Critical Diffusion Dynamics

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Rapid Reaction Task Force on
Dynamics of Critical Fluctuations: Theory – Phenomenology – HIC
GSI, Darmstadt, Germany, 11/Apr./2019
Time Evolution of Fluctuations

Distributions in $\Delta Y$ and $\Delta y$ are different due to “thermal blurring”.  
Ohnishi, MK, Asakawa, PRC(2016)

Fluctuations in $\Delta Y$ continue to change until kinetic f.o.
Distributions in $\Delta Y$ and $\Delta y$ are different due to “thermal blurring”.

Asakawa, Heinz, Muller, 2000
Jeon, Koch, 2000
Thermal distribution in y space

Blast wave squeezes the distribution in rapidity space

\[ w = \frac{m}{T} \]

\[ \begin{align*}
& \text{• pions} & \quad w & \approx 1.5 \\
& \text{• nucleons} & \quad w & \approx 9
\end{align*} \]

- blast wave
- flat freezeout surface

Ohnishi, MK, Asakawa
PRC, 2016
Rapidity distribution can be well approximated by Gaussian.

- blast wave
- flat freezeout surface

\[ w = \frac{m}{T} \]

- pions \( w \approx 1.5 \)
- nucleons \( w \approx 9 \)
Rapidity-window Dependence

Asakawa, Heinz, Muller, 2000
Jeon, Koch, 2000; Shuryak, Stephanov, 2001

Quark-Gluon Plasma

Hadronization

Freezeout

The larger $\Delta \eta$, the slower diffusion.
Resonance Decay

Neutral Particles

$\rho_0$

Decay into charged particles

$\langle \Delta N^2 \rangle$

$\Delta \eta$

$\Delta \eta$
Resonance Decay

Neutral Particles

\( \rho_0 \)

Decay into charged particles

The larger \( \Delta \eta \), the slower diffusion.
2nd Order @ ALICE

Net charge fluctuation

\[ D \simeq 4 \frac{\langle \delta N_Q^2 \rangle}{\langle \delta N_Q^2 \rangle_{\text{HRG}}} \]

ALICE, PRL2013
Net charge fluctuation has a suppression, but net-proton fluctuation does not. Why??
**Suggestion**

**Net charge fluctuation**
- Construct $\langle \delta N_B^2 \rangle$ ($\langle \delta N_N^2 \rangle$), $\langle \delta N_Q^2 \rangle$
- Then, take ratio $\frac{\langle \delta N_B^2 \rangle}{\langle \delta N_Q^2 \rangle}$
- Compare it with lattice

**Net proton fluctuation**
- First reliable comparison of LAT/HIC

Special thanks to F. Karsch

- Linear $T$ dependence near $T_c$!!
- Only 2$^{nd}$ order: reliable!!

**ALICE, PRL2013**

**HotQCD preliminary**
\[ \frac{\langle \delta N_B^2 \rangle}{\langle \delta N_Q^2 \rangle} \]

Prediction

LATTICE

\[ T_{pc} = (156.5 \pm 1.5) \text{ MeV} \]

ALICE

\[ \sqrt{s_{NN}} = 5.02 \text{ TeV} \]

\[ \sqrt{s_{NN}} = 2.76 \text{ TeV} \]

\[ \Delta \eta \]

\[ \Delta \eta \text{ dependence for tracing back the history!} \]

HotQCD preliminary

Special thanks to F. Karsch before continuum limit

\[ 1.6 \]
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Time Evolution of Fluctuations

Fluctuations in $\Delta Y$ continue to change until kinetic f.o.

Asakawa, Heinz, Muller, 2000
Jeon, Koch, 2000
Shuryak, Stephanov, 2001
<\delta N_B^2> \text{ and } <\delta N_p^2> \text{ @ LHC?}

\langle \delta N_Q^2 \rangle, \langle \delta N_B^2 \rangle, \langle \delta N_p^2 \rangle
should have different \Delta \eta dependence.

