Stress-Energy-Momentum Tensor on the Lattice

Masakiyo Kitazawa
(Osaka U.)
Energy-Momentum Tensor

\[ T_{\mu\nu} = \begin{bmatrix} T_{00} & T_{01} & T_{02} & T_{03} \\ T_{10} & T_{11} & T_{12} & T_{13} \\ T_{20} & T_{21} & T_{22} & T_{23} \\ T_{30} & T_{31} & T_{32} & T_{33} \end{bmatrix} \]

All components are important physical observables!
Definition of the operator is nontrivial because of the explicit breaking of Lorentz symmetry. Its measurement is extremely noisy due to high dimensionality and etc.

\[ T_{\mu \nu} = F_{\mu \rho} F_{\nu \rho} - \frac{1}{4} \delta_{\mu \nu} F F \]
Stress = Force per Unit Area
Stress = Force per Unit Area

Pressure

\[ \vec{P} = \frac{\vec{F}}{S} \]

\[ \vec{P} = P \hat{n} \]
Stress = Force per Unit Area

Pressure

\[ \vec{P} = \frac{\vec{F}}{S} \]

In thermal medium

\[ T_{ij} = P \delta_{ij} \]

Generally, F and n are not parallel

\[ \frac{F_i}{S} = \sigma_{ij} n_j \]

Stress Tensor

\[ \sigma_{ij} = -T_{ij} \]

Landau Lifshitz
Force

Action-at-a-distance

Newton
1687

\[ F = -G \frac{m_1 m_2}{r^2} \]

Local interaction

Faraday
1839

\[ F = -\frac{1}{4\pi \varepsilon_0} \frac{q_1 q_2}{r^2} \]
Maxwell Stress
(in Maxwell Theory)

\[ \sigma_{ij} = \varepsilon_0 E_i E_j + \frac{1}{\mu_0} B_i B_j - \frac{1}{2} \delta_{ij} \left( \varepsilon_0 E^2 + \frac{1}{\mu_0} B^2 \right) \]

\[ \vec{E} = (E, 0, 0) \]

\[ T_{ij} = \begin{pmatrix} -E^2 & 0 & 0 \\ 0 & E^2 & 0 \\ 0 & 0 & E^2 \end{pmatrix} \]

- Parallel to field: **Pulling**
- Vertical to field: **Pushing**
Maxwell Stress
(in Maxwell Theory)

\[ T_{ij} v_j^{(k)} = \lambda_k v_i^{(k)} \]

\[ (k = 1, 2, 3) \]

length: \( \sqrt{|\lambda_k|} \)

Definite physical meaning
- Distortion of field, line of the field
- Propagation of the force as local interaction
Quark-Anti-quark system

Formation of the flux tube $\rightarrow$ confinement

Previous Studies on Flux Tube

- Potential
- Action density
- Color-electric field

so many studies...

Cea+ (2012)
Cardoso+ (2013)
Stress Tensor in $Q\bar{Q}$ System

(a) SU(3) Yang-Mills

Lattice simulation
SU(3) Yang-Mills
$a=0.029$ fm
$R=0.69$ fm
$t/a^2=2.0$

Definite physical meaning
- Distortion of field, line of the field
- Propagation of the force as local interaction
- Manifestly gauge invariant

Yanagihara+, PLB, 2019
SU(3) Yang-Mills (quantum) vs Maxwell (classical)

Propagation of the force is clearly different in YM and Maxwell theories!
Casimir Effect
Casimir Effect

attractive force between two conductive plates
Casimir Effect

\[ T_{ij} = \begin{pmatrix} P_x & 0 & 0 \\ 0 & P_z & 0 \\ 0 & 0 & P_z \end{pmatrix} \]

- \( P_x < 0 \)
- \( P_z > 0 \)

Brown, Maclay 1969
Casimir Effect

\[ T_{ij} = \begin{pmatrix} P_x & 0 & 0 \\ 0 & P_z & 0 \\ 0 & 0 & P_z \end{pmatrix} \]

- \( P_x < 0 \)
- \( P_z > 0 \)
Casimir Effect

\[
T_{ij} = \begin{pmatrix}
P_x & 0 & 0 \\
0 & P_z & 0 \\
0 & 0 & P_z \\
\end{pmatrix}
\]

- \( P_x < 0 \)
- \( P_z > 0 \)
Pressure Anisotropy @ T≠0

Free scalar field

- $L_2 = L_3 = \infty$
- Periodic BC

MK, Mogliacci, Kolbe, Horowitz, PRD, 2019
Mogliacci+, 1807.07871
Pressure Anisotropy @ $T \neq 0$

Free scalar field
- $L_2 = L_3 = \infty$
- Periodic BC
- Only $t \to 0$ limit
- Error: stat. + sys.

