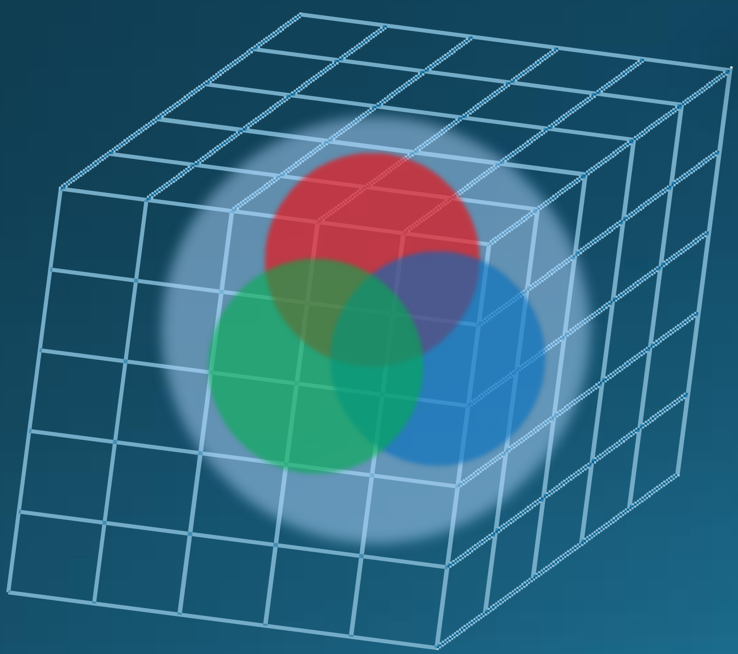
Nuclear Physics School (NPS2022), Busan, Korea, 2022/June/28

Lattice QCD and Physics at $T \neq 0$



Masakiyo Kitazawa
(Osaka U.)

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1. Why is Lattice so Difficult?
   1. Lattice field theory
   2. Observables
   3. Monte-Carlo simulations
   4. Nonzero temperature
   5. Dynamics

2. QCD at $T \neq 0$
   1. Equation of state
   2. QCD critical points & Columbia plot
   3. Gradient flow & energy-momentum tensor

QCD

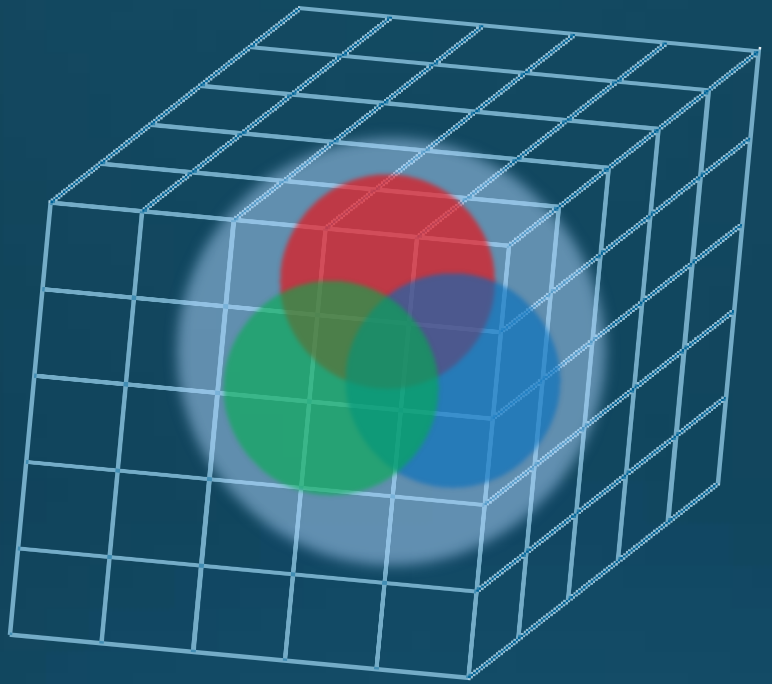
Fundamental Theory of Strong Interaction

$$\mathcal{L} = \bar{\psi}(i\not{D} - m)\psi - \frac{1}{4}F_{\mu\nu,a}F_a^{\mu\nu}$$

- Degrees of freedom
  - ψ: quark field
  - A_μ^a: gluon field

Properties

- **Asymptotic freedom**
  - ➔ Energy-scale dependent coupling constant
  - ➔ Violation of perturbation at low E scale
- Quark confinement
- Chiral symmetry breaking

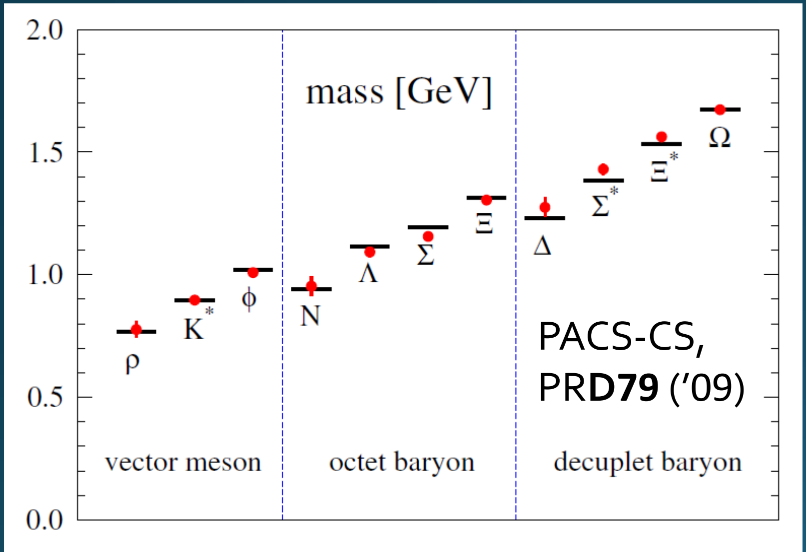


Lattice QCD numerical simulations are powerful tools to explore non-perturbative phenomena of QCD.