Baryon # cumulants are experimentally observable! MK, Asakawa, 2012
How does $\langle \delta N_Q^4 \rangle_c$ behave as a function of $D_{h}$? suppression or enhancement
(Non-Interacting) Brownian Particle Model

Initial condition (uniform)

Random walk

Cumulants: $\langle \bar{Q}^2 \rangle_c$, $\langle \bar{Q}^3 \rangle_c$, $\langle \bar{Q}^4 \rangle_c$

diffusion master equation: MK+, PLB(2014)

Probabilistic argument: Ohnishi+, PRC(2016)
(Non-Interacting) Brownian Particle Model

Initial condition (uniform)

\[ \Delta Y_{\text{drift}} \]

diffusion distance

\[ t \rightarrow \infty \]

Poisson distribution

\[ \Delta Y \]

Study \( \Delta Y \) dependence

cumulants: \( \langle \bar{Q}^2 \rangle_c, \langle \bar{Q}^3 \rangle_c, \langle \bar{Q}^4 \rangle_c \)

random walk

diffusion master equation: MK+, PLB(2014)

probabilistic argument: Ohnishi+, PRC(2016)
Baryons in Hadronic Phase

Baryons behave like Brownian pollens in water
Before the diffusion

\[ D_4 = 4, \ D_2 = 1 \]

Initial Condition

\[
D_4 = \frac{\langle Q_{(net)}^4 \rangle_c}{\langle Q_{(tot)} \rangle} = 4
\]

\[
b = \frac{\langle Q_{(net)}^2 \rangle_c}{\langle Q_{(net)} \rangle}
\]

\[
c = \frac{\langle Q_{(tot)}^2 \rangle_c}{\langle Q_{(tot)} \rangle}
\]

\[
D_2 = \frac{\langle Q_{(net)}^2 \rangle_c}{\langle Q_{(tot)} \rangle} = 1
\]
Cumulant at small $\Delta\eta$ is modified toward a Poisson value. Non-monotonic behavior can appear.
Time Evolution of Fluctuations

As a result of a simple random walk…
**Rapidity Window Dep.**

4\textsuperscript{th}-order cumulant  

**STAR Collab.** (X. Luo, CPOD2014)

- Initial Conditions
  
  \[ D_4 = \frac{\langle Q_{(net)}^4 \rangle_c}{\langle Q_{(tot)} \rangle} \quad b = \frac{\langle Q_{(net)}^2 Q_{(tot)} \rangle_c}{\langle Q_{(net)} \rangle} \]
  
  \[ D_2 = \frac{\langle Q_{(net)}^2 \rangle_c}{\langle Q_{(tot)} \rangle} \quad c = \frac{\langle Q_{(tot)}^2 \rangle_c}{\langle Q_{(tot)} \rangle} \]

- Is non-monotonic \( \Delta \eta \) dependence already observed?
- Different initial conditions give rise to different characteristic \( \Delta \eta \) dependence. → Study initial condition

 Finite volume effects: Sakaida+ , PRC90 (2015)
\( \Delta \eta \) Dependence: 4\textsuperscript{th} order

Initial Condition

\[ D_4 = \frac{\langle Q_{\text{net}}^4 \rangle_c}{\langle Q_{\text{tot}} \rangle} \]

\[ b = \frac{\langle Q_{\text{net}}^2 Q_{\text{tot}} \rangle_c}{\langle Q_{\text{net}}^2 \rangle} \]

\[ c = \frac{\langle Q_{\text{tot}}^2 \rangle_c}{\langle Q_{\text{tot}} \rangle} \]

\[ D_2 = \frac{\langle Q_{\text{net}}^2 \rangle_c}{\langle Q_{\text{tot}} \rangle} = 0.5 \]

Characteristic \( \Delta \eta \) dependences!
Cumulants with a \( \Delta \eta \) is not the initial value.
\( \Delta \eta \) Dependence: 4\(^{th}\) order

Initial Condition

\[
D_4 = \frac{\langle Q_{\text{net}}^4 \rangle_c}{\langle Q_{\text{tot}} \rangle_c}
\]

\[
b = \frac{\langle Q_{\text{net}}^2 Q_{\text{tot}} \rangle_c}{\langle Q_{\text{net}}^2 \rangle_c}
\]

\[
c = \frac{\langle Q_{\text{tot}}^2 \rangle_c}{\langle Q_{\text{tot}} \rangle_c}
\]

\[
D_2 = \frac{\langle Q_{\text{net}}^2 \rangle_c}{\langle Q_{\text{tot}} \rangle_c} = 0.5
\]

\[D \sim M^{-1}\]

\( \Delta \eta = 1.0 \) at ALICE

\( \Delta \eta = 1.6 \) at ALICE

\( \Delta \eta = 1.0 \) baryon #
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Critical Fluctuation

① Growth of critical fluctuation
  - Critical slowing down

② Decay by diffusion

③ At 1\textsuperscript{st} order transition
  - domain formation
Critical Fluctuation

① Growth of critical fluctuation
  • Critical slowing down

② Decay by diffusion

③ At 1st order transition
  • domain formation

Kibble-Zurek

Hadrons
Evolution of spatially uniform “σ” mode

Berdnikov, Rajagopal (2000)
Asakawa, Nonaka (2002)
Mukherjee+ (2015)

See also, Kapusta, Torres-Rincon (2012); Herold, Nahrgang, ... (2015); ...