Medium near $T_c$ is remarkably insensitive to finite size!

MK, Mogliacci, Kolbe, Horowitz, PRD, 2019

Mogliacci+, 1807.07871
Contents

1. Constructing EMT

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  FlowQCD, PRD90, 011501 (2014); WHOT-QCD, PRD96, 014509 (2017); Iritani+, PTEP 2019, 023B02 (2019)
  PRD94, 114512 (2016);

3. Flux Tube
  FlowQCD, PLB789, 210 (2019); Yanagihara+, in prep.

4. Static Quark Systems at Nonzero T
  FlowQCD, in prep.

5. Casimir Effect
  MK, Mogliacci, Horowitz, Kolbe, PRD, 2019

6. Correlation Function
  FlowQCD, PRD96, 111502 (2017)
Yang-Mills Gradient Flow

\[ \frac{\partial}{\partial t} A_\mu(t, x) = - \frac{\partial S_{YM}}{\partial A_\mu} \]

leading

\[ A_\mu(0, x) = A_\mu(x) \]

- diffusion equation in 4-dim space
- diffusion distance \( d \sim \sqrt{8t} \)
- “continuous” cooling/smearing
- No UV divergence at \( t>0 \)

Luscher 2010
Narayanan, Neuberger, 2006
Luscher, Weiss, 2011
Small Flow-Time Expansion

\[ \tilde{O}(t, x) \xrightarrow{t \to 0} \sum_i c_i(t) \tilde{O}_i^R(x) \]

- An operator at \( t > 0 \)
- Remormalized operators of original theory

Original 4-dim theory

\[ t \to 0 \] limit

2\sqrt{2t}
Constructing EMT 1

Gauge-invariant dimension 4 operators

\[ \tilde{O}(t, x) \xrightarrow{t \to 0} \sum_i c_i(t) O_i^R(x) \]

- \[ U_{\mu\nu}(t, x) = G_{\mu\rho}(t, x)G_{\nu\rho}(t, x) - \frac{1}{4}\delta_{\mu\nu}G_{\mu\nu}(t, x)G_{\mu\nu}(t, x) \]
- \[ E(t, x) = \frac{1}{4}\delta_{\mu\nu}G_{\mu\nu}(t, x)G_{\mu\nu}(t, x) \]

Suzuki, 2013
Constructing EMT

\[ U_{\mu\nu}(t, x) = \alpha_U(t) \left[ T_{\mu\nu}^R(x) - \frac{1}{4} \delta_{\mu\nu} T_{\rho\rho}^R(x) \right] + \mathcal{O}(t) \]

\[ E(t, x) = \langle E(t, x) \rangle + \alpha_E(t) T_{\rho\rho}^R(x) + \mathcal{O}(t) \]

**Remormalized EMT**

\[ T_{\mu\nu}^R(x) = \lim_{t \to 0} \left[ c_1(t) U_{\mu\nu}(t, x) + \delta_{\mu\nu} c_2(t) E(t, x)_{\text{subt.}} \right] \]

Perturbative coefficient:
Suzuki (2013); Makino, Suzuki (2014); Harlander+ (2018); Iritani, MK, Suzuki, Takaura (2019)
Perturbative Coefficients

\[ T_{\mu\nu}(t) = c_1(t)U_{\mu\nu}(t) + \delta_{\mu\nu}c_2(t)E(t) \]

<table>
<thead>
<tr>
<th></th>
<th>LO</th>
<th>1-loop</th>
<th>2-loop</th>
<th>3-loop</th>
</tr>
</thead>
<tbody>
<tr>
<td>( c_1(t) )</td>
<td></td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
</tbody>
</table>

\( c_2(t) \) is zero in 1-loop.

- Improved: \( \mu_d(t) = 1/\sqrt{8t} \)
- Improved: \( \mu_0(t) = 1/\sqrt{2e^{\gamma E}t} \)

\( g^2 \) Choice of the scale

Iritani, MK, Suzuki (2013)
Harlander+, 1808.09837
Iritani, MK, Suzuki, Takaura, PTEP 2019
Gradient Flow Method

Lattice regularized gauge theory

Smeared world by gradient flow

Continuum theory

Measure on the lattice

Take Extrapolation \((t,a) \to (0,0)\)

\[
\langle T_{\mu\nu}(t) \rangle_{\text{latt}} = \langle T_{\mu\nu}(t) \rangle_{\text{phys}} + C_{\mu\nu} t + D_{\mu\nu} \frac{a^2}{t} + \cdots
\]

\(O(t)\) terms in SFTE

Lattice discretization
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   FlowQCD, in prep.

