchi+ (WHOT) 2016; 2017
EMT in QCD

\[
T_{\mu\nu}(t, x) = c_1(t)U_{\mu\nu}(t, x) + c_2(t)\delta_{\mu\nu}(E(t, x) - \langle E \rangle_0) \\
+ c_3(t)(O_{3\mu\nu}(t, x) - 2O_{4\mu\nu}(t, x) - \text{VEV}) \\
+ c_4(t)(O_{4\mu\nu}(t, x) - \text{VEV}) + c_5(t)(O_{5\mu\nu}(t, x) - \text{VEV})
\]

\[
T_{\mu\nu}(x) = \lim_{t \to 0} T_{\mu\mu}(t, x)
\]

\[
\bar{O}_{3\mu\nu}^f(t, x) \equiv \varphi_f(t)\bar{\chi}_f(t, x) \left( \gamma_\mu \overrightarrow{D}_\nu + \gamma_\nu \overrightarrow{D}_\mu \right) \chi_f(t, x), \\
\bar{O}_{4\mu\nu}^f(t, x) \equiv \varphi_f(t)\delta_{\mu\nu} \bar{\chi}_f(t, x) \overrightarrow{D} \chi_f(t, x), \\
\bar{O}_{5\mu\nu}^f(t, x) \equiv \varphi_f(t)\delta_{\mu\nu} \bar{\chi}_f(t, x) \chi_f(t, x),
\]

\[
\varphi_f(t) \equiv -\frac{6}{(4\pi)^2t^2} \left( \bar{\chi}_f(t, x) \overrightarrow{D} \chi_f(t, x) \right)_0
\]

\[
c_1(t) = \frac{1}{\bar{g}(1/\sqrt{8t})^2} - \frac{1}{(4\pi)^2} \left[ 9(\gamma - 2\ln2) + \frac{19}{4} \right],
\]

\[
c_2(t) = \frac{1}{(4\pi)^2} \frac{33}{16},
\]

\[
c_3(t) = \frac{1}{4} \left\{ 1 + \frac{\bar{g}(1/\sqrt{8t})^2}{(4\pi)^2} \left[ 2 + \frac{4}{3}\ln(432) \right] \right\},
\]

\[
c_4(t) = \frac{1}{(4\pi)^2} \bar{g}(1/\sqrt{8t})^2,
\]

\[
c_5(t) = -\bar{m}_f(1/\sqrt{8t}) \left\{ 1 + \frac{\bar{g}(1/\sqrt{8t})^2}{(4\pi)^2} \left[ 4(\gamma - 2\ln2) + \frac{14}{3} + \frac{4}{3}\ln(432) \right] \right\}
\]
Agreement with integral method except for $N_t=4, 6$

$N_t=4, 6$: No stable extrapolation is possible

Statistical error is substantially suppressed!

Physical mass: Kanaya+ (WHOT-QCD), 1710.10015
Pressure anisotropy in finite system

Casimir effect

Finite system at nonzero $T$

MK, Mogliacci, Kolbe, Horowitz, in preparation

$V = L_x \times L_y \times L_z$

$L_x \ll L_y = L_z$

pressure anisotropy

$T_{11} \neq T_{22} = T_{33}$
Two Special Cases with PBC

1/\( T \ll L_x = L_y = L_z \)

\[
T_{11} = T_{22} = T_{33}
\]

1/\( T = L_x, \quad L_y = L_z \)

\[
T_{44} = T_{11}, \quad T_{22} = T_{33}
\]

In conformal (\( \Sigma_\mu T_{\mu\mu} = 0 \))

\[
\frac{p_1}{p_2} = 1
\]

\[
\frac{p_1}{p_2} = -1
\]
Pressure Anisotropy

Free scalar field

$\mathbf{L_2} = \mathbf{L_3} = \infty$

Mogliacci+, 1807.07871

$\frac{p_1}{p_2}$

$L_1 T = \frac{N_x}{N_t}$
Pressure Anisotropy

Free scalar field
- $L_2 = L_3 = \infty$
- Mogliacci+, 1807.07871

Lattice result
- Periodic BC
- $N_s^2 \times N_x \times N_t = 72^2 \times N_x \times 12$
- $N_x = 12, 14, 16, 18$
- Only $t \to 0$ limit (fixed $a$)

Medium near $T_c$ is remarkably insensitive to finite size!
How do we understand??