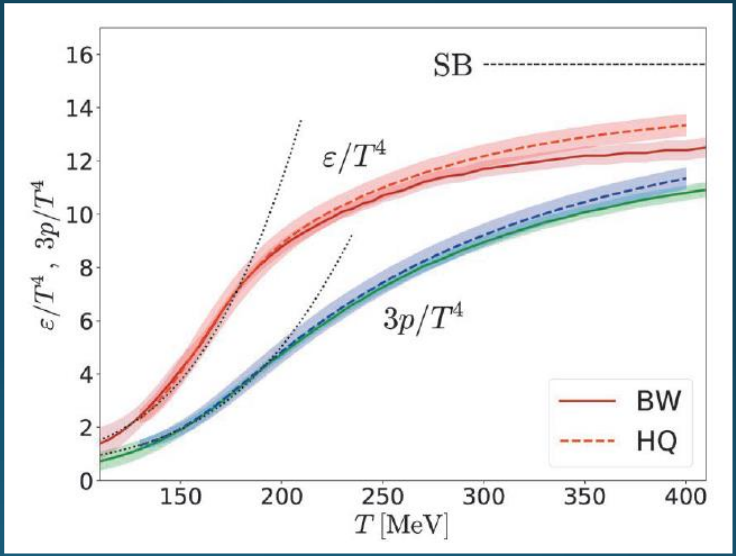
Yes, but it is not so useful…

Establishments

Hadron Spectroscopy

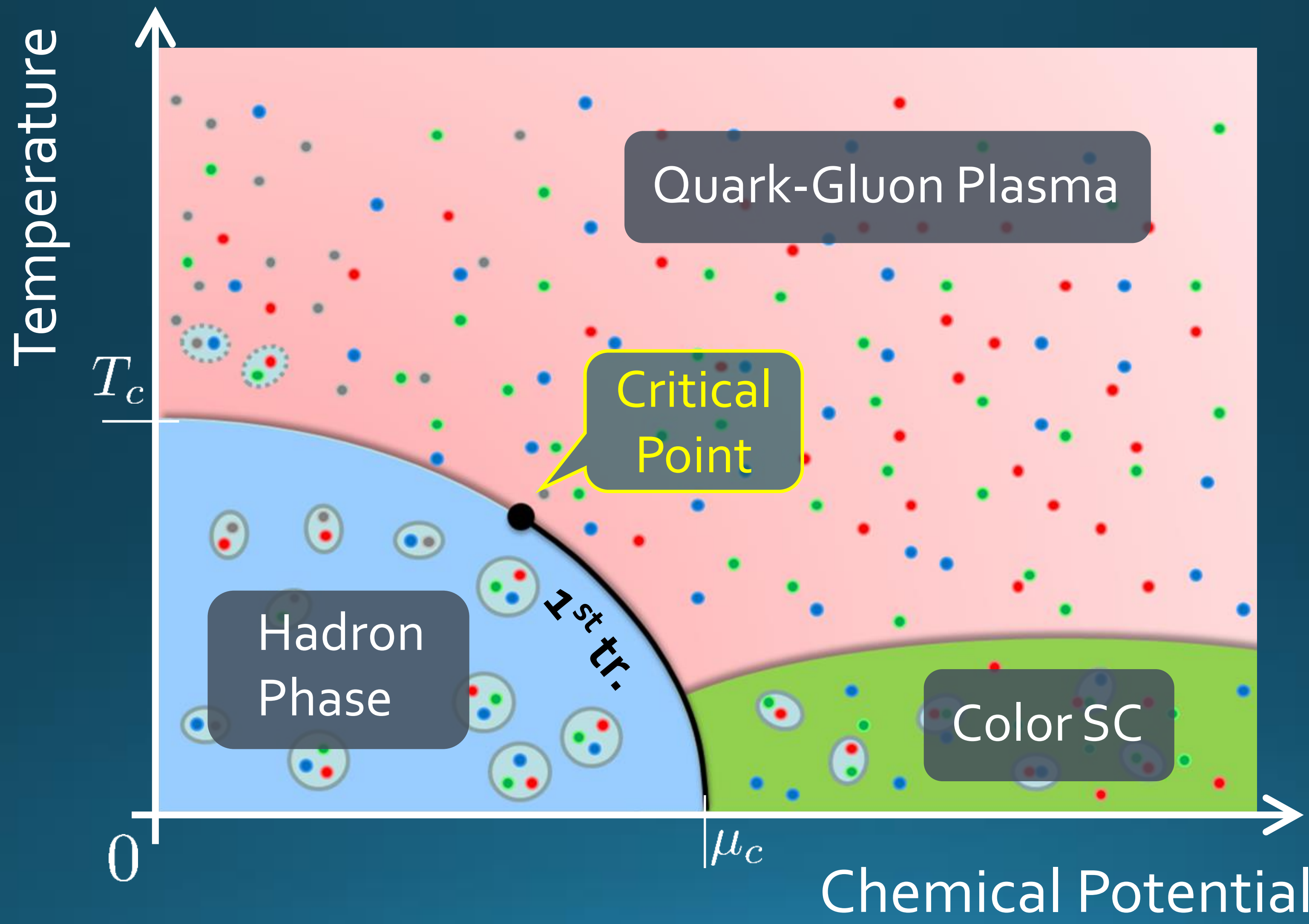


Thermodynamics

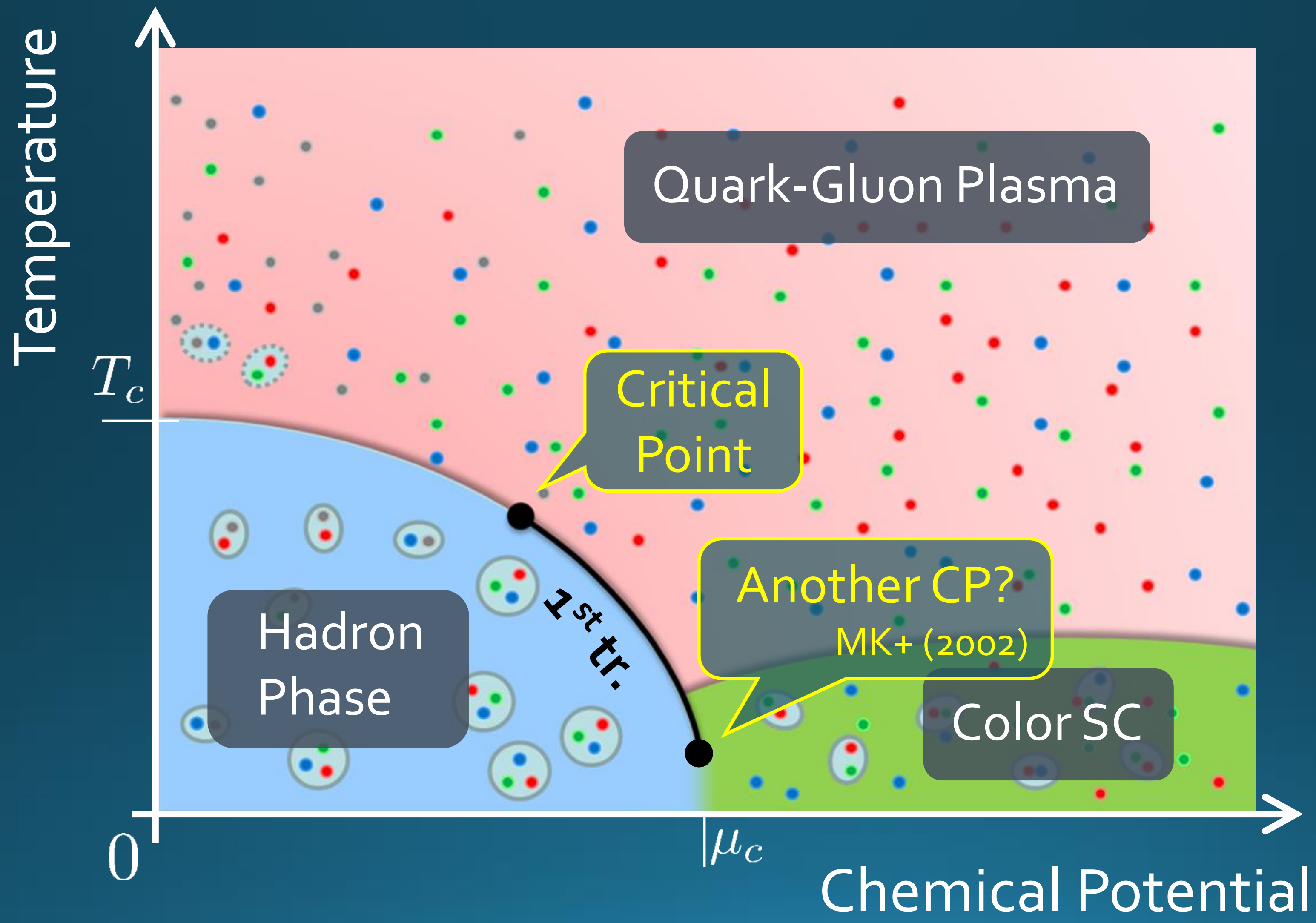


Budapest-Wuppertal; HotQCD, 2014

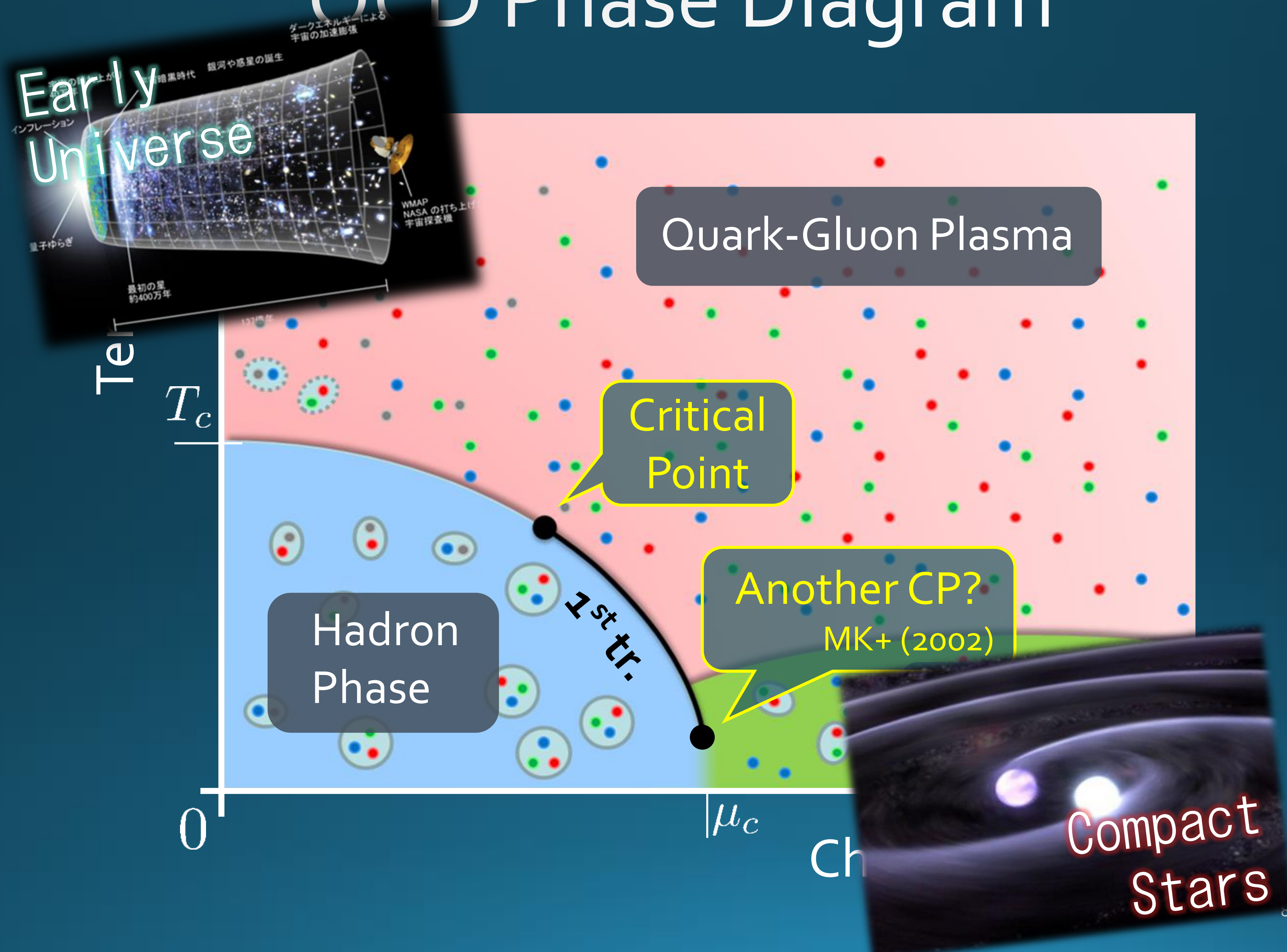
QCD Phase Diagram



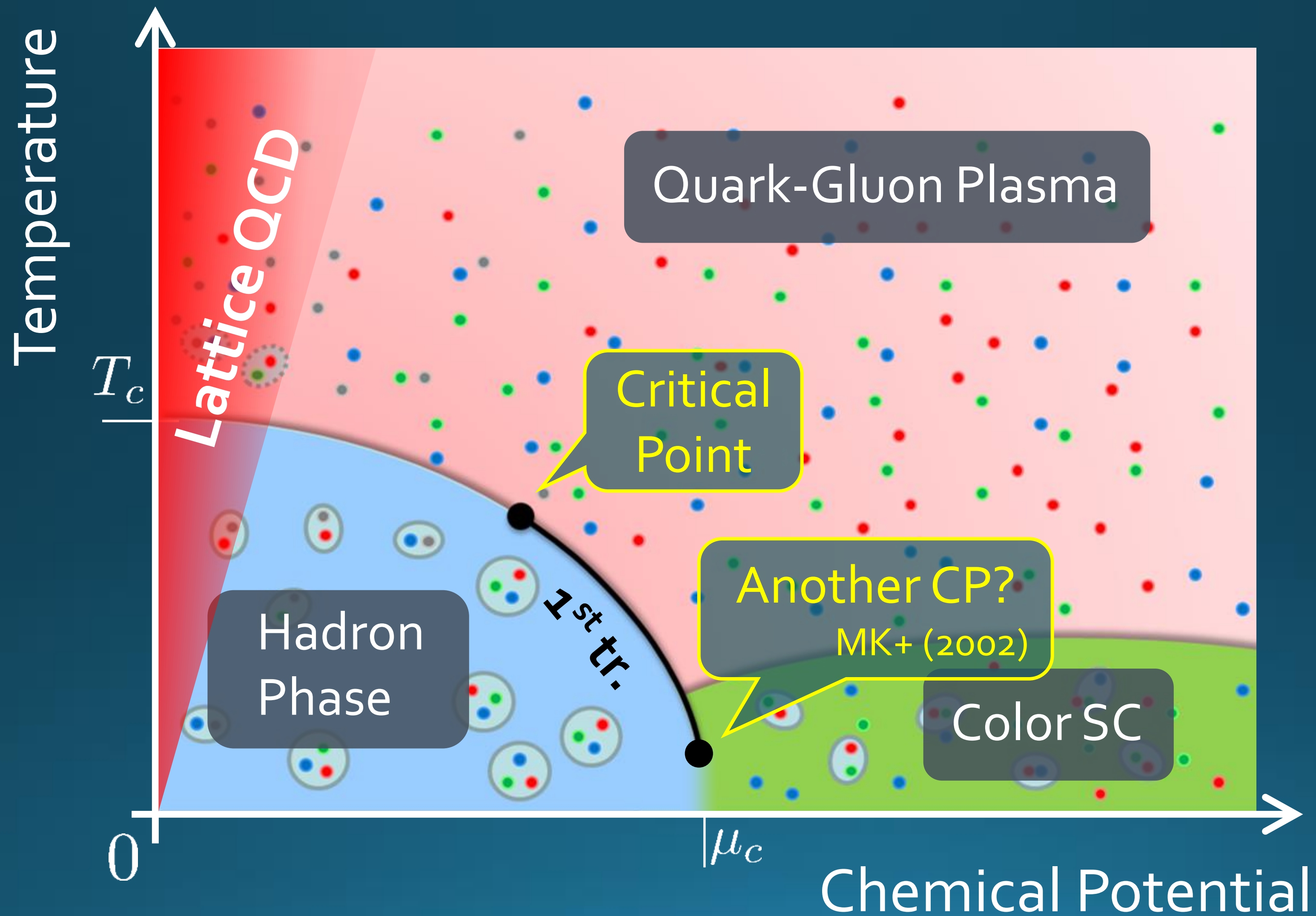