5. Casimir Effect

6. Correlation Function
   FlowQCD, PRD96, 111502 (2017)
Higher Order Coefficient: $\varepsilon + p$

- t dependence becomes milder with higher order coeff.
- Better $t \rightarrow 0$ extrapolation
- Systematic error: $\mu_0$ or $\mu_d$, uncertainty of $\Lambda$ ($\pm 3\%$), fit range
- Extrapolation func: linear, higher order term in $c_1$ ($\sim g^6$)

Iritani, MK, Suzuki, Takaura, PTEP 2019
Double Extrapolation
$t \to 0$, $a \to 0$

$$\langle T_{\mu\nu}(t) \rangle_{\text{latt}} = \langle T_{\mu\nu}(t) \rangle_{\text{phys}} + C_{\mu\nu}t + \left[ D_{\mu\nu}(t) \frac{a^2}{t} \right]$$

O(t) terms in SFTE lattice discretization

Continuum extrapolation
$$\langle T_{\mu\nu}(t) \rangle_{\text{cont}} = \langle T_{\mu\nu}(t) \rangle_{\text{lat}} + C(t)a^2$$

Small t extrapolation
$$\langle T_{\mu\nu} \rangle = \langle T_{\mu\nu}(t) \rangle + C't$$
Higher Order Coefficient: $\varepsilon+p$

- NLO (1-loop)
  - $T/T_C = 1.68$ (NLO) $\mu = \mu_0$
  - Parameter ranges:
    - $64^3 \times 12$
    - $96^3 \times 16$
    - $128^3 \times 20$

- $N^2$LO (2-loop)
  - $T/T_C = 1.68$ (N$^2$LO) $\mu = \mu_0$
  - Parameter ranges:
    - $64^3 \times 12$
    - $96^3 \times 16$
    - $128^3 \times 20$

- t dependence becomes milder with higher order coeff.
- Better $t \to 0$ extrapolation
- Systematic error: $\mu_0$ or $\mu_d$, uncertainty of $\Lambda$ ($\pm 3\%$), fit range
- Extrapolation func: linear, higher order term in $c_1$ ($\sim g^6$)

Iritani, MK, Suzuki, Takaura, PTEP 2019
Effect of Higher-Order Coeffs.

Systematic error: $\mu_0$ or $\mu_d$, $\Lambda$, $t \to 0$ function, fit range

More stable extrapolation with higher order $c_1$ & $c_2$ (pure gauge)
Gradient Flow for Fermions

\[ \partial_t \psi(t, x) = D_\mu D_\mu \psi(t, x) \]
\[ \partial_t \bar{\psi}(t, x) = \psi(t, x) \overleftarrow{D}_\mu \overrightarrow{D}_\mu \]

\[ D_\mu = \partial_\mu + A_\mu(t, x) \]

- Not “gradient” flow but a “diffusion” equation.
- Divergence in field renormalization of fermions.
- All observables are finite at t>0 once Z(t) is fixed.

\[ \tilde{\psi}(t, x) = Z(t) \psi(t, x) \]

- Energy-momentum tensor from SFTE Makino, Suzuki, 2014
\[ T_{\mu\nu}(t, x) = c_1(t) U_{\mu\nu}(t, x) + c_2(t) \delta_{\mu\nu} (E(t, x) - \langle E \rangle_0) + c_3(t) (O_{3\mu\nu}(t, x) - 2O_{4\mu\nu}(t, x) - \text{VEV}) + c_4(t) (O_{4\mu\nu}(t, x) - \text{VEV}) + c_5(t) (O_{5\mu\nu}(t, x) - \text{VEV}) \]

\[ T_{\mu\nu}(x) = \lim_{t \to 0} T_{\mu\mu}(t, x) \]

\[ \tilde{O}^f_{3\mu\nu}(t, x) \equiv \varphi_f(t) \bar{\chi}_f(t, x) \left( \gamma_\mu \overleftrightarrow{D}_\nu + \gamma_\nu \overleftrightarrow{D}_\mu \right) \chi_f(t, x), \]

\[ \tilde{O}^f_{4\mu\nu}(t, x) \equiv \varphi_f(t) \delta_{\mu\nu} \bar{\chi}_f(t, x) \overleftrightarrow{D} \chi_f(t, x), \]

\[ \tilde{O}^f_{5\mu\nu}(t, x) \equiv \varphi_f(t) \delta_{\mu\nu} \bar{\chi}_f(t, x) \chi_f(t, x), \]

\[ \varphi_f(t) \equiv \frac{-6}{(4\pi)^2 t^2 \langle \bar{\chi}_f(t, x) \overleftrightarrow{D} \chi_f(t, x) \rangle_0}. \]
Agreement with integral method except for $N_t=4, 6$

- $N_t=4, 6$: No stable extrapolation is possible
- Statistical error is substantially suppressed!