MK, Mogliacci, Kolbe, Horowitz, in prep.
Constructing EMT on the lattice

Thermodynamics

- direct measurement of expectation values
  \[ \langle T_{00} \rangle, \langle T_{ii} \rangle \]

EMT Correlation Function

- fluctuations and correlations
  - viscosity, specific heat, ...
  \[ \eta = \int_0^\infty dt \langle T_{12}^* T_{12} \rangle \]
  \[ c_V \sim \langle \delta T_{00}^2 \rangle \]

Stress distribution in \( \bar{q}q \) system

- flux tube / hadrons
- stress distribution
EMT Correlator: Motivation

- Transport Coefficient
  Kubo formula \(\rightarrow\) viscosity
  \[\eta = \int_0^\infty dt \int_0^{1/T} d\tau \int d^3x \langle T_{12}(x, -i\tau) T_{12}(0, t) \rangle\]
  
  Karsch, Wyld, 1987
  Nakamura, Sakai, 2005
  Meyer; 2007, 2008
  ...
  Borsanyi+; 2018
  Astrakhantsev+; 2018

- Energy/Momentum Conservation
  \[\langle \bar{T}_{0\mu}(\tau) T_{\rho\sigma}(0) \rangle : \tau\text{-independent constant}\]

- Fluctuation-Response Relations
  \[c_V = \frac{\langle \delta E^2 \rangle}{VT^2}\]
  \[E + p = \frac{\langle \bar{T}_{01}^2 \rangle}{VT} = \frac{\langle \bar{T}_{11} \bar{T}_{00} \rangle}{VT}\]
EMT Euclidean Correlator

FlowQCD, PR D96, 111502 (2017)

- \(\langle \tilde{T}_{44}(\tau)\tilde{T}_{44}(0) \rangle\)
- \(\langle \tilde{T}_{44}(\tau)\tilde{T}_{11}(0) \rangle\)
- \(\langle \tilde{T}_{41}(\tau)\tilde{T}_{41}(0) \rangle\)

- \(\tau\)-independent plateau in all channels \(\Rightarrow\) conservation law
- Confirmation of fluctuation-response relations
- New method to measure \(c_V\)
  - Similar result for (41;41) channel: Borsanyi+, 2018
  - Perturbative analysis: Eller, Moore, 2018
Fluctuation-Response Relations

New measurement of $c_V$

<table>
<thead>
<tr>
<th>$T/T_c$</th>
<th>$C_{44;44}(\tau_m)$</th>
<th>Ref.[19]</th>
<th>Ref.[11]</th>
<th>ideal gas</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.68</td>
<td>$17.7(8)(^{+2.1}_{-0.4})$</td>
<td>22.8(7)*</td>
<td>17.7</td>
<td>21.06</td>
</tr>
<tr>
<td>2.24</td>
<td>$17.5(0.8)(^{+0}_{-0.1})$</td>
<td>$17.9(7)**$</td>
<td>18.2</td>
<td>21.06</td>
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</tbody>
</table>

$E + p = \frac{\langle T_{01}^2 \rangle}{VT} = \frac{\langle T_{11} T_{00} \rangle}{VT}$

2+1 QCD: Taniguchi+ (WHOT-QCD), 1711.02262
Contents

Constructing EMT on the lattice

Thermodynamics

- Thermodynamics
  - direct measurement of expectation values
    \( \langle T_{00} \rangle, \langle T_{ii} \rangle \)

EMT Correlation Function

- Fluctuations and Correlations
  - viscosity, specific heat, ...
    \[ \eta = \int_0^\infty dt \langle T_{12}:T_{12} \rangle \]
    \[ c_V \sim \langle \delta T_{00}^2 \rangle \]

