The future of lattice studies in Korea, PKNU, Busan, Korea, 2019/Sep./6

Lattice QCD & QGP

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(Osaka U.)

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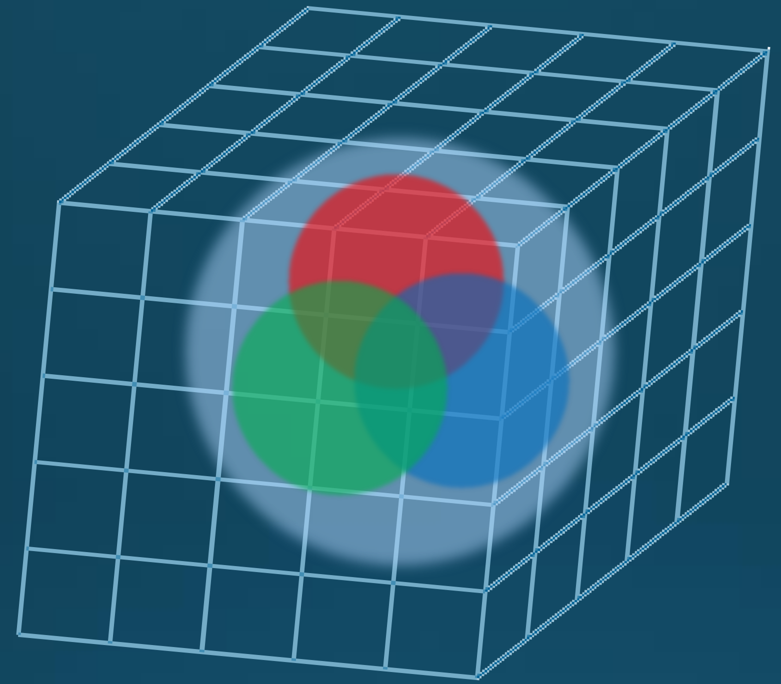
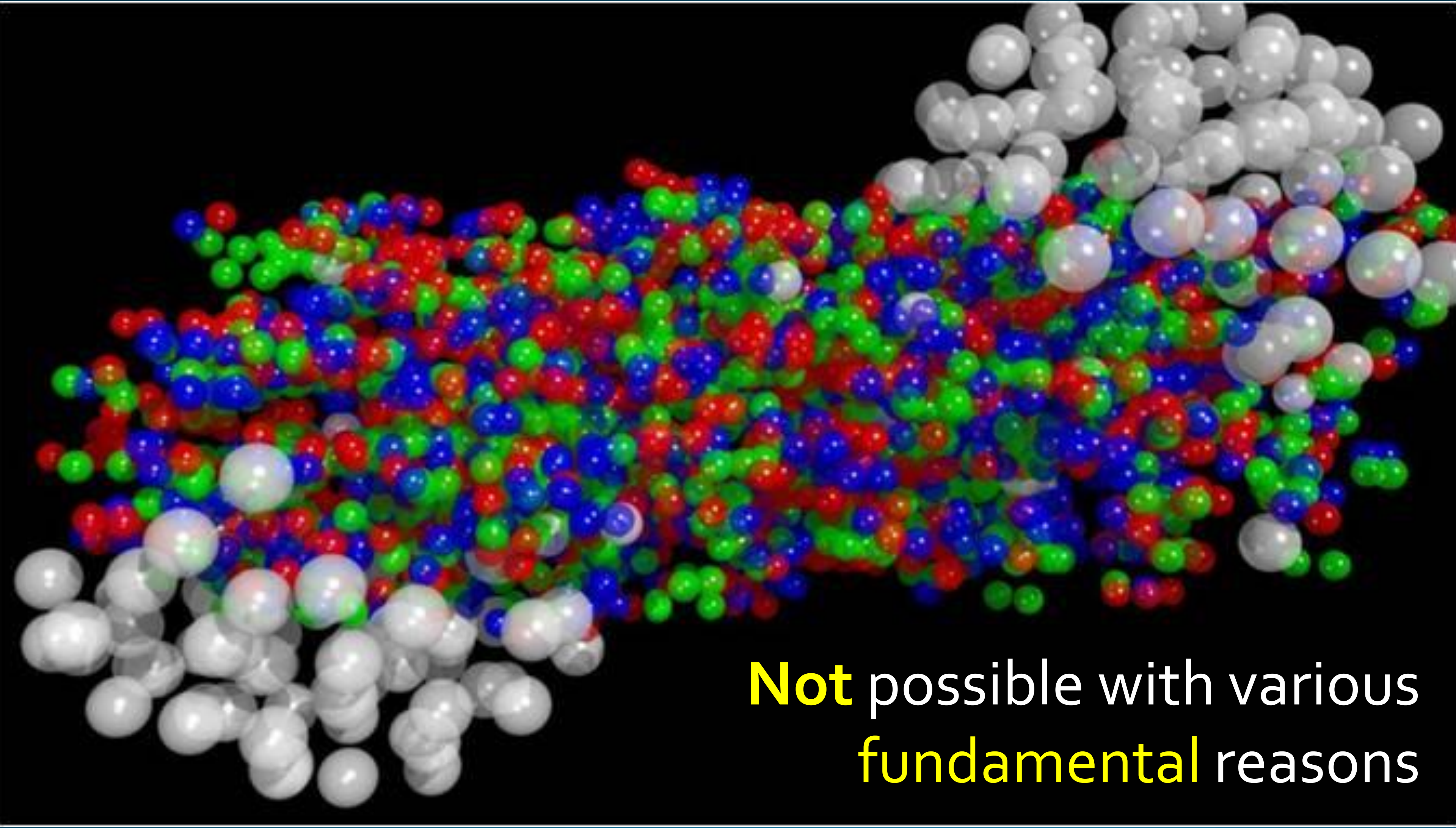


Lattice QCD is a powerful tool to study non-perturbative phenomena of QCD?

Yes, but it is not so useful…

Reproducing HIC on the lattice?



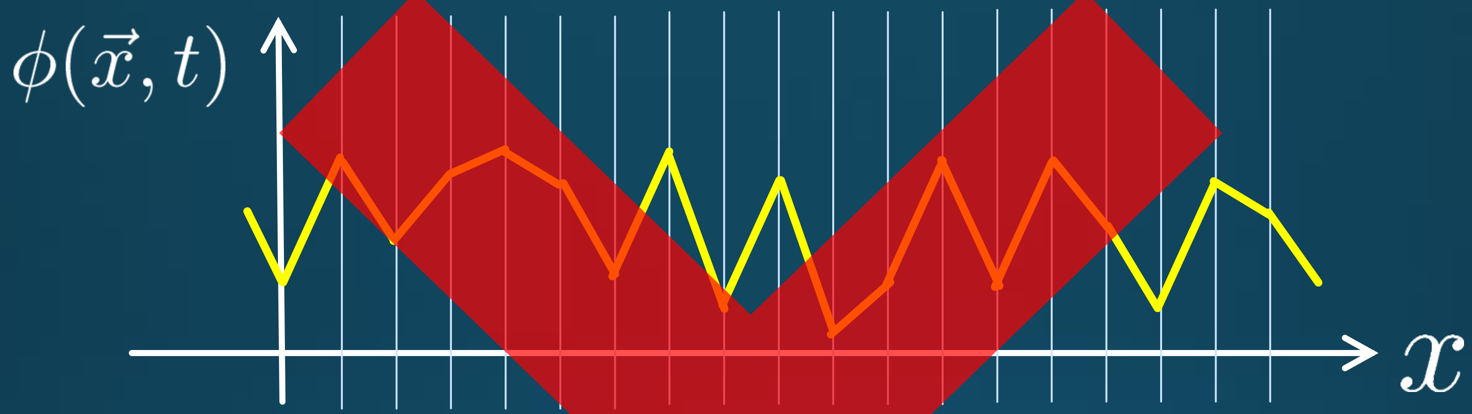
Not possible with various fundamental reasons

Difficulties

- ☐ Too many degrees of freedom
- ☐ Ignorance about physical states
- ☐ Ignorance about physical operators

Lattice simulations are accessible only to correlation functions of specific operators in Euclidean space-time.

Lattice Simulations



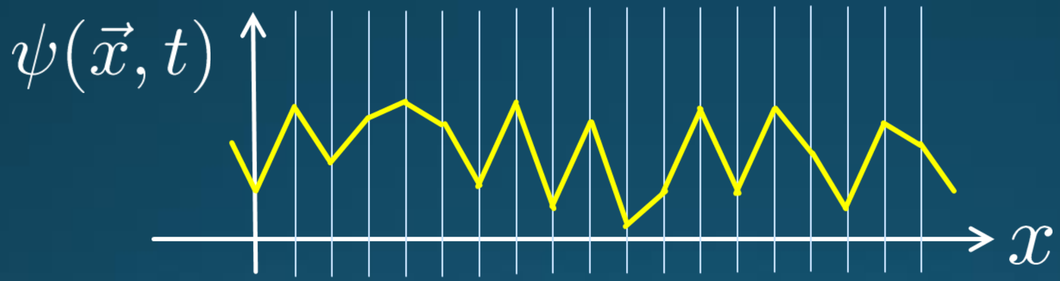
Diffusion eq.: $$\frac{\partial}{\partial t}\phi(\vec{x}, t) = D\nabla^2\phi(\vec{x}, t)$$

$$\frac{d^2\phi(x)}{dx^2} = \frac{1}{a^2}\{\phi(x-a) - 2\phi(x) + \phi(x+a)\}$$

$$\phi(x+\Delta t) = \phi(x) + \Delta t D \frac{d^2\phi(x)}{dx^2}$$

QCD is a Quantum Theory.

$$i\frac{\partial}{\partial t}\psi(\vec{x},t) = -\frac{\hbar^2\nabla^2}{2m}\psi(\vec{x},t) + V(x)\psi(\vec{x},t)$$



Time evolution can be simulated.
(Eigenvalue problem would be easier.)

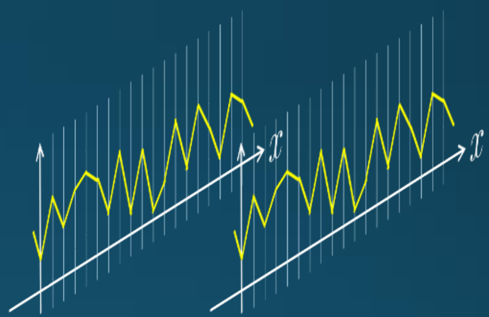
QCD is a Quantum Field Theory

Quantum Field Theory

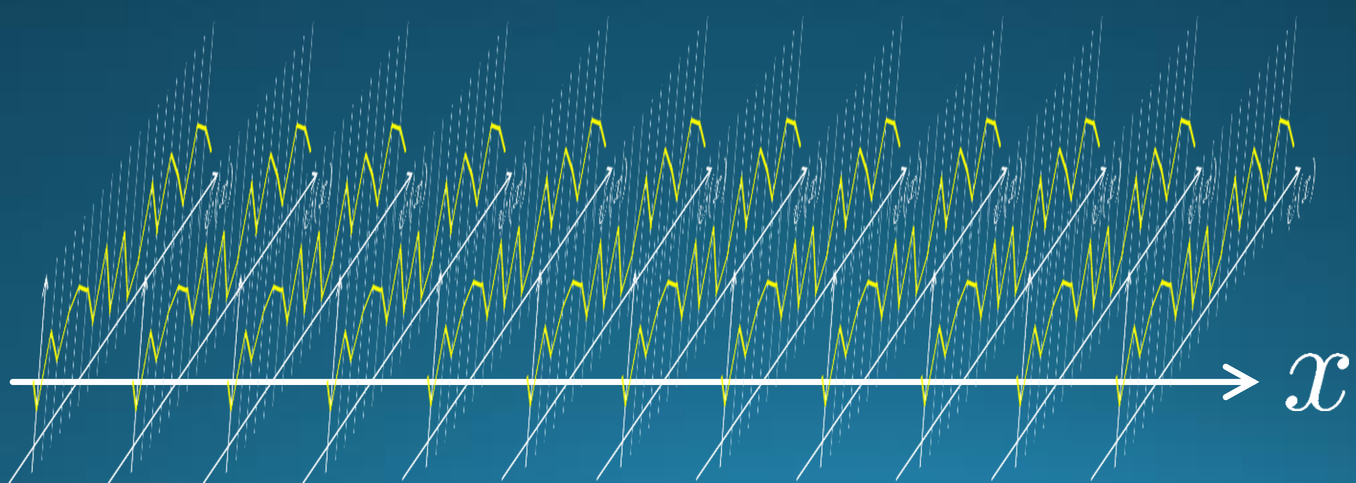
ϕ(x) at every space-time points are arguments of wave func.

Spin ½ system:

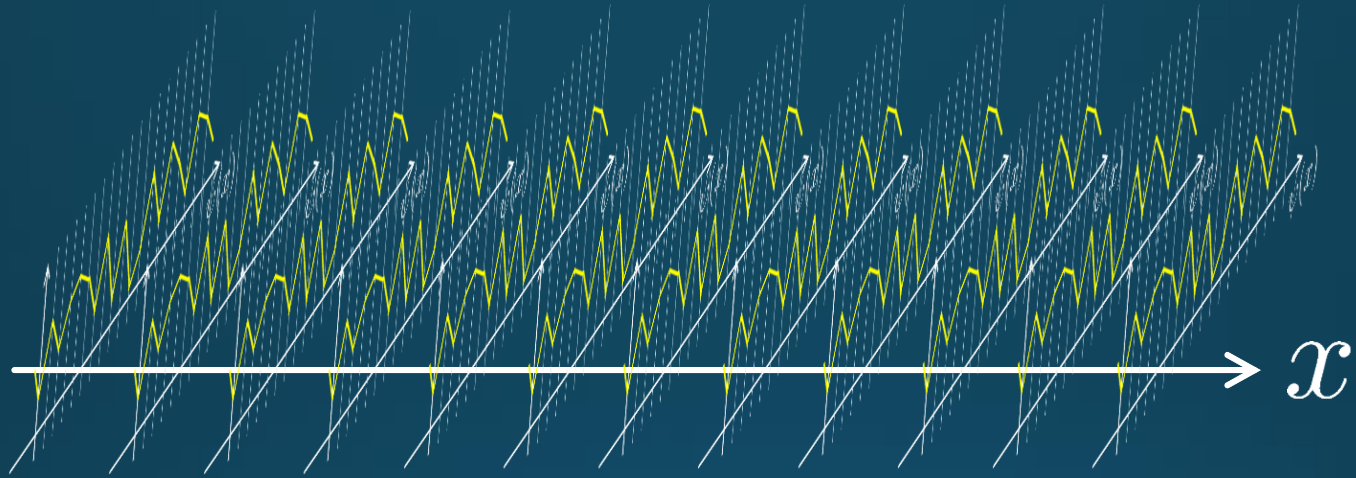
$$\Psi(x) = (\psi_{\uparrow}(x), \psi_{\downarrow}(x))$$



QFT:



Physical States



$$\Psi[\psi(x)]$$

Functional of ψ
So many d.o.f

Numerical simulation of time evolution is too difficult!

Initial Conditions

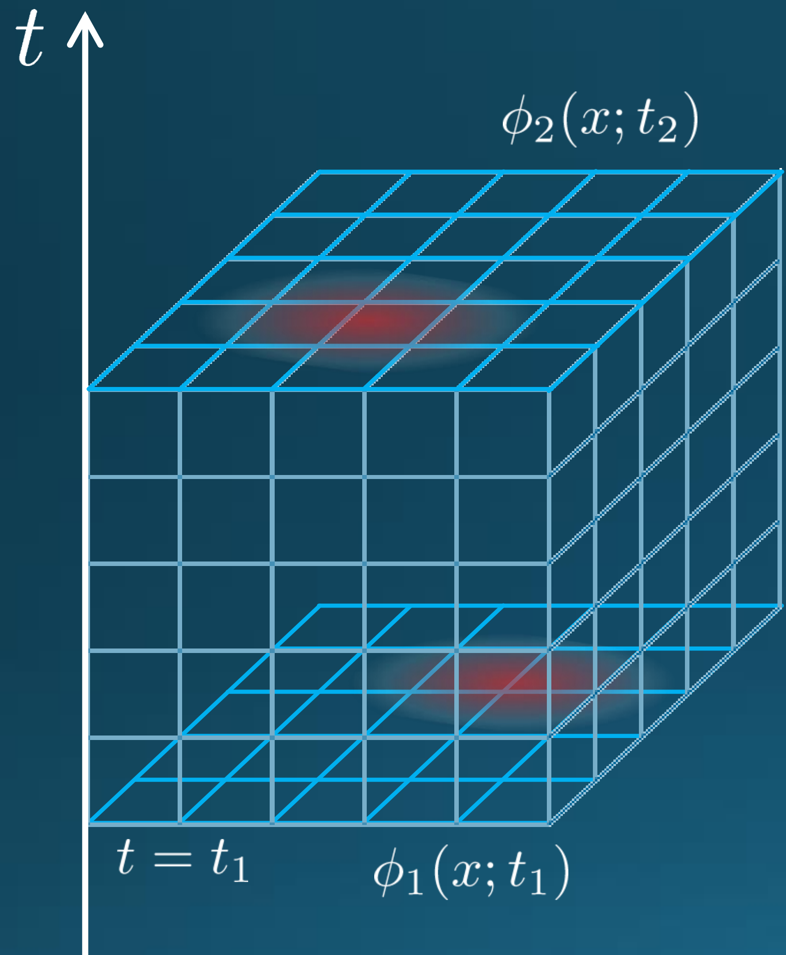
Initial conditions having physical meaning?

- Vacuum $|0\rangle$
- 1-particle state $a_p^\dagger|0\rangle$
- 2-particle state $a_{p_1}^\dagger c_{p_2}^\dagger|0\rangle$

$|0\rangle$ Vacuum state: unknown

$a_p^\dagger$ Creation operators: unknown

Path Integral



Transition amplitudes between two states can be calculated as

$$\langle \phi_2(x), t_2 | \phi_1(x), t_1 \rangle$$
$$= \lim_{a \to 0} \left[\prod_x \int d\phi(x) \right] e^{iS[\phi(x)]/\hbar}$$
$$= \int \mathcal{D}\phi e^{iS(\phi)/\hbar}$$

Lattice field theory is constructed by the space-time discretization

Problems: ①What are physical states?
②How to carry out path integral numerically?

QFT in Euclidean SpaceTime

- Action becomes real: Importance sampling

$$\int \mathcal{D}x e^{iS[x(t)]/\hbar} \Rightarrow \int \mathcal{D}x e^{-S_{\mathrm{E}}[x(\tau)]/\hbar}$$

- Vacuum state is created by taking $\tau \to \pm\infty$

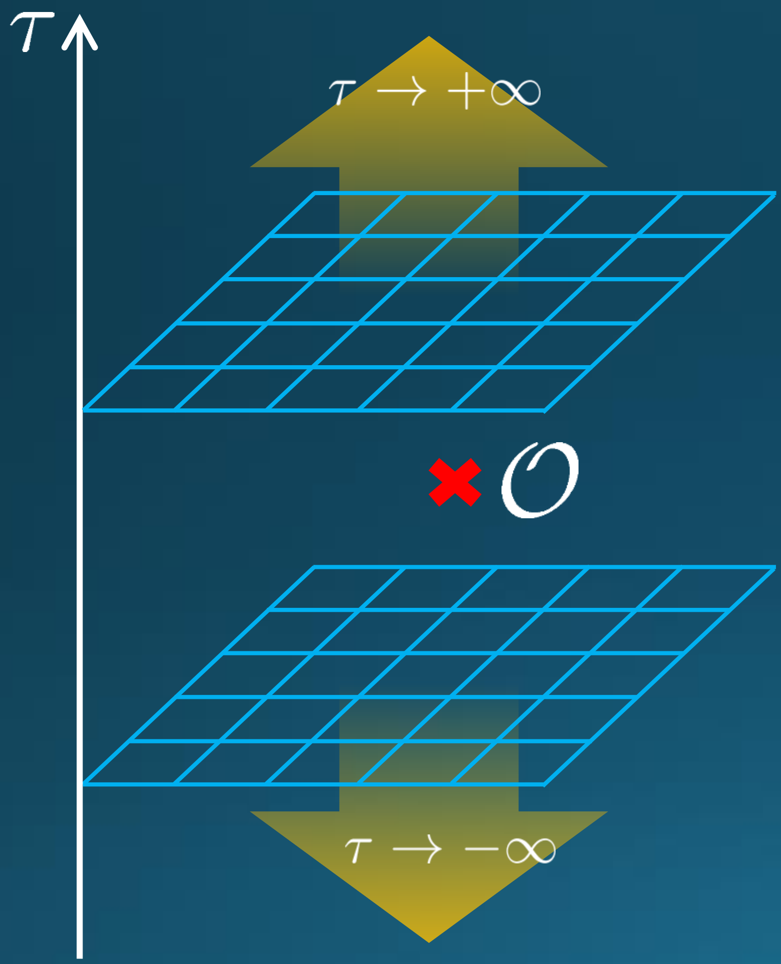
$$\int \mathcal{D}x e^{-\int_{-\tau_1}^{0} d\tau L[x(\tau)]} \sim e^{-H\tau_1}|x, -\tau_1\rangle \underset{\tau_1 \to \infty}{\longrightarrow} |0\rangle$$

$$\Downarrow$$

$$\langle 0|f(\hat{x})|0\rangle \sim \int_{-\infty}^{\infty} \mathcal{D}x f(x)_{\tau=0} e^{-S/\hbar}$$

Note: One may apply the periodic BC.

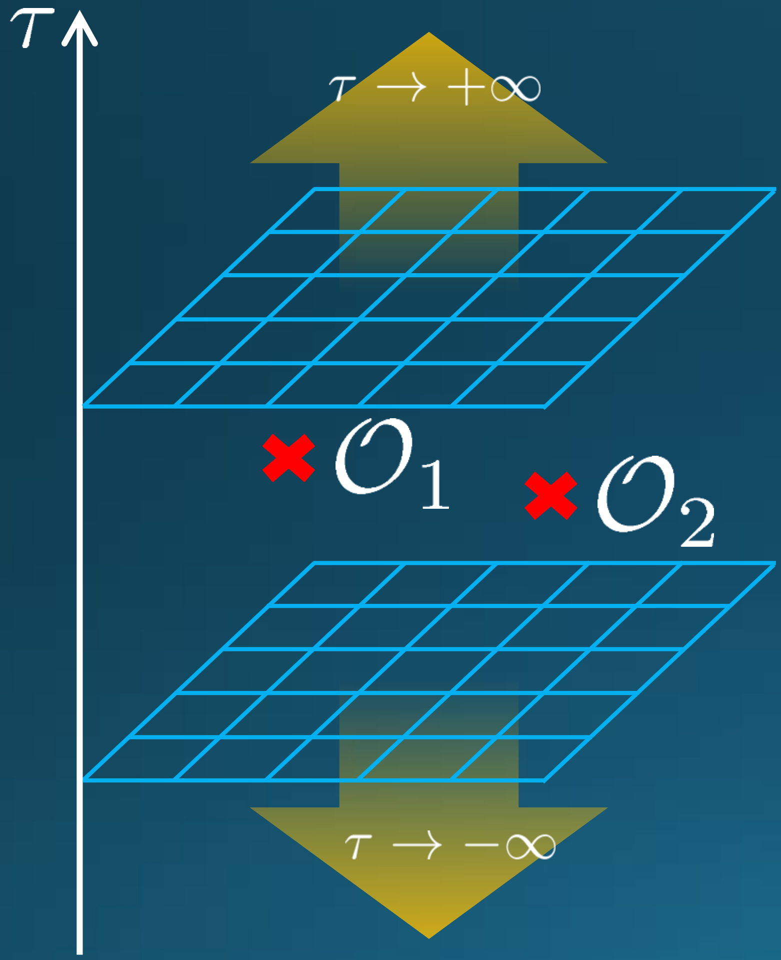
Calculating Operators



Lattice Simulations can calculate vacuum correlation funcs.

$$\langle 0 | \mathcal{O}(x) | 0 \rangle$$

Calculating Operators



Lattice Simulations can calculate vacuum correlation funcs.

$$\langle 0|\mathcal{O}(x)|0\rangle$$

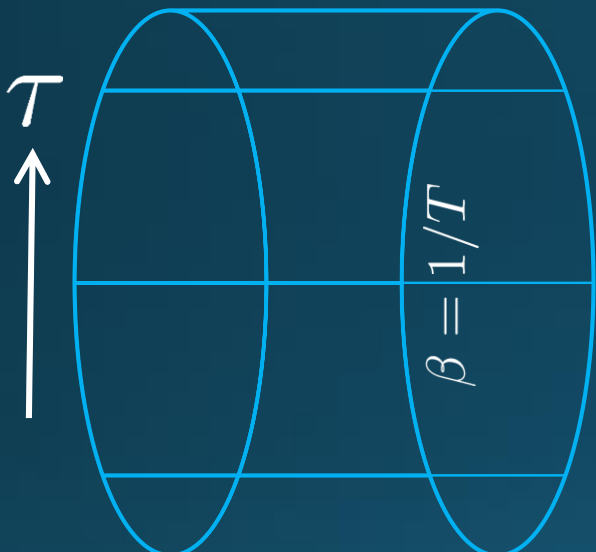
$$\langle 0|\mathcal{O}_1(x)\mathcal{O}_2(y)|0\rangle$$

$$\cdots$$

These are almost everything that lattice simulations can do.

QFT @ Nonzero T

$$Z = \mathrm{Tr} e^{-\beta H} = \sum_n \langle n | e^{-\beta H} | n \rangle$$
$$= \int \mathcal{D}\phi e^{-S_T}$$



(Anti-)Periodic BC
= Nonzero T system

$$\langle \mathcal{O} \rangle_T = \int \mathcal{D}\phi \mathcal{O} e^{-S_T}$$

Thermodynamics

$$\begin{cases} \text{Energy density: } \langle T_{00} \rangle_T \\ \text{Pressure: } \langle T_{11} \rangle_T \end{cases}$$

Suzuki,2013; FlowQCD, 2014

New Physics on the Lattice

- New operators
- New usage of operators
- Cleverer measurement
- New Environment

  Nonzero *T*, magnetic field,
  boundary conditions, finite density,
  N_c, N_f, …

Polyakov loop,
Wilson loop,

$\bar{\psi}\psi, \ \bar{\psi}\Gamma\psi, \ \cdots$

$T_{\mu\nu}$

Energy-Momentum Tensor

energy momentum

$$T_{\mu\nu} = \begin{bmatrix} T_{00} & T_{01} & T_{02} & T_{03} \\ T_{10} & T_{11} & T_{12} & T_{13} \\ T_{20} & T_{21} & T_{22} & T_{23} \\ T_{30} & T_{31} & T_{32} & T_{33} \end{bmatrix}$$

pressure

stress

All components are important physical observables!

EMT on the Lattice: Conventional

Lattice EMT Operator

Caracciolo+, 1990

$$T_{\mu\nu} = Z_6 T^{[6]}_{\mu\nu} + Z_3 T^{[3]}_{\mu\nu} + Z_1 \left(T^{[1]}_{\mu\nu} - \langle T^{[1]}_{\mu\nu} \rangle \right)$$

$$T^{[6]}_{\mu\nu} = (1-\delta_{\mu\nu}) F^a_{\mu\rho} F^a_{\nu\rho}, \quad T^{[3]}_{\mu\nu} = \delta_{\mu\nu} \left(F^a_{\mu\rho} F^a_{\nu\rho} - \frac{1}{4} F^a_{\rho\sigma} F^a_{\rho\sigma} \right), \quad T^{[1]}_{\mu\nu} = \delta_{\mu\nu} F^a_{\rho\sigma} F^a_{\rho\sigma}$$

- Fit to thermodynamics: Z_3, Z_1
- Shifted-boundary method: Z_6, Z_3

Giusti, Meyer, 2011; 2013;
Giusti, Pepe, 2014~; Borsanyi+, 2018

Multi-level algorithm

- effective in reducing statistical error of correlator

Meyer, 2007;
Borsanyi, 2018;
Astrakhantsev+, 2018

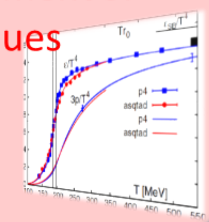
EMT with Gradient Flow
"SFtE Method"

New measurement of the renormalized EMT on the lattice.
Suzuki 2013; FlowQCD 2014~; WHOT-QCD 2017~

Thermodynamics

direct measurement of expectation values

$\langle T_{00} \rangle, \langle T_{ii} \rangle$



Fluctuations and Correlations

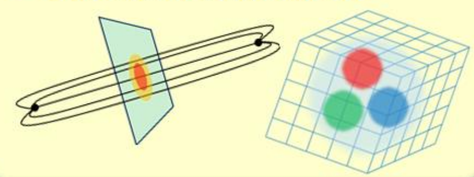
viscosity, specific heat, ...

$\eta = \int_0^\infty dt \langle T_{12}; T_{12} \rangle$

$c_V \sim \langle \delta T_{00}^2 \rangle$

Hadron Structure

- flux tube / hadrons
- stress distribution



Thermodynamics

Quantum Statistical Mechanics

$$\rho = \frac{1}{Z} e^{-\beta(H-\mu N)}$$ Density Matrix

$$Z = \mathrm{Tr} e^{-\beta(H-\mu N)}$$ Partition Function

$$\langle O \rangle = \mathrm{Tr}[O\rho]$$

Thermodynamics on the Lattice

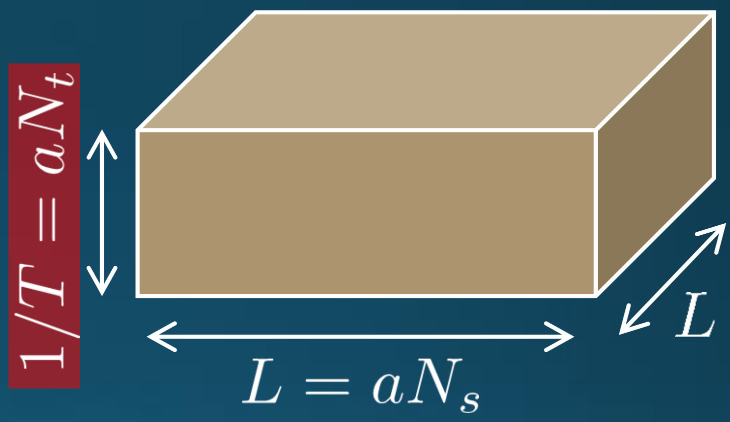
Thermodynamic Relations

$$\varepsilon = \frac{T^2}{V}\frac{\partial \ln Z}{\partial T} \qquad p = T\frac{\partial \ln Z}{\partial V}$$

e and *p* are obtained by *T* & *V* derivatives of ln *Z*.