Physical mass: Kanaya+ (WHOT-QCD), 1710.10015
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so many studies...

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(a) SU(3) Yang-Mills

$\lambda_k < 0$
$\lambda_k > 0$

Definite physical meaning
- Distortion of field, line of the field
- Propagation of the force as local interaction
- Manifestly gauge invariant

Yanagihara+, 1803.05656 PLB, in press
Lattice simulation
SU(3) Yang-Mills
a=0.029 fm
R=0.69 fm
t/a²=2.0

Pulling
Pushing
SU(3) Yang-Mills (Quenched)

Wilson gauge action

Clover operator

APE smearing / multi-hit

Fine lattices (a=0.029-0.06 fm)

Continuum extrapolation

Simulation: bluegene/Q@KEK

\[
\langle O(x) \rangle_{Q\bar{Q}} = \lim_{T \to \infty} \frac{\langle \delta O(x) \delta W(R, T) \rangle}{\langle W(R, T) \rangle}
\]

<table>
<thead>
<tr>
<th>(\beta)</th>
<th>(a) [fm]</th>
<th>(N_{size}^4)</th>
<th>(N_{conf})</th>
<th>(R/a)</th>
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<tbody>
<tr>
<td>6.304</td>
<td>0.058</td>
<td>48^4</td>
<td>140</td>
<td>8</td>
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<tr>
<td>6.465</td>
<td>0.046</td>
<td>48^4</td>
<td>440</td>
<td>10</td>
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<tr>
<td>6.513</td>
<td>0.043</td>
<td>48^4</td>
<td>600</td>
<td>–</td>
</tr>
<tr>
<td>6.600</td>
<td>0.038</td>
<td>48^4</td>
<td>1,500</td>
<td>12</td>
</tr>
<tr>
<td>6.819</td>
<td>0.029</td>
<td>64^4</td>
<td>1,000</td>
<td>16</td>
</tr>
</tbody>
</table>

\(R\) [fm]: 0.46, 0.69, 0.92

\(R=0.92\) fm

\(R=0.69\) fm

\(R=0.46\) fm
Continuum Extrapolation at mid-point

- $a \to 0$ extrapolation with fixed $t$
\( \alpha \to 0 \) Extrapolation at mid-point

- \( \alpha \to 0 \) extrapolation with fixed \( t \)
- Then, \( t \to 0 \) with three ranges

\[ -\langle T_{zz}(t, 0) \rangle_{\text{lat}}^{\text{Qq}} [\text{GeV}/\text{fm}^3] \]

\[ t/\bar{a}^2 \]

\[ a = 0.029 \text{ fm} \]
\[ a = 0.038 \text{ fm} \]
\[ a = 0.046 \text{ fm} \]
\[ a = 0.058 \text{ fm} \]

Continuum limit

Range 1
Range 2
Range 3
From rotational symm. & parity

EMT is diagonalized in Cylindrical Coordinates

\[
T_{cc'}(r) = \begin{pmatrix}
T_{rr} & T_{r\theta} \\
T_{\theta r} & T_{\theta\theta} \\
T_{zz} & T_{z4} \\
T_{44} & T_{44}
\end{pmatrix}
\]

\[
T_{rr} = \vec{e}_r^T T \vec{e}_r,
\]

\[
T_{\theta\theta} = \vec{e}_\theta^T T \vec{e}_\theta
\]

Degeneracy in Maxwell theory

\[
T_{rr} = T_{\theta\theta} = -T_{zz} = -T_{44}
\]
Degeneracy: $T_{44} \simeq T_{zz}$, $T_{rr} \simeq T_{\theta\theta}$

Separation: $T_{zz} \neq T_{rr}$

Nonzero trace anomaly $\sum T_{cc} \neq 0$

In Maxwell theory

$T_{rr} = T_{\theta\theta} = -T_{zz} = -T_{44}$

Continuum Extrapolated!