QCD Phase Diagram



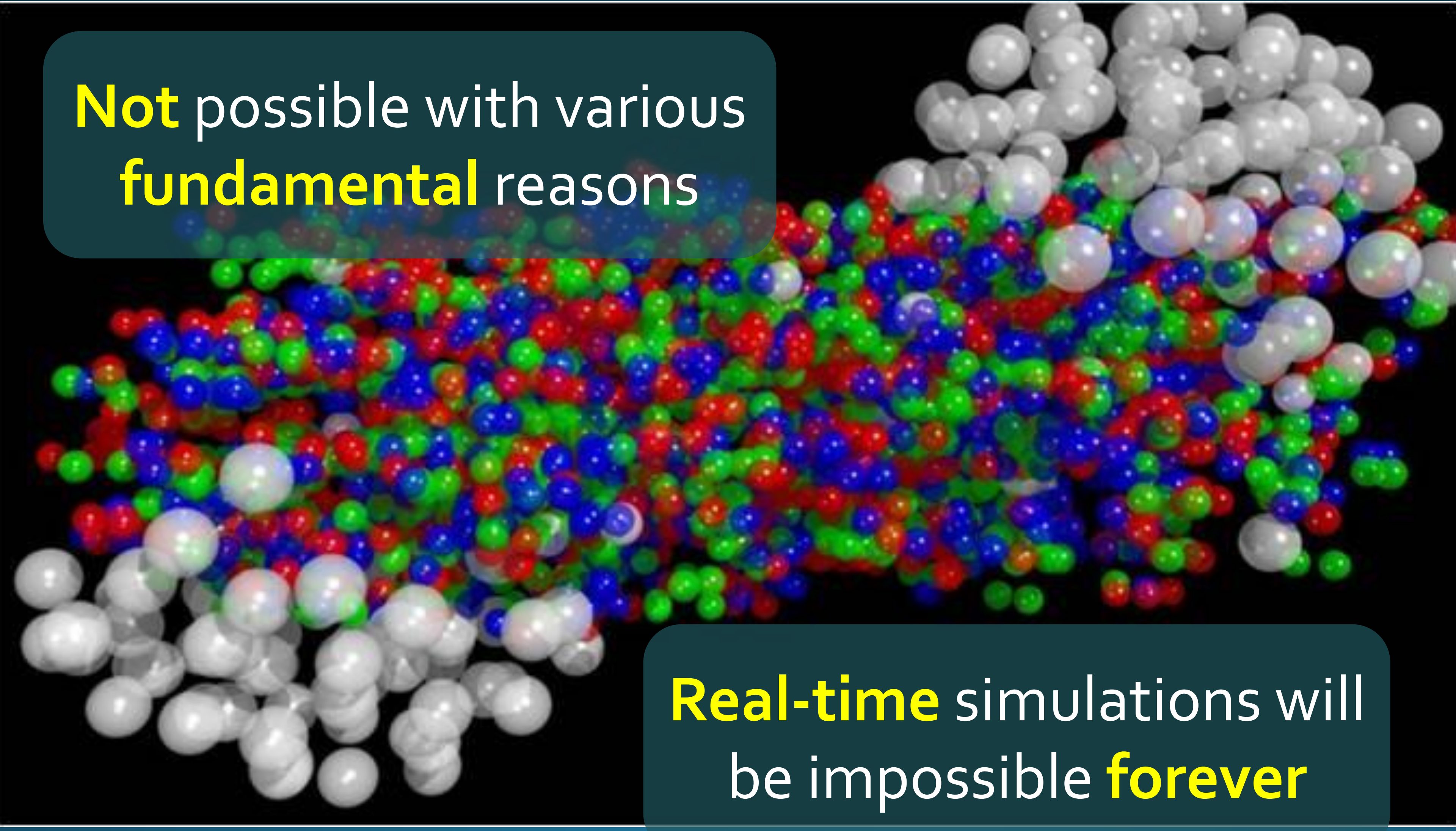
QCD Phase Diagram



QCD Phase Diagram



Reproducing HIC on the lattice?



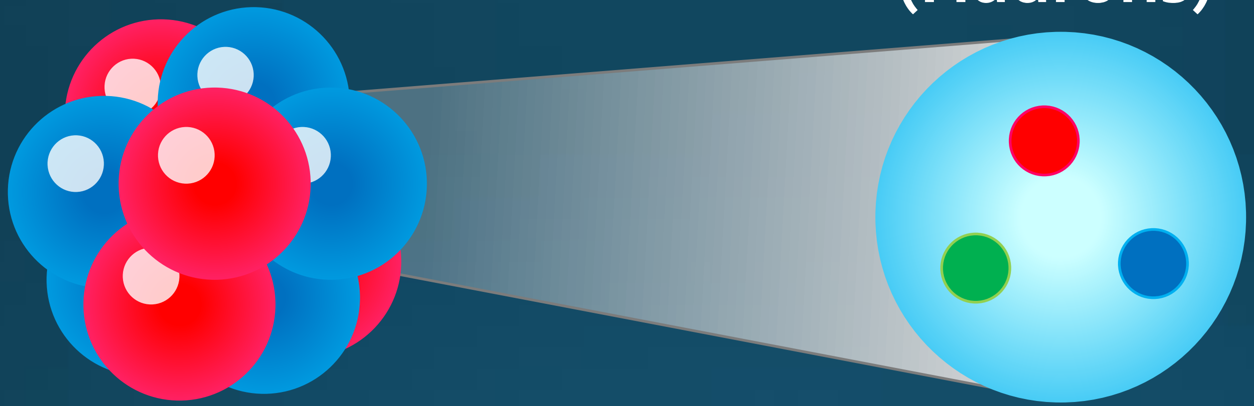
Not possible with various **fundamental** reasons

Real-time simulations will be impossible **forever**

Nuclear Structure on the Lattice?