Hadron Structure

- flux tube / hadrons
- stress distribution

Stress distribution in \( \bar{q}q \) system
Propagation of the force is clearly different in YM and Maxwell theories!
Lattice Setup

- SU(3) Yang-Mills (Quenched)
- Wilson gauge action
- Clover operator

- APE smearing / multi-hit

- Fine lattices (a=0.029-0.06 fm)
- Continuum extrapolation

Simulation: bluegene/Q@KEK

Yanagihara+, 1803.05656
\[
\langle T_{\mu\nu}(x) \rangle_{Q\bar{Q}} = \lim_{T \to \infty} \frac{\langle W(T, R)T_{\mu\nu}(x) \rangle}{\langle W(T, R) \rangle} - \langle T_{\mu\nu}(x) \rangle
\]

- **EMT** \( T_{\mu\nu}(x) \)
  - via gradient flow method

- **Wilson Loop** \( W(R, T) \)
  - No gradient flow for \( W(R, T) \)
  - APE smearing for spatial links
  - Multi-hit for temporal links
Continuum Extrapolation at mid-point

- (c) $-\langle T_{22}(t,0) \rangle_{Q\bar{Q}}^{\text{lat}}$ [GeV/fm$^3$] vs $t/\bar{a}^2$

- $a \to 0$ extrapolation with fixed $t$
$a \to 0$ extrapolation with fixed $t$

Then, $t \to 0$ with three ranges

- Range 1
- Range 2
- Range 3
Stress Distribution on Mid-Plane

From rotational symm. & parity

EMT is diagonalized in Cylindrical Coordinates

\[
T_{cc'}(r) = \begin{pmatrix} T_{rr} & T_{\theta\theta} \\ T_{\theta\theta} & T_{zz} \\ T_{zz} & T_{44} \end{pmatrix}
\]

\[
T_{rr} = \bar{e}_r^T T \bar{e}_r,
\]

\[
T_{\theta\theta} = \bar{e}_\theta^T T \bar{e}_\theta
\]

Degeneracy in Maxwell theory

\[
T_{rr} = T_{\theta\theta} = -T_{zz} = -T_{44}
\]
Mid-Plane

Degeneracy: \( T_{44} \simeq T_{zz} \), \( T_{rr} \simeq T_{\theta\theta} \)

Separation: \( T_{zz} \neq T_{rr} \)

Nonzero trace anomaly: \( \sum T_{cc} \neq 0 \)

In Maxwell theory:

\[
T_{rr} = T_{\theta\theta} = -T_{zz} = -T_{44}
\]
Degeneracy: $T_{44} \simeq T_{zz}$, $T_{rr} \simeq T_{\theta\theta}$

Separation: $T_{zz} \neq T_{rr}$

Nonzero trace anomaly $\sum T_{cc} \neq 0$
Force

**Force from Potential**

\[ F_{\text{pot}} = -\frac{dV}{dR} \]

**Force from Stress**

\[ F_{\text{stress}} = \int_{\text{mid.}} d^2 x T_{zz}(x) \]
Force from Potential

\[ F_{\text{pot}} = -\frac{dV}{dR} \]

Force from Stress

\[ F_{\text{stress}} = \int_{\text{mid.}} d^2x T_{zz}(x) \]

Newton
1687

Faraday
1839
Force

Force from Potential

\[ F_{\text{pot}} = -\frac{dV}{dR} \]

Force from Stress

\[ F_{\text{stress}} = \int_{\text{mid.}} d^2x T_{zz}(x) \]

Good Agreement!
Abelian-Higgs Model

\[ \mathcal{L}_{AH} = -\frac{1}{4} F_{\mu\nu}^2 + \left| (\partial_\mu + igA_\mu) \phi \right|^2 - \lambda (\phi^2 - \nu^2)^2 \]