Mid-Plane
Degeneracy: $T_{44} \simeq T_{zz}$, $T_{rr} \simeq T_{\theta\theta}$

Separation: $T_{zz} \neq T_{rr}$

Nonzero trace anomaly $\sum T_{cc} \neq 0$
Momentum Conservation

\[ \partial_i T_{ij} = 0 \implies \partial_r (r T_{rr}) = T_{\theta\theta} - r \partial_z T_{r\theta} \]

- In cylindrical coordinates,

- For infinitely-long flux tube

\[ \partial_r (r T_{rr}) = T_{\theta\theta} \implies T_{rr} \text{ and } T_{\theta\theta} \text{ must separate!} \]

Effect of boundaries is important for the flux tube at \( R = 0.92 \text{ fm} \)

\[ (\text{c}) R = 0.92 \text{ fm} \]

- \( \langle T_{44}^R(r) \rangle_{QQ} \text{ [GeV/fm}^3\rangle \)
- \( \langle T_{zz}^R(r) \rangle_{QQ} \text{ [GeV/fm}^3\rangle \)
- \( \langle T_{rr}^R(r) \rangle_{QQ} \text{ [GeV/fm}^3\rangle \)
- \( \langle T_{\theta\theta}^R(r) \rangle_{QQ} \text{ [GeV/fm}^3\rangle \)

\( r \text{ [fm]} \)
Force

Force from Potential

\[ F_{\text{pot}} = -\frac{dV}{dR} \]

Force from Stress

\[ F_{\text{stress}} = \int_{\text{mid.}} d^2 x T_{zz}(x) \]
Force from Potential

\[ F_{\text{pot}} = - \frac{dV}{dR} \]

Force from Stress

\[ F_{\text{stress}} = \int_{\text{mid.}} d^2 x T_{zz}(x) \]

Newton
1687

Faraday
1839
Force from Potential

\[ F_{\text{pot}} = -\frac{dV}{dR} \]

Force from Stress

\[ F_{\text{stress}} = \int_{\text{mid.}} d^2x T_{zz}(x) \]

Good Agreement!
Dual Superconductor Picture

QCD Vacuum

- Quark
- Anti-quark
- Flux Tube

Superconductor

- Monopole
- Anti-Monopole
- Magnetic Vortex

Dual (E⇔B)

Nambu, 1970
Nielsen, Olesen, 1973
’t Hooft, 1981
...

Quark
Anti-quark
Abelian-Higgs Model

\[ \mathcal{L}_{AH} = -\frac{1}{4} F_{\mu\nu}^2 + \left| (\partial_\mu + igA_\mu) \phi \right|^2 - \lambda (\phi^2 - \nu^2)^2 \]

**GL parameter:** \( \kappa = \sqrt{\lambda/g} \)

- **type-I:** \( \kappa < 1/\sqrt{2} \)
- **type-II:** \( \kappa > 1/\sqrt{2} \)
- **Bogomol’nyi bound:** \( \kappa = 1/\sqrt{2} \)

**Infinitely long tube**

- degeneracy
  \[ T_{zz}(r) = T_{44}(r) \] Luscher, 1981
- momentum conservation
  \[ \frac{d}{dr} (r T_{rr}) = T_{\theta\theta} \]
Stress Tensor in AH Model
ininitely-long flux tube

Bogomol’nyi bound: $\kappa = 1/\sqrt{2}$

$$T_{rr} = T_{\theta\theta} = 0$$

de Vega, Schaposnik, PRD14, 1100 (1976).
Stress Tensor in AH Model
infinitely-long flux tube

Type-I
\[ \kappa = 0.1 \]

- \[ T_{44}(r) \]
- \[ T_{zz}(r) \]
- \( T_{rr}(r) \)
- \( T_{\theta\theta}(r) \)

\[ T_{rr} > 0 \]

- No degeneracy bw \( T_{rr} \) & \( T_{\theta\theta} \)
- \( T_{\theta\theta} \) changes sign

Type-II
\[ \kappa = 3.0 \]

- \[ T_{44}(r) \]
- \[ T_{zz}(r) \]
- \( T_{rr}(r) \)
- \( T_{\theta\theta}(r) \)

\[ T_{rr} < 0 \]

conservation law
\[ \frac{d}{dr} (r T_{rr}) = T_{\theta\theta} \]
Stress Tensor in AH Model
ininitely-long flux tube

Type-I

$\kappa = 0.1$

- $T_{44}(r)$
- $T_{zz}(r)$
- $T_{rr}(r)$
- $T_{\theta\theta}(r)$

$T_{rr} > 0$

- No degeneracy bw $T_{rr}$ & $T_{\theta\theta}$
- $T_{\theta\theta}$ changes sign