Nucleus

Nucleon (Hadrons)

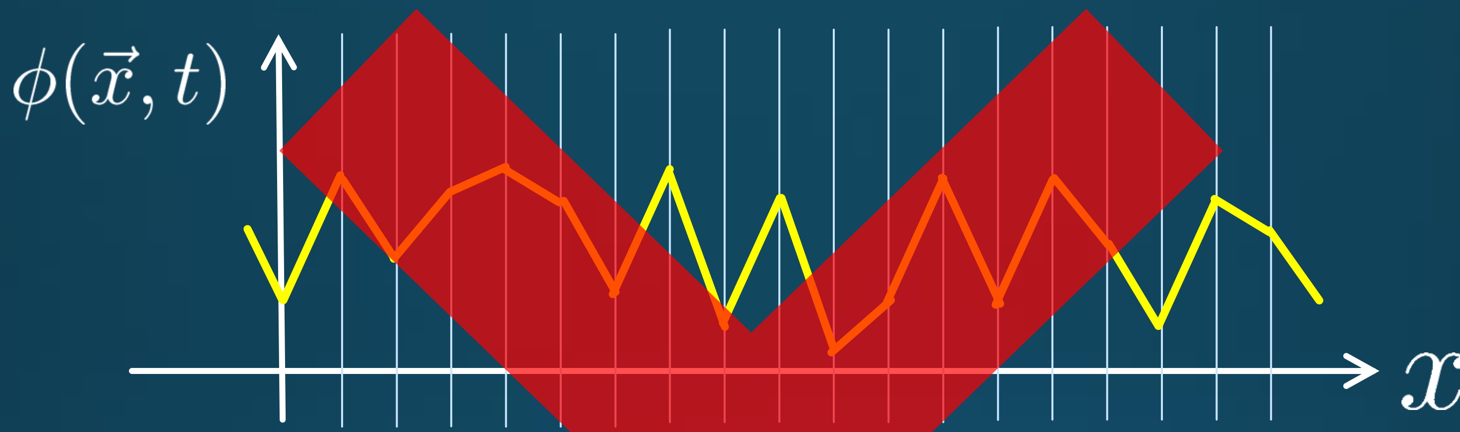


- Difficult to treat.
- Even a reliable measurement of deuteron mass has not been achieved.

- Masses have been measured.
- Other properties, such as charge distribution, are still difficult to measure.

Contents

Lattice Simulations



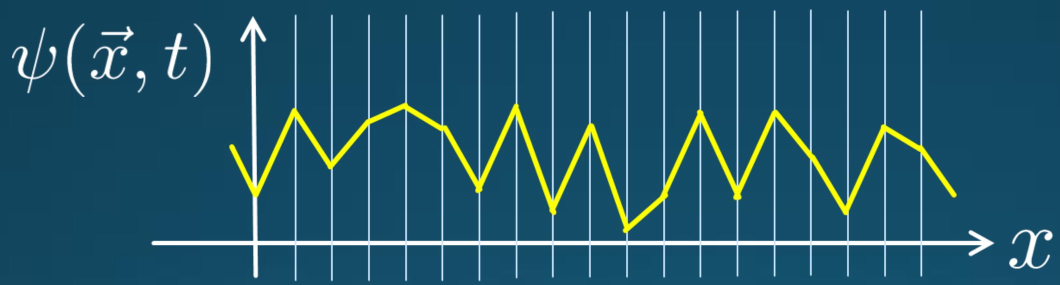
Diffusion eq.: $$\frac{\partial}{\partial t}\phi(\vec{x},t) = D\nabla^2\phi(\vec{x},t)$$

$$\frac{d^2\phi(x)}{dx^2} = \frac{1}{a^2}\{\phi(x-a) - 2\phi(x) + \phi(x+a)\}$$

$$\phi(x+\Delta t) = \phi(x) + \Delta t D\frac{d^2\phi(x)}{dx^2}$$

QCD is a Quantum Theory

$$i\frac{\partial}{\partial t}\psi(\vec{x},t) = -\frac{\hbar^2\nabla^2}{2m}\psi(\vec{x},t) + V(x)\psi(\vec{x},t)$$



Time evolution can be simulated,
but the eigenvalue problem would be better.

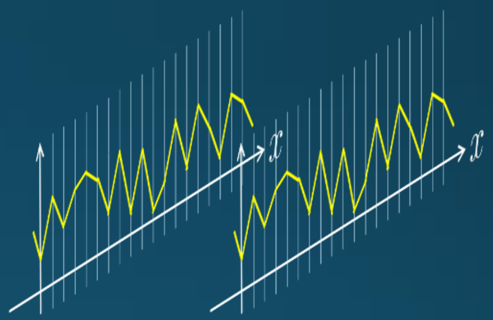
QCD is a Quantum Field Theory

Quantum Field Theory

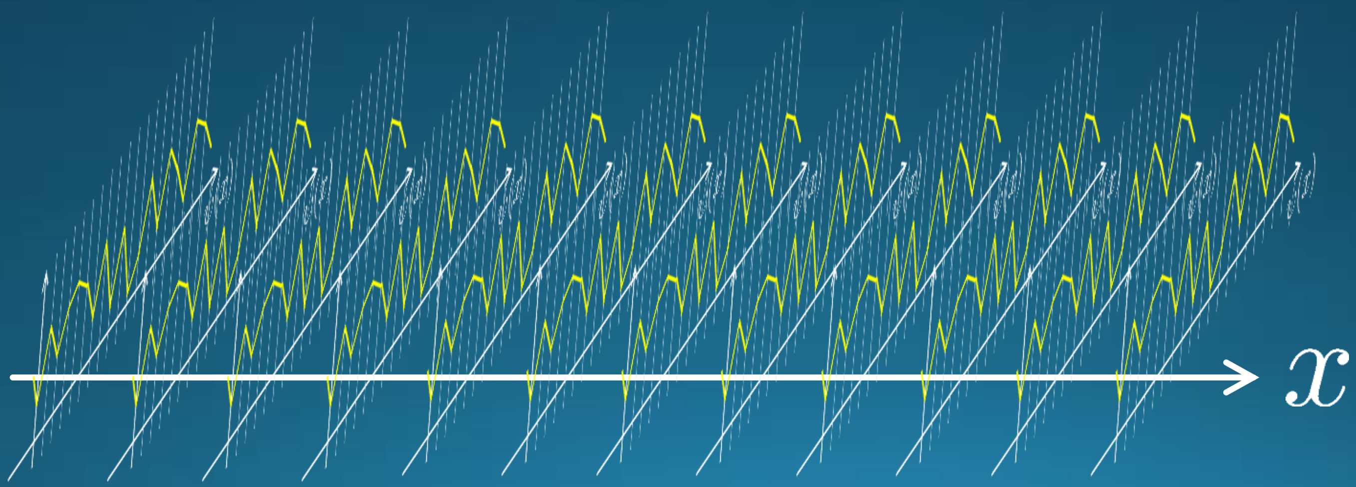
ϕ(x) at every space-time points are arguments of wave func.

Spin ½ system:

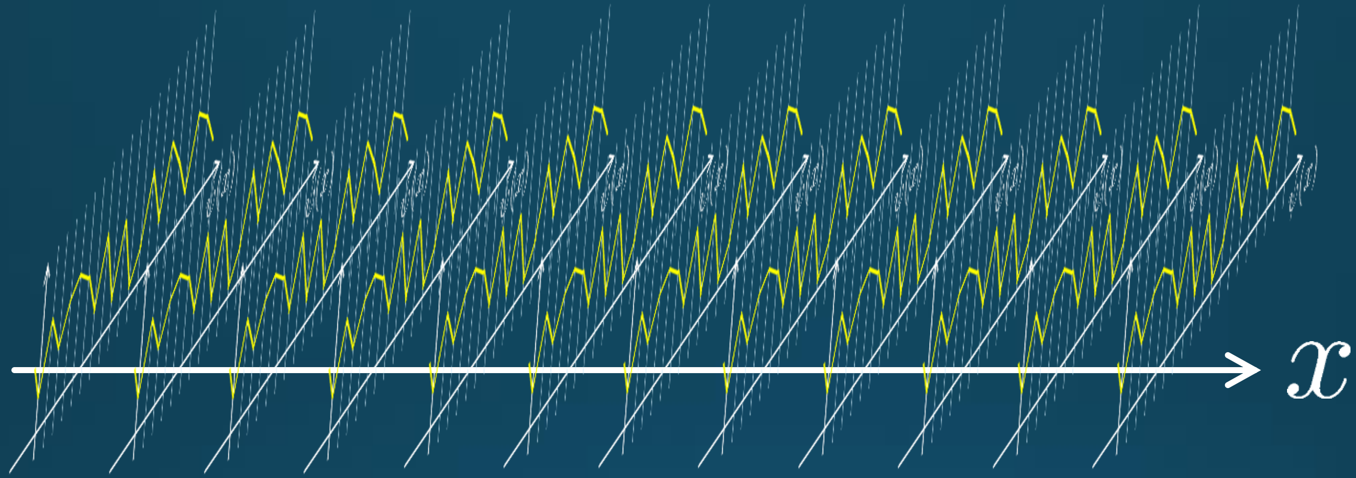
$$\Psi(x) = (\psi_{\uparrow}(x), \psi_{\downarrow}(x))$$



QFT:



Physical States



$$\Psi[\psi(x)]$$

Functional of ψ
So many d.o.f

Numerical simulation of time evolution
is too difficult to handle!

Initial Conditions

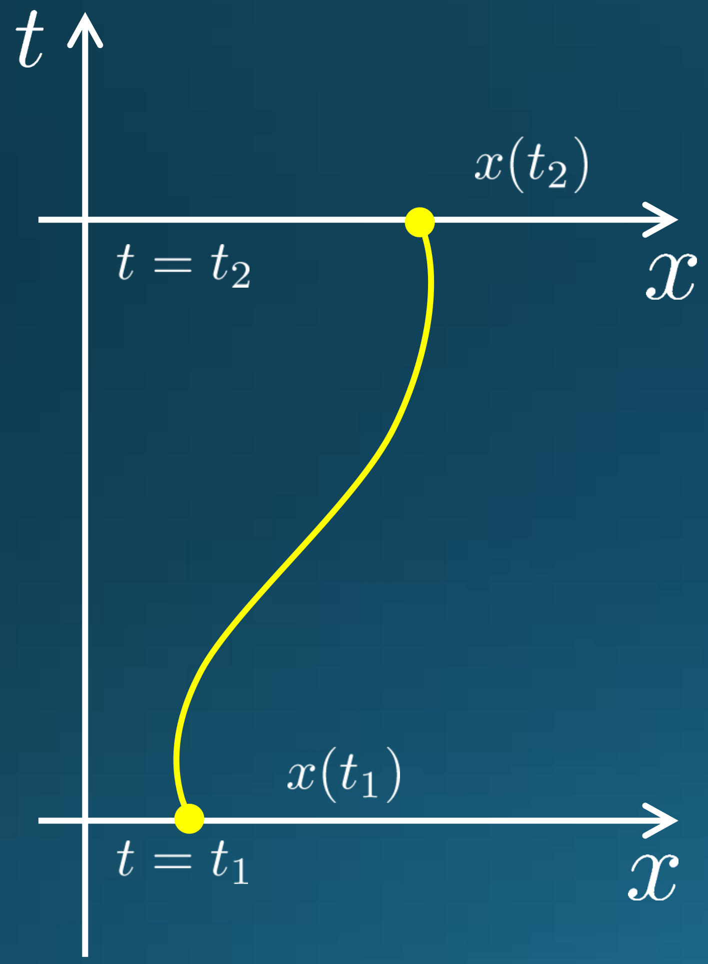
Initial conditions having physical meaning?