Derivative w.r.t. *a* → V & 1/T changes

$$a\frac{\partial \ln Z}{\partial a} \sim \frac{V}{T}(\varepsilon - 3p)$$

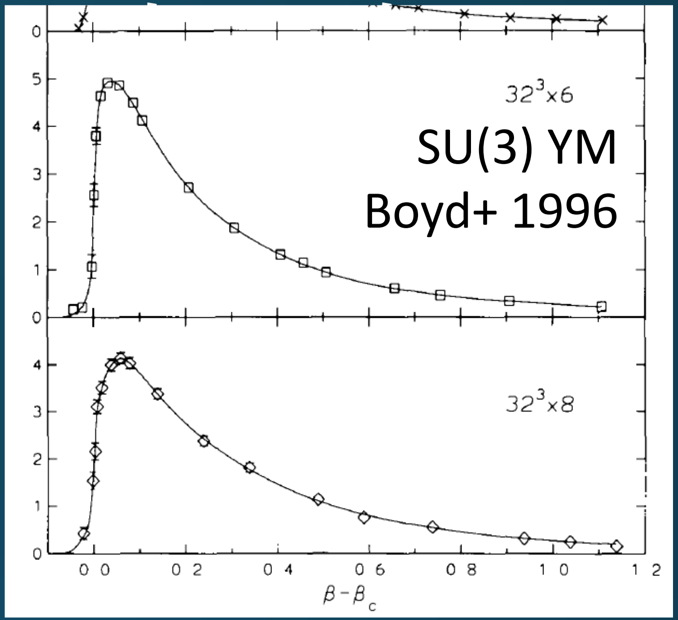


Integral Method

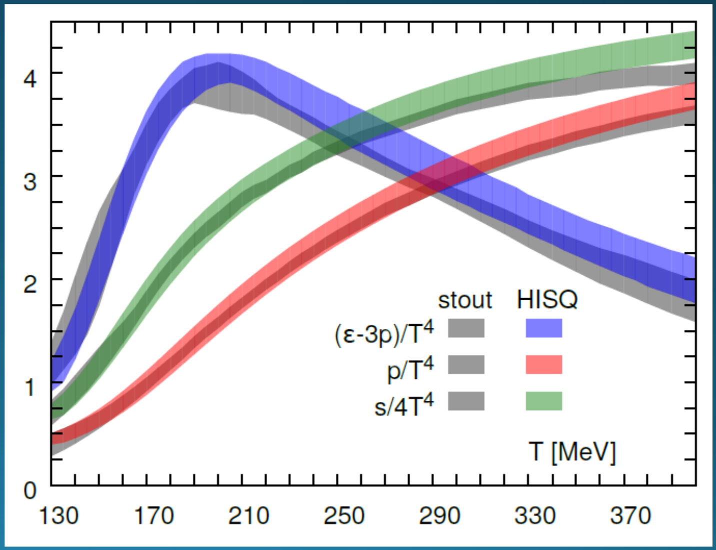
$$\frac{\partial \ln Z}{\partial a} = \frac{\partial \beta}{\partial a}\frac{\partial \ln Z}{\partial \beta} \sim \frac{\partial \beta}{\partial a}\langle S \rangle$$

$$T\frac{\partial (p/T^4)}{\partial T} = \frac{\varepsilon - 3p}{T^4}$$

$$\frac{p}{T^4} = \int_{T_0}^{T} dT \frac{\varepsilon - 3p}{T^5}$$



Full QCD, BW; HotQCD (2014)



Thermodynamics of SU(3) YM

- **Integral method**
  - Most conventional / established
  - Use themodynamic relations

    Boyd+ 1995; Borsanyi, 2012

$$p = \frac{T}{V} \ln Z$$

$$T \frac{\partial (p/T^4)}{\partial T} = \frac{\varepsilon - 3p}{T^4}$$

- Gradient-flow method
  - Take expectation values of EMT

    FlowQCD, 2014, 2016

$$\begin{cases} \varepsilon = \langle T_{00} \rangle \\ p = \langle T_{11} \rangle \end{cases}$$

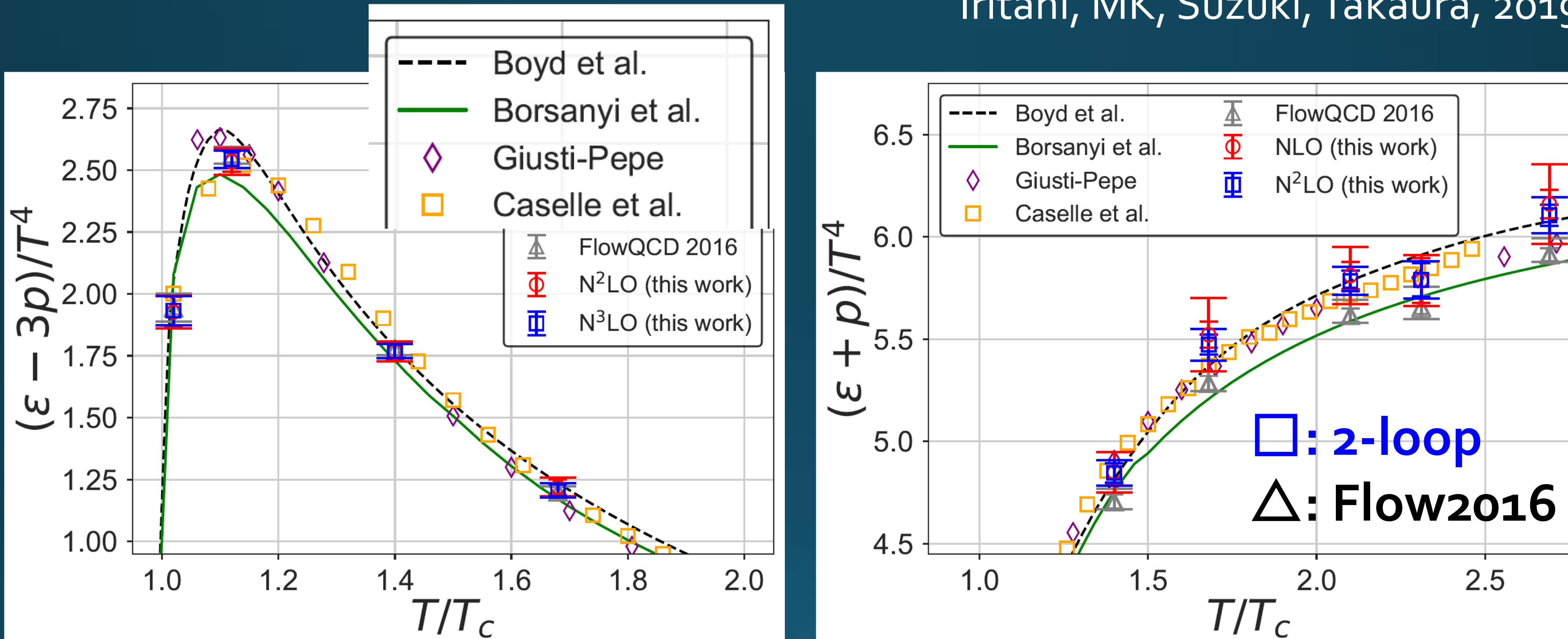
- Moving-frame method

  Giusti, Pepe, 2014~
- Non-equilibrium method
  - Use Jarzynski's equality Caselle+, 2016;2018
- Differential method

  Shirogane+(WHOT-QCD), 2016~

SU(3) Thermodynamics: Comparison

Iritani, MK, Suzuki, Takaura, 2019



Boyd+:1996 / Borsanyi+: 2012

- All results agree well.
- But, the results of integral method has a discrepancy. (Older result looks better…)

Future Study

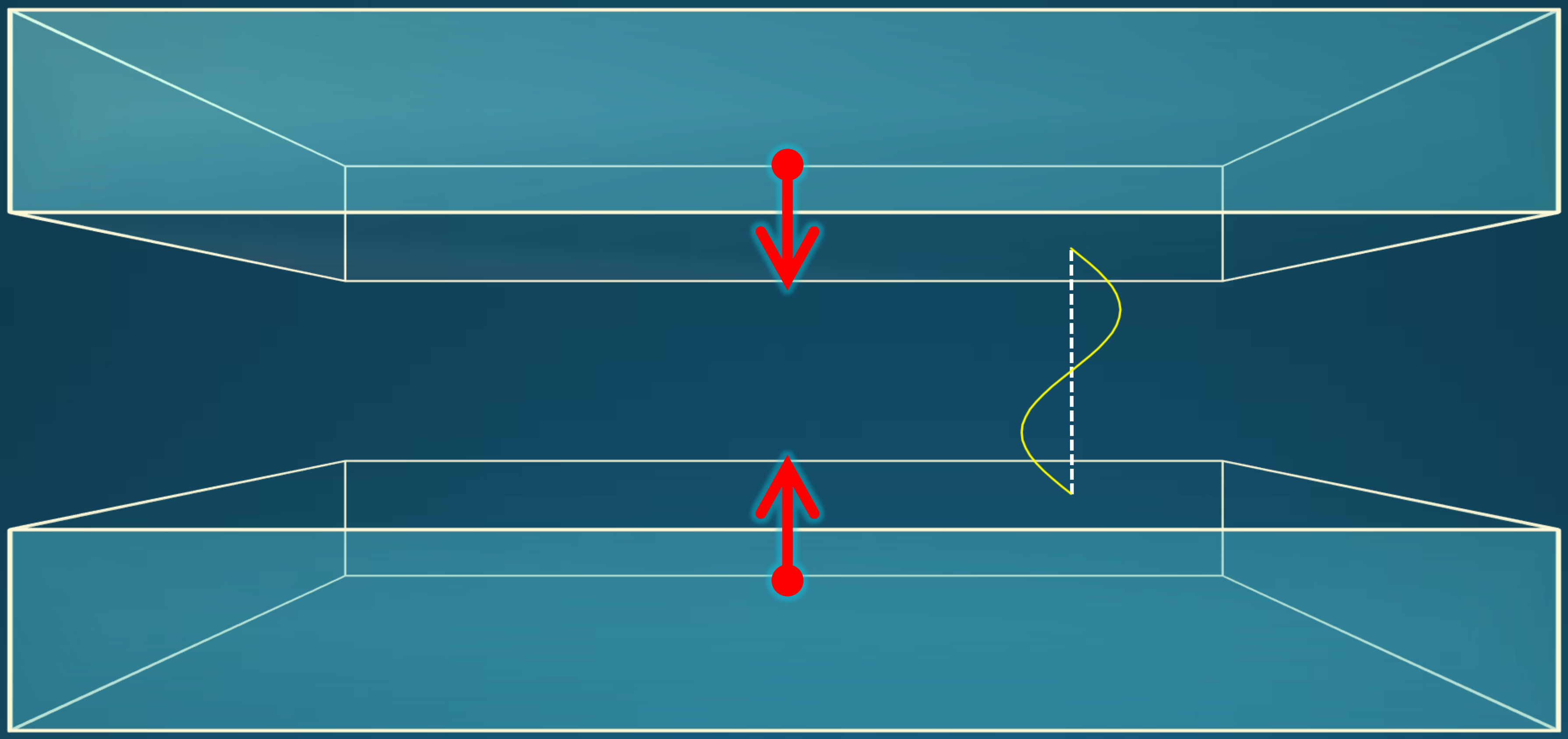
- Thermodynamics in SU(3) YM: Understand discrepancy between various analyses especially in two integral methods.
- Invent other methods

Casimir Effect

of SU(3) YM @ T>0

Casimir Effect

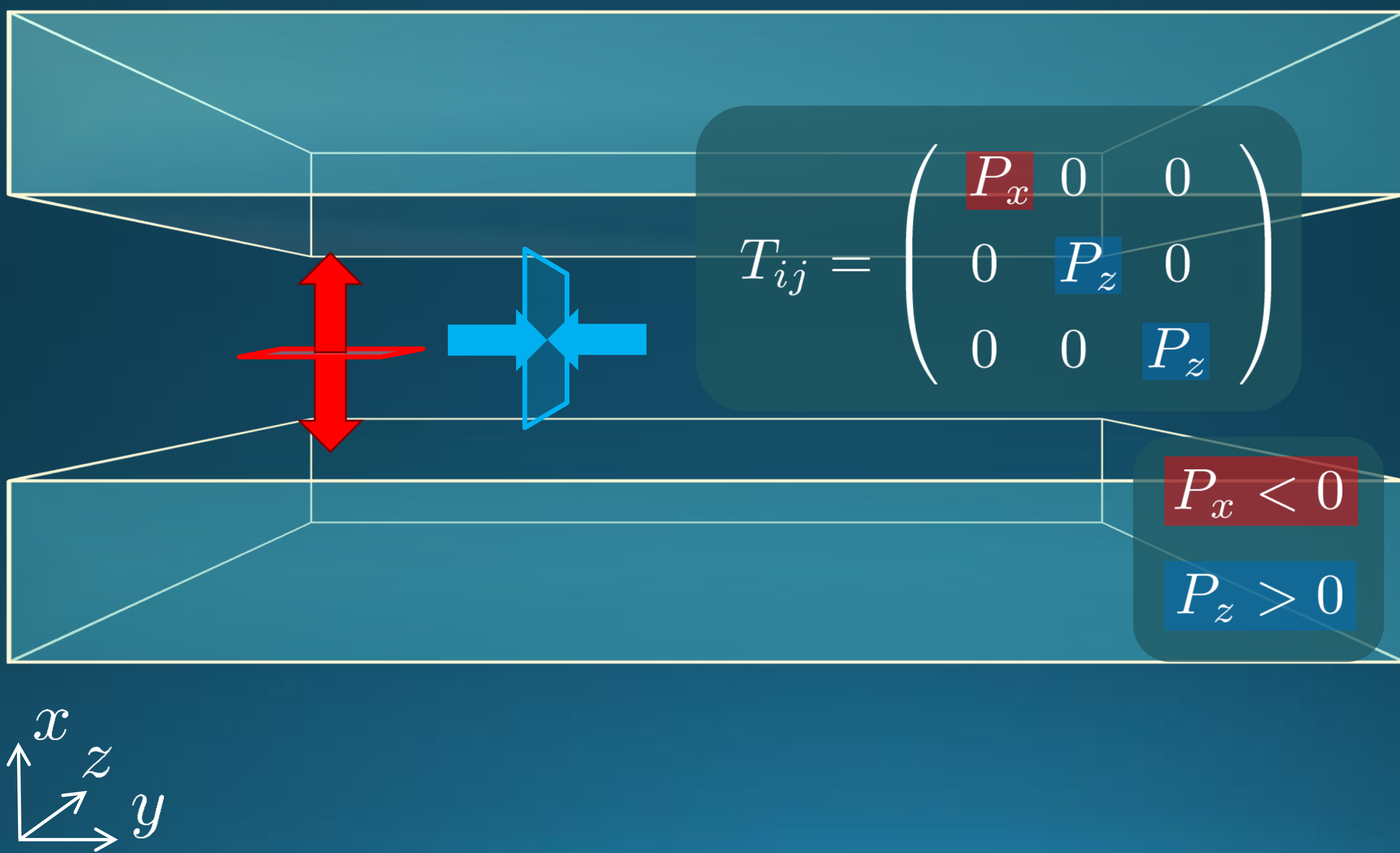
Casimir Effect



attractive force between two conductive plates

Casimir Effect

Brown, Maclay
1969



$$T_{ij} = \begin{pmatrix} P_x & 0 & 0 \\ 0 & P_z & 0 \\ 0 & 0 & P_z \end{pmatrix}$$

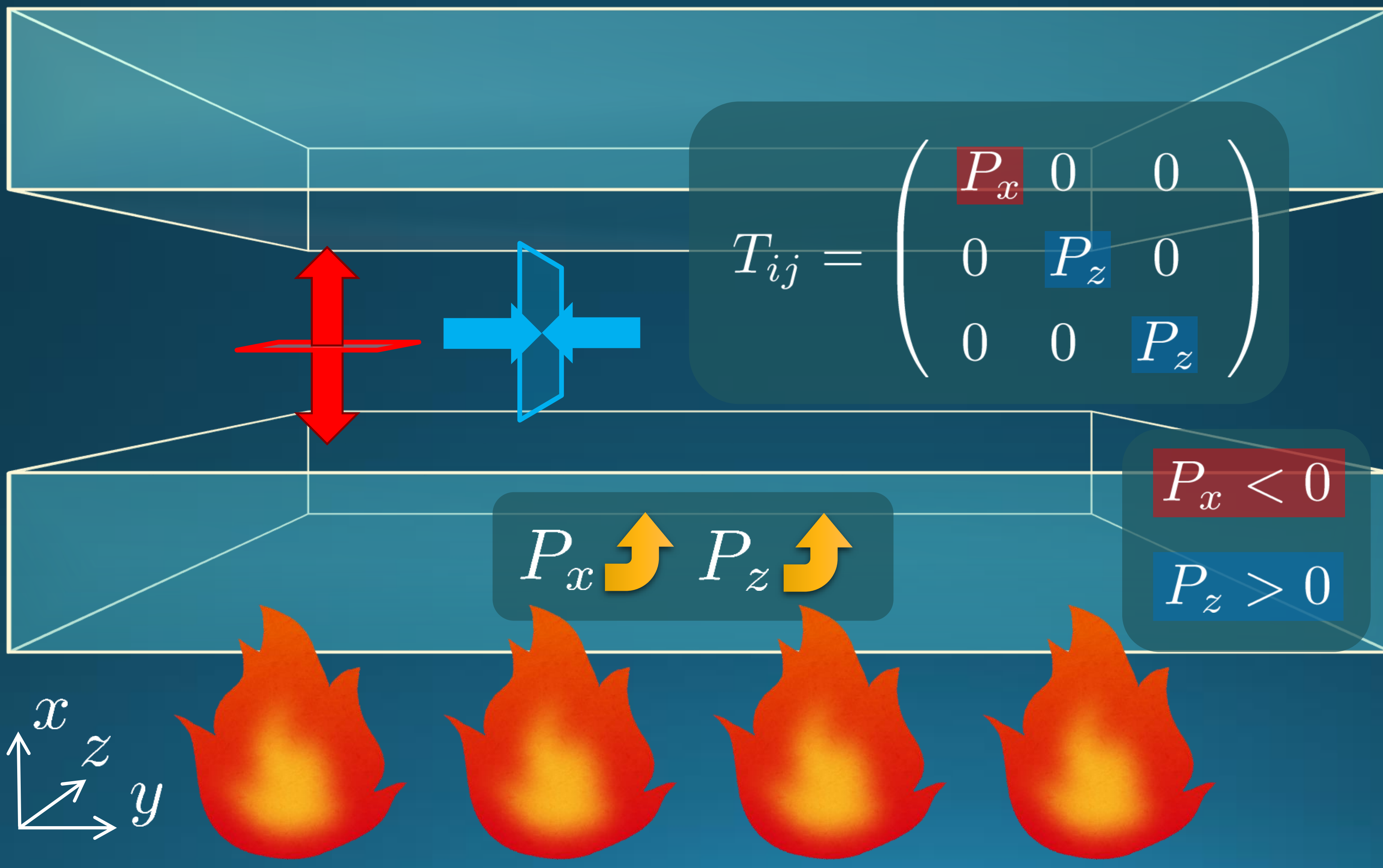
$$P_x < 0$$

$$P_z > 0$$

Casimir Effect

Brown, Maclay
1969

$$T_{ij} = \begin{pmatrix} P_x & 0 & 0 \\ 0 & P_z & 0 \\ 0 & 0 & P_z \end{pmatrix}$$



$P_x < 0$

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Brown, Maclay
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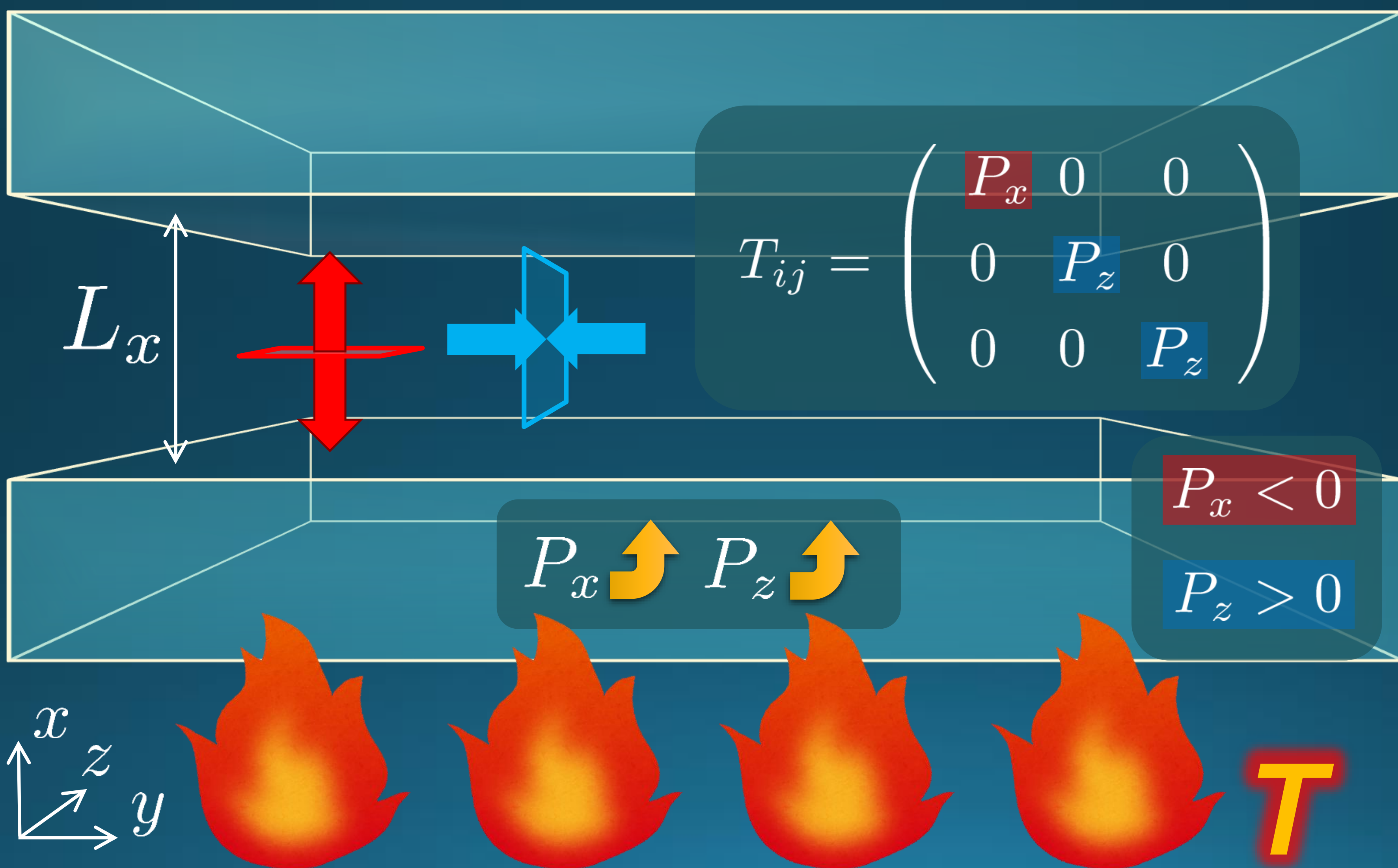


Thermodynamics on the Lattice

Various Methods

- **Integral**, differential, moving frame, non-equilibrium, ...
- rely on thermodynamic relations valid in V→∞

$$P = \frac{T}{V} \ln Z \qquad sT = \varepsilon + P$$

⇒ **Not applicable to anisotropic systems**

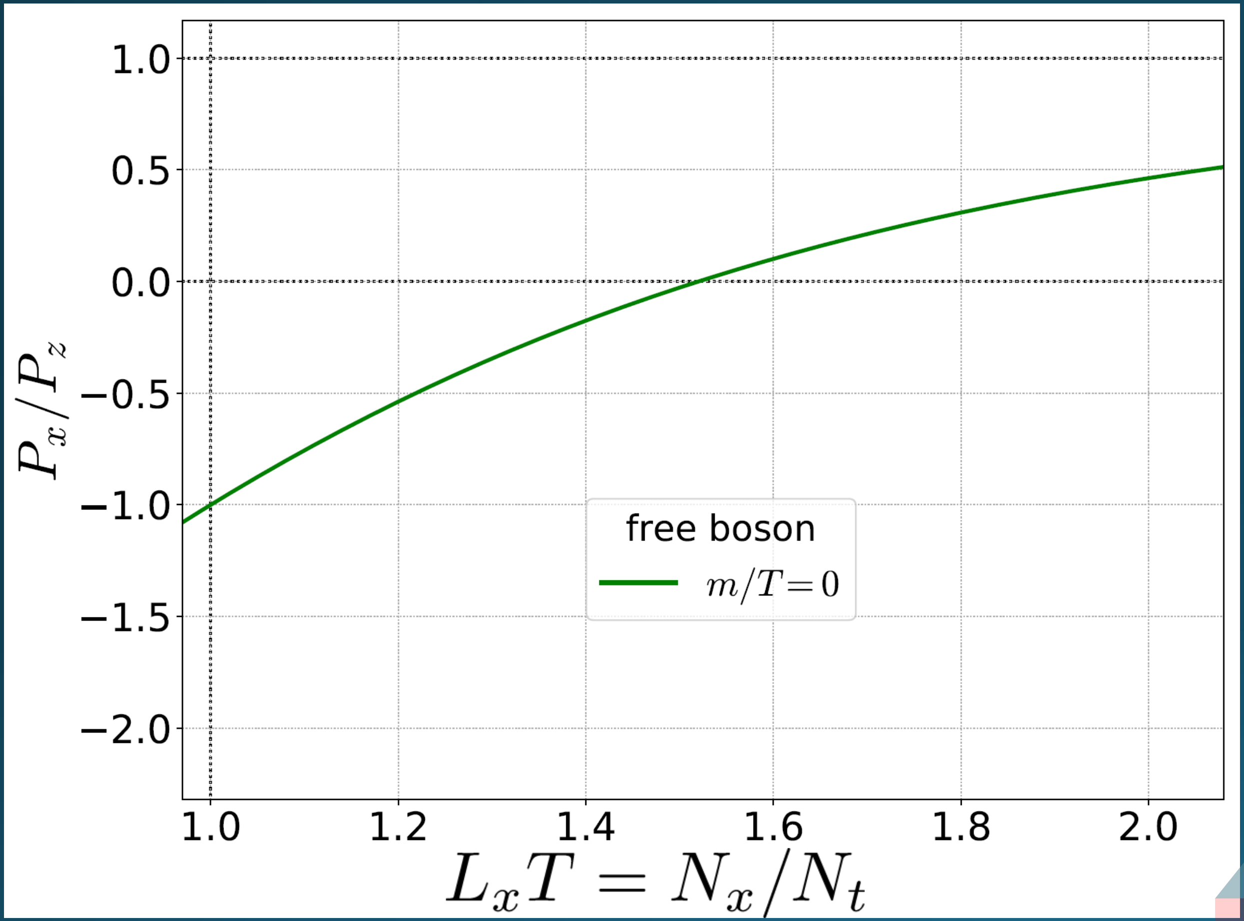
- We employ **Gradient Flow Method**

$$\varepsilon = \langle T_{00} \rangle \quad P = \langle T_{11} \rangle$$

Components of EMT are directly accessible!

Pressure Anisotropy @ T≠0

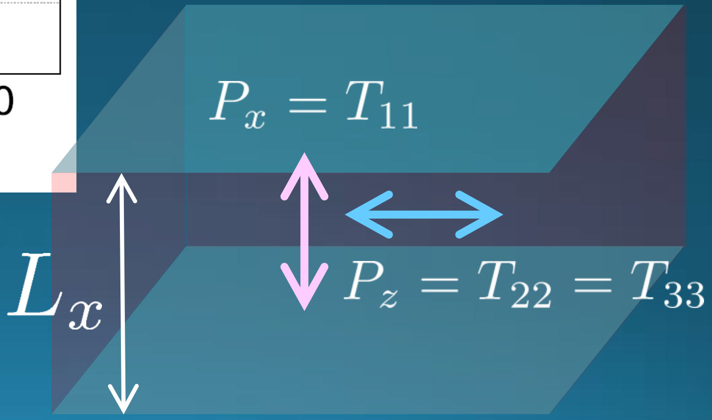
MK, Mogliacci, Kolbe, Horowitz, 1904.00241



Free scalar field

- $L_2 = L_3 = \infty$
- Periodic BC

Mogliacci+, 1807.07871



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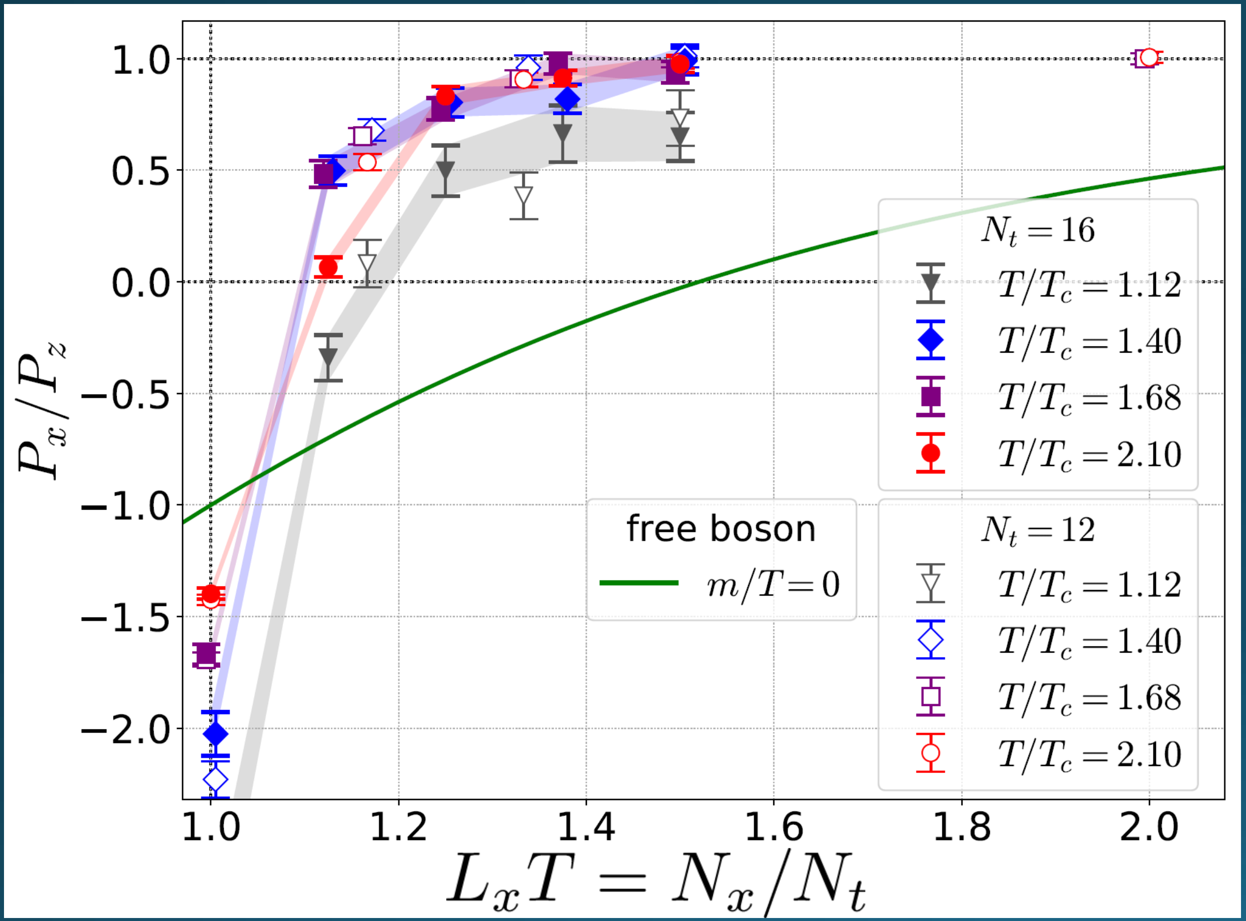


Free scalar field

- $L_2 = L_3 = \infty$
- Periodic BC

Mogliacci+, 1807.07871

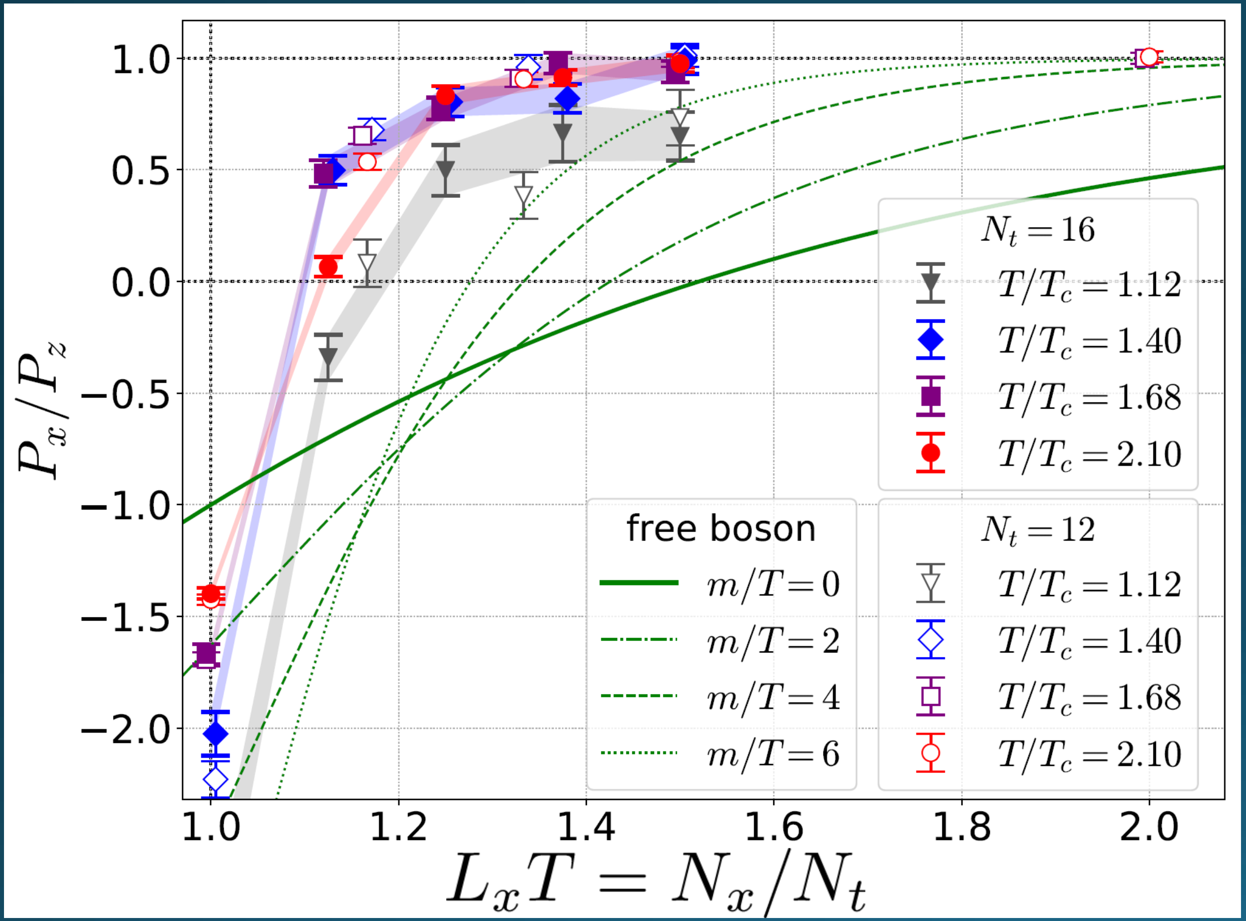
Lattice result

- Periodic BC
- Only t→0 limit
- Error: stat.+sys.