Inconsistent with lattice result

$T_{rr} \simeq T_{\theta\theta}$

Lattice

(c) $R = 0.92$ fm

- $\langle T_{44}^R(r) \rangle_{QQ}$ [GeV/fm$^3$]
- $\langle T_{zz}^R(r) \rangle_{QQ}$ [GeV/fm$^3$]
- $\langle T_{rr}^R(r) \rangle_{QQ}$ [GeV/fm$^3$]
- $\langle T_{\theta\theta}^R(r) \rangle_{QQ}$ [GeV/fm$^3$]
Flux Tube with Finite Length

R=0.92 fm

**Left:** $T_{zz}(0), T_{rr}(0)$ reproduce lattice result

**Right:** A parameter satisfying $T_{rr} \approx T_{\theta\theta}$

No parameters to reproduce lattice data at R=0.92 fm.
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Screening of $Q\bar{Q}$ Force above $T_c$

$Q$-$Q$bar force is screened in the deconfined phase.
Temperature Dependence

Vacuum
(Current Universe)

High Temperature
(Early Universe)

\[
\langle T_{\mu\nu}(x) \rangle_{QQ} = \frac{\langle \delta T_{\mu\nu}(x) \delta \Omega(y) \Omega^\dagger(z) \rangle}{\langle \Omega(y) \Omega^\dagger(z) \rangle}
\]
Temperature Dependence

Vacuum
(Current Universe)

High Temperature
(Early Universe)

\( T = 1.44T_c \)

- Singlet projection for \( T = 1.44T_c \)
- Flux-tube structure is screened above \( T_c \).
Mid Plane

$T=1.44 T_c, R=0.46 \text{ fm}$

Vacuum, $R=0.46 \text{ fm}$

- Separation b/w $T_{44}$ & $T_{zz}$?
Stress Tensor around a Quark in a deconfined phase
Pressure inside the Earth
Pressure inside Hadrons

EMT distribution inside hadrons now accessible??

Pressure @ proton

EMT distribution @ pion

arXiv:1810.07589

Kumano, Song, Teryaev
Stress Tensor around A Quark

in a deconfined phase

\[ \langle T_{\mu\nu}(x) \rangle_Q = \frac{\langle \delta T_{\mu\nu}(x) \delta \Omega(0) \rangle}{\langle \Omega \rangle} \]

Yanagihara+, in prep. Quenched QCD

48^3 \times 12 \ (T \approx 1.4T_c)

fixed \ t, \ a

Spherical Coordinates

- Energy density
  \[ - \langle T_{44} \rangle = \varepsilon \]
- Longitudinal pressure
  \[ - \langle T_{rr} \rangle = -p(r) \]
- Transverse pressure
  \[ - \langle T_{tt} \rangle \]

Not reliable

Screening mass

Strong coupling const.
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Casimir Effect

\[ T_{ij} = \begin{pmatrix} P_x & 0 & 0 \\ 0 & P_z & 0 \\ 0 & 0 & P_z \end{pmatrix} \]

- \( P_x < 0 \)
- \( P_z > 0 \)
Pressure Anisotropy \( @ T \neq 0 \)

**Free scalar field**
- \( L_2 = L_3 = \infty \)
- Periodic BC
- Only \( t \to 0 \) limit
- Error: stat. + sys.

**Lattice result**
- Periodic BC
- Only \( t \to 0 \) limit
- Error: stat. + sys.

Medium near \( T_c \) is remarkably insensitive to finite size!
Thermodynamics on the Lattice

Various Methods

- Integral, differential, moving frame, non-equilibrium, ...
- rely on thermodynamic relations valid in $V \to \infty$

$$P = \frac{T}{V} \ln Z$$

$sT = \varepsilon + P$

Not applicable to anisotropic systems

- We employ Gradient Flow Method

$$\varepsilon = \langle T_{00} \rangle \quad P = \langle T_{11} \rangle$$

Components of EMT are directly accessible!
SU(3) YM theory
Wilson gauge action

- $N_t = 16, 12$
- $N_z/N_t = 6$
- 2000~4000 confs.
- Even $N_x$
- No Continuum extrap.