- Vacuum $|0\rangle$
- 1-particle state $a_p^\dagger |0\rangle$
- 2-particle state $a_{p_1}^\dagger c_{p_2}^\dagger |0\rangle$

$|0\rangle$ Vacuum state: **unknown**

$a_p^\dagger$ Creation operators: **unknown**

Path Integral

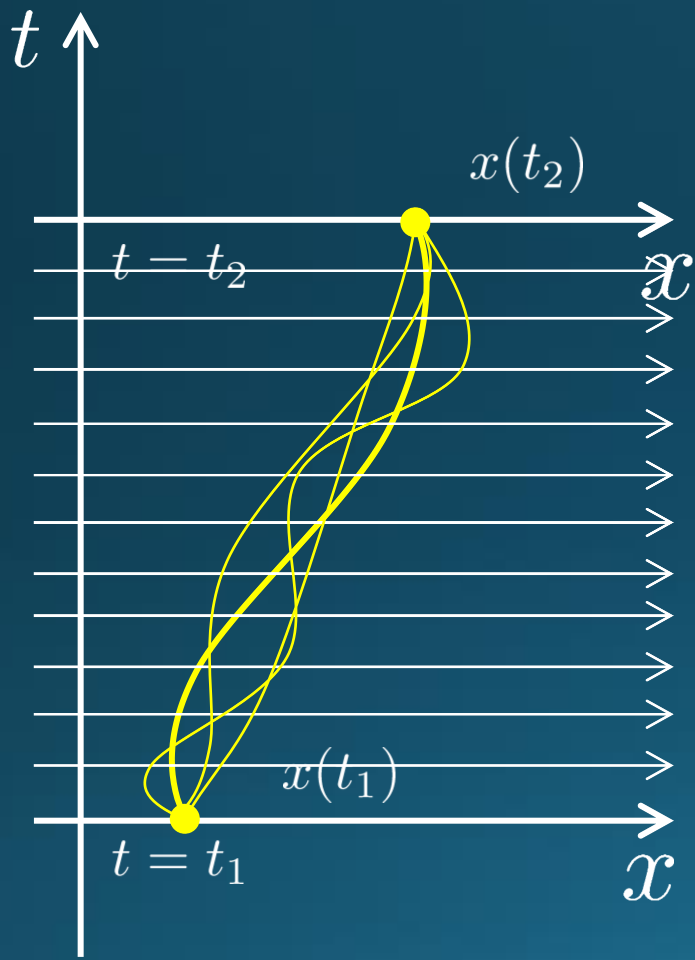


Classical mechanics:
Principle of least action

Trajectory that minimize the action S is realized as a classical path between x_1 and x_2.

$$S[x(t)] = \int_{t_1}^{t_2} dt \mathcal{L}(x(t), \dot{x}(t))$$

Path Integral



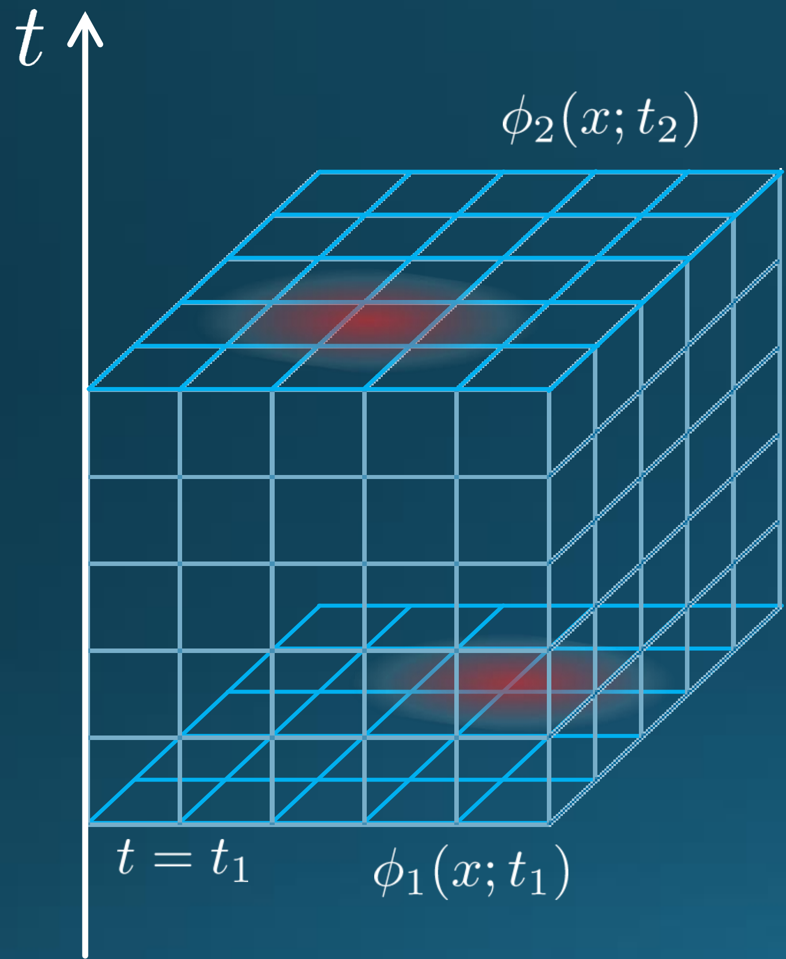
Quantum mechanics:
Path integral

Transition amplitude $\langle x_1, t_1 | x_2, t_2 \rangle$ is given by the sum of all trajectories with the weight e^{iS}.

$$\langle x_2, t_2 | x_1, t_1 \rangle$$
$$= \lim_{\Delta t \to 0} \left[\prod_n \int dx(t_n) \right] e^{iS[x(t)]/\hbar}$$
$$= \int \mathcal{D}x e^{iS/\hbar}$$

Note: QM states are labeled only by the coordinate x.

Path Integral in QFT



Transition amplitude between two states can be calculated as

$$\langle \phi_2(x), t_2 | \phi_1(x), t_1 \rangle$$
$$= \lim_{a \to 0} \left[\prod_x \int d\phi(x) \right] e^{iS[\phi(x)]/\hbar}$$
$$= \int \mathcal{D}\phi e^{iS(\phi)/\hbar}$$

Lattice field theory is constructed by the space-time discretization

Problems: ①What are physical states?
②How to carry out path integral numerically?

Problems

① Quantum states

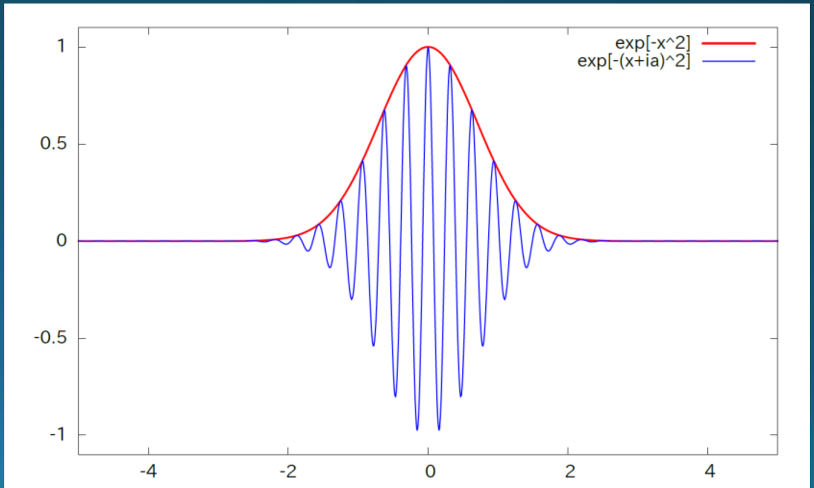
- QM: $\langle x_2, t_2 | x_1, t_1 \rangle$: Not very useful…
- QFT: We don't know meaningful quantum states

$|\phi(x)\rangle$?