Medium near T_c is remarkably insensitive to finite size!

Pressure Anisotropy @ T≠0

MK, Mogliacci, Kolbe, Horowitz, 1904.00241



Free scalar field

- $L_2 = L_3 = \infty$
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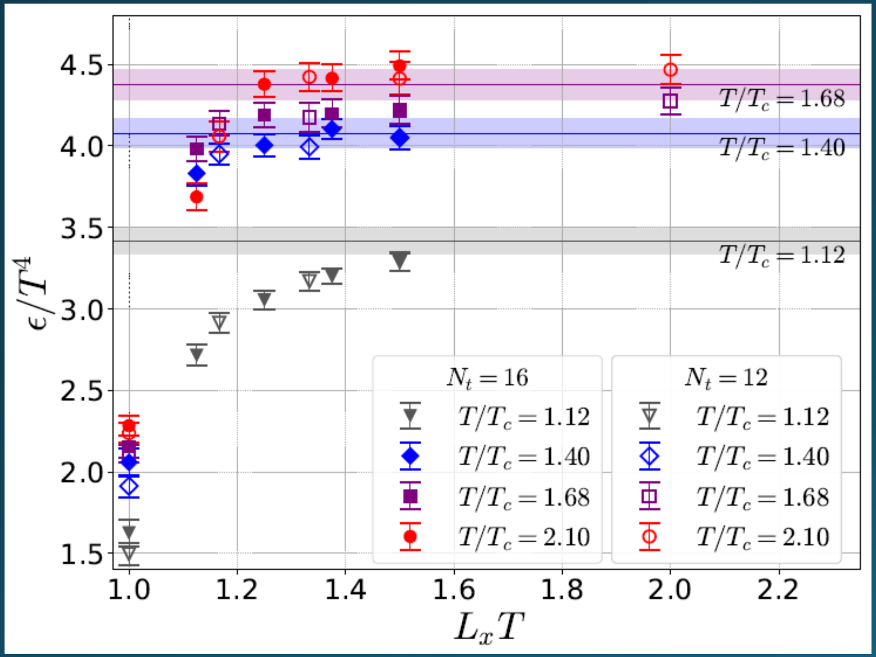
Lattice result

- Periodic BC
- Only $t \rightarrow 0$ limit
- Error: stat.+sys.

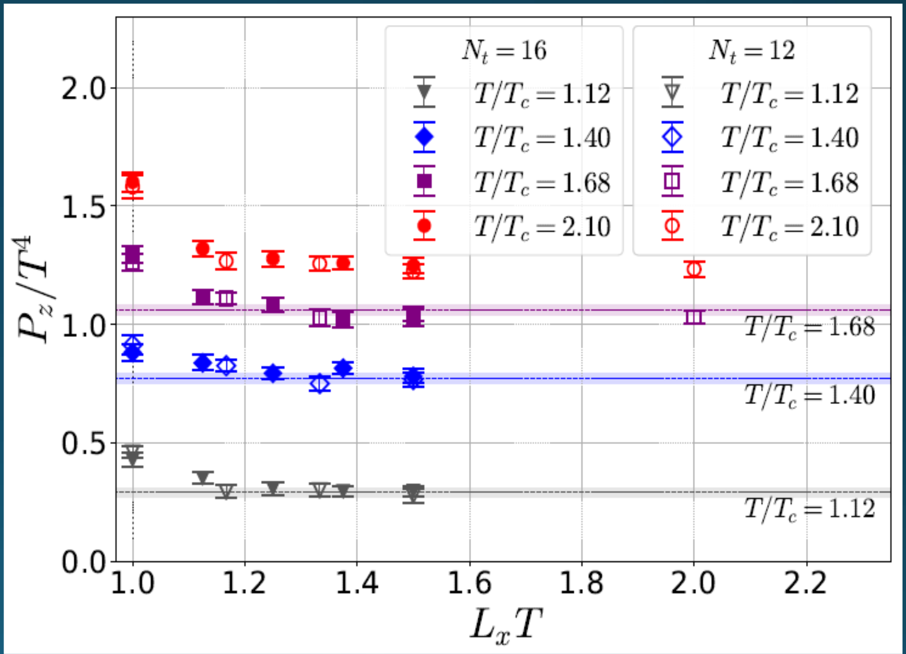
Medium near T_c is remarkably insensitive to finite size!

Energy densty / transverse P

Energy Density



Transverse Pressure P_z



Higher T

High-T limit: massless free gluons

How does the anisotropy approach this limit?

Difficulties

- ☐ Vacuum subtraction requires large-volume simulations.
- ☐ Lattice spacing not available → $c_1(t)$, $c_2(t)$ are not determined.

Higher T

High-T limit: massless free gluons

How does the anisotropy approach this limit?

Difficulties

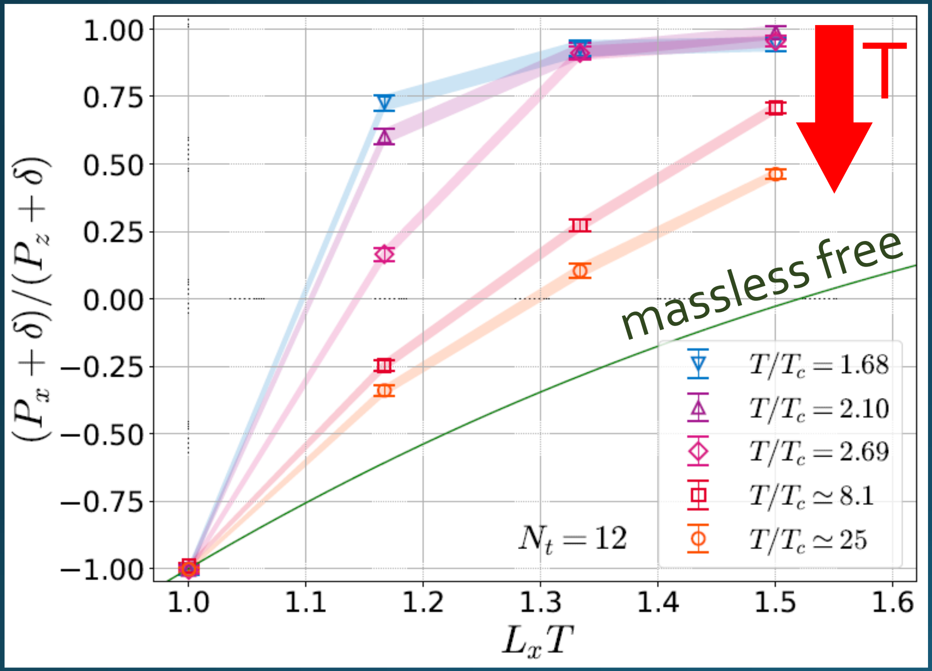
- ☐ Vacuum subtraction requires large-volume simulations.
- ☐ Lattice spacing not available → $c_1(t)$, $c_2(t)$ are not determined.

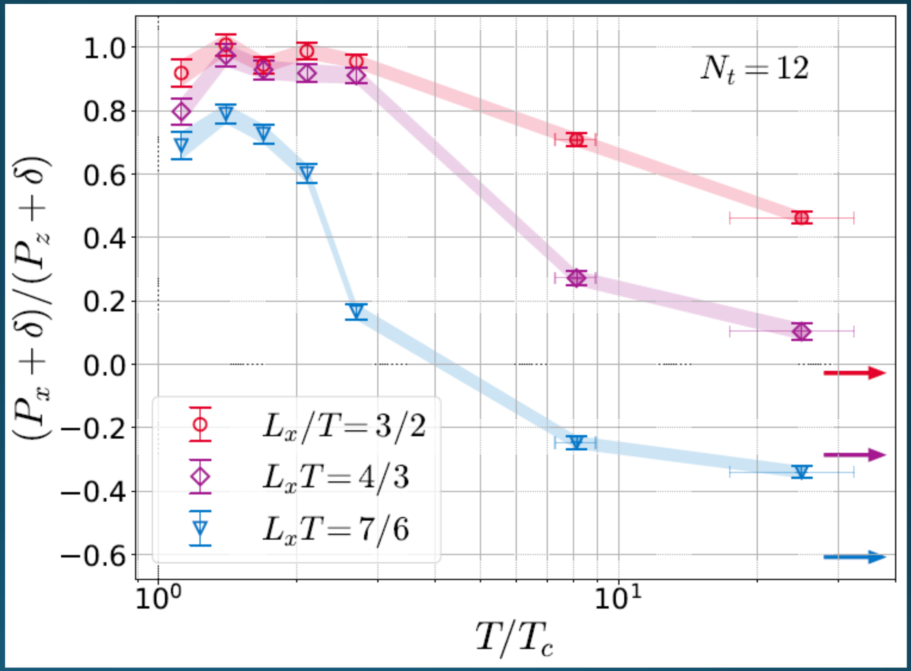
We study

$$\frac{P_x + \delta}{P_z + \delta} \qquad \delta = -\frac{1}{4}\sum_{\mu} T^{\mathrm{E}}_{\mu\mu}$$

No vacuum subtr. nor Suzuki coeffs. necessary!

$$\frac{P_x + \delta}{P_z + \delta}$$





$T/T_c \cong 8.1$ (β=8.0) / $T/T_c \cong 25$ (β=9.0)

- ☐ Ratio approaches the asymptotic value.
- ☐ But, large deviation exists even at $T/T_c \sim 25$.

Future Study

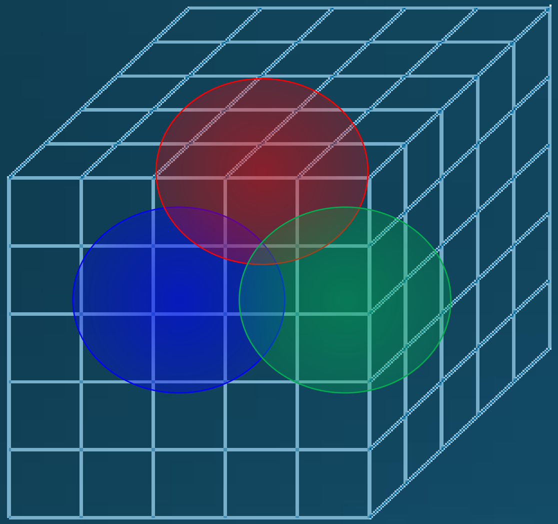
- Why SU(3) YM theory near but above Tc is so insensitive to the existence of the boundary?

- Much higher temperature
- Other boundary conditions (anti-PBC and etc.)

Dynamics
(Real-time Evolution)

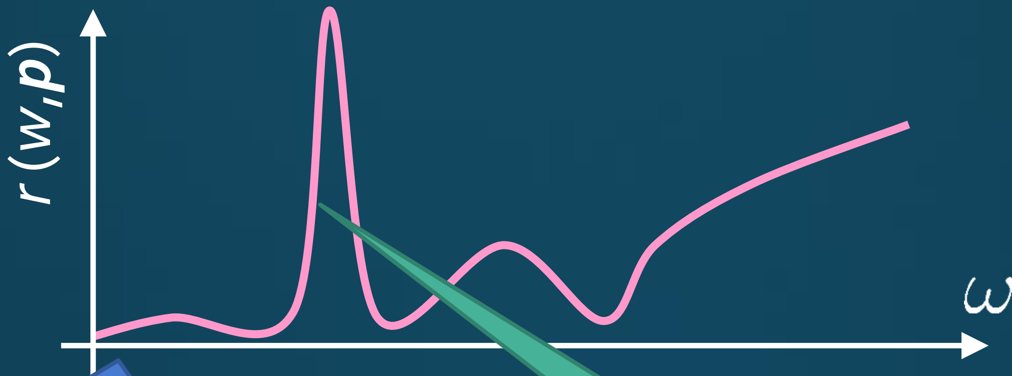
Analytic Continuation

- **Lattice:** imaginary time
- **Dynamics:** real time



Real-time info. have to be extracted from the correlation funcs. in imaginary time.

Spectral Function



slope at the origin

→ transport coefficients

Kubo formulae $\eta \sim \lim_{\omega \to 0} \frac{1}{\omega} \rho(\omega)$

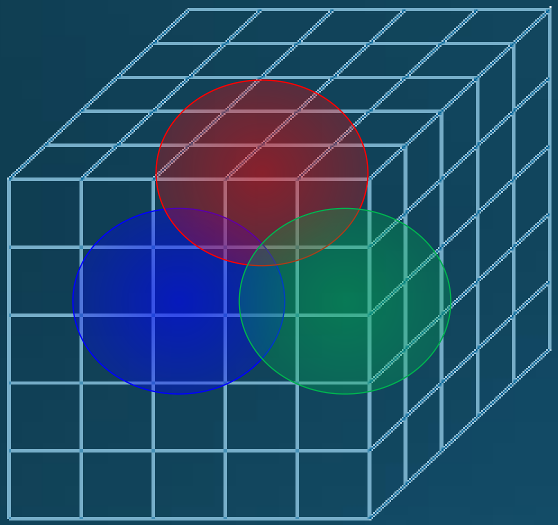
- shear viscosity : T_{12}
- bulk viscosity : T_{mm}
- electric conductivity : J_{ii}

peaks

quasi-particle excitation
width ~ decay rate

Analytic Continuation

- **Lattice:** imaginary time
- **Dynamics:** real time



$\tilde{G}(\tau, \boldsymbol{k})$

discrete and noisy

$\rho(\omega, \boldsymbol{k})$

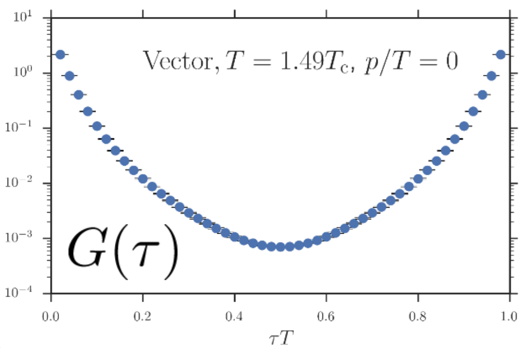
continuous

$$\tilde{G}(\tau) = \int d\omega \frac{e^{(\beta/2-\tau)\omega}}{e^{\beta\omega/2} + e^{-\beta\omega/2}} \rho(\omega)$$

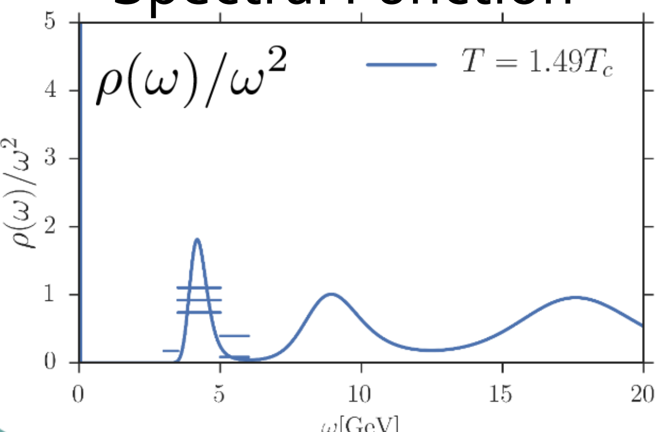
Maximum Entropy Method

Asakawa, Nakahara
Hatsuda, 2001

Lattice data



Spectral Function



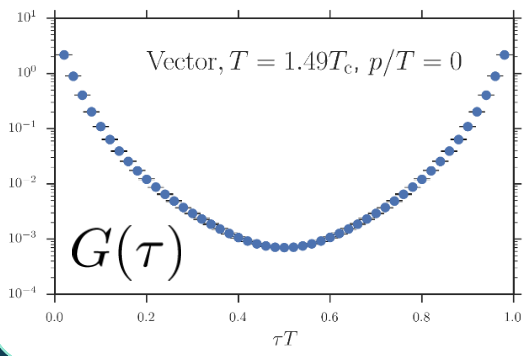
$$G(\tau) = \int_0^\infty d\omega \frac{\cosh(1/2T - \tau)\omega)}{\sinh(\omega/2T)} \rho(\omega)$$

"ill-posed problem"

Maximum Entropy Method

Asakawa, Nakahara
Hatsuda, 2001

Lattice data



Bayes theorem ×

Prior probability

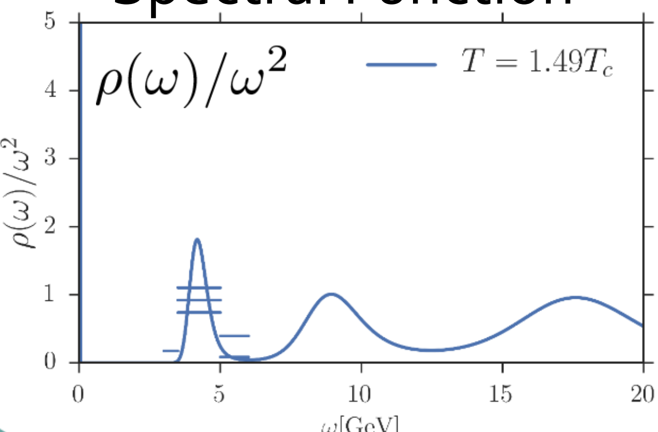
- Shannon-Jaynes entropy
- default model $m(\omega)$

=

Probability of $\rho(\omega)$

$P[\rho(\omega), \alpha]$

Spectral Function



Maximum Entropy Method

Asakawa, Nakahara
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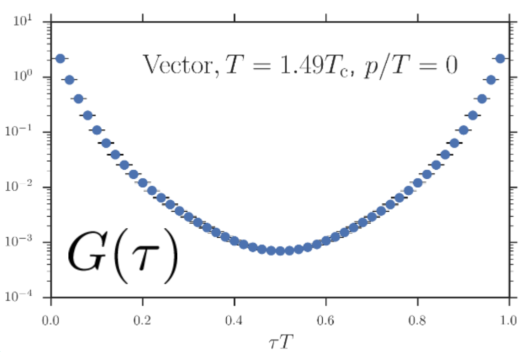
Lattice data



Bayes theorem ×

Prior probability

- Shannon-Jaynes entropy
- default model $m(\omega)$

=

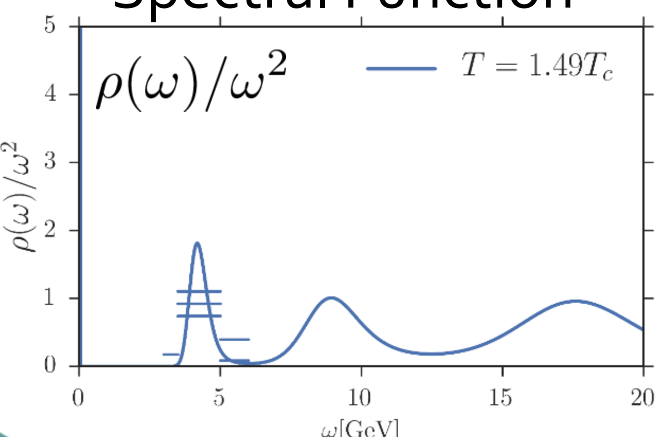
Probability of $\rho(\omega)$

$P[\rho(\omega), \alpha]$

expectation value
$\langle \rho(\omega) \rangle_P$

$$\langle \mathcal{O} \rangle_P = \int d\alpha \int [d\rho] P[\rho, \alpha] \mathcal{O}$$

Spectral Function

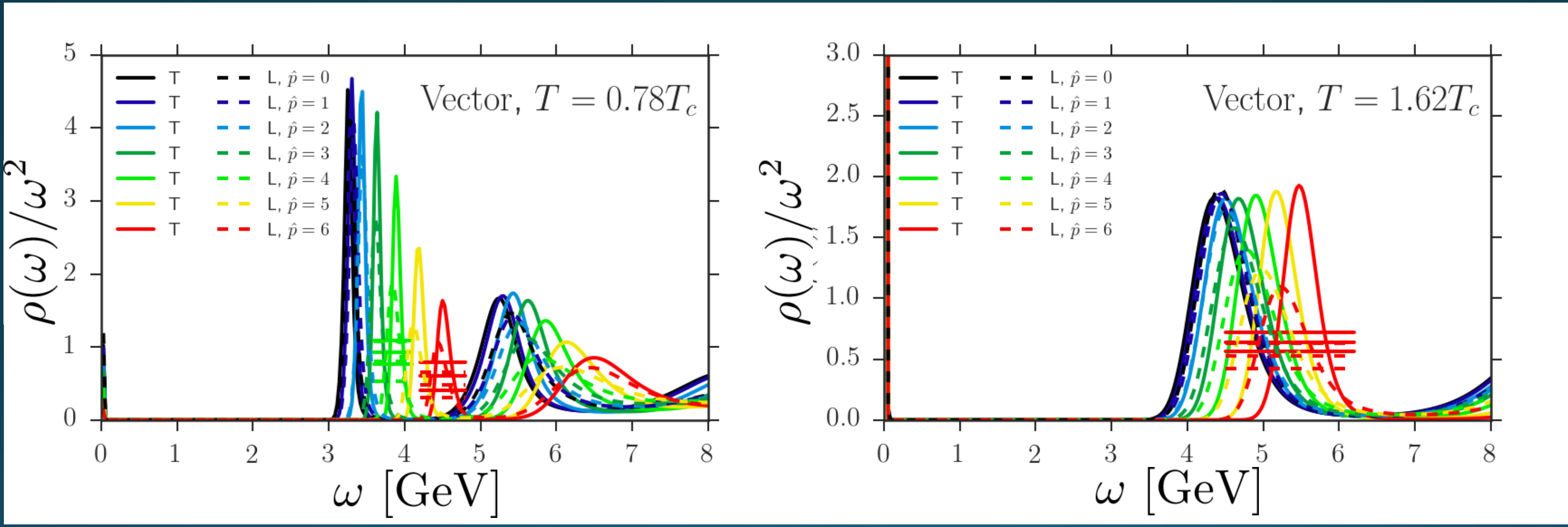


- ☐ Output of MEM is jus an expectation value.
- ☐ Error analysis is necessary!!!

Charmonium SPC

Ikeda, Asakawa, MK
PRD 2017

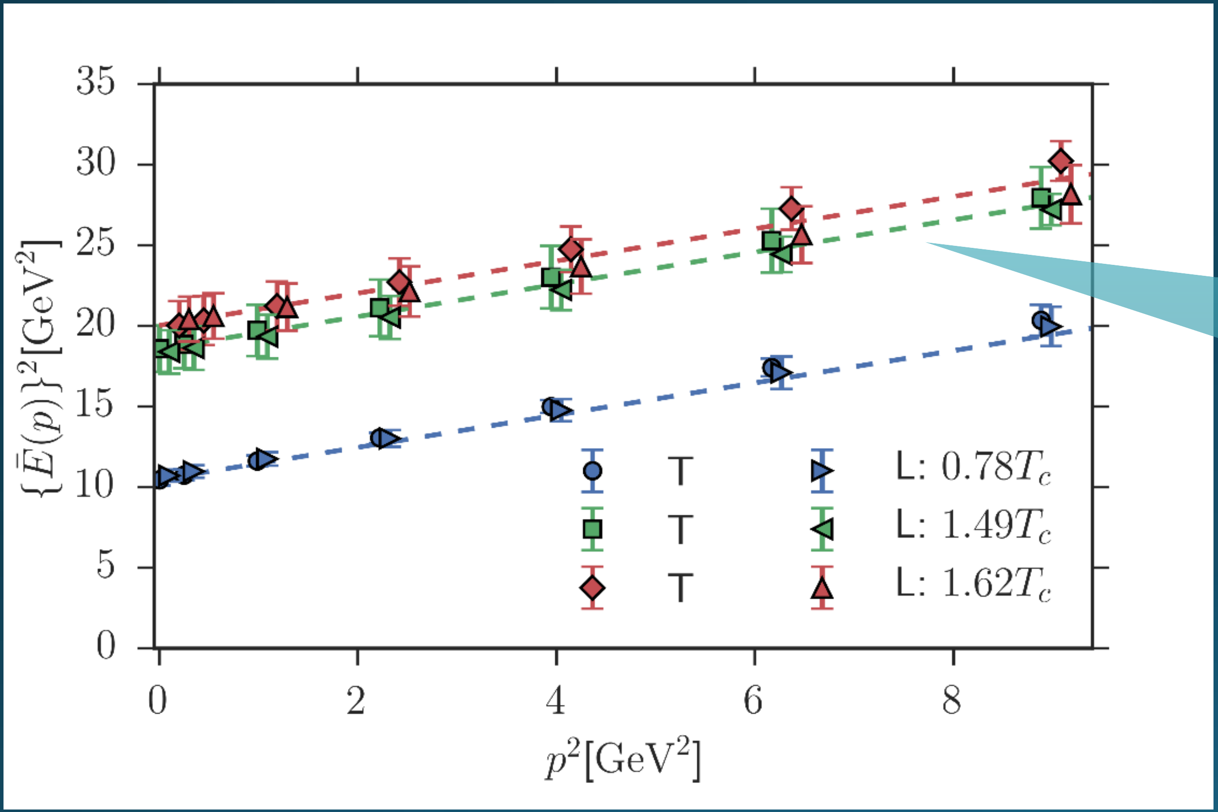
Spectral function of *J/ψ*



- ☐ Transverse/longitudinal decomposed
- ☐ Mass enhancement in medium?

Dispersion Relation of Charmonia

Ikeda, Asakawa, MK
PRD 2017



Disp. Rel. in vacuum

$$E = \sqrt{p^2 + m^2}$$

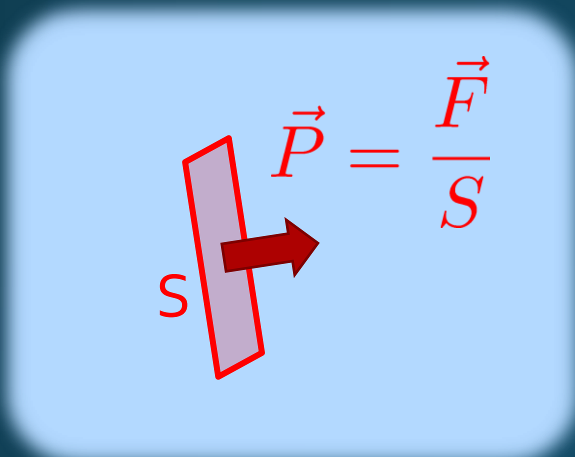
- Large mass enhancement at nonzero T.
- Disp. Rel. of J/ψ is unchanged from the vacuum one.