Same Spatial volume
- $12 \times 72^2 \times 12 \sim 16 \times 96^2 \times 16$
- $18 \times 72^2 \times 12 \sim 24 \times 96^2 \times 16$

Simulations on OCTOPUS/Reedbush
Extrapolations $t \to 0$, $a \to 0$

\[
\langle T_{\mu \nu}(t) \rangle_{\text{latt}} = \langle T_{\mu \nu}(t) \rangle_{\text{phys}} + C_{\mu \nu} t + \left( D_{\mu \nu}(t) \frac{a^2}{t} \right)
\]

$O(t)$ terms in SFTE lattice discretization

FlowQCD2016

This Study

1. Continuum
2. Small $t$ extrapolation

Strong discretization effect
Small-t Extrapolation

\[ T/T_c = 1.68 \]

Filled: \( N_t = 16 \) / Open: \( N_t = 12 \)

Small-t extrapolation
- Solid: \( N_t = 16 \), Range-1
- Dotted: \( N_t = 16 \), Range-2,3
- Dashed: \( N_t = 12 \), Range-1

- Stable small-t extrapolation
- No \( N_t \) dependence within statistics for \( L_x T = 1, 1.5 \)
Small-t Extrapolation

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Pressure Anisotropy @ T≠0

Free scalar field
- L_2 = L_3 = \infty
- Periodic BC
  - Mogliacci+, 1807.07871

Lattice result
- Periodic BC
- Only t \to 0 limit
- Error: stat.+sys.

Medium near T_c is remarkably insensitive to finite size!

MK, Mogliacci, Kolbe, Horowitz, 1904.00241

\[ \pi L_2 = L_3 = \infty \]

\[ \text{Periodic BC} \]

\[ \text{Only } t \to 0 \text{ limit} \]

\[ \text{Error: stat.} + \text{sys.} \]
Pressure Anisotropy @ T≠0

Free scalar field

- L₂ = L₃ = ∞
- Periodic BC

Lattice result

- Periodic BC
- Only t→0 limit
- Error: stat.+sys.

Medium near T_c is remarkably insensitive to finite size!
Energy density / transverse pressure $P_z$

**Energy Density**

![Graph showing energy density vs $L_x T$ for different $T/T_c$ and $N_t$ values.](image)

**Transverse Pressure $P_z$**

![Graph showing transverse pressure $P_z$ vs $L_x T$ for different $T/T_c$ and $N_t$ values.](image)
Higher T

High-T limit: massless free gluons
How does the anisotropy approach this limit?

Difficulties
- Vacuum subtraction requires large-volume simulations.
- Lattice spacing not available $\rightarrow c_1(t), c_2(t)$ are not determined.
Difficulties
- Vacuum subtraction requires large-volume simulations.
- Lattice spacing not available \( \rightarrow c_1(t), c_2(t) \) are not determined.

We study

\[
\frac{P_x + \delta}{P_z + \delta}
\]

\[
\delta = -\frac{1}{4} \sum_\mu T^{\text{E}}_{\mu\mu}
\]

No vacuum subtr. nor Suzuki coeffs. necessary!
\( \frac{P_x + \delta}{P_z + \delta} \)

- Ratio approaches the asymptotic value.
- But, large deviation exists even at \( T/T_c \approx 25 \).
Contents

1. Constructing EMT
2. Thermodynamics
   FlowQCD, PRD90, 011501 (2014); WHOT-QCD, PRD96, 014509 (2017); Iritani+, PTEP 2019, 023B02 (2019)
3. Flux Tube
   FlowQCD, PLB789, 210 (2019); Yanagihara+, in prep.
4. Static Quark Systems at Nonzero T
   FlowQCD, in prep.
5. Casimir Effect
6. Correlation Function
   FlowQCD, PRD96, 111502 (2017)
EMT Correlator: Motivation

- Transport Coefficient
  - Kubo formula $\rightarrow$ viscosity
    \[
    \eta = \int_0^\infty dt \int_0^{1/T} d\tau \int d^3x \langle T_{12}(x, -i\tau)T_{12}(0, t) \rangle
    \]
    - Karsch, Wyld, 1987
    - Nakamura, Sakai, 2005
    - Meyer; 2007, 2008
    - ... Borsanyi+, 2018
    - Astrakhantsev+, 2018

- Energy/Momentum Conservation
  \[
  \langle \bar{T}_{0\mu}(\tau)\bar{T}_{\rho\sigma}(0) \rangle: \tau\text{-independent constant}
  \]

- Fluctuation-Response Relations
  \[
  c_V = \frac{\langle \delta E^2 \rangle}{VT^2} \quad E + p = \frac{\langle \bar{T}_{01}^2 \rangle}{VT} = \frac{\langle \bar{T}_{11}\bar{T}_{00} \rangle}{VT}
  \]
EMT Euclidean Correlator

FlowQCD, PR D96, 111502 (2017)