**GL parameter:** \( \kappa = \sqrt{\frac{\lambda}{g}} \)

- **type-I:** \( \kappa < 1/\sqrt{2} \)
- **type-II:** \( \kappa > 1/\sqrt{2} \)
- **Bogomol’nyi bound:** \( \kappa = 1/\sqrt{2} \)

**Infinitely long tube**

- degeneracy
  \[ T_{zz}(r) = T_{44}(r) \quad \text{Luscher, 1981} \]
- momentum conservation
  \[ \frac{d}{dr} (r T_{rr}) = T_{\theta\theta} \]
Stress Tensor in AH Model
ininitely-long flux tube

Bogomol’nyi bound: \( \kappa = 1/\sqrt{2} \)

\[
T_{rr} = T_{\theta\theta} = 0
\]

Stress Tensor in AH Model
infinitely-long flux tube

Type-I $\kappa = 0.1$

- $T_{44}(r)$
- $T_{zz}(r)$
- $T_{rr}(r)$
- $T_{\theta\theta}(r)$

No degeneracy bw $T_{rr}$ & $T_{\theta\theta}$

$T_{rr} > 0$

Type-II $\kappa = 3.0$

- $T_{44}(r)$
- $T_{zz}(r)$
- $T_{rr}(r)$
- $T_{\theta\theta}(r)$

$T_{rr} < 0$

conservation law

$$\frac{d}{dr} (rT_{rr}) = T_{\theta\theta}$$

- $T_{\theta\theta}$ changes sign

- No degeneracy bw $T_{rr}$ & $T_{\theta\theta}$
Stress Tensor in AH Model
infinitely-long flux tube

Type-I

$\kappa = 0.1$

No degeneracy bw $T_{rr}$ & $T_{\theta \theta}$

$T_{rr}$ changes sign

Inconsistent with lattice result

$T_{rr} \simeq T_{\theta \theta}$
Flux Tube with Finite Length

R=0.92 fm

Left: $T_{zz}(0), T_{rr}(0)$ reproduce lattice result

Right: A parameter satisfying $T_{rr} \approx T_{\theta\theta}$

No parameter can reproduce lattice data at R=0.92 fm.
The analysis of energy-momentum tensor on the lattice is now available, and various studies are ongoing!
- gradient flow method
- higher-order perturbative coefficients

Summary
The analysis of energy-momentum tensor on the lattice is now available, and various studies are ongoing!
- gradient flow method
- higher-order perturbative coefficients

So many future studies
- Flux tube at nonzero temperature
- EMT distribution inside hadrons
- viscosity / other operators / instantons / full QCD
backup
EMT on the Lattice: Conventional

Lattice EMT Operator  Caracciolo+, 1990

\[
T_{\mu\nu} = Z_6 T_{\mu\nu}^{[6]} + Z_3 T_{\mu\nu}^{[3]} + Z_1 \left( T_{\mu\nu}^{[1]} - \langle T_{\mu\nu}^{[1]} \rangle \right)
\]

\[
T_{\mu\nu}^{[6]} = (1 - \delta_{\mu\nu}) F_{\mu\rho}^a F_{\nu\rho}^a, \quad T_{\mu\nu}^{[3]} = \delta_{\mu\nu} \left( F_{\mu\rho}^a F_{\nu\rho}^a - \frac{1}{4} F_{\rho\sigma}^a F_{\rho\sigma}^a \right), \quad T_{\mu\nu}^{[1]} = \delta_{\mu\nu} F_{\rho\sigma}^a F_{\rho\sigma}^a
\]

- Fit to thermodynamics: \( Z_3, Z_1 \)

- Shifted-boundary method: \( Z_6, Z_3 \)  Giusti, Meyer, 2011; 2013; Giusti, Pepe, 2014~; Borsanyi+, 2018

Multi-level algorithm

- effective in reducing statistical error of correlator  Meyer, 2007; Borsanyi, 2018; Astrakhantsev+, 2018
Smeared world by gradient flow