EMT Distribution inside Hadrons

Stress = Force per Unit Area

Stress = Force per Unit Area

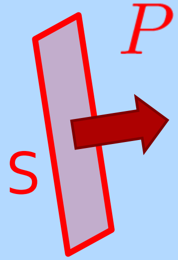
Pressure



$$\vec{P} = P\vec{n}$$

Stress = Force per Unit Area

Pressure



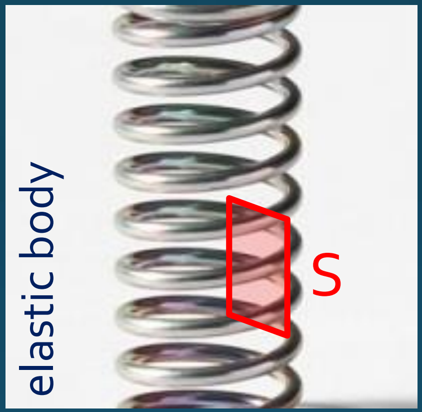
$$\vec{P} = \frac{\vec{F}}{S}$$

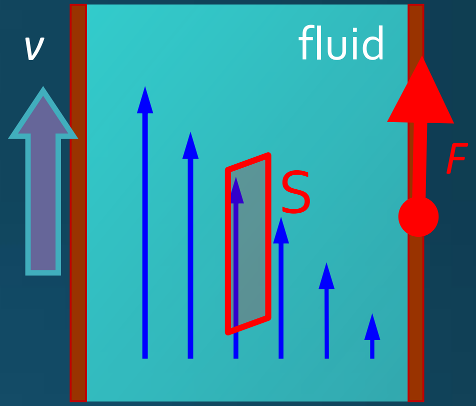
$$\vec{P} = P\vec{n}$$

In thermal medium

$$T_{ij} = P\delta_{ij}$$

Generally, F and n are not parallel





$$\frac{F_i}{S} = \boxed{\sigma_{ij}} n_j$$

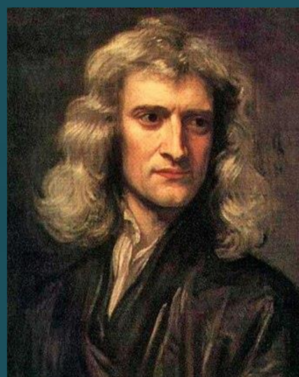
Stress Tensor

$$\sigma_{ij} = -T_{ij}$$

Landau
Lifshitz

Force

Action-at-a-distance



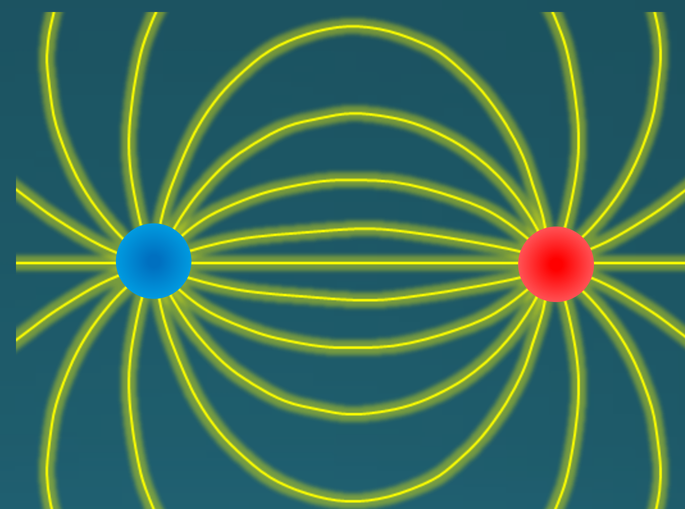
Newton
1687

$m_1,\ q_1$ $\qquad$ $m_2,\ q_2$

$$F = -G\frac{m_1 m_2}{r^2} \qquad F = -\frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r^2}$$

Local interaction

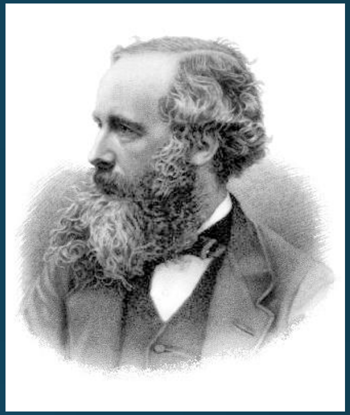
Faraday
1839



Maxwell Stress

(in Maxwell Theory)

$$\sigma_{ij} = \varepsilon_0 E_i E_j + \frac{1}{\mu_0} B_i B_j - \frac{1}{2}\delta_{ij}\left(\varepsilon_0 E^2 + \frac{1}{\mu_0} B^2\right)$$

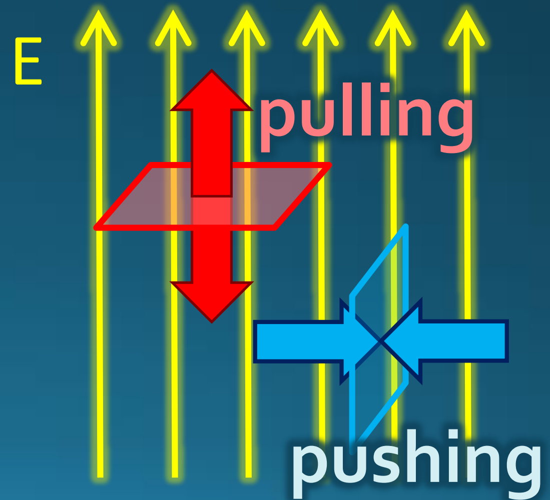


Maxwell

$$\vec{E} = (E, 0, 0)$$

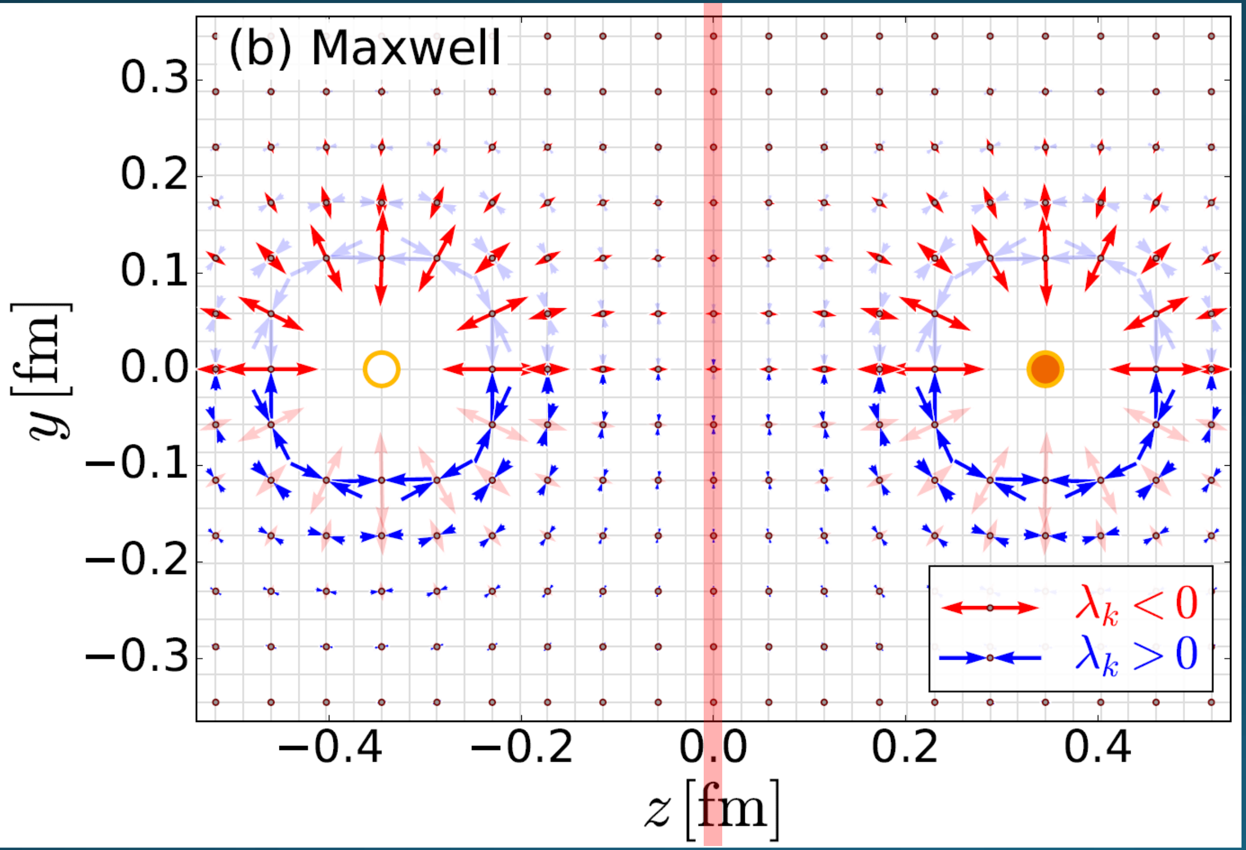
$$T_{ij} = \begin{pmatrix} -E^2 & 0 & 0 \\ 0 & E^2 & 0 \\ 0 & 0 & E^2 \end{pmatrix}$$

- Parallel to field: **Pulling**
- Vertical to field: **Pushing**



Maxwell Stress

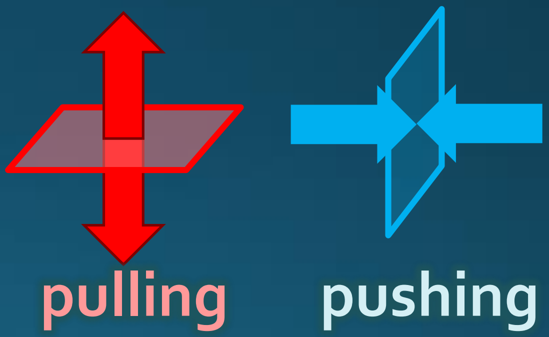
(in Maxwell Theory)



$$T_{ij} v_j^{(k)} = \lambda_k v_i^{(k)}$$

$$(k = 1, 2, 3)$$

length: $\sqrt{|\lambda_k|}$

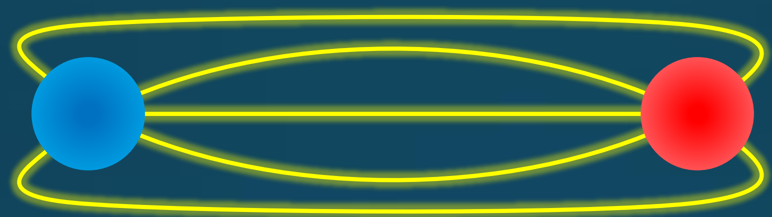


Definite physical meaning

- ☐ Distortion of field, line of the field
- ☐ Propagation of the force as local interaction

Quark-Anti-quark system

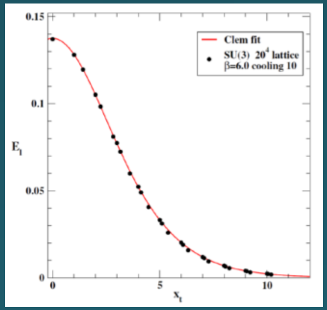
Formation of the flux tube → confinement



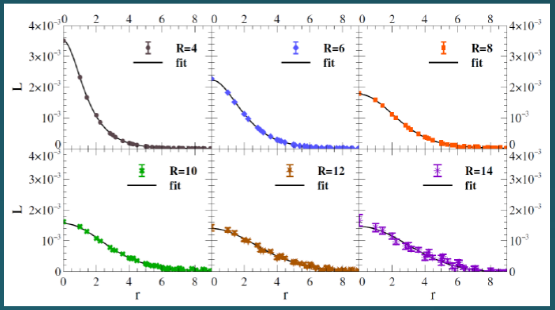
Previous Studies on Flux Tube

- ☐ Potential
- ☐ Action density
- ☐ Color-electric field

so many studies…

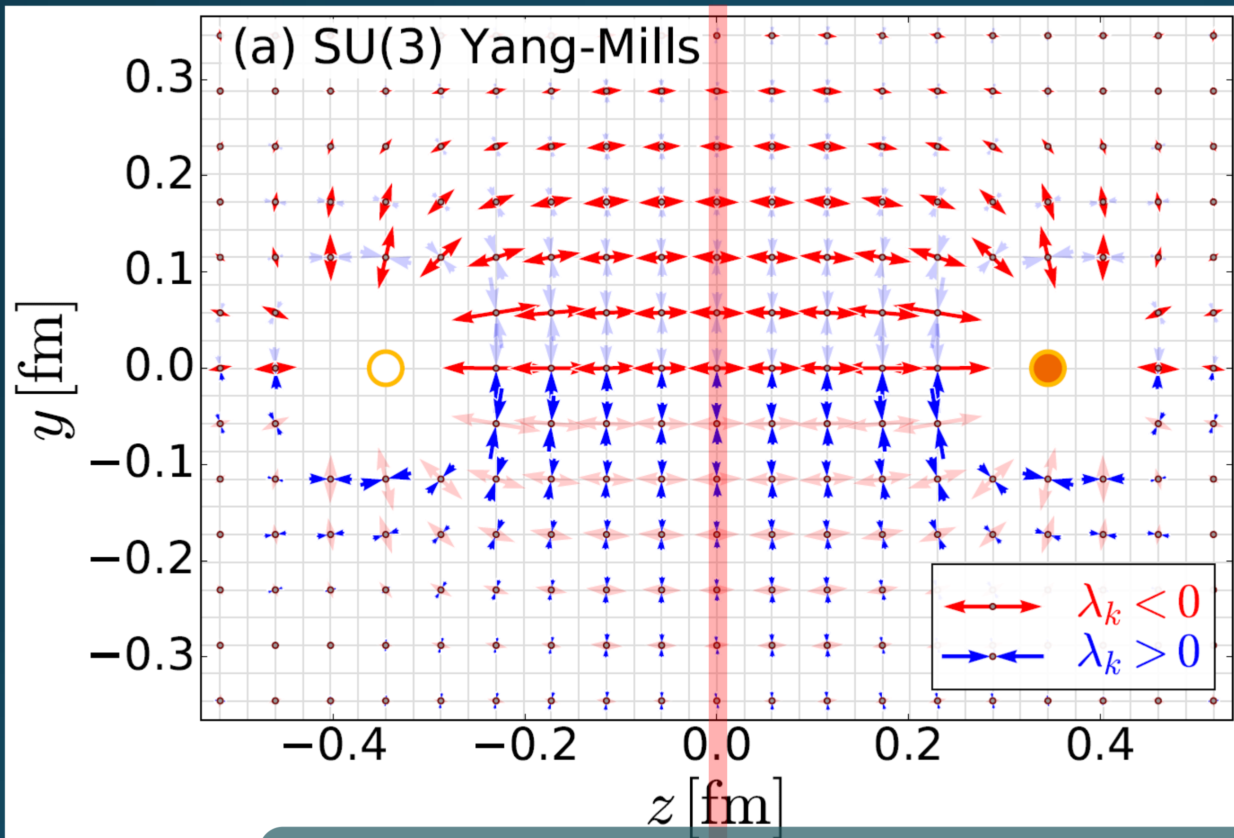


Cea+ (2012)



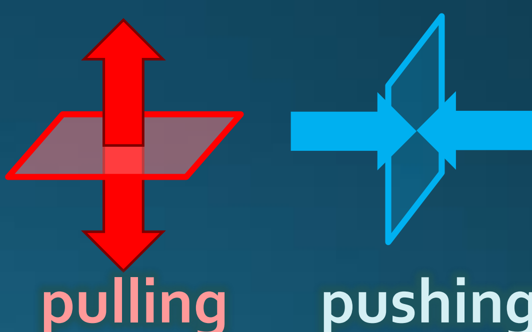
Cardoso+ (2013)

Stress Tensor in $Q\bar{Q}$ System



Yanagihara+, 1803.05656
PLB, in press

Lattice simulation
SU(3) Yang-Mills
a=0.029 fm
R=0.69 fm
t/a^2=2.0



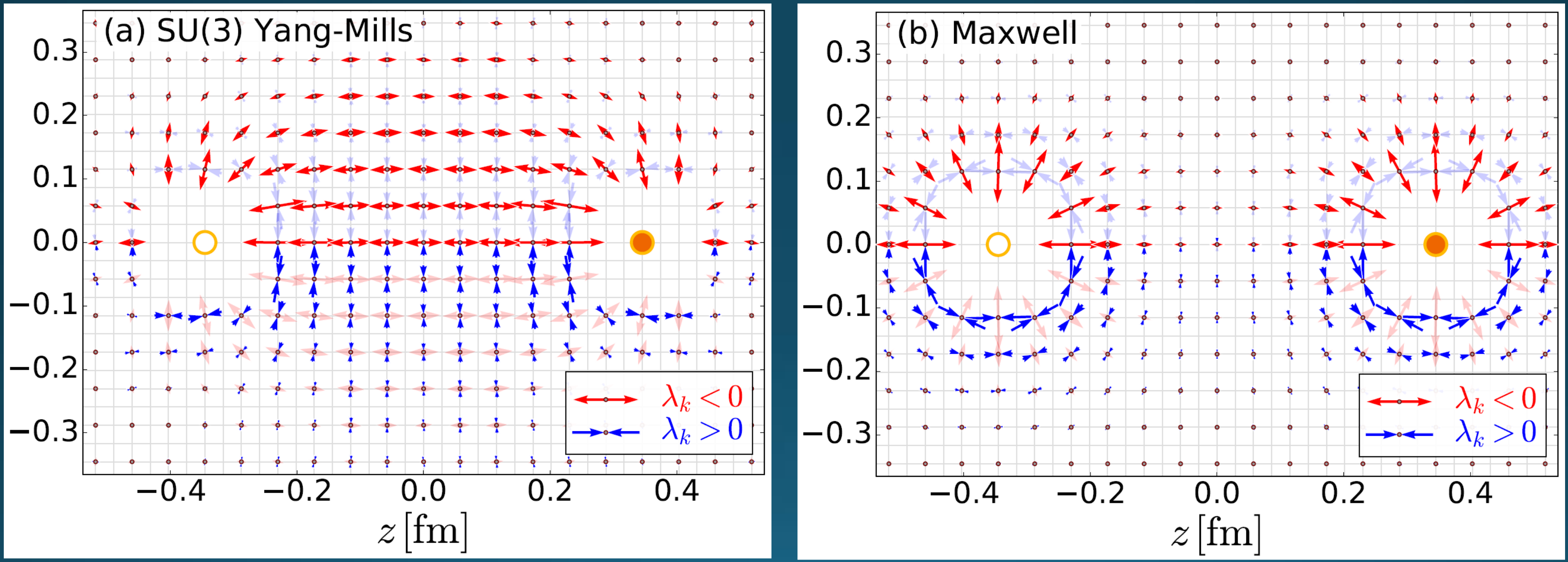
Definite physical meaning

- ☐ Distortion of field, line of the field
- ☐ Propagation of the force as local interaction
- ☐ Manifestly gauge invariant

SU(3) YM vs Maxwell

SU(3) Yang-Mills
(quantum)

Maxwell
(classical)



Propagation of the force is clearly different
in YM and Maxwell theories!

Stress Distribution on Mid-Plane

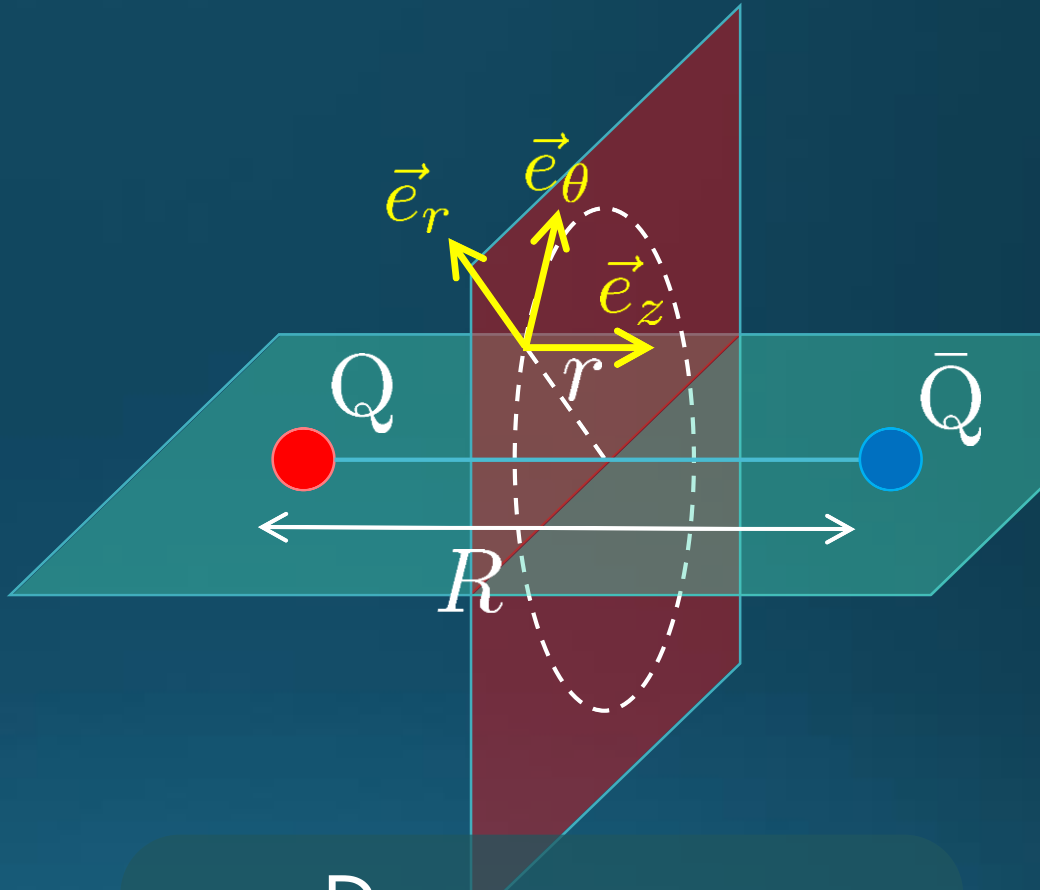
From rotational symm. & parity

EMT is diagonalized
in Cylindrical Coordinates

$$T_{cc'}(r) = \begin{pmatrix} T_{rr} & & & \\ & T_{\theta\theta} & & \\ & & T_{zz} & \\ & & & T_{44} \end{pmatrix}$$

$$T_{rr} = \vec{e}_r^{\,T} T \vec{e}_r$$

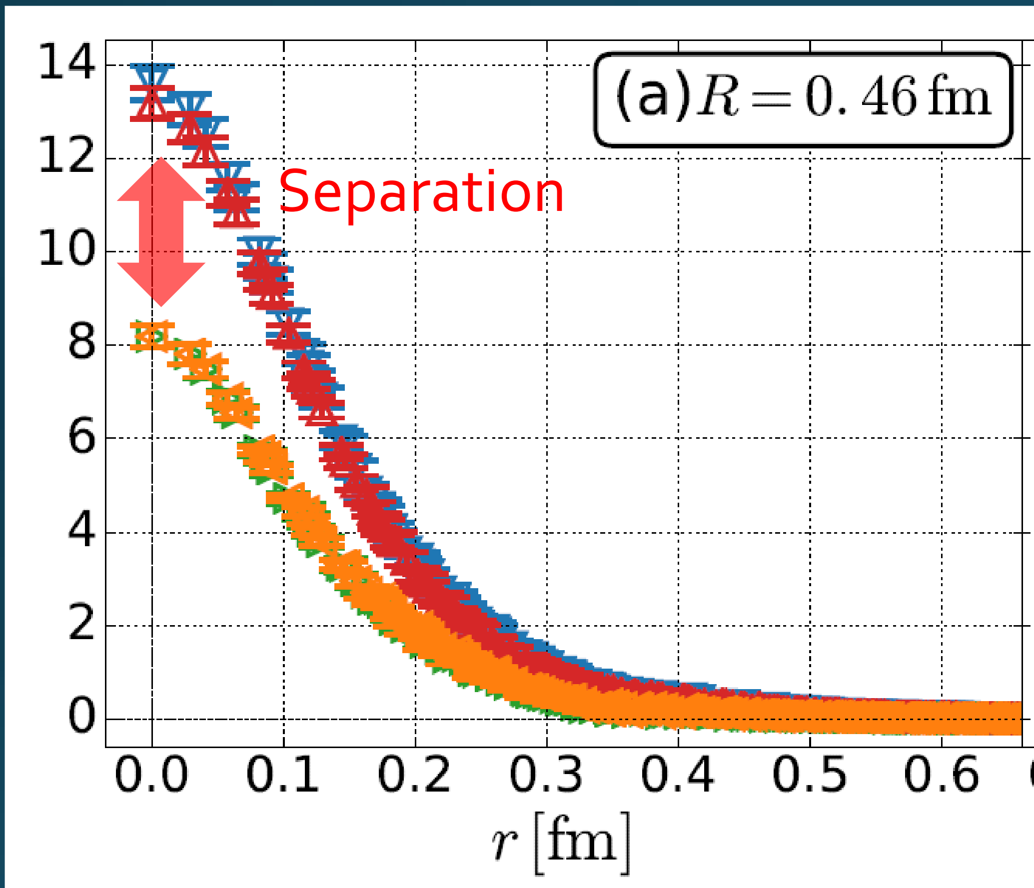
$$T_{\theta\theta} = \vec{e}_\theta^{\,T} T \vec{e}_\theta$$

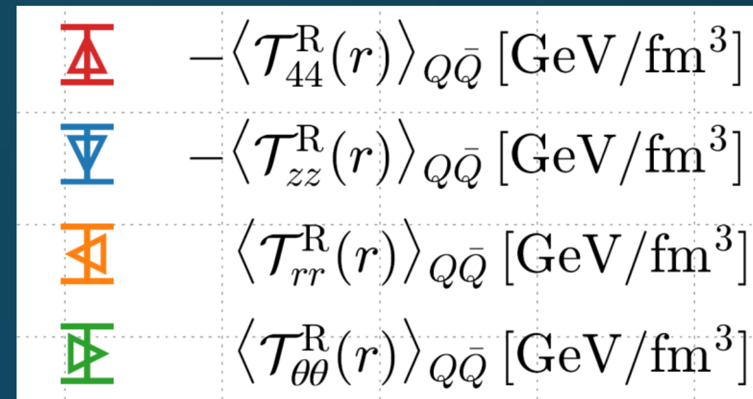


Degeneracy
in Maxwell theory

$$T_{rr} = T_{\theta\theta} = -T_{zz} = -T_{44}$$

Mid-Plane





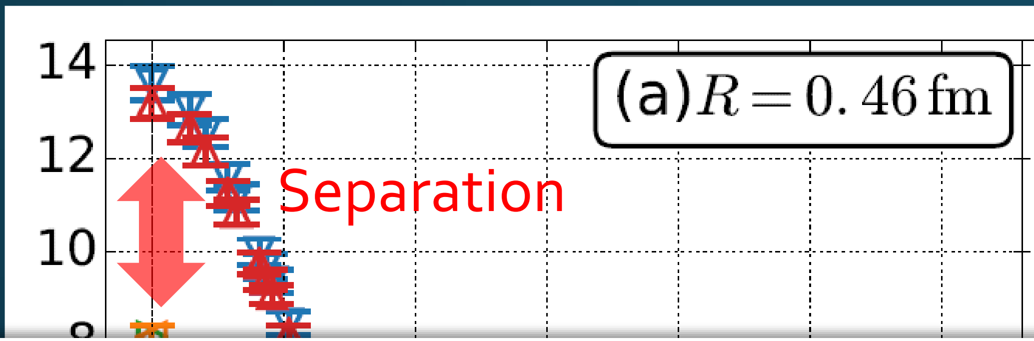
Continuum Extrapolated!

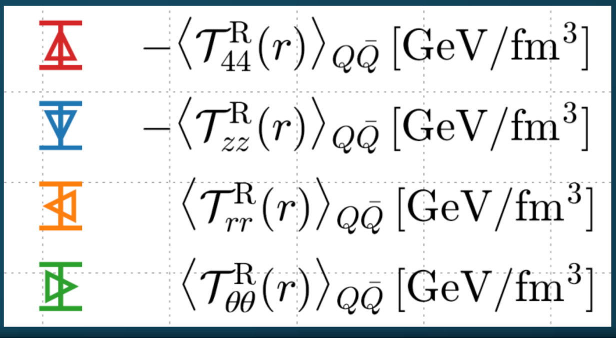
In Maxwell theory

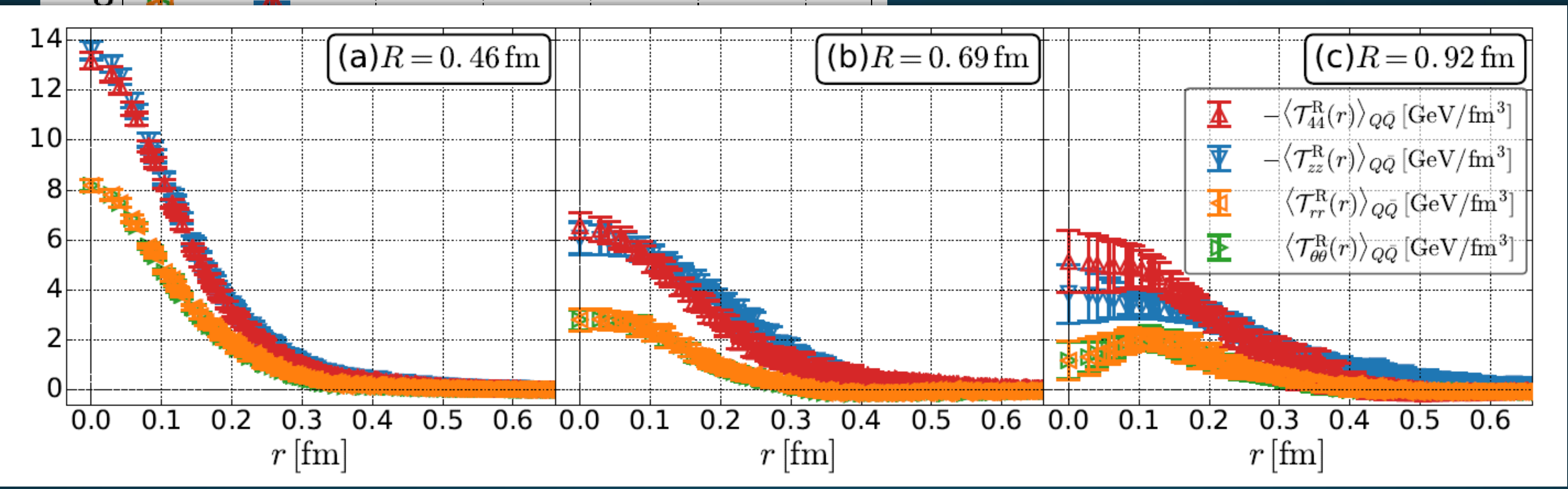
$$T_{rr} = T_{\theta\theta} = -T_{zz} = -T_{44}$$

- ☐ Degeneracy: $T_{44} \simeq T_{zz}, \quad T_{rr} \simeq T_{\theta\theta}$
- ☐ Separation: $T_{zz} \neq T_{rr}$
- ☐ Nonzero trace anomaly $\sum T_{cc} \neq 0$

Mid-Plane







- Degeneracy: $T_{44} \simeq T_{zz}, \quad T_{rr} \simeq T_{\theta\theta}$
- Separation: $T_{zz} \neq T_{rr}$
- Nonzero trace anomaly $\sum T_{cc} \neq 0$

Gradient Flow and EMT

SF*t*E Method

Yang-Mills Gradient Flow

Luscher 2010
Narayanan, Neuberger, 2006
Luscher, Weiss, 2011

$$\frac{\partial}{\partial t} A_\mu(t,x) = -\frac{\partial S_{\mathrm{YM}}}{\partial A_\mu}$$

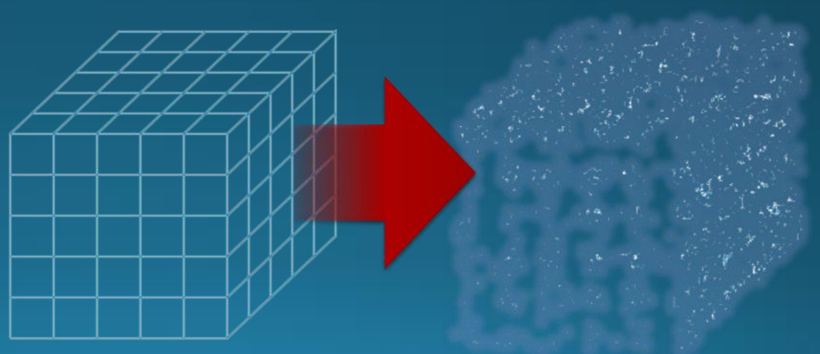
t: "flow time"
dim:[length^2]

$$A_\mu(0,x) = A_\mu(x)$$

leading

$$\boxed{\partial_t A_\mu} = D_\nu G_{\mu\nu} = \boxed{\partial_\nu \partial_\nu A_\mu} + \cdots$$

- diffusion equation in 4-dim space
- diffusion distance $d \sim \sqrt{8t}$
- "continuous" cooling/smearing
- No UV divergence at t>0



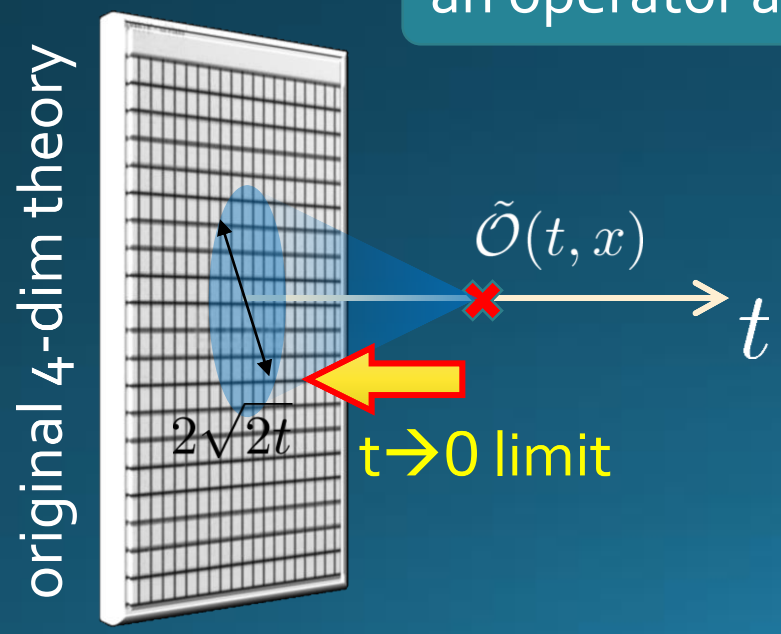
Small Flow-Time Expansion

Luescher, Weisz, 2011
Suzuki, 2013

$$\tilde{\mathcal{O}}(t,x) \xrightarrow[t\to 0]{} \sum_i c_i(t)\mathcal{O}_i^R(x)$$

an operator at t>0

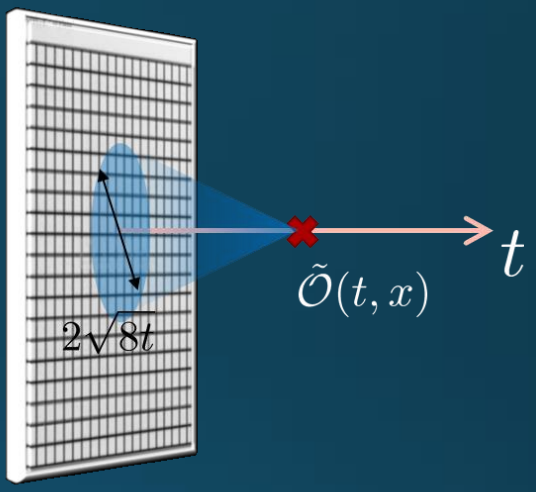
remormalized operators of original theory



Constructing EMT 1

Suzuki, 2013

$$\tilde{\mathcal{O}}(t,x) \xrightarrow[t \to 0]{} \sum_i c_i(t) \mathcal{O}_i^R(x)$$



Gauge-invariant dimension 4 operators

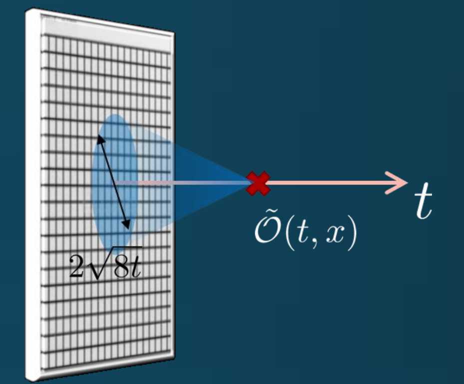
$$\begin{cases} U_{\mu\nu}(t,x) = G_{\mu\rho}(t,x)G_{\nu\rho}(t,x) - \frac{1}{4}\delta_{\mu\nu}G_{\mu\nu}(t,x)G_{\mu\nu}(t,x) \\ E(t,x) = \frac{1}{4}\delta_{\mu\nu}G_{\mu\nu}(t,x)G_{\mu\nu}(t,x) \end{cases}$$

Constructing EMT

Suzuki, 2013

$$U_{\mu\nu}(t,x) = \alpha_U(t) \left[T^R_{\mu\nu}(x) - \frac{1}{4}\delta_{\mu\nu} T^R_{\rho\rho}(x) \right] + \mathcal{O}(t)$$

$$E(t,x) = \langle E(t,x) \rangle + \alpha_E(t) T^R_{\rho\rho}(x) + \mathcal{O}(t)$$



vacuum subtr.