- \( \langle \bar{T}_{44}(\tau)\bar{T}_{44}(0) \rangle \)
- \( \langle \bar{T}_{44}(\tau)\bar{T}_{11}(0) \rangle \)
- \( \langle \bar{T}_{41}(\tau)\bar{T}_{41}(0) \rangle \)

- \( \tau \)-independent plateau in all channels \( \Rightarrow \) conservation law
- Confirmation of fluctuation-response relations
- New method to measure \( c_V \)

- Similar result for \((41;41)\) channel: Borsanyi+, 2018
- Perturbative analysis: Eller, Moore, 2018
New measurement of $c_V$

\[
\langle T_{44}(\tau)T_{44}(0) \rangle \quad \langle T_{44}(\tau)T_{11}(0) \rangle \quad \langle T_{41}(\tau)T_{41}(0) \rangle
\]

Confirmation of FRR

\[
E + p = \frac{\langle T_{01}^2 \rangle}{VT} = \frac{\langle T_{11} T_{00} \rangle}{VT}
\]

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<thead>
<tr>
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</thead>
<tbody>
<tr>
<td>1.68</td>
<td>$17.7(8)(^{+0.2}_{-0.4})$</td>
<td>22.8(7)*</td>
<td>17.7</td>
<td>21.06</td>
<td></td>
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<tr>
<td>2.24</td>
<td>$17.5(0.8)(^{+0.0}_{-0.1})$</td>
<td>17.9(7)**</td>
<td>18.2</td>
<td>21.06</td>
<td></td>
</tr>
</tbody>
</table>

2+1 QCD:
Taniguchi+ (WHOT-QCD), 1711.02262
Successful analysis of energy-momentum tensor on the lattice is now available, and various studies are ongoing!
- gradient flow method
- higher-order perturbative coefficients

So many future studies
- Flux tube at nonzero temperature
- EMT distribution inside hadrons
- viscosity / other operators / instantons / full QCD
backup
Two Special Cases with PBC

\[ \frac{1}{T} \ll L_x = L_y = L_z \]

\[ T_{11} = T_{22} = T_{33} \]

\[ \frac{p_1}{p_2} = 1 \]

\[ \frac{1}{T} = L_x, \quad L_y = L_z \]

\[ T_{44} = T_{11}, \quad T_{22} = T_{33} \]

In conformal (\( \Sigma_\mu T_{\mu\mu} = 0 \))

\[ \frac{p_1}{p_2} = -1 \]
EMT on the Lattice: Conventional

Lattice EMT Operator  

\[ T_{\mu\nu} = Z_6 T_{\mu\nu}^{[6]} + Z_3 T_{\mu\nu}^{[3]} + Z_1 \left( T_{\mu\nu}^{[1]} - \langle T_{\mu\nu}^{[1]} \rangle \right) \]

- Fit to thermodynamics: \( Z_3, Z_1 \)
- Shifted-boundary method: \( Z_6, Z_3 \)

Multi-level algorithm

- effective in reducing statistical error of correlator  

Caracciolo+, 1990
Giusti, Meyer, 2011; 2013;
Giusti, Pepe, 2014~; Borsanyi+, 2018
Meyer, 2007;
Borsanyi, 2018;
Astrakhantsev+, 2018
Fermion Propagator

\[ S(t, x; s, y) = \langle \chi(t, x) \bar{\chi}(s, y) \rangle = \sum_{v,w} K(t, x; 0, v) S(v, w) K(s, y; 0, w)^\dagger \]

\[
\left( \partial_t - D_\mu D_\mu \right) K(t, x) = 0
\]

- propagator of flow equation
- Inverse propagator is needed
\[ N_f=2+1 \text{ QCD Thermodynamics} \]

- \( N_f=2+1 \) QCD, Iwasaki gauge + NP-clover
- \( m_{PS}/m_V \approx 0.63 \) / almost physical \( s \) quark mass

- \( T=0 \): CP-PACS+JLQCD (\( \beta=2.05, 28^3 \times 56, a\approx 0.07 \text{fm} \))
- \( T>0 \): \( 32^3 \times N_t, N_t = 4, 6, \ldots, 14, 16 \):
  - \( T \approx 174 \text{-} 697 \text{MeV} \)

- \( t \rightarrow 0 \) extrapolation only (No continuum limit)

\[
\begin{array}{c|c|c|c|c|c|c}
N_t & 16 & 14 & 12 & 10 & 8 & 6 \\
\hline
m_s/m_s & 0.6 \\
(\beta=2.05) & & & & & & \\
\end{array}
\]

\[
T_{pc} \approx 190 \text{ MeV}
\]