② Numerical Integration

$$\lim_{\Delta t \to 0} \left[\prod_n \int dx(t_n) \right] e^{iS[x(t)]/\hbar}$$

The phase oscillates rapidly.
➔Difficult to handle in numerical integration



Solution: Wick Rotation
$(t \to \tau = -it)$

□ Minkowski → Euclid spacetime

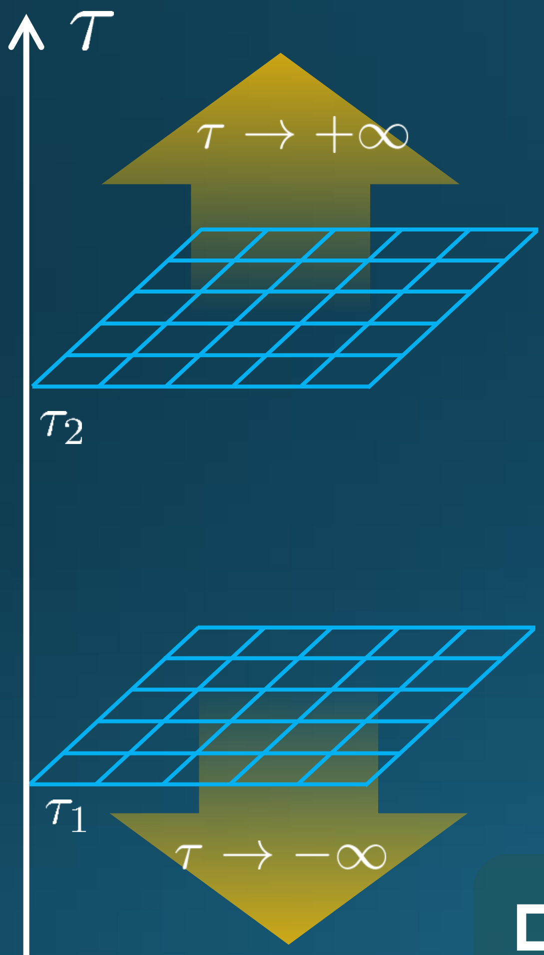
□ $S[x(t)] = \int_{t_1}^{t_2} dt \mathcal{L}(x(t), \dot{x}(t))$

$$\Rightarrow S_{\mathrm{E}}[x(\tau)] = \int_{\tau_1}^{\tau_2} d\tau \mathcal{L}_{\mathrm{E}}(x, \dot{x})$$

□ $\int \mathcal{D}x e^{iS[x(t)]/\hbar} \Rightarrow \int \mathcal{D}x e^{-S_{\mathrm{E}}[x(\tau)]/\hbar}$

Integrand becomes real → Numerically feasible

Solution: Wick Rotation
$(t \to \tau = -it)$



☐ Vacuum expectation value

Take the limit: $\tau_1 \to -\infty, \ \tau_2 \to \infty$

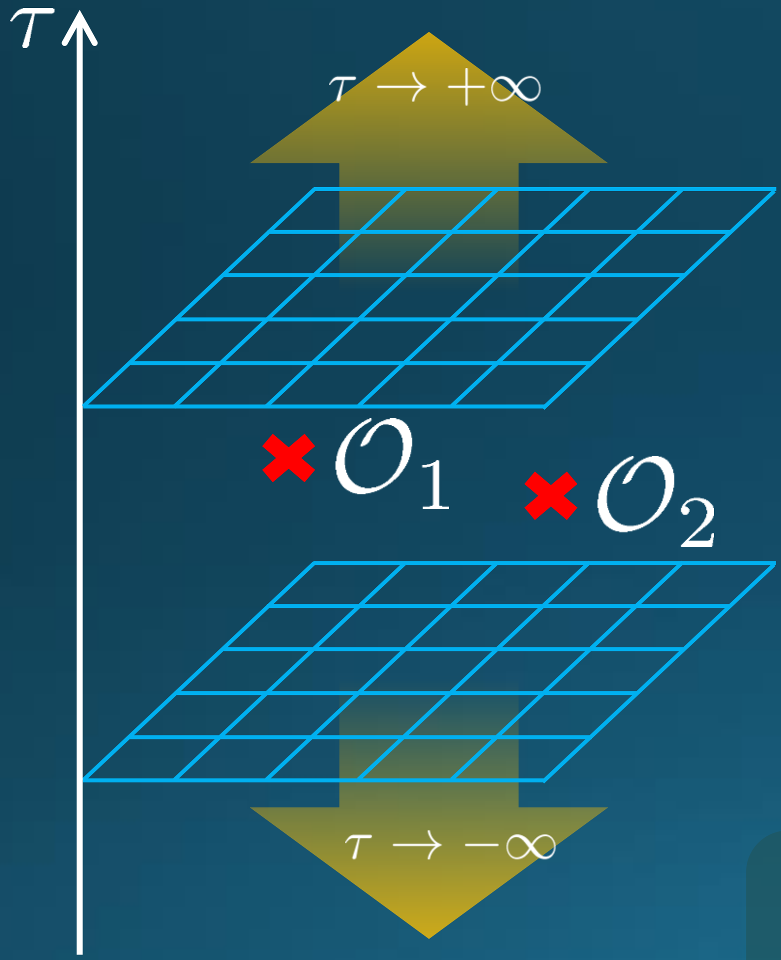
$$\int \mathcal{D}x e^{-\int_{-\tau_1}^{0} d\tau L[x(\tau)]} \sim e^{-H\tau_1}|x\rangle \xrightarrow[\tau_1 \to \infty]{} |0\rangle$$

vacuum state

$$\langle 0|f(\hat{x})|0\rangle \sim \int_{-\infty}^{\infty} \mathcal{D}x f(x)_{\tau=0} e^{-S/\hbar}$$

- ☐ Expectation values w.r.t. $|0\rangle$ can be evaluated!
- ☐ Note: periodic BC is also possible.

Calculating Operators



Lattice Simulations can calculate vacuum expectation values and correlation funcs.

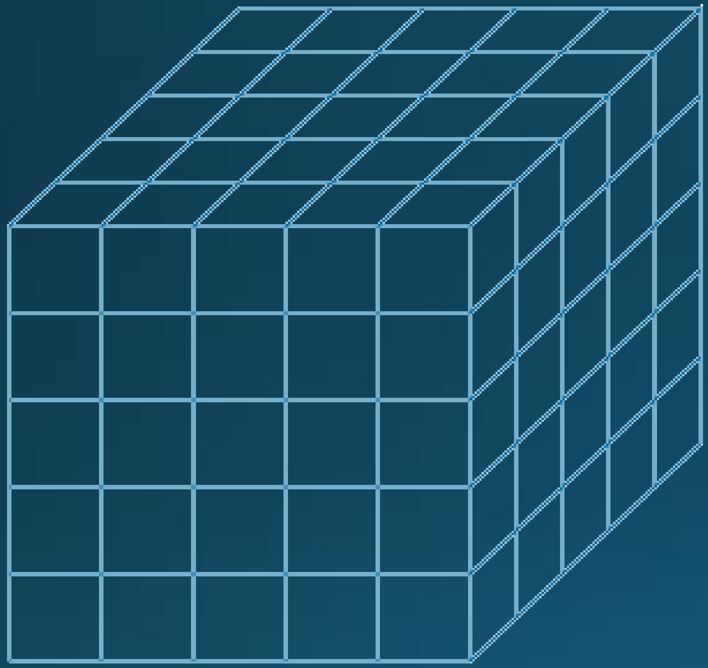
$$\langle 0|\mathcal{O}(x)|0\rangle$$

$$\langle 0|\mathcal{O}_1(x)\mathcal{O}_2(y)|0\rangle$$

$$\cdots$$

These are **almost everything** that lattice simulations can do.

Plane-Wave Solution of QCD?



$$\langle 0|\mathcal{O}(x)|0\rangle = \frac{1}{Z}\int \mathcal{D}\phi\,\mathcal{O}(x)e^{-S_E}$$

Q.
Are states having translational symmetry (such as plane waves) of QCD analyzed in lower dimensional simulations?
Then, such a simulation will reduce numerical costs drastically.

A.
No. Gauge configurations are not translationally symmetric.

General Comments

- Another advantage of lattice FT: removal of ultraviolet divergence thanks to finite d.o.f. on the lattice.
- Lattice provides us with a **non-perturbative construction of the QFT**.
- Continuum extrapolation ($a \rightarrow 0$ limit) must be taken at the end.

- Numerical simulations were not the original purpose of introducing lattice gauge theory by K. Wilson.

Summary so far

- A real-time simulation of QFT is quite difficult.
- Ignorance of physical states is one of the reasons.

- Lattice FT in Euclidean spacetime enables
  - Stable numerical integral.
    - ← real integrand of path integral.
  - Calculation of vacuum expectation values.

- Lattice calculates vacuum expectation values (correlation functions / Green functions).

$$\langle 0|\phi(x_1)\phi(x_2)|0\rangle, \quad \cdots$$

- Physical information are extracted from them.

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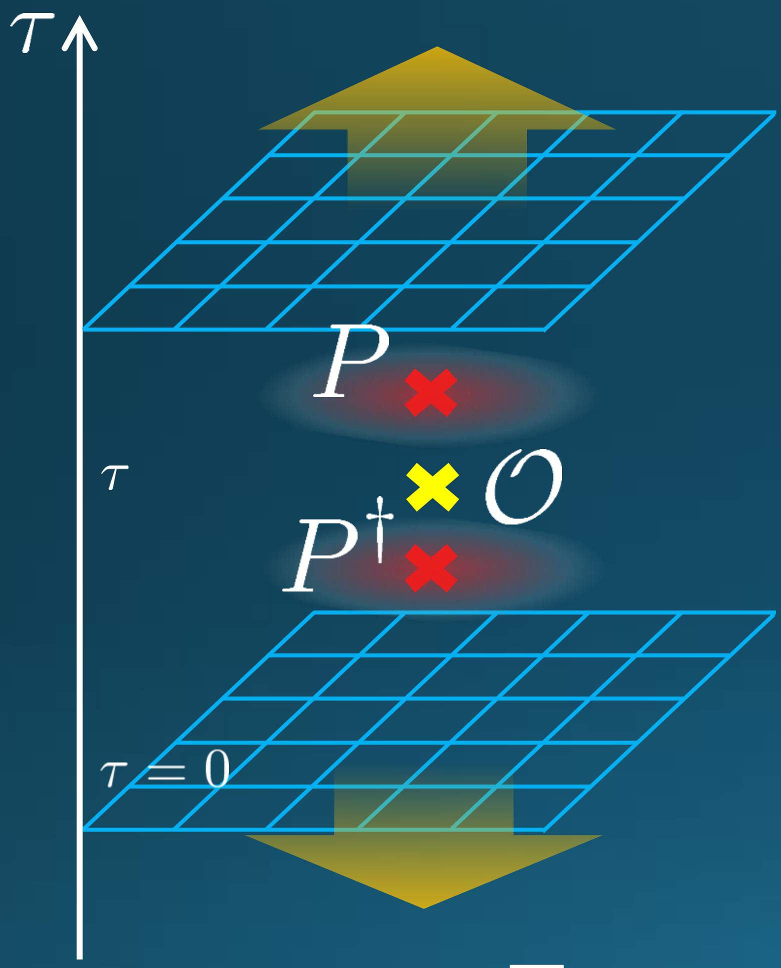
Expectation Value of Physical States