THIS STUDY

Evolution of conserved charge fluctuations

Sakaida+, PRC2017; Murata, MK, in prep.

1. Conserved charges are directly observable.
2. Soft mode at QCD-CP is a conserved mode.
Fluctuations of $\sigma$ and $n_B$ are coupled around the CP!

$$\delta \sigma \simeq \delta n_B$$

$\sigma \sim M_q$

$F(\sigma, n) = A\sigma^2 + B\sigma n + Cn^2 + \cdots$

Critical soft mode
Analysis of 2\textsuperscript{nd}-order Cumulant

Evolution of baryon number density

**Stochastic Diffusion Equation**

\[
\partial_t n = D(t) \partial_x^2 n + \partial_x \xi
\]

\[
\langle \xi(x_1, t_1) \xi(x_2, t_2) \rangle = 2D \chi_2 \delta^{(2)}(1 - 2)
\]

\(D(t), \chi_2(t)\): parameters characterizing criticality

- Analytic solution is obtained.
- Study 2\textsuperscript{nd} order cumulant & correlation function.

**Our Main Conclusion**

Non-monotonicity in cumulants or correlation func. = Signal of QCD-CP
Bjorken Expansion

Cartesian coordinates

$$\partial_t n = D(t) \partial_x^2 n + \partial_x \xi$$

Milne coordinates

$$\partial_{\tau} n = \frac{D(t)}{\tau^2} \partial_Y^2 n + \frac{1}{\tau} \partial_Y \xi - \frac{n}{\tau}$$

suppression of diffusion
density reduction
Parametrizing $D(\tau)$ and $\chi(\tau)$

- Critical behavior
  - 3D Ising $(r,h)$
  - model H

Berdnikov, Rajagopal (2000)
Stephanov (2011); Mukherjee+ (2015)

- Temperature dep.
Assumptions

Evolution of baryon number density

**Stochastic Diffusion Equation**

\[ \partial_t n = D(t) \partial_x^2 n + \partial_x \xi \]

\[ \langle \xi(x_1, t_1) \xi(x_2, t_2) \rangle = 2D \chi_2 \delta^{(2)}(1 - 2) \]

\( D(t), \chi_2(t) \) : parameters characterizing criticality

- Uniform / infinitely long system
- Near equilibrium: \( \delta N_\mu \ll N_0 \)
- Short correlation length
- Slow diffusion
Resonance Decay

The larger $D_h$, the slower diffusion.

Neutral Particles

Decay into charged particles

The larger $\Delta \eta$, the slower diffusion.
Crossover / Cumulant

\[ K(\Delta y) = \frac{\langle \delta Q^2 \rangle}{\langle \delta Q^2 \rangle_{\text{eq.}}} \]

- \( T = 220 \)
- \( T = 170 \)
- \( T = 160 \)
- \( T = 150 \)
- \( T = 100 \)
- Blurring

- monotonically decreasing

**Analytic result**

\[ \chi(\tau) \]

monotonically increasing

\[ K(\Delta y) \]

monotonically decreasing

ALICE
PRL 2013
Critical Point / Cumulant

\[ K(\Delta y) = \frac{\langle \delta Q^2 \rangle}{\langle \delta Q^2 \rangle_{\text{eq.}}} \]

- non-monotonic \( \Delta y \) dep.

Analytic result: \( K(\Delta y) \) non-monotonic \( \leftrightarrow \) \( \chi(\tau) \) non-monotonic

See also,
Wu, Song
arXiv: 1903.06075
Cumulants and Correlation Function

\[ Q = \int_V dx n(x) \]

(total charge)

\[ \langle \delta Q^2 \rangle = \int_V dx dy \langle \delta n(x) \delta n(y) \rangle \]

(2\textsuperscript{nd} order cumulant (fluctuation))

\[ \langle \delta Q^2 \rangle_{\Delta y} = \int_{\Delta y} dy (\Delta y - |y|) \langle \delta n(y) \delta n(0) \rangle \]

(1-dim case)

1-to-1 correspondence

charge density

correlation function
Criticap Point / Correlation Func.

\[ C(\bar{y}) = \left\langle \delta n(\bar{y}) \delta n(0) \right\rangle / \chi_{\text{hadron}} \]

- **non-monotonic** $\Delta y$ dep.