Measure on the lattice

Take Extrapolation \((t,a) \rightarrow (0,0)\)

\[
\langle T_{\mu\nu}(t) \rangle_{\text{latt}} = \langle T_{\mu\nu}(t) \rangle_{\text{phys}} + C_{\mu\nu} t + D_{\mu\nu} \frac{a^2}{t} + \ldots
\]

\(O(t)\) terms in SFTE  lattice discretization
Numerical Simulation

- Expectation values of $T_{\mu\nu}$
- SU(3) YM theory
- Wilson gauge action
- Parameters:
  - $N_t = 12, 16, 20-24$
  - aspect ratio $5.3 < N_s/N_t < 8$
  - 1500~2000 configurations

- Scale from gradient flow
  $$\rightarrow aT_c \text{ and } a\Lambda_{\text{MS}}$$

FlowQCD, PRD94, 114512 (2016)

FlowQCD, 1503.06516

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Fermion Propagator

\[ S(t, x; s, y) = \langle \chi(t, x)\bar{\chi}(s, y) \rangle \]
\[ = \sum_{v, w} K(t, x; 0, v)S(v, w)K(s, y; 0, w)^\dagger \]

\[ (\partial_t - D_\mu D_\mu)K(t, x) = 0 \]

- propagator of flow equation
- Inverse propagator is needed
$N_f=2+1$ QCD Thermodynamics

- $N_f=2+1$ QCD, Iwasaki gauge + NP-clover
- $m_{PS}/m_V \approx 0.63$ / almost physical s quark mass

- $T=0$: CP-PACS+JLQCD ($\beta=2.05$, $28^3 \times 56$, $a \approx 0.07$ fm)
- $T>0$: $32^3 \times N_t$, $N_t = 4, 6, \ldots, 14, 16$:
  - $T \approx 174\text{-}697$ MeV
- $t \to 0$ extrapolation only (No continuum limit)

Taniguchi+ (WHOT-QCD), PRD96, 014509 (2017)
Preparing Static Q\bar{Q}

\[ V(R) = - \lim_{T \to \infty} \log \langle W(R, T) \rangle \]

\[ \langle O(x) \rangle_{Q\bar{Q}} = \lim_{T \to \infty} \frac{\langle \delta O(x) \delta W(R, T) \rangle}{\langle W(R, T) \rangle} \]

- APE smearing for spatial links
- Multi-hit for temporal links
- No gradient flow for \( W(R, T) \)

Potential at \( b=6.6 \)
\( (a=0.038 \text{ fm}) \)
Ground State Saturation of $W$

Effective Mass Plot of $W(R,T)$

- APE smearing for spatial links
- Multi-hit for temporal links
- About 100 confs.
- Use translational symmetry

With an optimal choice of $N_{\text{APE}}$, ground-state saturation is already established at $T/a=5$ ($C_o > 99.5\%$)
Ground State Saturation

\[ \beta = 6.819 \ (a = 0.029 \ \text{fm}), \ R = 0.46 \ \text{fm} \]

Appearance of plateau for \( t/a^2 < 4, \ T/a > 15 \)

Grand state saturation under control
Grand state saturation under control

Appearance of plateau for $t/a^2 < 4$, $T/a > 15$

$\beta = 6.819$ (a=0.029 fm), $R = 0.46$ fm

Wilson loop $W(R, T)$

Ground state saturation
Abelian-Higgs Model

\[ \mathcal{L}_{AH} = -\frac{1}{4} F_{\mu\nu}^2 + |(\partial_\mu + igA_\mu)\phi|^2 - \lambda(\phi^2 - \nu^2)^2 \]

**GL parameter:** \( \kappa = \sqrt{\frac{\lambda}{g}} \)

- **type-I:** \( \kappa < 1/\sqrt{2} \)
- **type-II:** \( \kappa > 1/\sqrt{2} \)
- **Bogomol’nyi bound:** \( \kappa = 1/\sqrt{2} \)

**Infinitely long tube**

- degeneracy
  \[ T_{zz}(r) = T_{44}(r) \]  Luscher, 1981
- conservation law
  \[ \frac{d}{dr} (r T_{rr}) = T_{\theta\theta} \]