Pion creation operator: $P^\dagger(p=0,\tau)$

➡ 1π state: $|\pi\rangle = P^\dagger|0\rangle$

- **Mass**

$$\langle 0|P(\tau)P^\dagger(0)|0\rangle$$
$$= \langle \pi(\tau)|\pi(0)\rangle \sim e^{-m_\pi \tau}$$

- **Charge density**

$$\lim_{\tau\to 0}\langle \pi(\vec{0},\tau)|\hat{\rho}(\vec{x})|\pi(\vec{0},0)\rangle$$

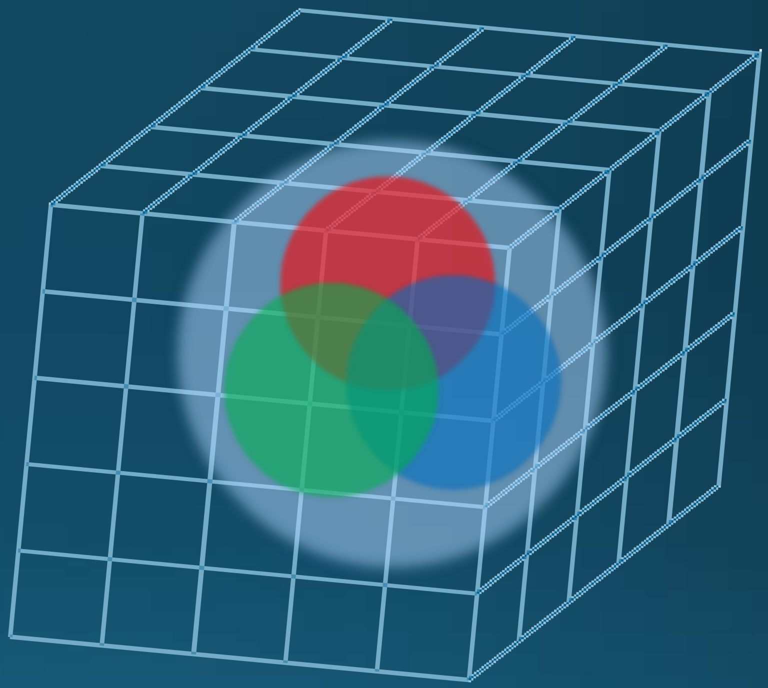
- **Energy density**

$$\langle \pi(\tau)|T_{00}(x)|\pi(0)\rangle \Rightarrow \int d^3x \langle T_{00}(x)\rangle = m_\pi$$

No Operators of Hadrons!!

- We cannot represent hadrons in terms of quark and gluon fields.

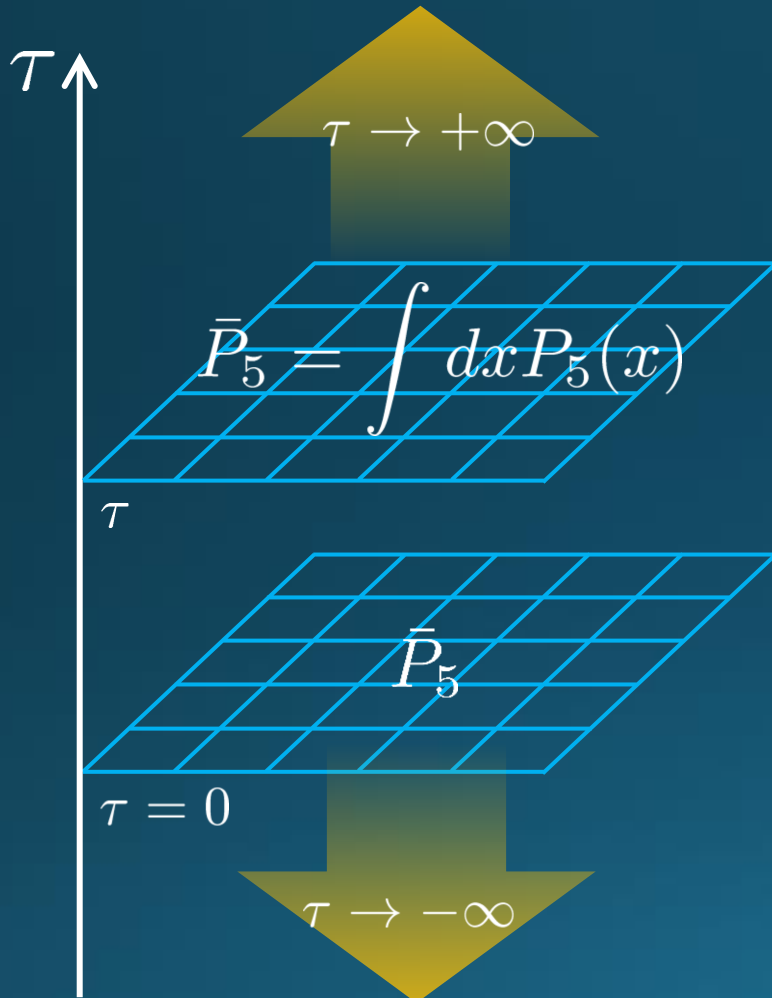
  ➡ We don't know their operators in QCD.



- Constructing operators of observables is also nontrivial.

  Ex. energy-momentum tensor
  ➔cannot be defined as Noether current
  (Recent progress: gradient flow mothod)

How to Create Hadrons on the Lattice?



Use an operator having the same quantum number as poins; ex.:

$$P_5(x) = \bar{\psi}(x)\gamma_5\psi(x)$$

$$P_5(-\tau)|0\rangle$$

$$= c_0 e^{-\tau m_\pi}|\pi\rangle + c_1 e^{-\tau m'_\pi}|\pi'\rangle + \cdots$$

$\tau \to \infty$ limit: $|\bar{P}_5\rangle \sim |\pi\rangle$

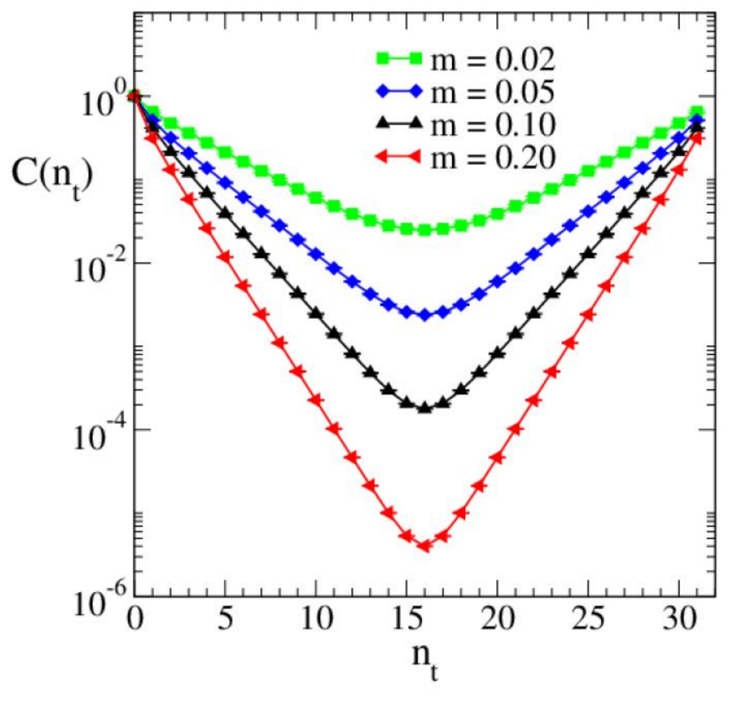
$$\langle \bar{P}_5(\tau)|\bar{P}_5(0)\rangle \to e^{-m_\pi \tau}$$

Evaluation of the lowest energy eigenvalue

Correlation Functions: Example

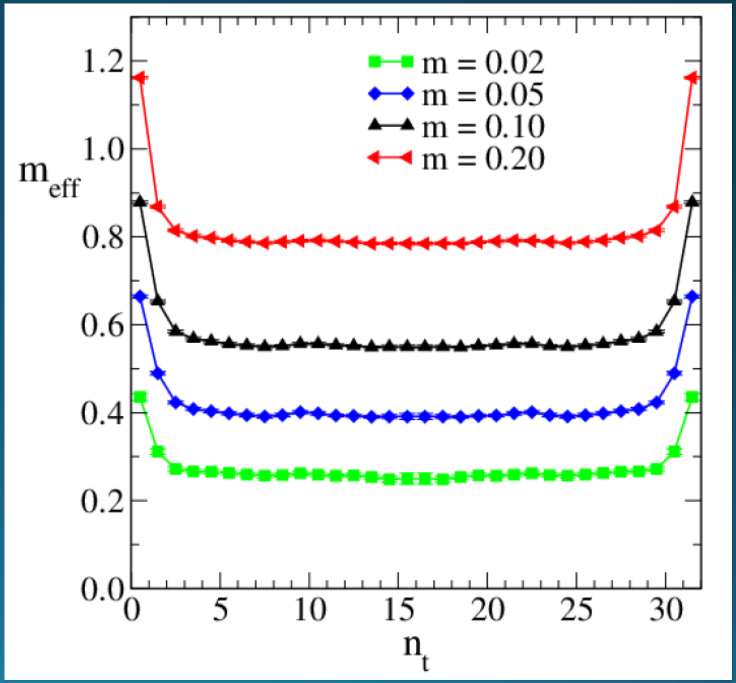
$$C(\tau) = \langle \bar{P}(\tau) | \bar{P}(0) \rangle \to e^{-m\tau}$$