Remormalized EMT

$$T^R_{\mu\nu}(x) = \lim_{t\to 0} \left[c_1(t) U_{\mu\nu}(t,x) + \delta_{\mu\nu} c_2(t) E(t,x)_{\text{subt.}} \right]$$

Perturbative coefficient:
Suzuki (2013); Makino, Suzuki (2014); Harlander+ (2018); Iritani, MK, Suzuki, Takaura (2019)

Perturbative Coefficients

Suzuki, PTEP 2013, 083B03
Harlander+, 1808.09837
Iritani, MK, Suzuki, Takaura, PTEP 2019

$$T_{\mu\nu}(t) = c_1(t)U_{\mu\nu}(t) + \delta_{\mu\nu}c_2(t)E(t)$$

	LO	1-loop	2-loop	3-loop
$c_1(t)$	○	○	○	
$c_2(t)$	× zero	○	○	○

Suzuki (2013) Harlander+(2018)

Iritani, MK, Suzuki, Takaura, 2019

□ Choice of the scale of g²

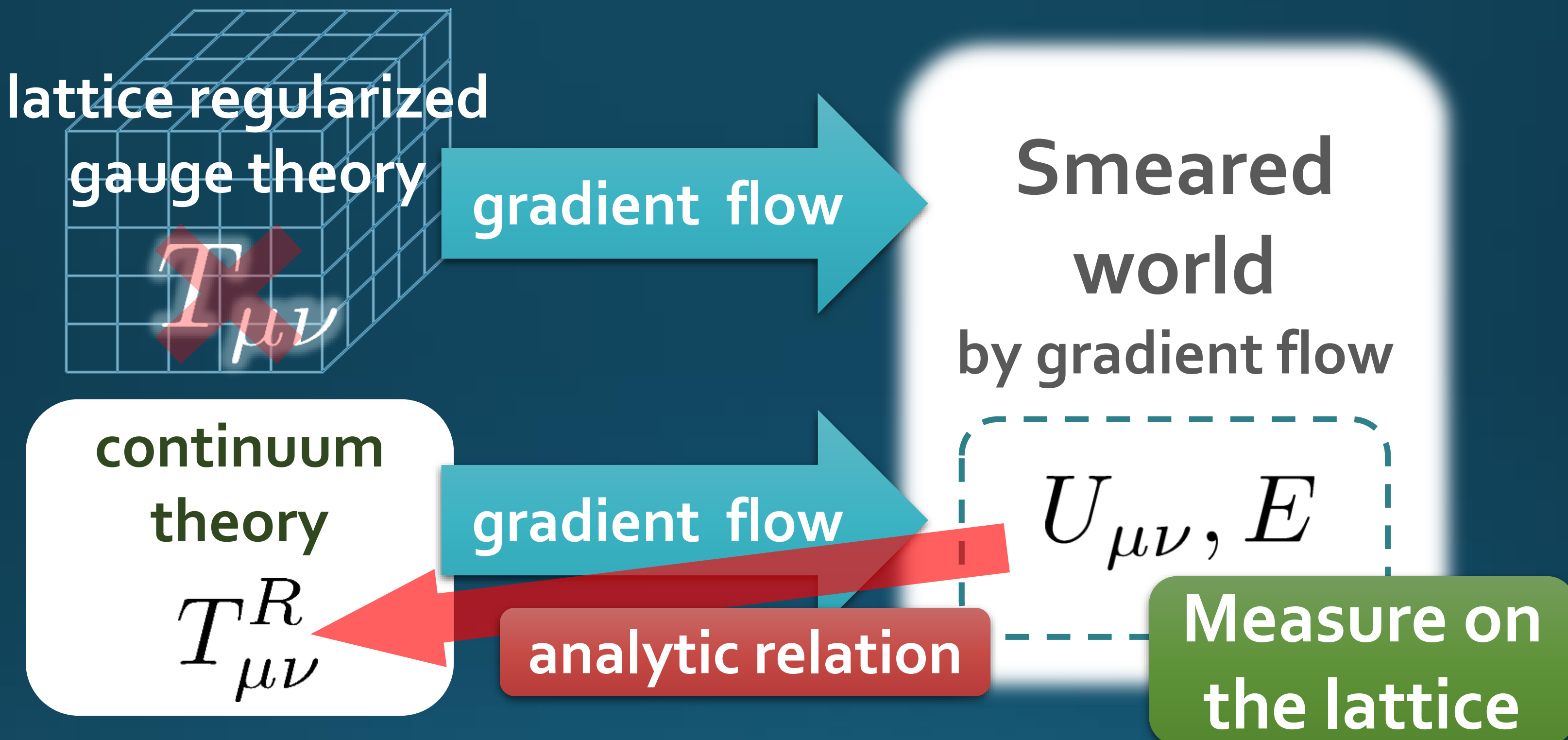
$$c_1(t) = c_1\left(g^2(\mu(t))\right)$$

Previous: $\mu_d(t) = 1/\sqrt{8t}$

Improved: $\mu_0(t) = 1/\sqrt{2e^{\gamma_E}t}$

Harlander+ (2018)

Gradient Flow Method



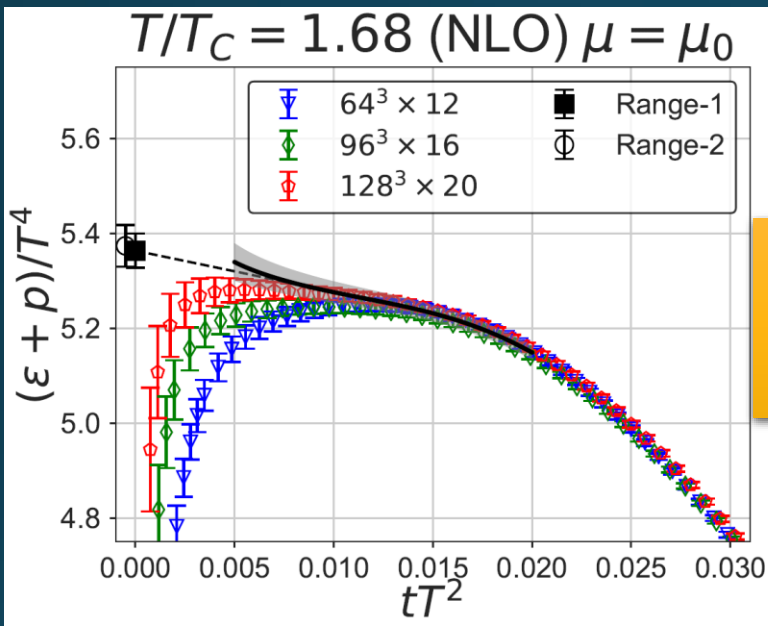
Take Extrapolation (t,a)→(0,0)

$$\langle T_{\mu\nu}(t)\rangle_{\rm latt} = \langle T_{\mu\nu}(t)\rangle_{\rm phys} + \boxed{C_{\mu\nu}t} + \boxed{D_{\mu\nu}\frac{a^2}{t}} + \cdots$$

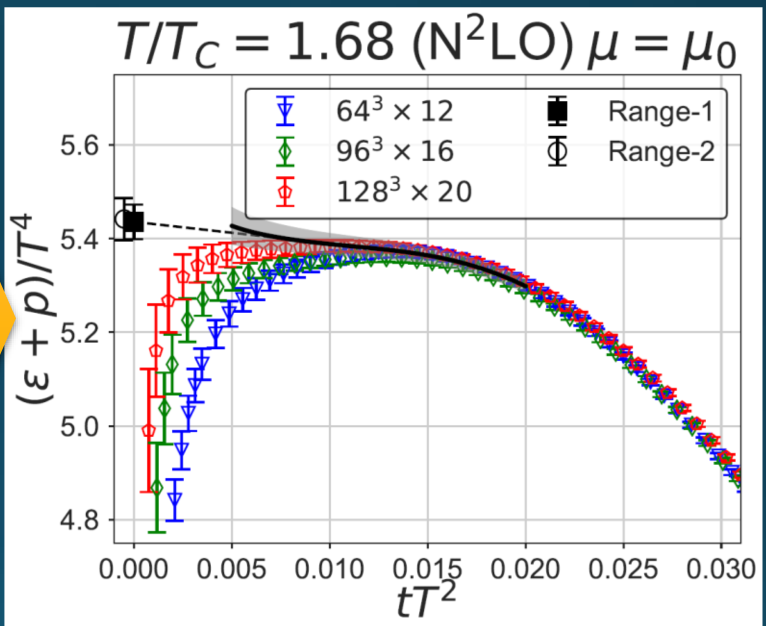
O(t) terms in SFTE   lattice discretization

Higher Order Coefficient: ε+p

NLO (1-loop)



N²LO (2-loop)



Iritani, MK, Suzuki, Takaura, PTEP 2019

- ☐ t dependence becomes milder with higher order coeff.
- ☐ Better t→0 extrapolation

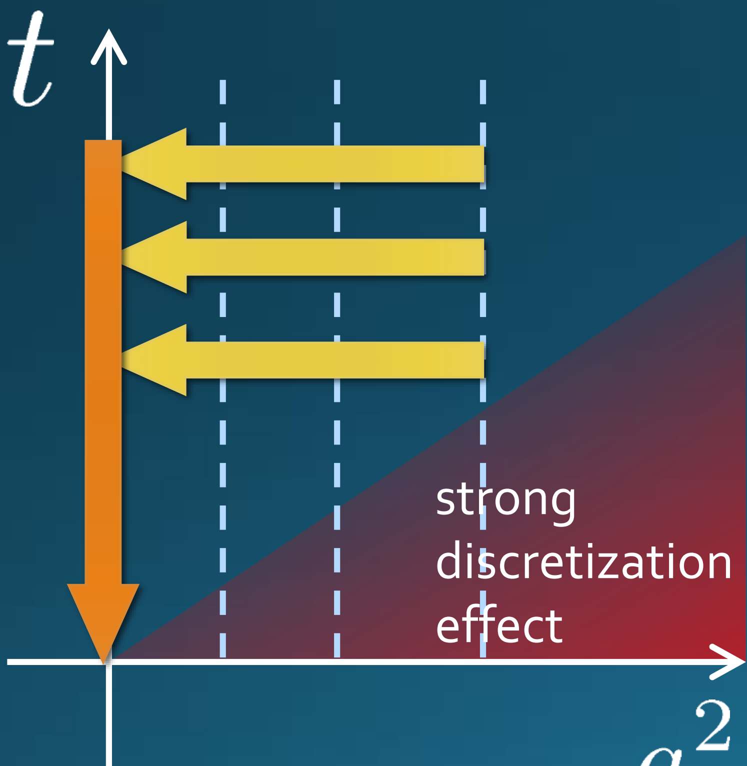
- ☐ Systematic error: μ_0 or μ_d, uncertaintyof Λ (±3%), fit range
- ☐ Extrapolation func: linear, higher order term in c_1 ($\sim g^6$)

Double Extrapolation

t→0, a→0

$$\langle T_{\mu\nu}(t)\rangle_{\mathrm{latt}} = \langle T_{\mu\nu}(t)\rangle_{\mathrm{phys}} + \boxed{C_{\mu\nu}t} + \boxed{D_{\mu\nu}(t)\frac{a^2}{t}}$$

O(t) terms in SFTE lattice discretization



Continuum extrapolation

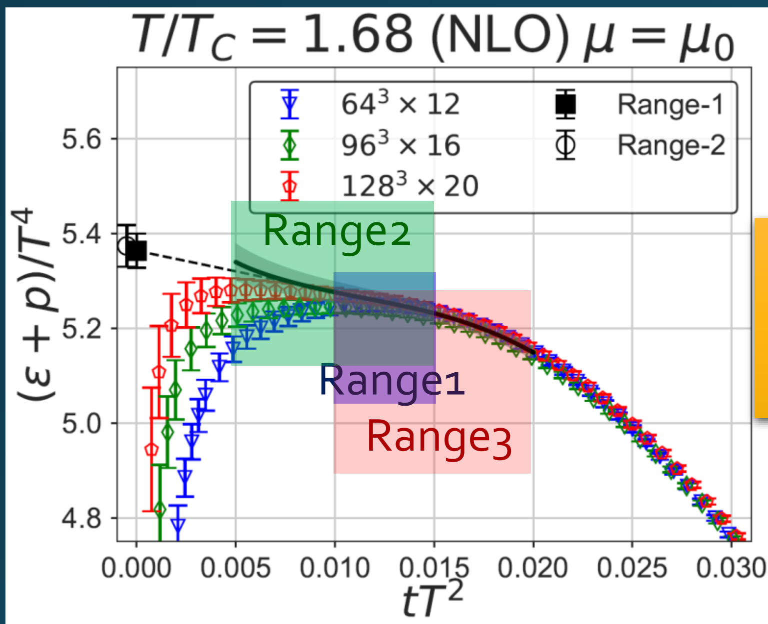
$$\langle T_{\mu\nu}(t)\rangle_{\mathrm{cont}} = \langle T_{\mu\nu}(t)\rangle_{\mathrm{lat}} + C(t)a^2$$

Small t extrapolation

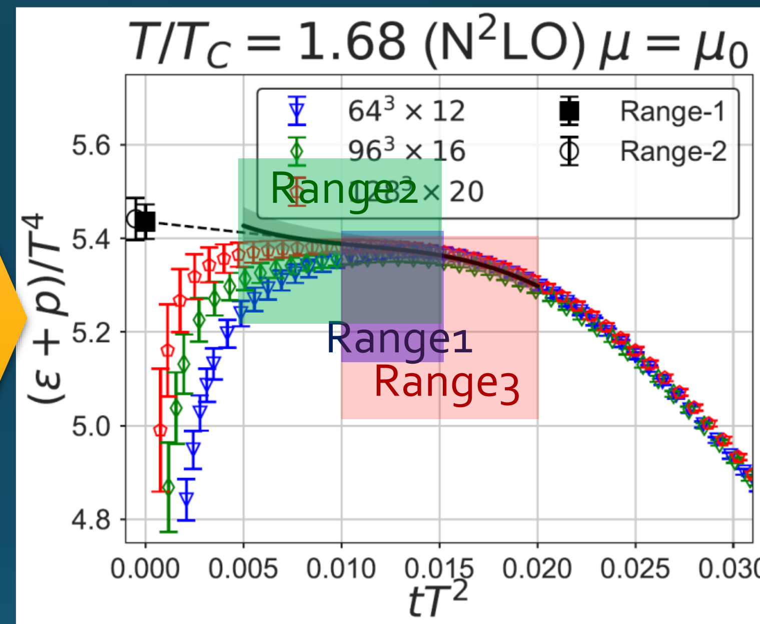
$$\langle T_{\mu\nu}\rangle = \langle T_{\mu\nu}(t)\rangle + C't$$

Higher Order Coefficient: ε+p

NLO (1-loop)



N²LO (2-loop)



Iritani, MK, Suzuki, Takaura, PTEP 2019

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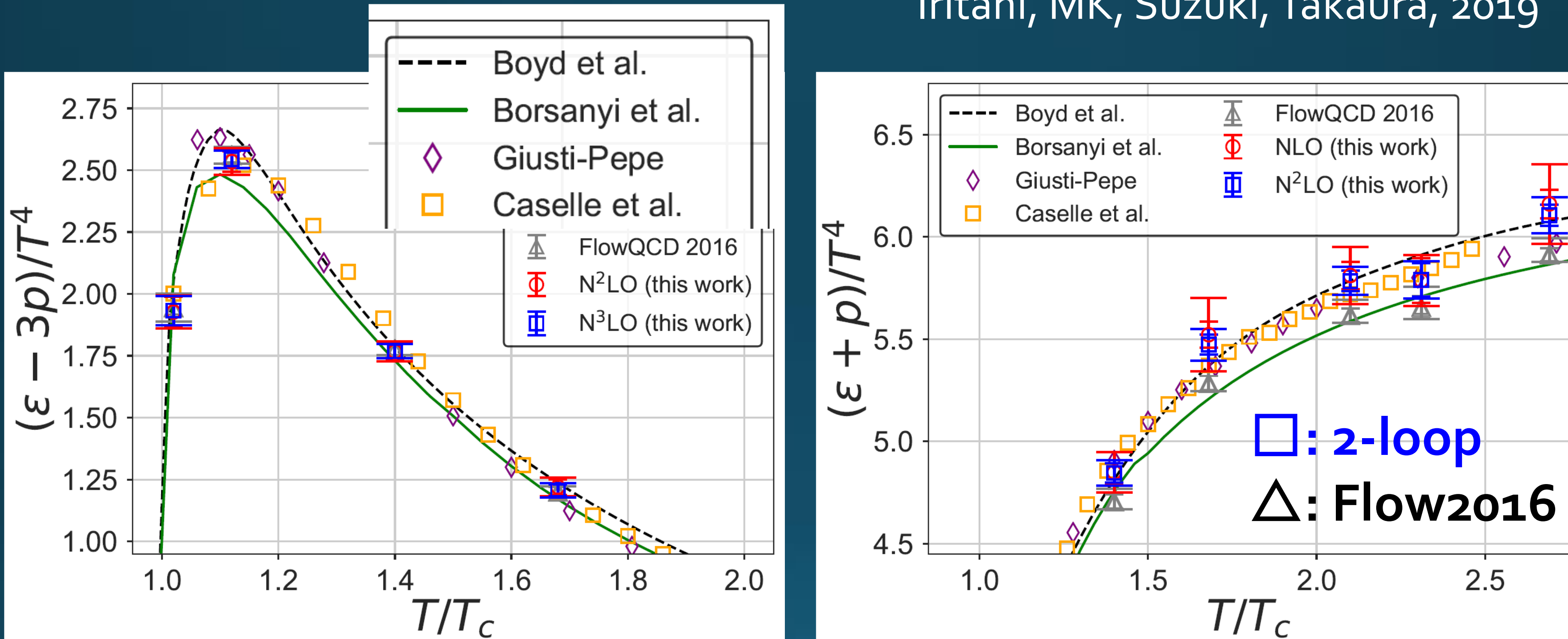
- ☐ Extrapolation func: linear, higher order term in c_1 (~g^6)

Effect of Higher-Order Coeffs.

Iritani, MK, Suzuki, Takaura, 2019



Systematic error: μ_0 or μ_d, Λ, t→0 function, fit range

More stable extrapolation with higher order c_1 & c_2 (pure gauge)

Gradient Flow for Fermions

$$\partial_t \psi(t,x) = D_\mu D_\mu \psi(t,x)$$

$$\partial_t \bar{\psi}(t,x) = \psi(t,x) \overleftarrow{D}_\mu \overleftarrow{D}_\mu$$

$$D_\mu = \partial_\mu + A_\mu(t,x)$$

Luscher, 2013
Makino, Suzuki, 2014
Taniguchi+ (WHOT) 2016; 2017

- Not "gradient" flow but a "diffusion" equation.

- Divergence in field renormalization of fermions.
- All observables are finite at t>0 once Z(t) is fixed.

$$\tilde{\psi}(t,x) = Z(t)\psi(t,x)$$

- Energy-momentum tensor from SFTE Makino, Suzuki, 2014

EMT in QCD

$$
\begin{aligned}
T_{\mu\nu}(t,x) =& c_1(t)U_{\mu\nu}(t,x) + c_2(t)\delta_{\mu\nu}\big(E(t,x) - \langle E\rangle_0\big) \\
&+ c_3(t)\big(O_{3\mu\nu}(t,x) - 2O_{4\mu\nu}(t,x) - \mathrm{VEV}\big) \\
&+ c_4(t)\big(O_{4\mu\nu}(t,x) - \mathrm{VEV}\big) + c_5(t)\big(O_{5\mu\nu}(t,x) - \mathrm{VEV}\big)
\end{aligned}
$$

$$T_{\mu\nu}(x) = \lim_{t\to 0} T_{\mu\mu}(t,x)$$

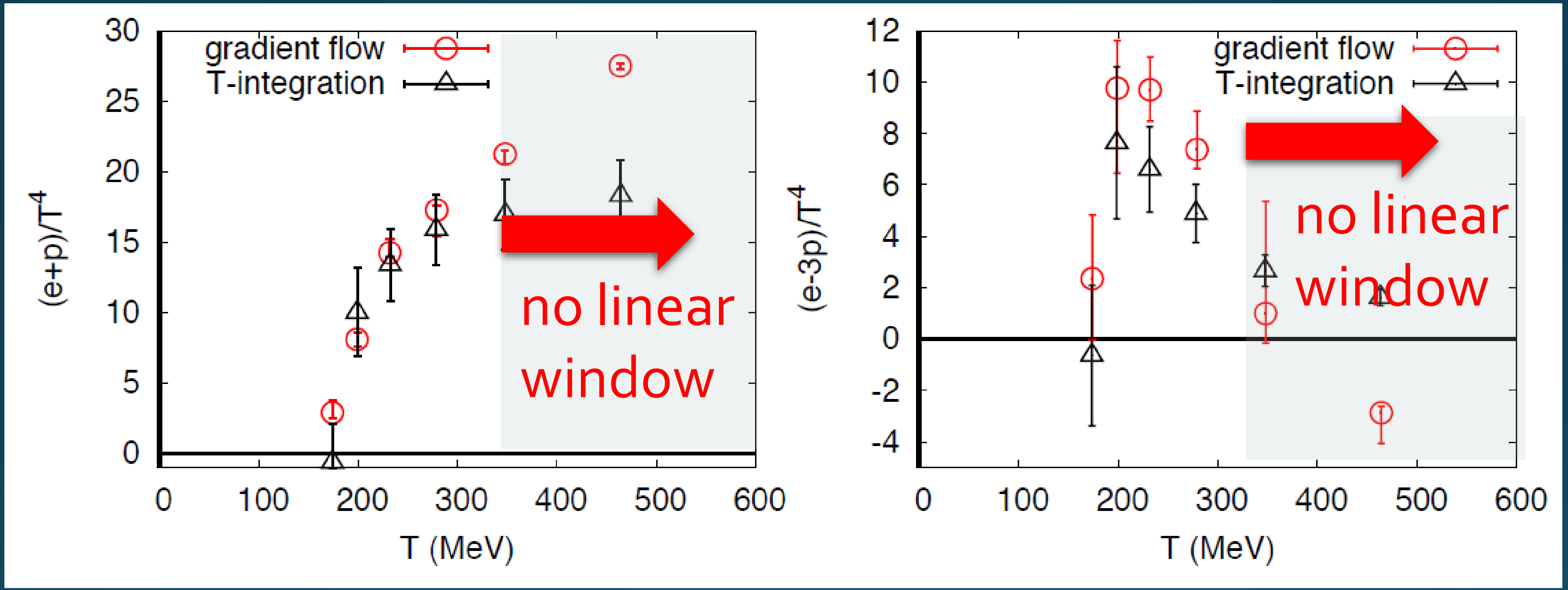
$$
\begin{aligned}
\tilde{O}^f_{3\mu\nu}(t,x) &\equiv \varphi_f(t)\bar{\chi}_f(t,x)\left(\gamma_\mu \overleftrightarrow{D}_\nu + \gamma_\nu \overleftrightarrow{D}_\mu\right)\chi_f(t,x), \\
\tilde{O}^f_{4\mu\nu}(t,x) &\equiv \varphi_f(t)\delta_{\mu\nu}\bar{\chi}_f(t,x)\overleftrightarrow{\not{D}}\chi_f(t,x), \\
\tilde{O}^f_{5\mu\nu}(t,x) &\equiv \varphi_f(t)\delta_{\mu\nu}\bar{\chi}_f(t,x)\chi_f(t,x),
\end{aligned}
$$

$$\varphi_f(t) \equiv \frac{-6}{(4\pi)^2 t^2 \left\langle \bar{\chi}_f(t,x)\overleftrightarrow{\not{D}}\chi_f(t,x)\right\rangle_0}.$$

2+1 QCD EoS from Gradient Flow

Taniguchi+ (WHOT-QCD), PR**D96**, 014509 (2017)

$m_{PS}/m_V \approx 0.63$



- Agreement with integral method except for N_t=4, 6
- N_t=4, 6: No stable extrapolation is possible
- Statistical error is substantially suppressed!

Physical mass: Kanaya+ (WHOT-QCD), 1710.10015

Lattice Setup

Yanagihara+, 1803.05656

- SU(3) Yang-Mills (Quenched)
- Wilson gauge action
- Clover operator

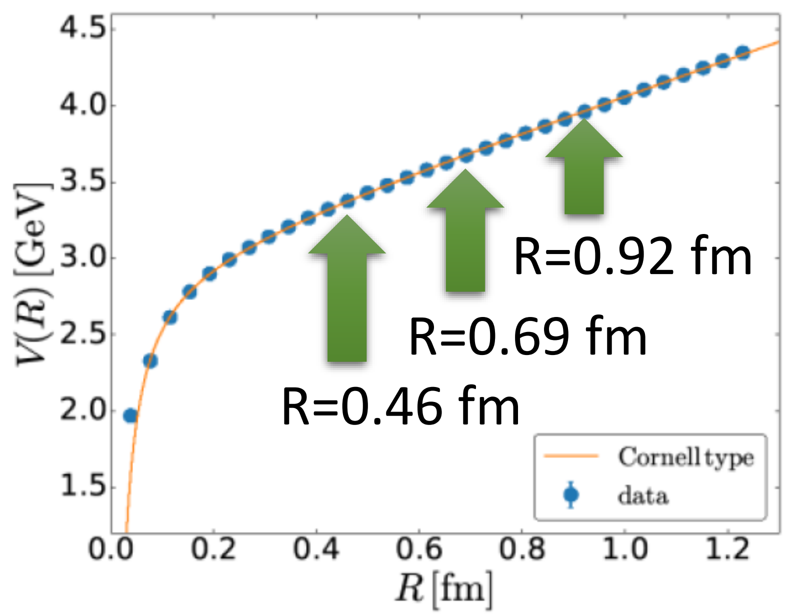
- APE smearing / multi-hit

- fine lattices (a=0.029-0.06 fm)
- continuum extrapolation

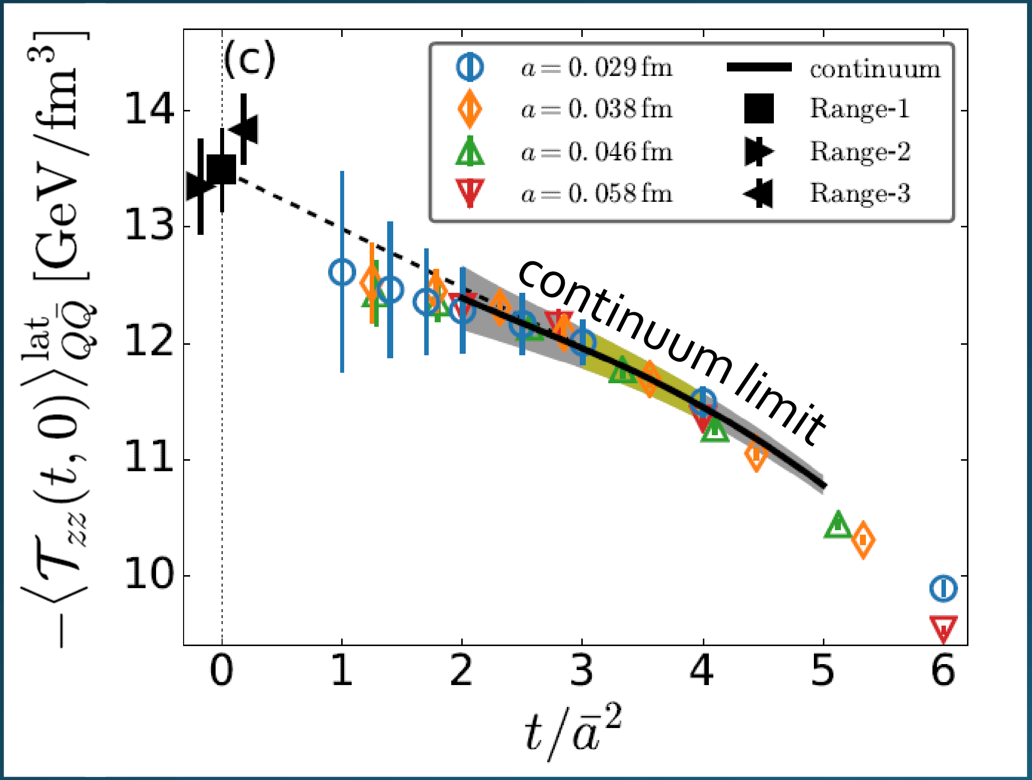
- Simulation: bluegene/Q@KEK

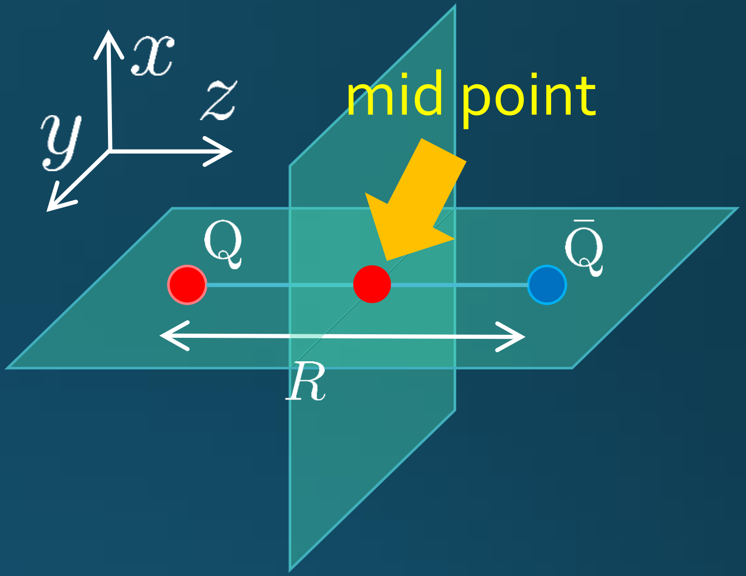
$$\langle O(x)\rangle_{\mathrm{Q\bar{Q}}} = \lim_{T\to\infty} \frac{\langle \delta O(x)\delta W(R,T)\rangle}{\langle W(R,T)\rangle}$$

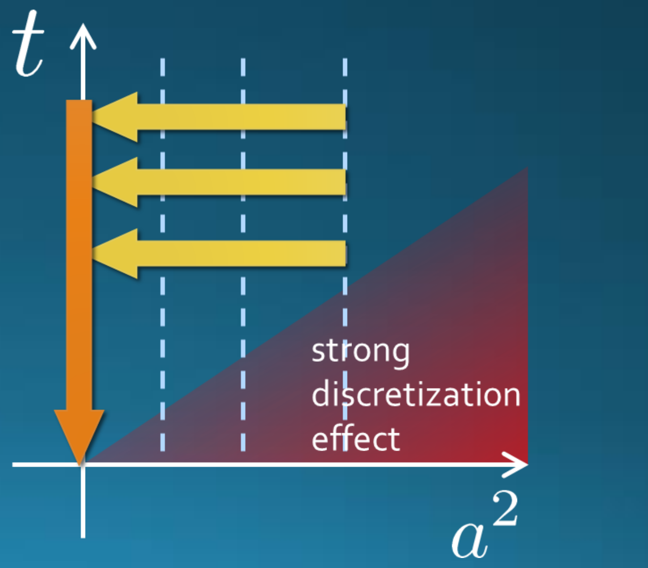
β	a [fm]	N_{size}^4	N_{conf}	R/a		
6.304	0.058	48^4	140	8	12	16
6.465	0.046	48^4	440	10	–	20
6.513	0.043	48^4	600	–	16	–
6.600	0.038	48^4	1,500	12	18	24
6.819	0.029	64^4	1,000	16	24	32
			R [fm]	0.46	0.69	0.92