Analytic result: 
- $C(\Delta y)$ non-monotonic 
- $\chi(\tau)$ non-monotonic

See also, Wu, Song
arXiv: 1903.06075
Away from the CP

\[ K(\Delta y) = \frac{\langle \delta Q^2 \rangle}{\langle \delta Q^2 \rangle_{\text{eq}}} \]

- Signal of the critical enhancement can be clearer on a path away from the CP.

Away from the CP \( \rightarrow \) Weaker critical slowing down
Describing Non-Gaussianity

**Diffusion Eq. with Non-linear Terms**

\[
\partial_\tau n = \Gamma(n) \partial_Y^2 \frac{\delta F[n]}{\delta n(Y)} + \partial_Y \xi
\]

\[
\langle \xi(Y_1, \tau_1) \xi(Y_2, \tau_2) \rangle = 2A \delta^{(2)}(1 - 2)
\]

\[
f(n) = k(\nabla n)^2 + a\Delta n^2 + b\Delta n^3 + c\Delta n^4 + \cdots
\]

Application to 1\(^{st}\) order transition:
Nonaka, Akamatsu, Bluhm, MK, Nahrgang, Wednesday

- Proper description of higher order cumulants
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1st-Order Transition

- Domain formation
- Non-uniform system

Herold, Nahrgang, et al. (2011~); Steinheimer, Randrup (2012; 2013)
Including Non-Linearity

Nahrgang, Bluhm, Schafer, Bass (2018)

\[ \partial_\tau n = \frac{D}{\tau^2} \partial^2_Y n + \frac{1}{\tau} \partial_Y \xi(\eta, \tau) - \frac{n}{\tau} \]

Include non-linear effects

\[ \partial_\tau n = \frac{\Gamma(n)}{\tau^2} \partial^2_Y \frac{\delta F[n]}{\delta n(x)} + \frac{1}{\tau} \partial_Y \xi(\eta, \tau) - \frac{n}{\tau} \]

\[ F[n(x)] = \int dx f(x) \]

- Diffusion equation: \( f(n) = \frac{a}{2} n^2, \quad D = \Gamma a \)
- solve numerically
Free Energy

- At 1st transition point

- Large and small n

\[ \chi(n) = \frac{\partial^2 f}{\partial n^2} \rightarrow \chi_{\text{QGP}} \ (n \rightarrow \infty) \]

\[ \rightarrow \chi_{\text{hadron}} \ (n \rightarrow 0) \text{ Poisson} \]
Modeling 1st Transition

\[ \partial_\tau n = \Gamma(n) \partial_Y^2 \frac{\delta F[n]}{\delta n(Y)} + \partial_Y \xi \]

\[ \langle \xi(Y_1, \tau_1)\xi(Y_2, \tau_2) \rangle = 2A\delta^{(2)}(1 - 2) \]

- \( f(n) = \frac{1}{2}a(n - n_s)^2 + \frac{1}{4}b(n - n_s)^4 \]
  \[ + c(\tau)n + k(\partial_Y n)^2 \]

- \( \Gamma \): positive
- adjust \( \Gamma \) and \( A \) to reproduce the behavior of \( D \) at small and large \( n \)

\[ \tilde{D} = \Gamma \left( \frac{\partial^2 f}{\partial n^2} + X \right) \quad A = 2D\chi_2 \]
Configurations in Equilibrium

\[ \partial_\tau n = \frac{\Gamma(n)}{\tau^2} \partial^2_Y \frac{\delta F[n]}{\delta n(x)} + \frac{1}{\tau} \partial_Y \xi(\eta, \tau) \frac{n}{\tau} \]

- Domain formation
- Surface: thickness \( v(2k/a) \), surface tension
Time Evolution

- Dynamical domain formation
- Domains survive even after 1st transition
Time Evolution

- Weaker 1ˢᵗ transition can also lead to formation of domains.
Correlation Function

\[ C(\bar{y}) = \langle \delta n(\bar{y}) \delta n(0) \rangle / \chi_{\text{hadron}} \]

- Domain leads to a peak structure in \( C(y) \).
- The peak can survive even in the final state.
Summary

- Fluctuations observed in HIC are not in equilibrium.

- Plenty of information in rapidity window dependences of higher-order cumulants.

- 2nd-order cumulant (correlation function) already contains interesting information.

Future
- Evolution of higher-order cumulants around the critical point / 1st transition
- Combination to momentum (model-H)
- More realistic model (dimension, Y dependence, ...
Δη dependence for tracing back the history!