Continuum Extrapolation
at mid-point

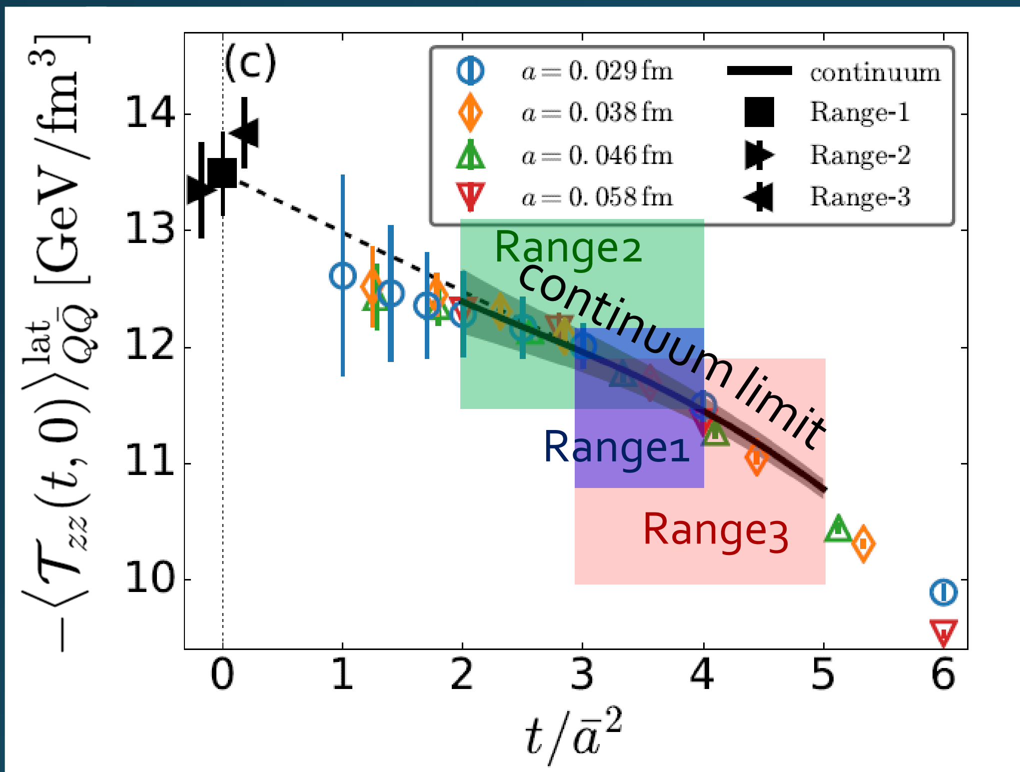


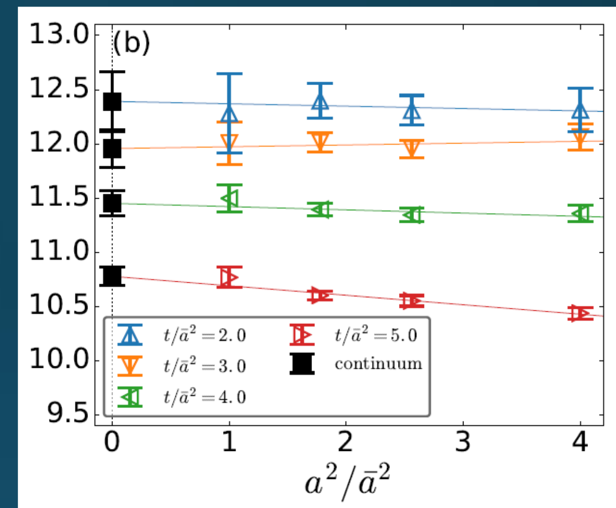


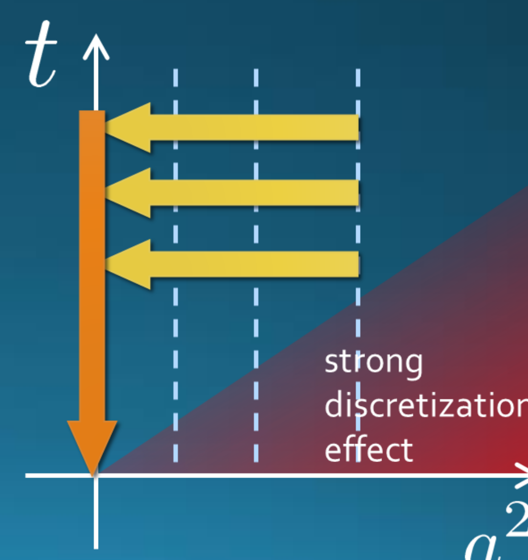


- ☐ a→0 extrapolation with fixed t

t→0 Extrapolation
at mid-point

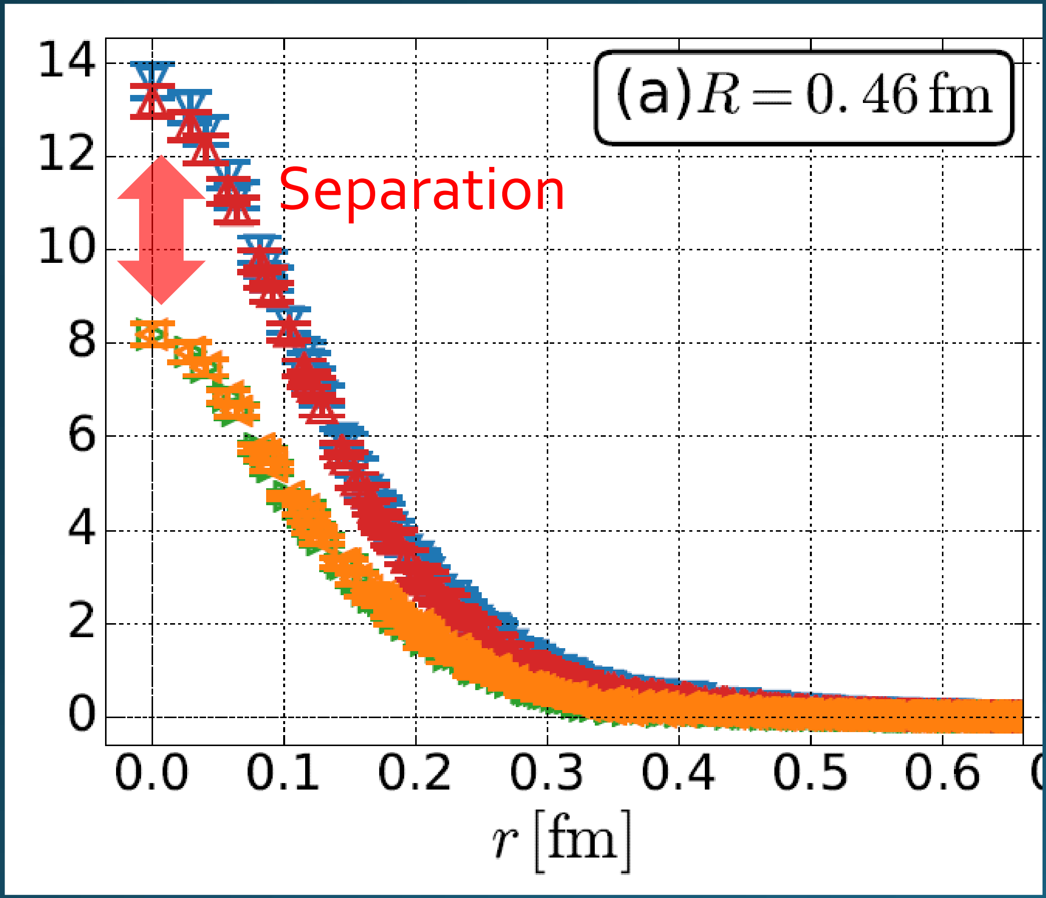


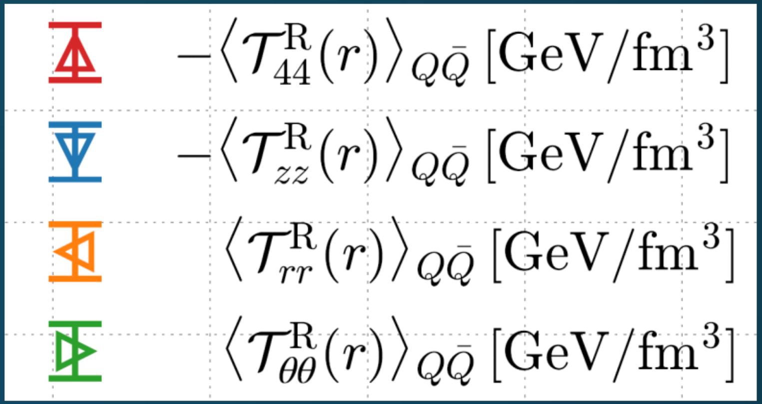




- ☐ a→0 extrapolation with fixed t
- ☐ Then, t→0 with three ranges

Mid-Plane





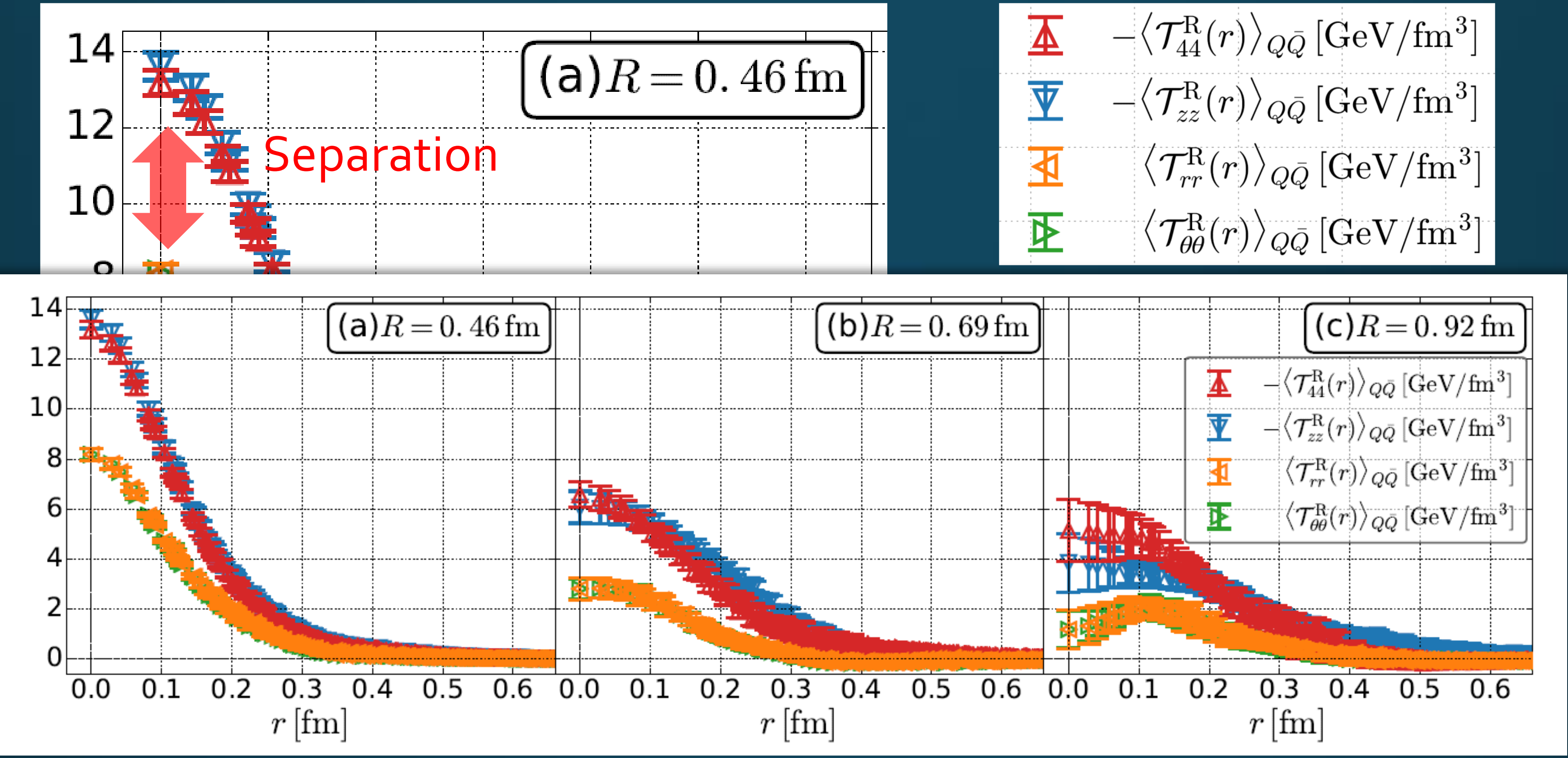
Continuum Extrapolated!

In Maxwell theory

$$T_{rr} = T_{\theta\theta} = -T_{zz} = -T_{44}$$

- ☐ Degeneracy: $T_{44} \simeq T_{zz}, \quad T_{rr} \simeq T_{\theta\theta}$
- ☐ Separation: $T_{zz} \neq T_{rr}$
- ☐ Nonzero trace anomaly $\sum T_{cc} \neq 0$

Mid-Plane



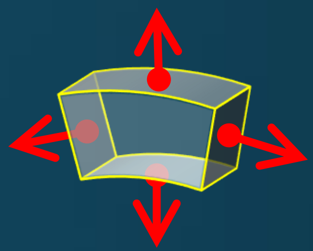
- Degeneracy: $T_{44} \simeq T_{zz}, \quad T_{rr} \simeq T_{\theta\theta}$
- Separation: $T_{zz} \neq T_{rr}$
- Nonzero trace anomaly $\sum T_{cc} \neq 0$

Momentum Conservation

Yanagihara+, in prep.

- In cylindrical coordinats,

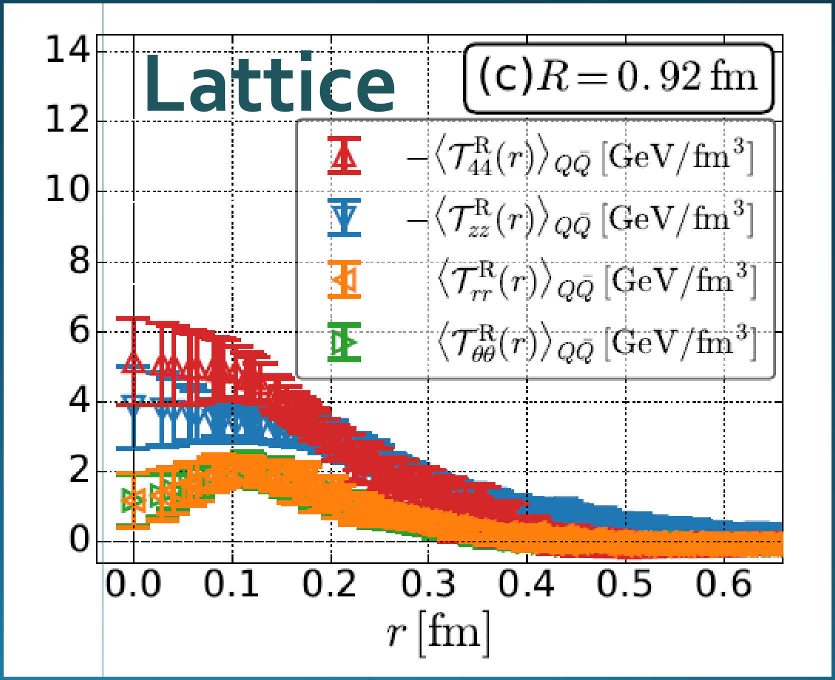
$$\partial_i T_{ij} = 0 \Rightarrow \partial_r (r T_{rr}) = T_{\theta\theta} - r\partial_z T_{rz}$$



- For infinitely-long flux tube

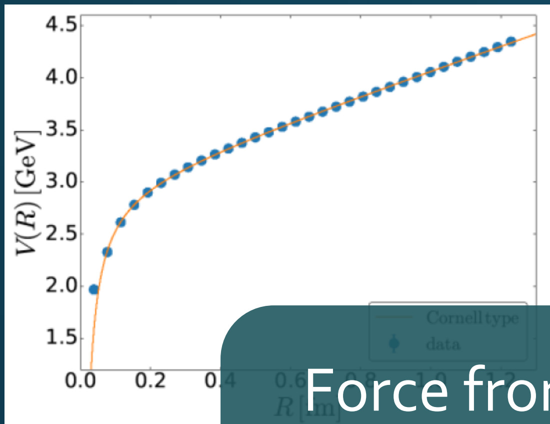
$$\partial_r (r T_{rr}) = T_{\theta\theta}$$

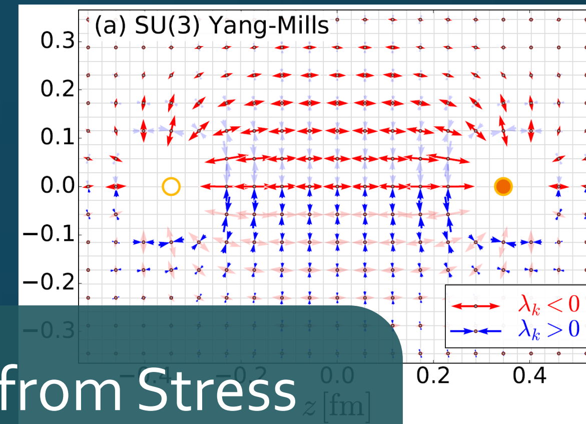
⇒ **T_{rr} and $T_{\theta\theta}$ must separate!**



Effect of boundaries is important for the flux tube at R=0.92fm

Force





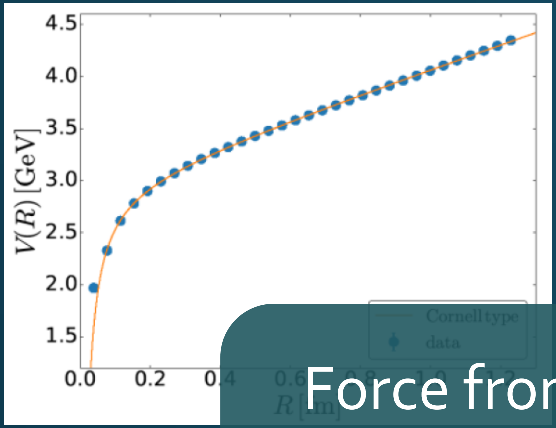
Force from Potential

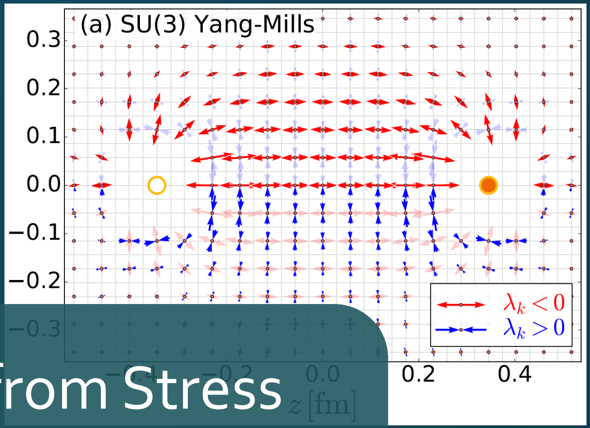
$$F_{\text{pot}} = -\frac{dV}{dR}$$

Force from Stress

$$F_{\text{stress}} = \int_{\text{mid.}} d^2x T_{zz}(x)$$

Force



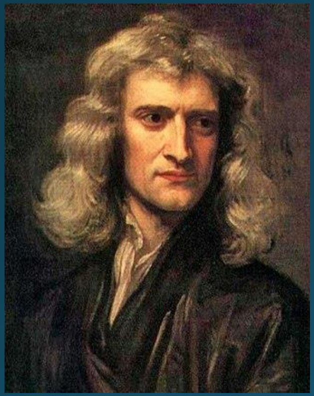


Force from Potential

$$F_{\text{pot}} = -\frac{dV}{dR}$$

Force from Stress

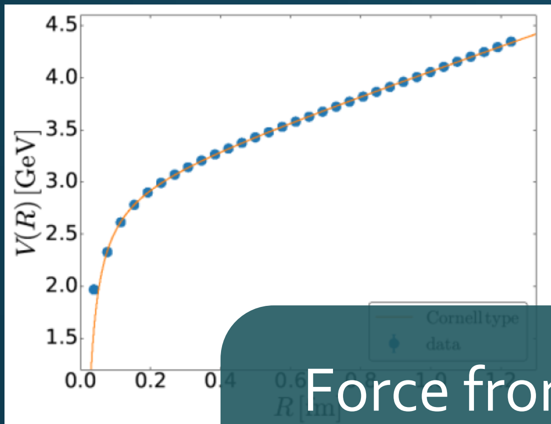
$$F_{\text{stress}} = \int_{\text{mid.}} d^2 x T_{zz}(x)$$

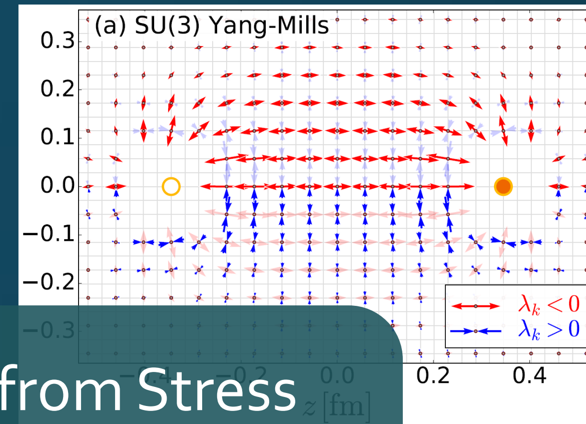


Newton
1687

Faraday
1839

Force



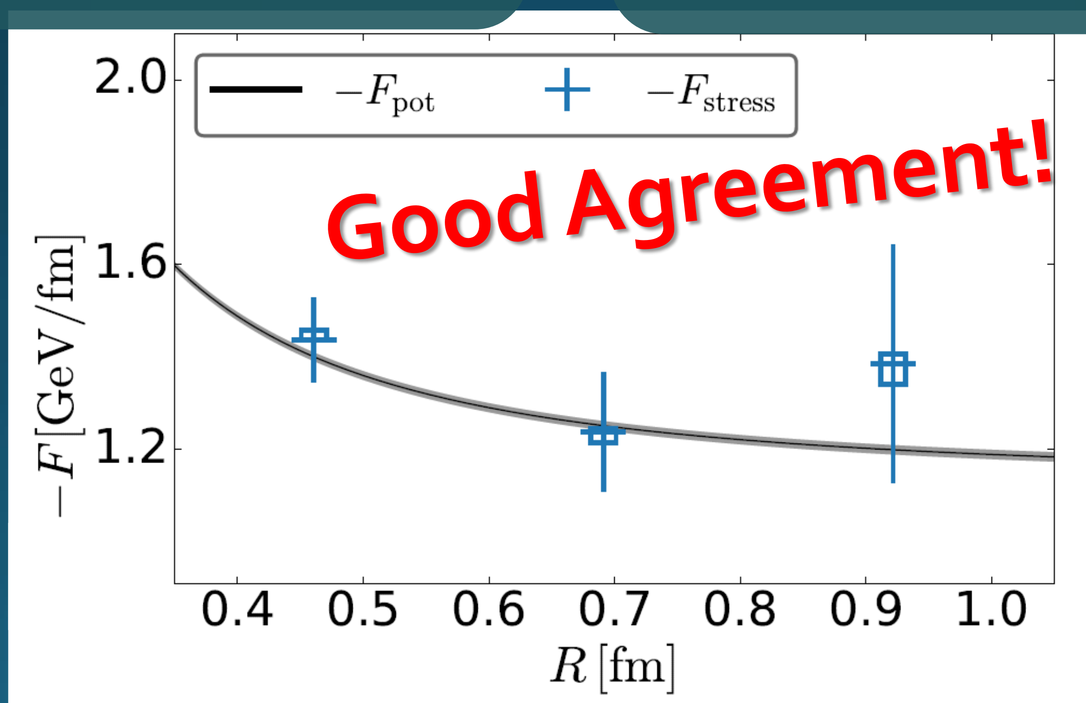


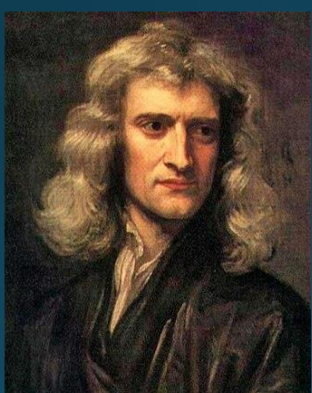
Force from Potential

$$F_{\text{pot}} = -\frac{dV}{dR}$$

Force from Stress

$$F_{\text{stress}} = \int_{\text{mid.}} d^2x T_{zz}(x)$$

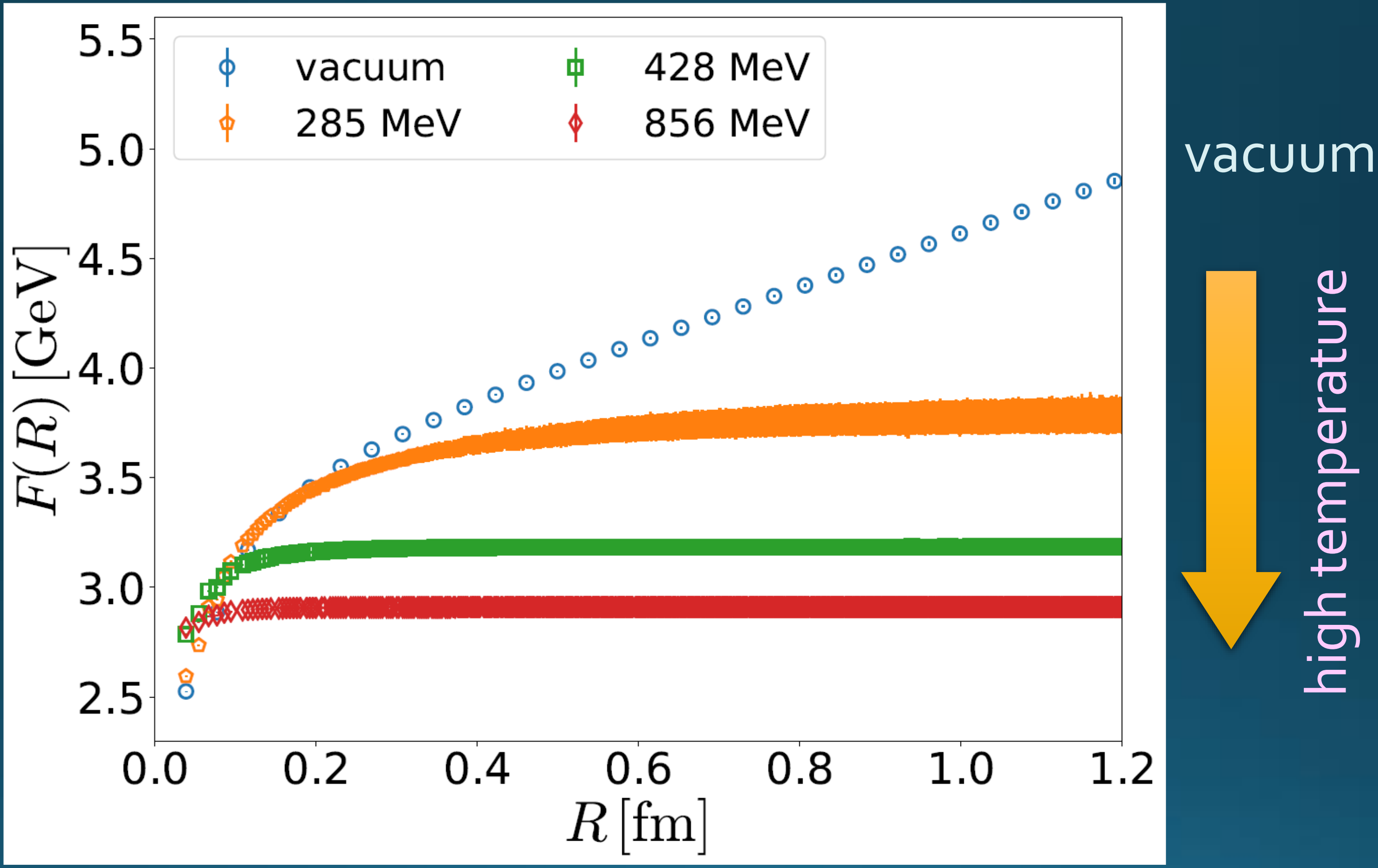




Newton
1687

Faraday
1839

Screening of $Q\bar{Q}$ Force above T_c

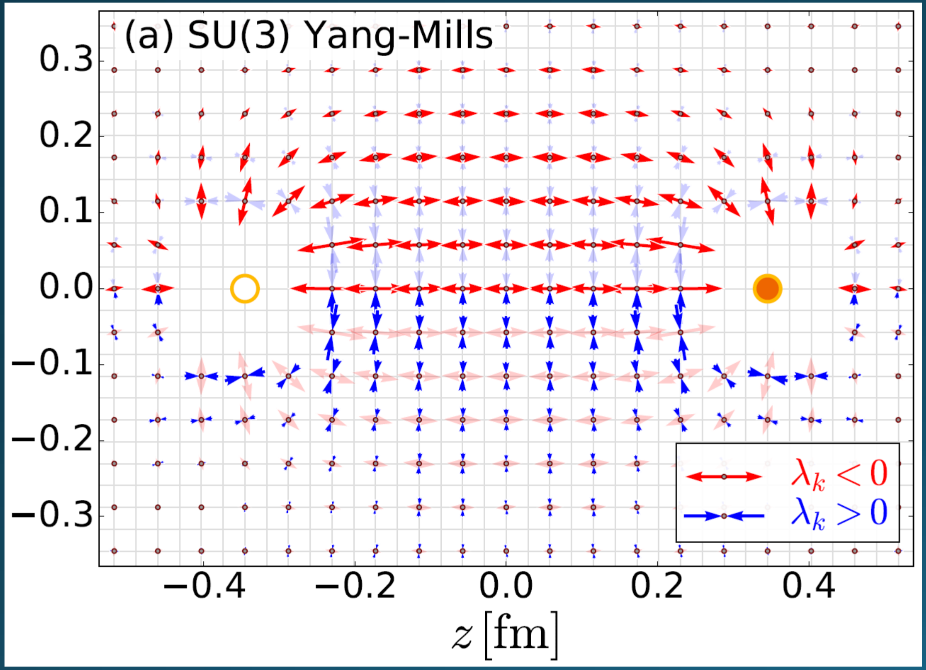


Q-Qbar force is screened in the deconfined phase.

Temperature Dependence

Vacuum

(Current Universe)



High Temperature

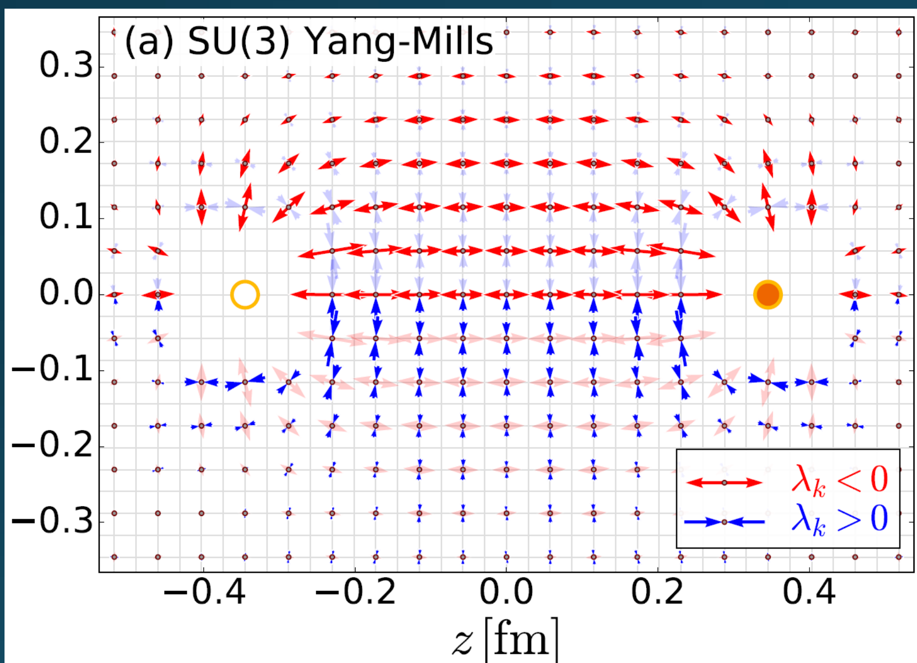
(Early Universe)

?

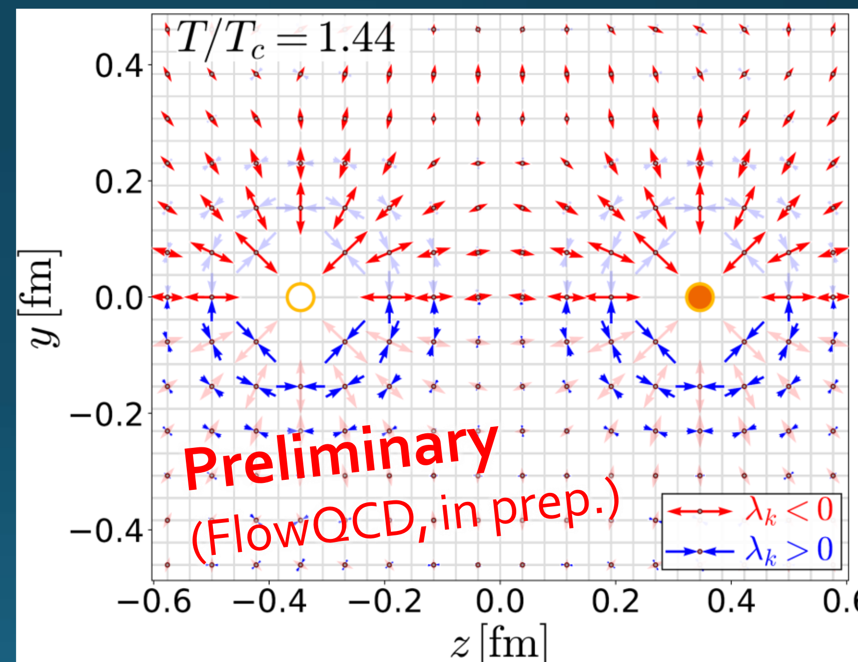
$$\langle T_{\mu\nu}(x)\rangle_{\mathrm{Q\bar{Q}}} = \frac{\langle \delta T_{\mu\nu}(x)\delta\Omega(y)\Omega^\dagger(z)\rangle}{\langle \Omega(y)\Omega^\dagger(z)\rangle}$$

Temperature Dependence

Vacuum
(Current Universe)



High Temperature
(Early Universe)

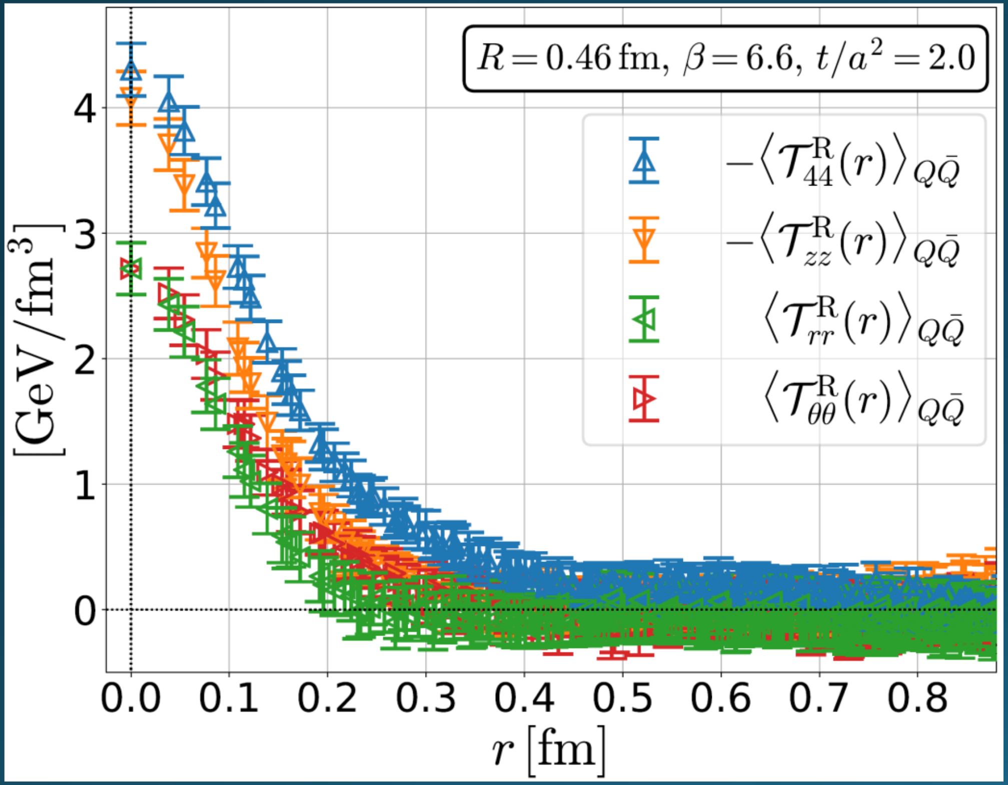


T=1.44T$_c$

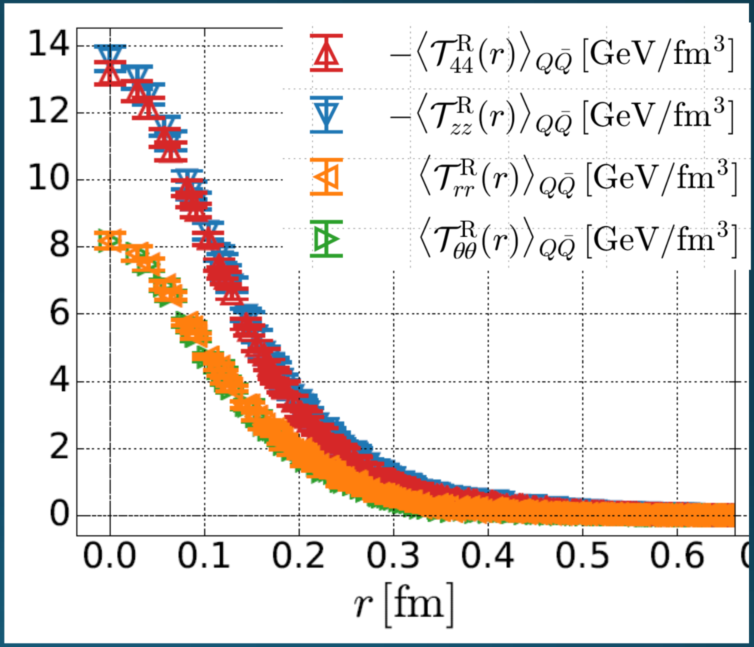
- ☐ Singlet projection for T=1.44T$_c$
- ☐ Flux-tube structure is screened above T$_c$.

Mid Plane

T=1.44Tc, R=0.46 fm



Vacuum, R=0.46 fm



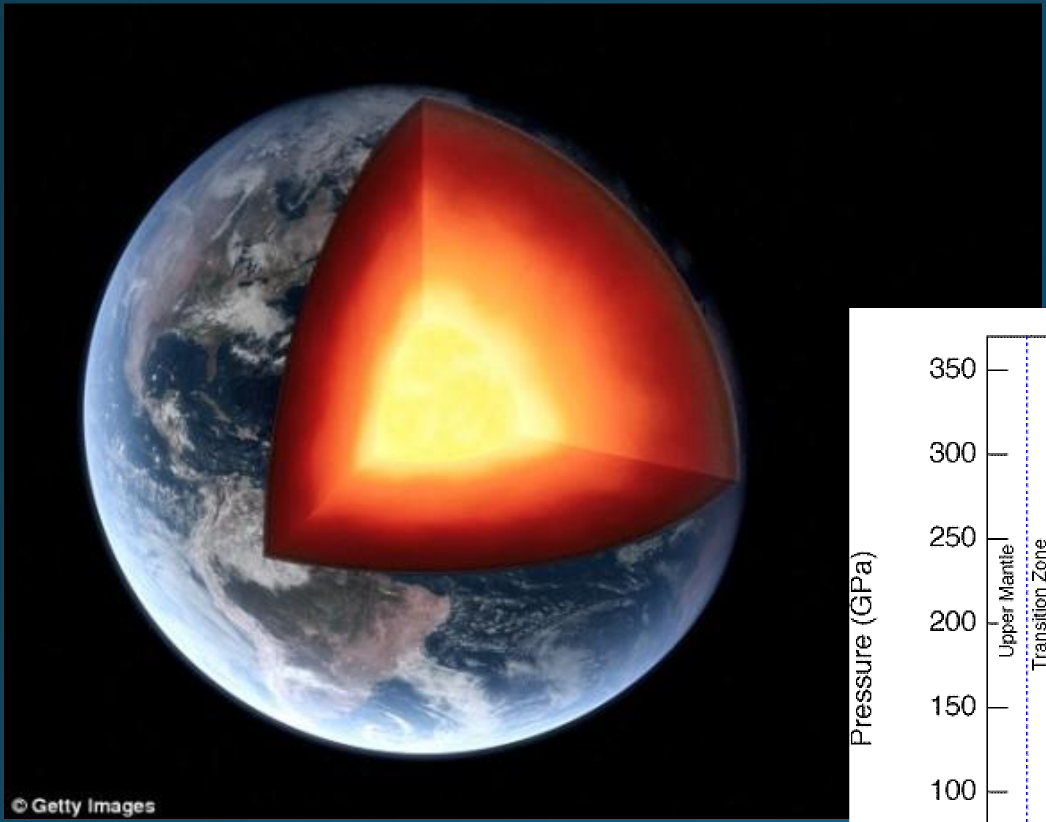
- ☐ Separation b/w T_{44} & T_{zz}?

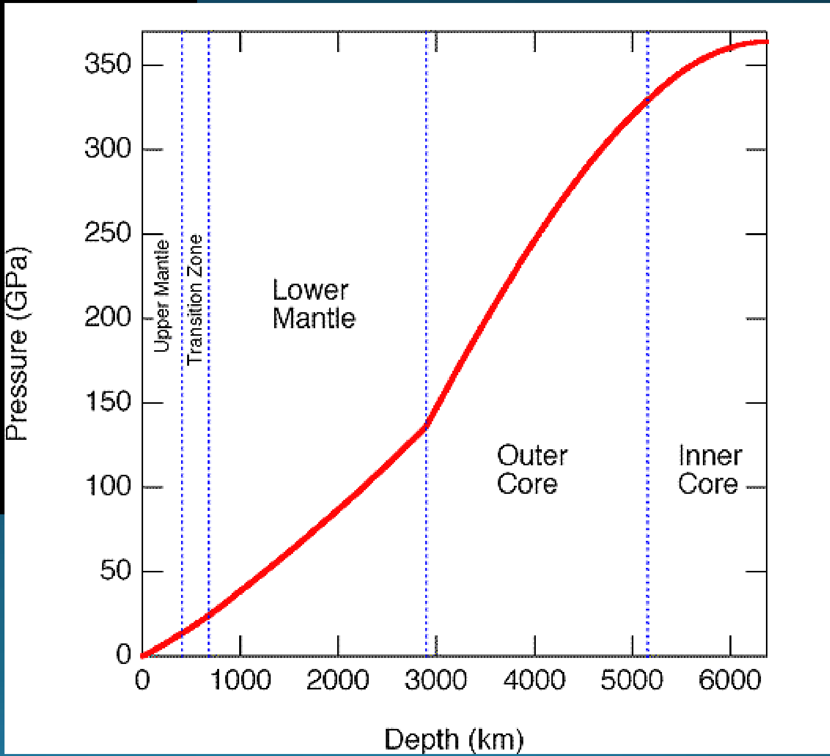
Stress Tensor around A Quark

in a deconfined phase

Q

Pressure inside the Earth

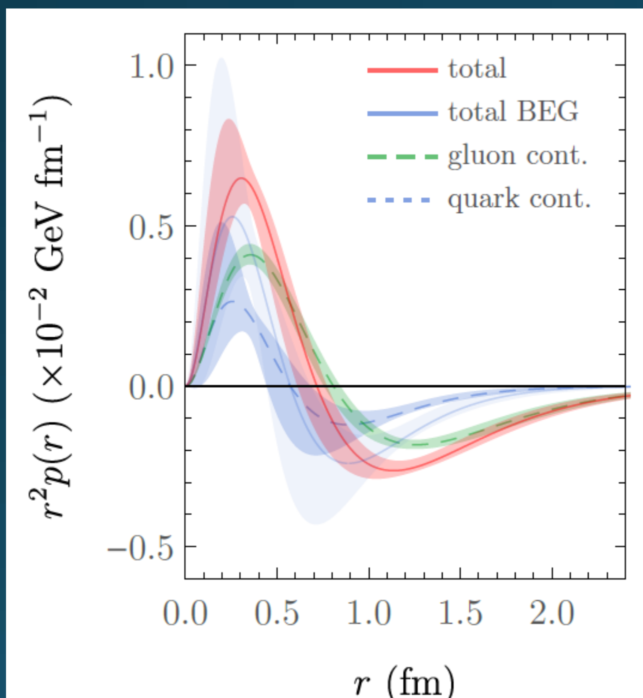




Pressure inside Hadrons

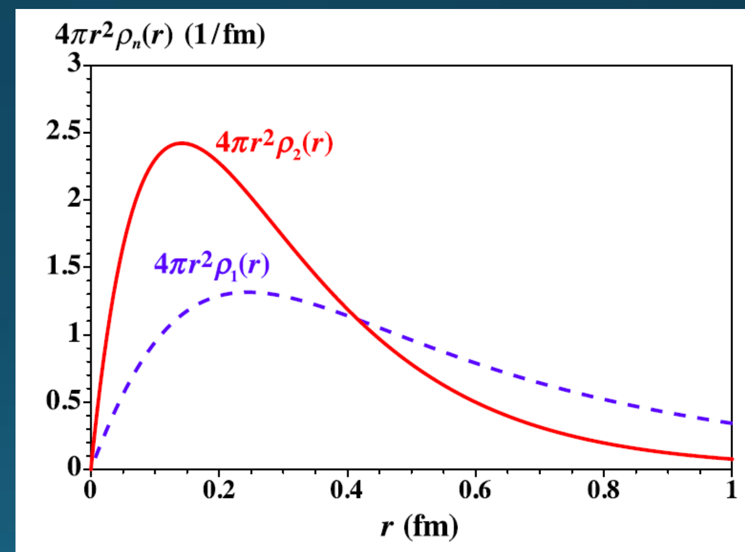
EMT distribution inside hadrons now accessible??

Pressure @ proton



arXiv:1810.07589
Nature, 557, 396 (2018)

EMT distribution @ pion

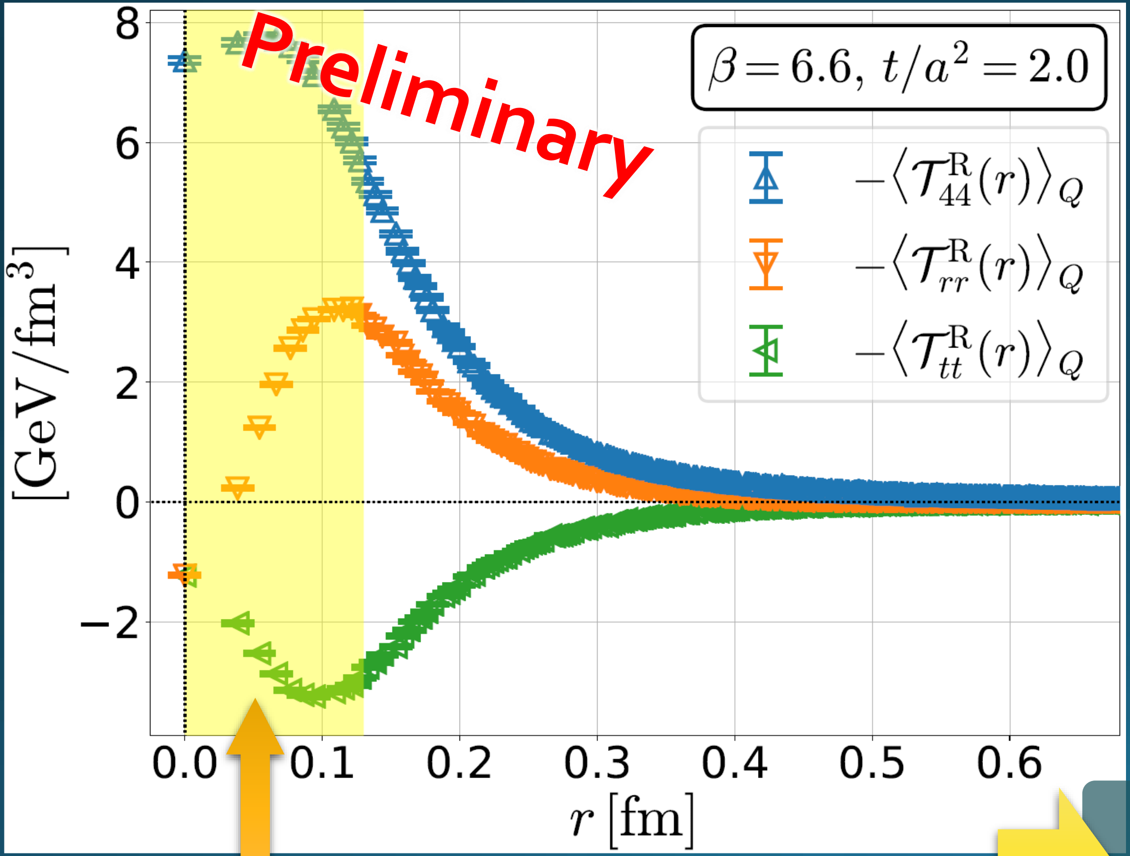


Kumano, Song, Teryaev
Phys. Rev. D 97, 014020 (2018)

Stress Tensor around A Quark

in a deconfined phase

$$\langle T_{\mu\nu}(x)\rangle_{\mathrm{Q}} = \frac{\langle \delta T_{\mu\nu}(x)\delta\Omega(0)\rangle}{\langle\Omega\rangle}$$



Not reliable

Yanagihara+, in prep.
Quenched QCD
48^3x12 ($T \approx 1.4T_c$)
fixed t , a

Spherical Coordinates

- Energy density
  $-\langle T_{44}\rangle = \varepsilon$
- Longitudinal pressure
  $-\langle T_{rr}\rangle = -p(r)$
- Transverse pressure
  $-\langle T_{tt}\rangle$

☐ Screening mass
☐ Strong coupling const.

Correlation Functions

EMT Correlator: Motivation

- Transport Coefficient

Kubo formula → viscosity

$$\eta = \int_0^\infty dt \int_0^{1/T} d\tau \int d^3x \langle T_{12}(x,-i\tau) T_{12}(0,t) \rangle$$

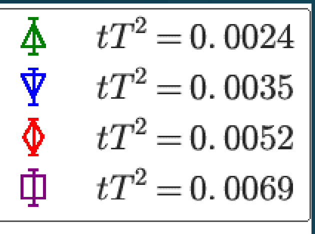
Karsch, Wyld, 1987
Nakamura, Sakai, 2005
Meyer; 2007, 2008
...
Borsanyi+, 2018
Astrakhantsev+, 2018

- Energy/Momentum Conservation

$\langle \bar{T}_{0\mu}(\tau) \bar{T}_{\rho\sigma}(0) \rangle$: τ-independent constant

- Fluctuation-Response Relations

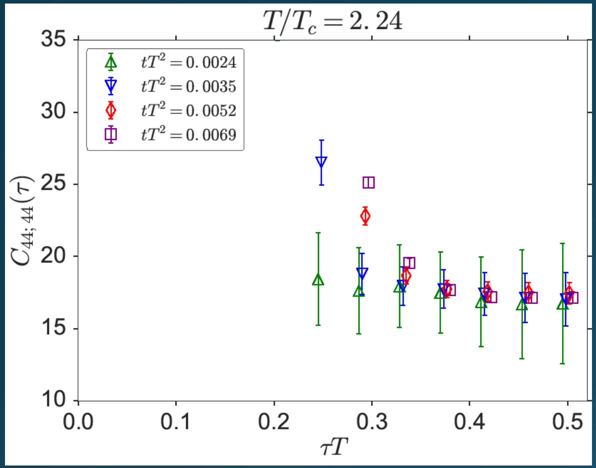
$$c_V = \frac{\langle \delta E^2 \rangle}{VT^2} \qquad E + p = \frac{\langle \bar{T}_{01}^2 \rangle}{VT} = \frac{\langle \bar{T}_{11} \bar{T}_{00} \rangle}{VT}$$

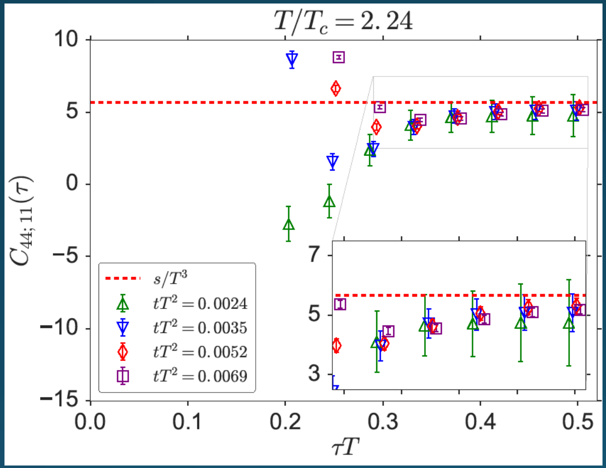


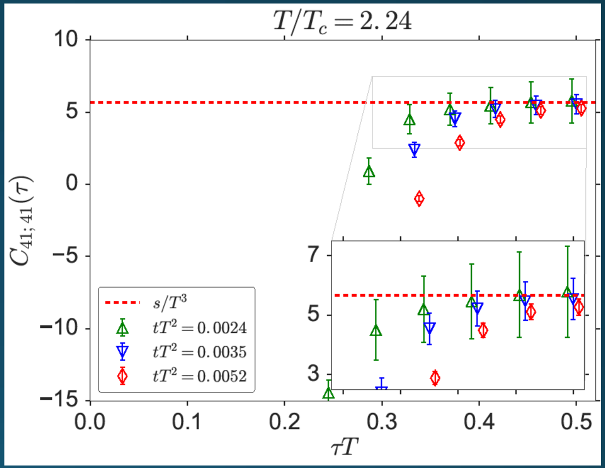
EMT Euclidean Correlator

FlowQCD, PR **D96**, 111502 (2017)

$\langle \bar{T}_{44}(\tau)\bar{T}_{44}(0) \rangle$ $\langle \bar{T}_{44}(\tau)\bar{T}_{11}(0) \rangle$ $\langle \bar{T}_{41}(\tau)\bar{T}_{41}(0) \rangle$





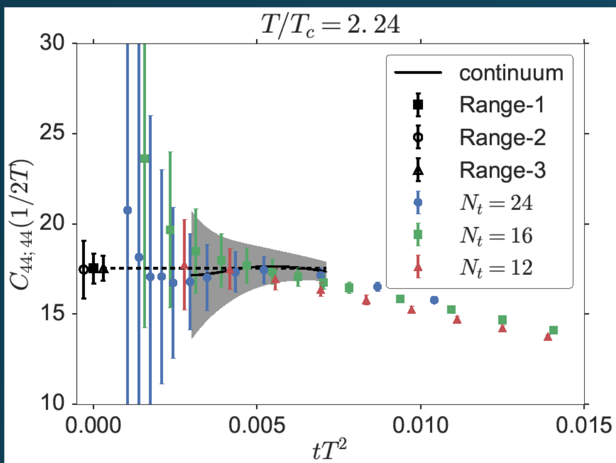


- ☐ τ-independent plateau in all channels ➔ conservation law
- ☐ Confirmation of fluctuation-response relations
- ☐ New method to measure c_V

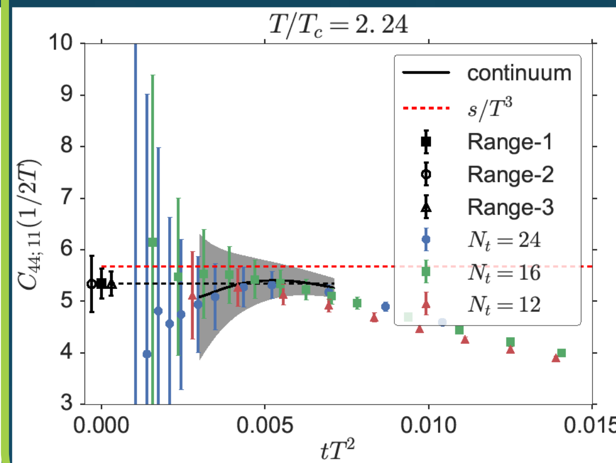
- ☐ Similar result for (41;41) channel: Borsanyi+, 2018
- ☐ Perturbative analysis: Eller, Moore, 2018

Fluctuation-Response Relations

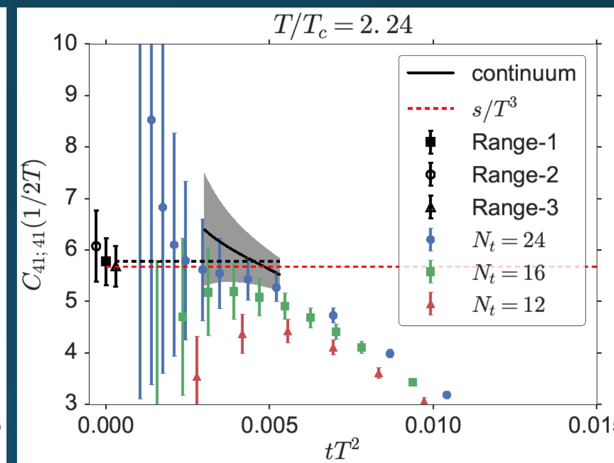
$\langle T_{44}(\tau) T_{44}(0) \rangle$



$\langle T_{44}(\tau) T_{11}(0) \rangle$



$\langle T_{41}(\tau) T_{41}(0) \rangle$



New measurement of c_V

c_V/T^3

T/T_c	$C_{44;44}(\tau_m)$	Ref.[19]	Ref.[11]	ideal gas
1.68	$17.7(8)(^{+2.1}_{-0.4})$	$22.8(7)^*$	17.7	21.06
2.24	$17.5(0.8)(^{+0}_{-0.1})$	$17.9(7)^{**}$	18.2	21.06

Confirmation of FRR

$$E + p = \frac{\langle \bar{T}_{01}^2 \rangle}{VT} = \frac{\langle \bar{T}_{11} \bar{T}_{00} \rangle}{VT}$$

2+1 QCD:
Taniguchi+ (WHOT-QCD), 1711.02262

Future Study

- ☐ Shear and bulk channel
- ☐ Correlation function at nonzero momentum
- ☐ Controlling flow time dependence

- ☐ Viscosity

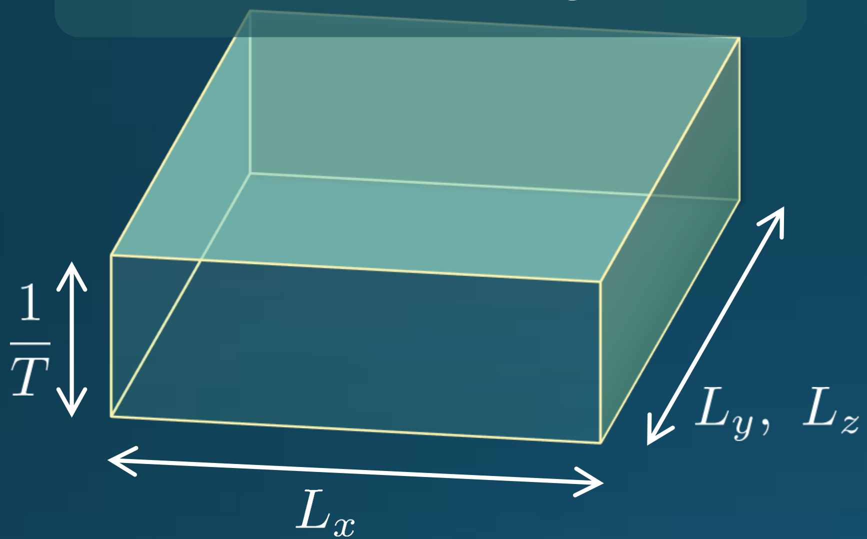
Summary

- Lattice simulations are not simple subjects.
- There are plenty of subjects in this community.
  - Thermodynamics
  - Thermodynamics under various conditions
  - EMT distribution inside hadrons
  - …

backup

Two Special Cases with PBC

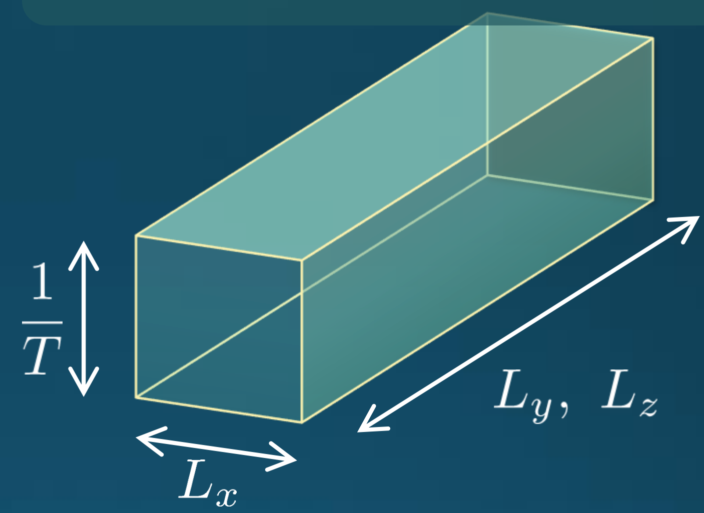
$$1/T \ll L_x = L_y = L_z$$



$$T_{11} = T_{22} = T_{33}$$

$$\frac{p_1}{p_2} = 1$$

$$1/T = L_x,\ L_y = L_z$$



$$T_{44} = T_{11},\ T_{22} = T_{33}$$

In conformal ($\Sigma_\mu T_{\mu\mu}=0$)

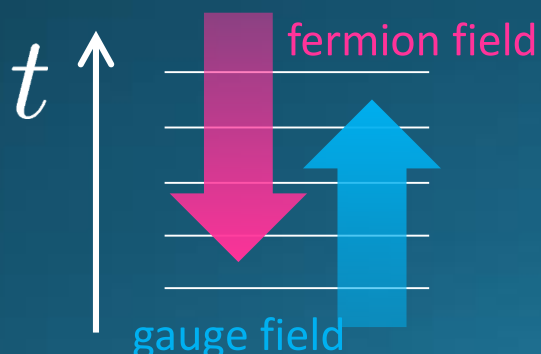
$$\frac{p_1}{p_2} = -1$$

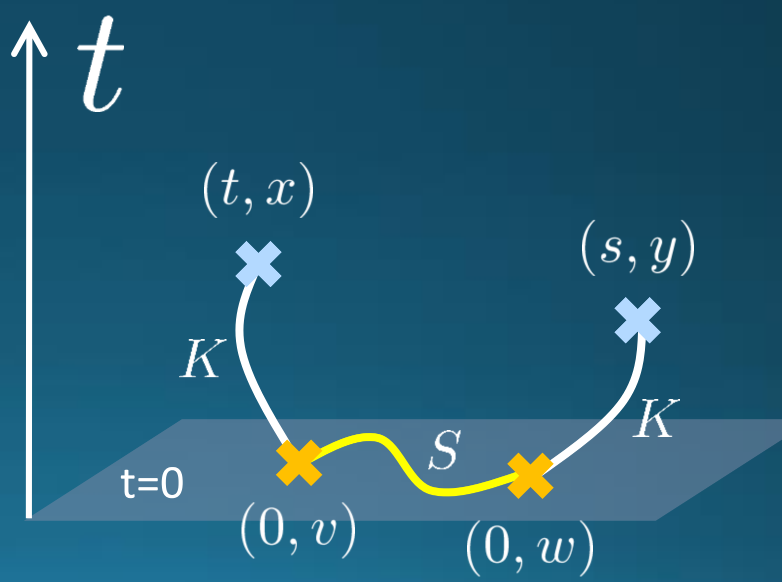
Fermion Propagator

$$S(t,x;s,y) = \langle \chi(t,x)\bar{\chi}(s,y)\rangle$$
$$= \sum_{v,w} K(t,x;0,v)S(v,w)K(s,y;0,w)^\dagger$$

$$\Big(\partial_t - D_\mu D_\mu\Big)K(t,x) = 0$$

- propagator of flow equation
- Inverse propagator is needed





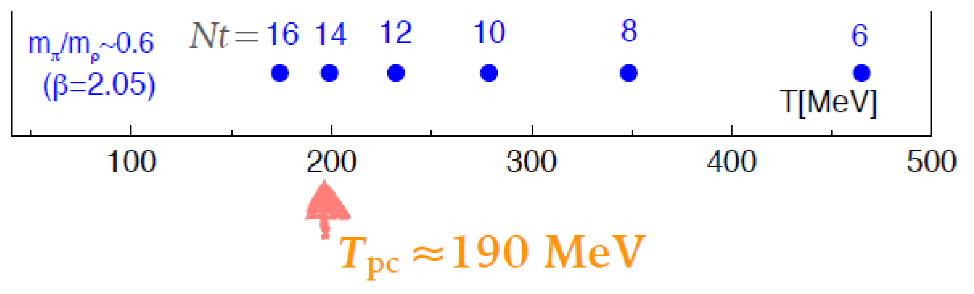
N_f=2+1 QCD Thermodynamics

Taniguchi+ (WHOT-QCD),
PRD**96**, 014509 (2017)

- N_f=2+1 QCD, Iwasaki gauge + NP-clover
- $m_{PS}/m_V \approx 0.63$ / almost physical s quark mass

- T=0: CP-PACS+JLQCD (ß=2.05, 28^3x56, a≈0.07fm)
- T>0: 32^3xN_t, N_t = 4, 6, … , 14, 16):
- T≈174-697MeV

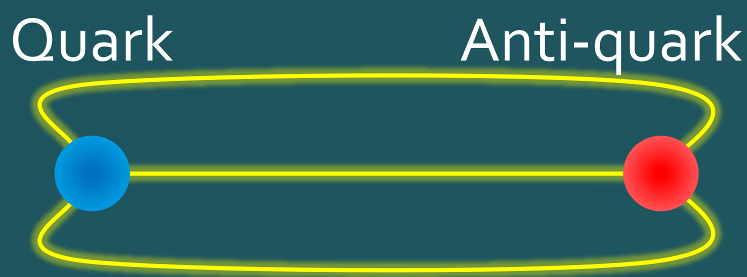
- t→0 extrapolation only (No continuum limit)



Dual Superconductor Picture

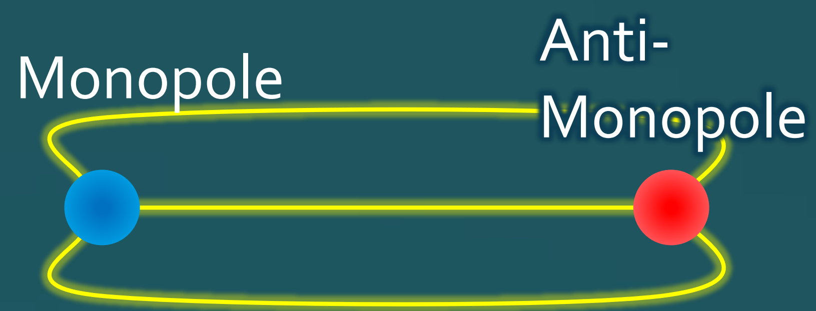
Nambu, 1970
Nielsen, Olesen, 1973
t 'Hooft, 1981
...

QCD Vacuum



Flux Tube

Superconductor



Magnetic Vortex

Dual (E⇔B)

Abelian-Higgs Model

Yanagihara, Iritani, MK, in prep.

Abelian-Higgs Model

$$\mathcal{L}_{\mathrm{AH}} = -\frac{1}{4}F_{\mu\nu}^2 + |(\partial_\mu + igA_\mu)\phi|^2 - \lambda(\phi^2 - v^2)^2$$

GL parameter: $\kappa = \sqrt{\lambda}/g$

- type-I : $\kappa < 1/\sqrt{2}$
- type-II : $\kappa > 1/\sqrt{2}$
- Bogomol'nyi bound : $\kappa = 1/\sqrt{2}$

Infinitely long tube

- degeneracy

$T_{zz}(r) = T_{44}(r)$ Luscher, 1981

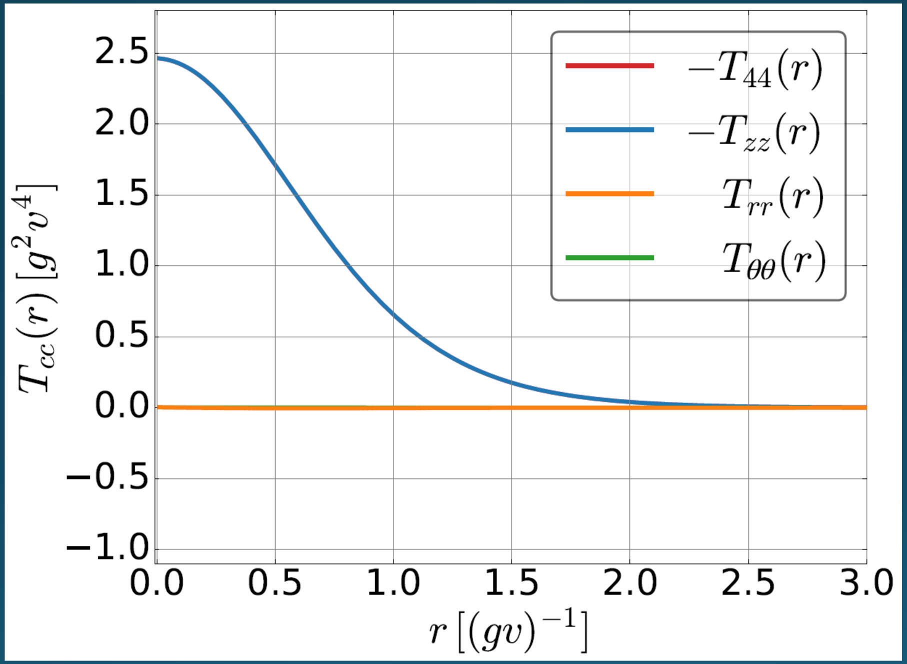
- momentum conservation

$$\frac{d}{dr}(rT_{rr}) = T_{\theta\theta}$$

Stress Tensor in AH Model

infinitely-long flux tube

Bogomol'nyi bound : $\kappa = 1/\sqrt{2}$



$$T_{rr} = T_{\theta\theta} = 0$$

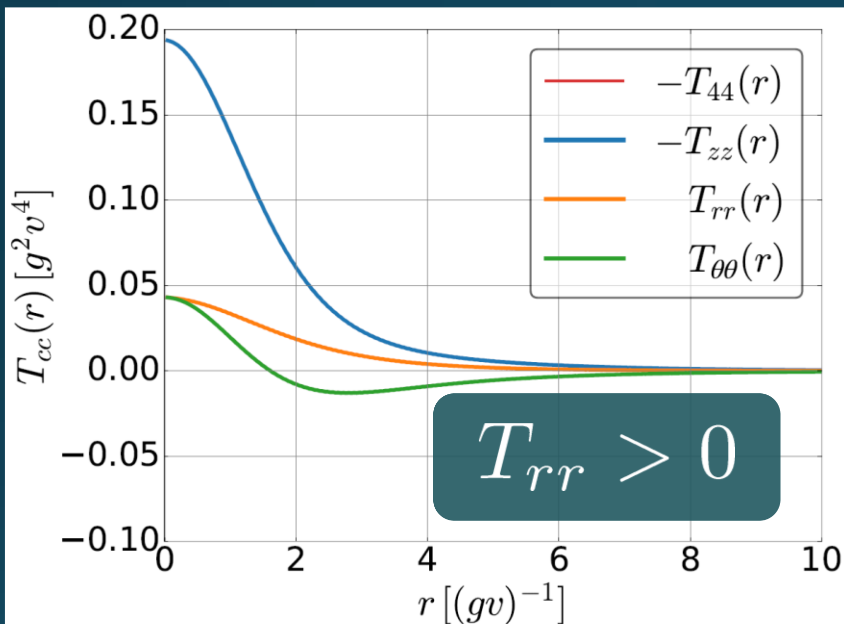
de Vega, Schaposnik, PR**D14**, 1100 (1976).

Stress Tensor in AH Model

infinitely-long flux tube

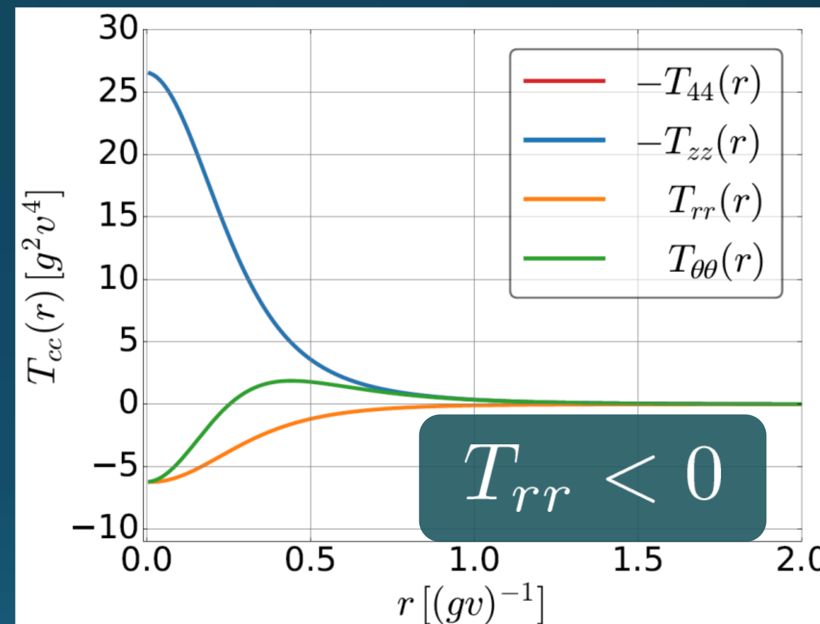
Type-I

$\kappa = 0.1$



Type-II

$\kappa = 3.0$



- ☐ No degeneracy bw T_{rr} & $T_{\theta\theta}$
- ☐ $T_{\theta\theta}$ changes sign

conservation law

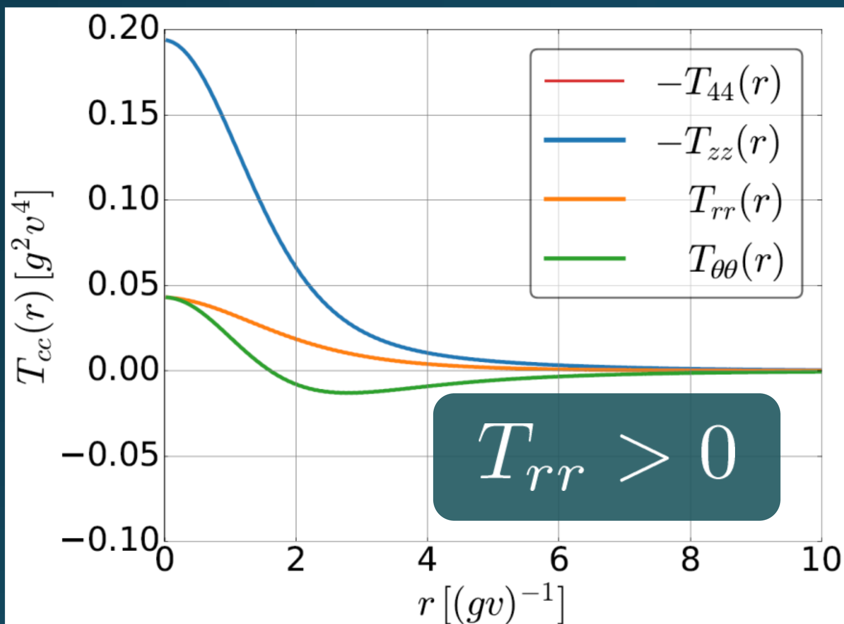
$$\frac{d}{dr}(rT_{rr}) = T_{\theta\theta}$$

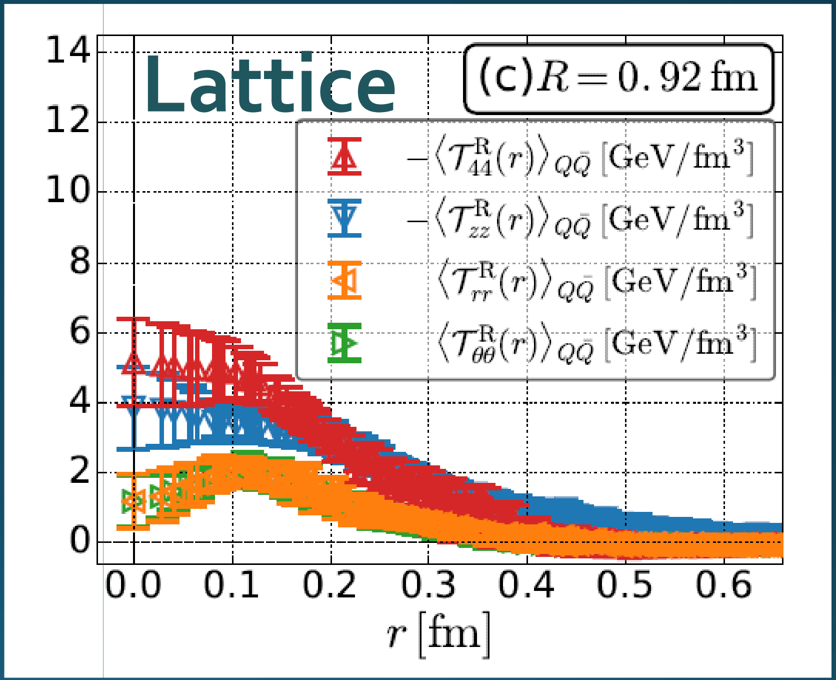
Stress Tensor in AH Model

infinitely-long flux tube

Type-I

$\kappa = 0.1$





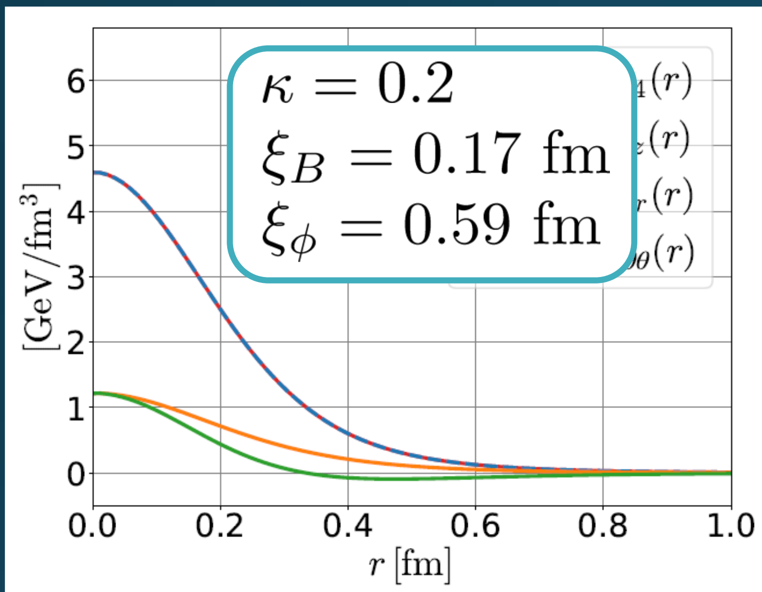
- ☐ No degeneracy bw T_{rr} & $T_{\theta\theta}$
- ☐ $T_{\theta\theta}$ changes sign

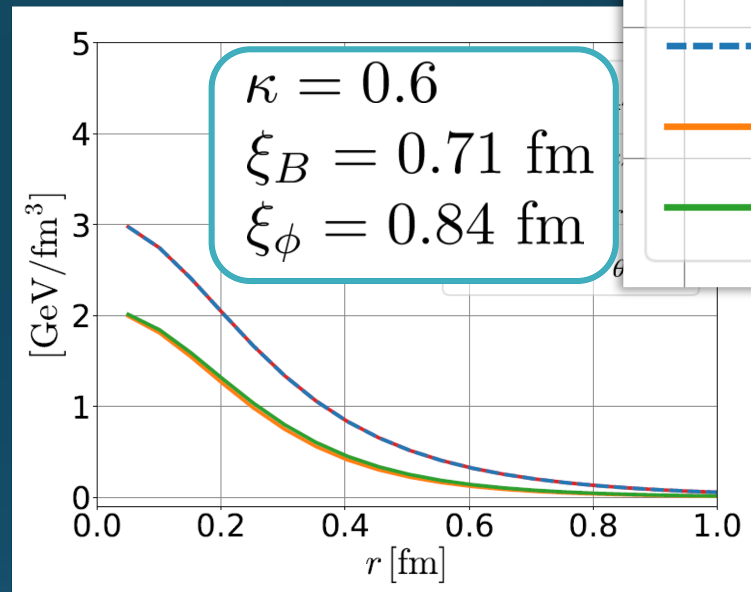
Inconsistent with lattice result

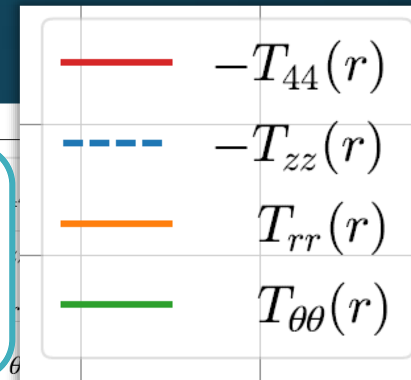
$$T_{rr} \simeq T_{\theta\theta}$$

Flux Tube with Finite Length

R=0.92 fm



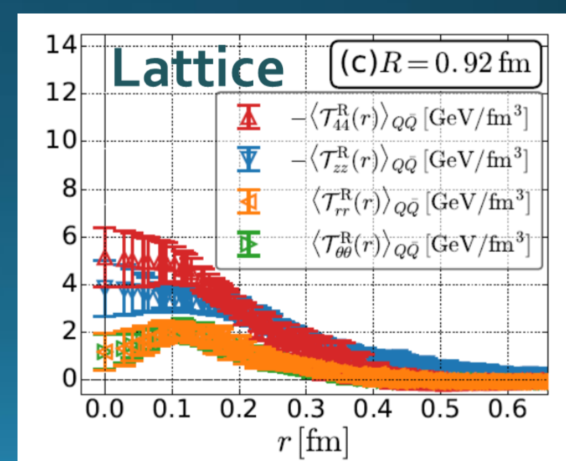




Left: $T_{zz}(0)$, $T_{rr}(0)$ reproduce lattice result

Right: A parameter satisfying $T_{rr} \approx T_{\theta\theta}$

No parameters to reproduce lattice data at R=0.92fm.



Numerical Setup

- SU(3) YM theory
- Wilson gauge action

- N_t = 16, 12
- N_z/N_t=6
- 2000~4000 confs.
- Even N_x

- No Continuum extrap.

T/T_c	β	N_z	N_τ	N_x	$N_{\rm vac}$
1.12	6.418	72	12	12, 14, 16, 18	64
	6.631	96	16	16, 18, 20, 22, 24	96
1.40	6.582	72	12	12, 14, 16, 18	64
	6.800	96	16	16, 18, 20, 22, 24	128
1.68	6.719	72	12	12, 14, 16, 18, 24	64
	6.719	96	12	14, 18	64
	6.941	96	16	16, 18, 20, 22, 24	96
2.10	6.891	72	12	12, 14, 16, 18, 24	72
	7.117	96	16	16, 18, 20, 22, 24	128
2.69	7.086	72	12	12, 14, 16, 18	-
$\simeq 8.1$	8.0	72	12	12, 14, 16, 18	-
$\simeq 25$	9.0	72	12	12, 14, 16, 18	-

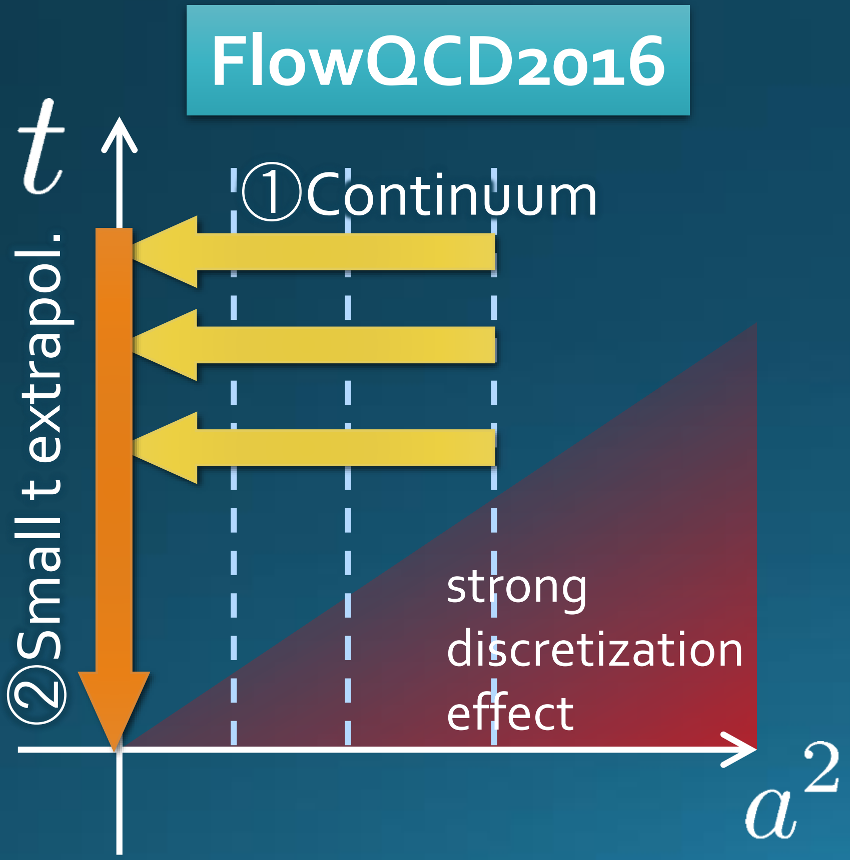
- Same Spatial volume
  - $12 \times 72^2 \times 12 \sim 16 \times 96^2 \times 16$
  - $18 \times 72^2 \times 12 \sim 24 \times 96^2 \times 16$

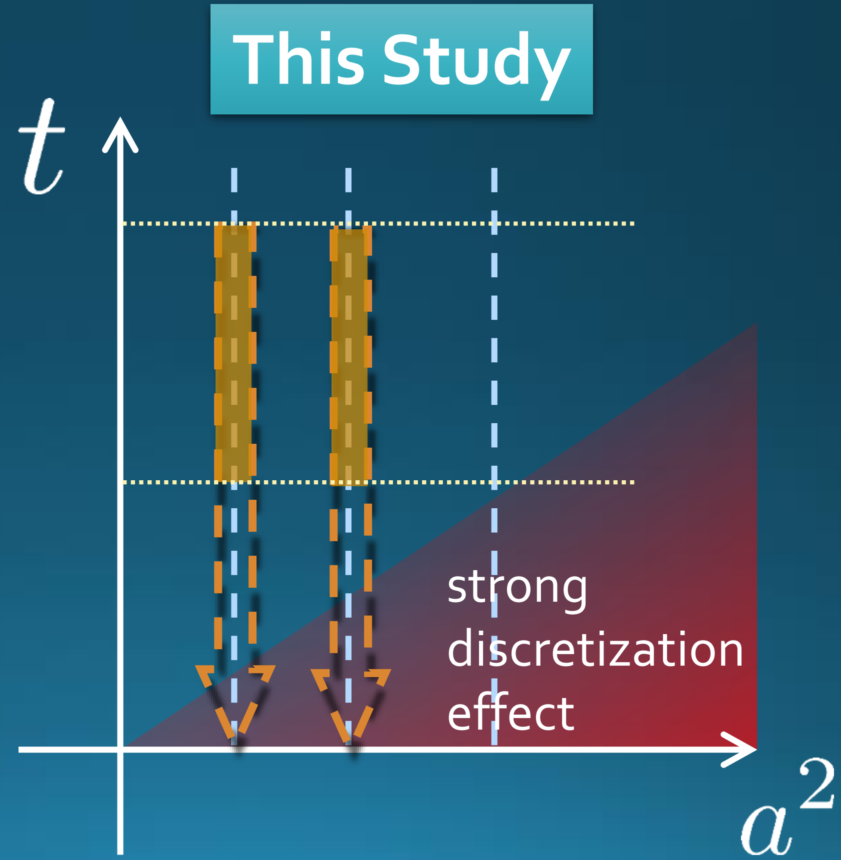
Simulations on OCTOPUS/Reedbush

Extrapolations t→0, a→0

$$\langle T_{\mu\nu}(t)\rangle_{\rm latt} = \langle T_{\mu\nu}(t)\rangle_{\rm phys} + \boxed{C_{\mu\nu} t} + \boxed{D_{\mu\nu}(t)\frac{a^2}{t}}$$

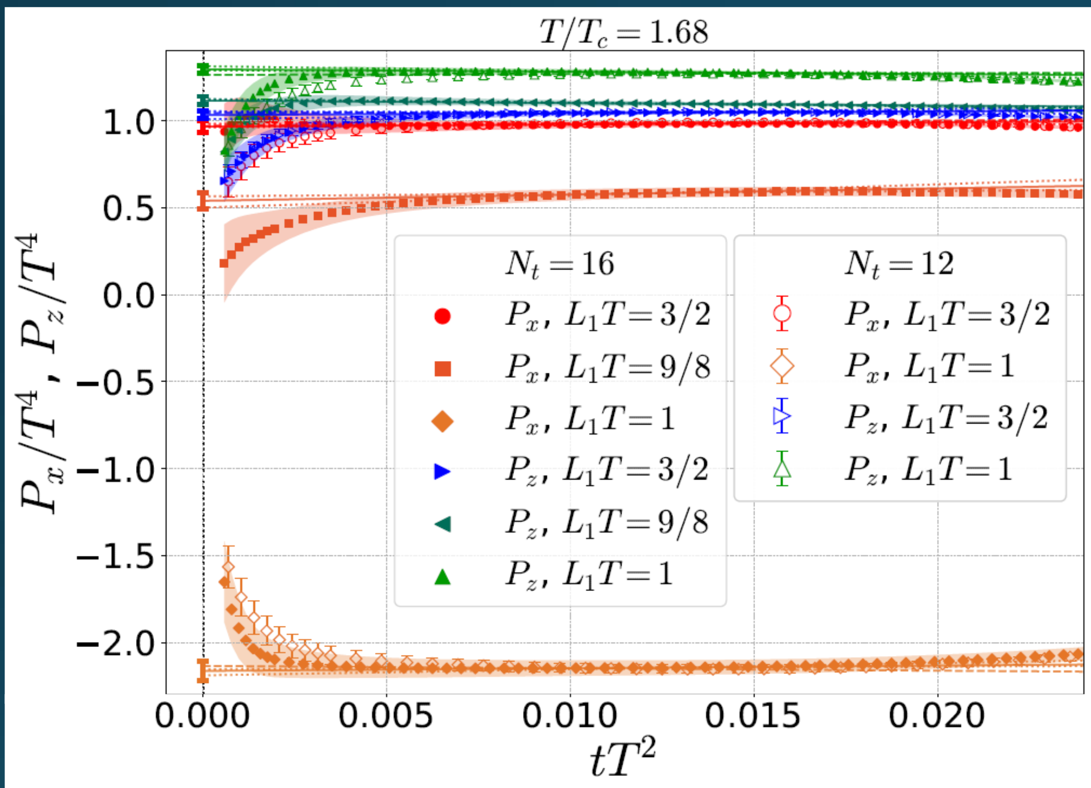
O(t) terms in SFTE lattice discretization

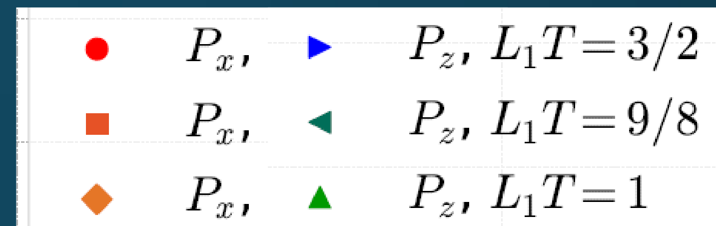




Small-t Extrapolation

$T/T_c = 1.68$





Filled: N_t=16 / Open: N_t=12

Small-t extrapolation

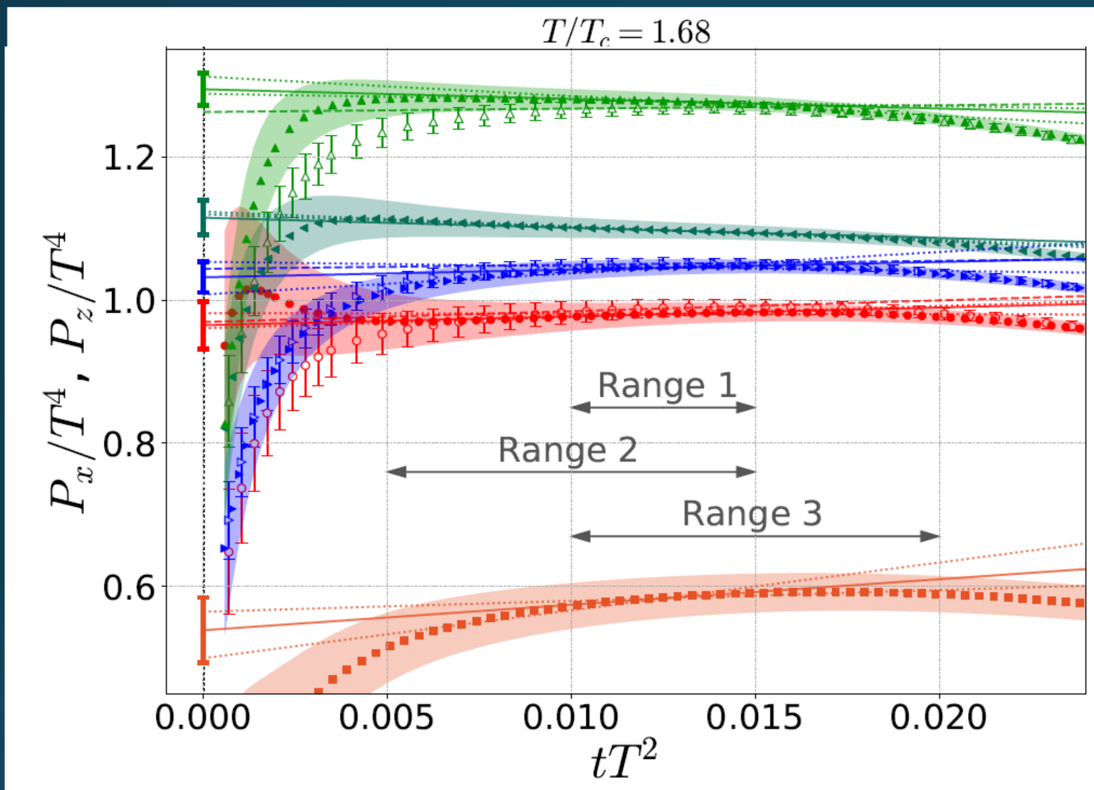
- Solid: N_t=16, Range-1
- Dotted: N_t=16, Range-2,3
- Dashed: N_t=12, Range-1

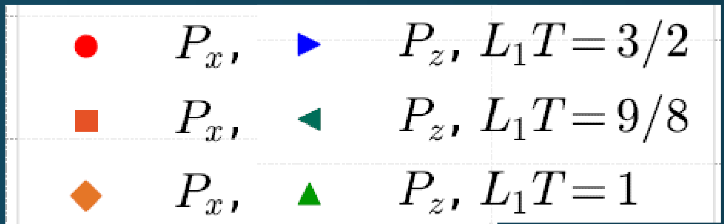
☐ Stable small-t extrapolation

☐ No N_t dependence within statistics for L_xT=1, 1.5

Small-t Extrapolation

$T/T_c = 1.68$





Filled: N_t=16 / Open: N_t=12

Small-t extrapolation

- Solid: N_t=16, Range-1
- Dotted: N_t=16, Range-2,3
- Dashed: N_t=12, Range-1

☐ Stable small-t extrapolation

☐ No N_t dependence within statistics for L_xT=1, 1.5