Figs from C.B. Lang
http://physik.uni-graz.at/~cbl/teaching/lgtped_c.pdf



- Effective-mass Plot

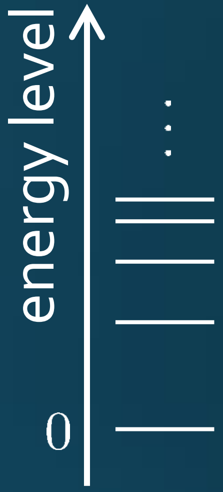
$$m_{\mathrm{eff}} = \ln \frac{C(n)}{C(n+1)}$$

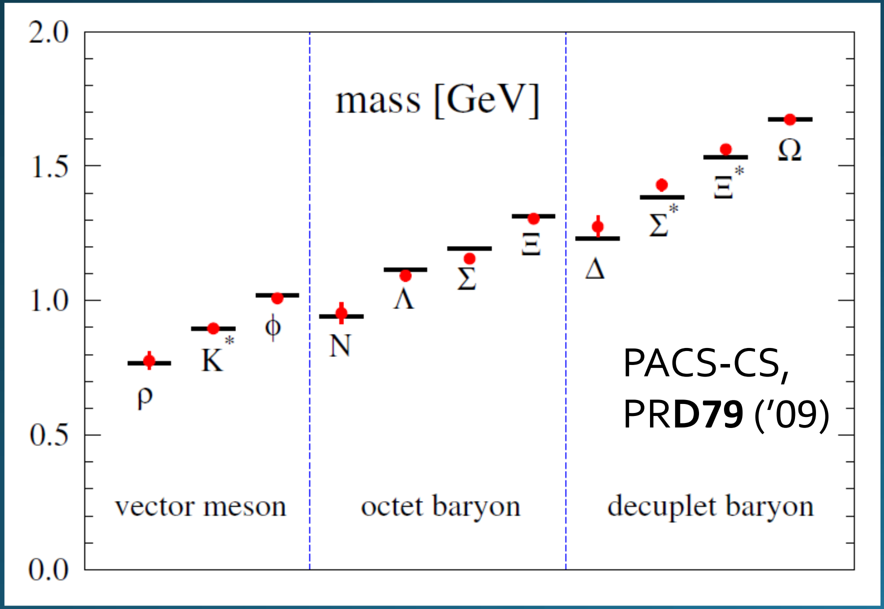


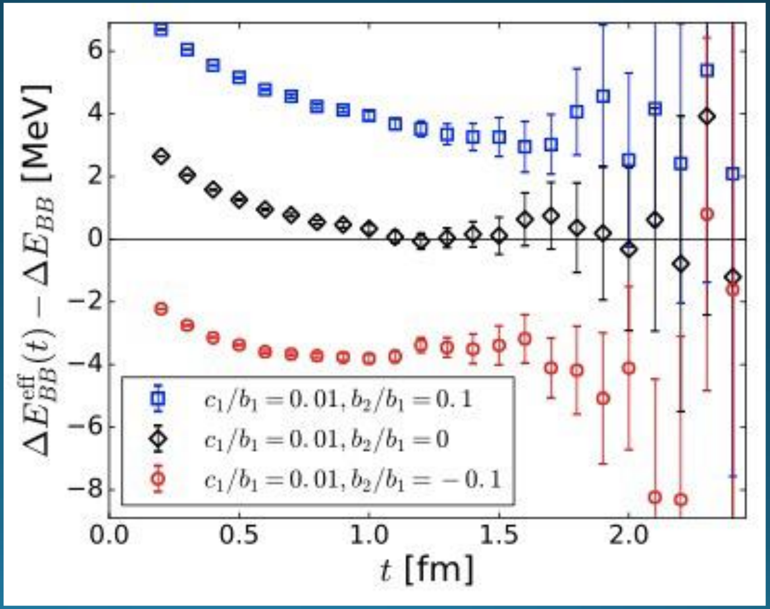
Mass of hadrons are obtained from the plateau of effective mass

Caveats

- Successful analysis only for the lowest-energy state.
- More sophisticated treatment is required for
  - Excited states.
  - Systems with small energy gaps: ex. multi-hadron states, etc.
- The “plateau” region should be determined carefully.

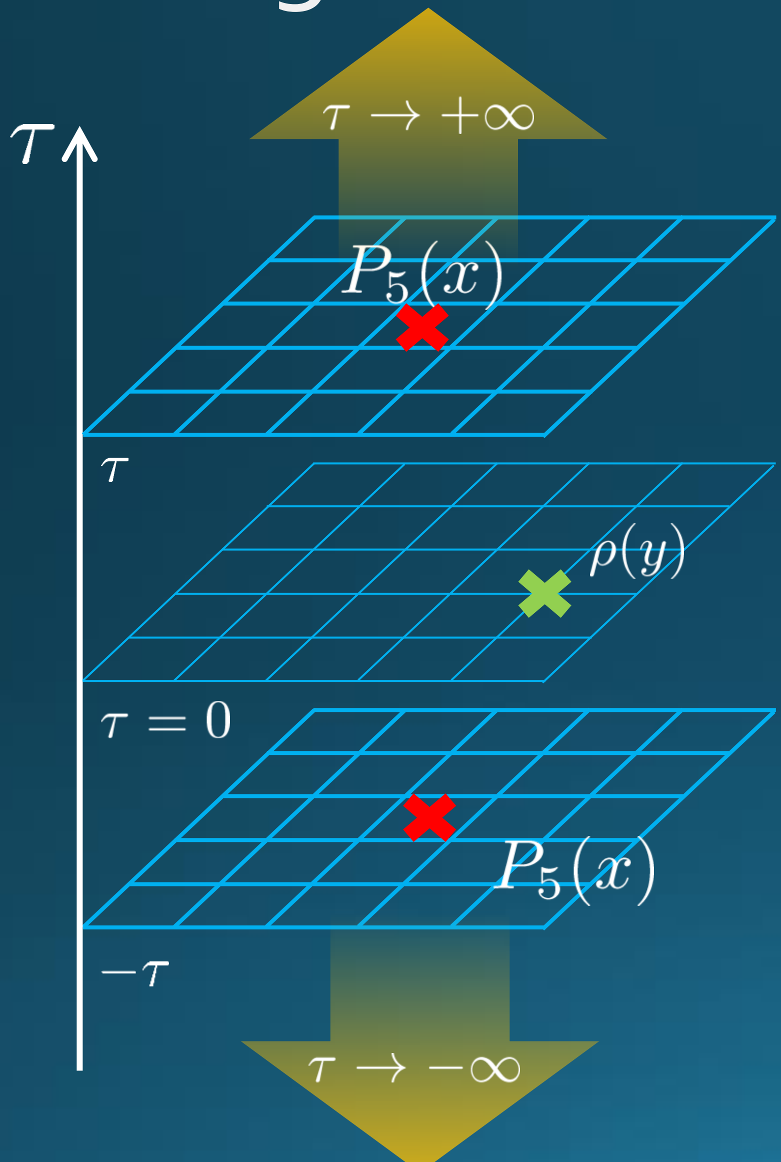






HAL-QCD Collab. 2016

Charge Distribution inside Hadrons?



$$\langle 0|P_5(\vec{x},\tau)\rho(\vec{y},0)P_5(\vec{x},-\tau)|0\rangle$$

Charge distribution & radius?

✕ The hadron state is not the eigenstate of coordinate x.

A hadron at position x cannot be created on the lattice.

○ form factor: $\langle \pi(\vec{p}_1)|V_\mu(\vec{q})|\pi(\vec{p}_2)\rangle$

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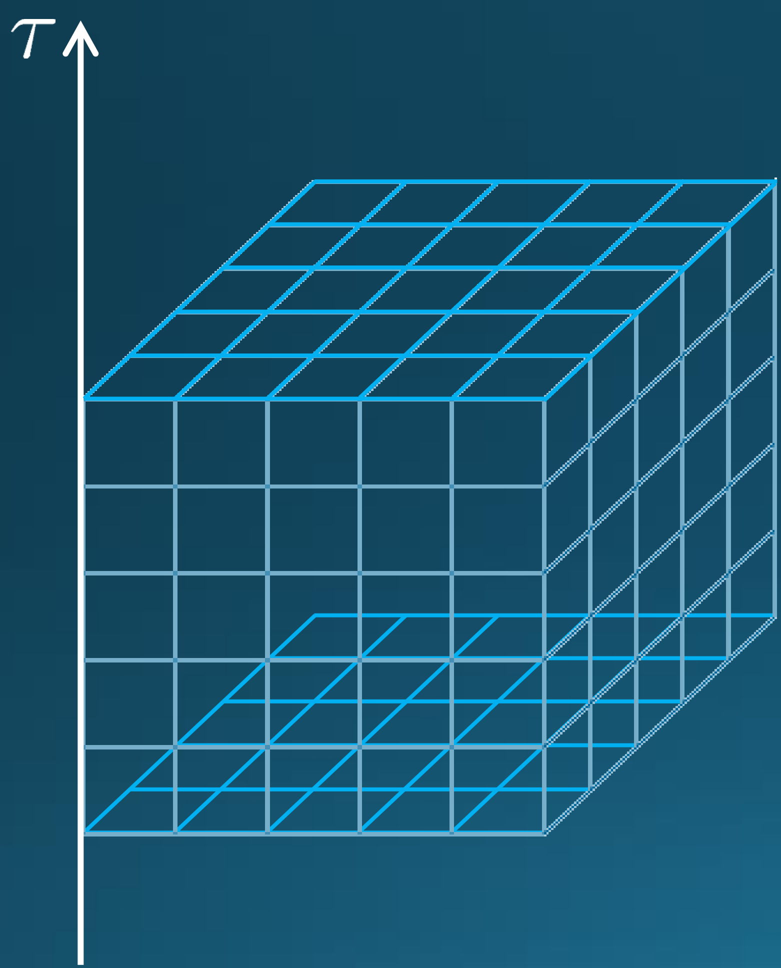
DoF of Path Integral



$$\int \mathcal{D}\phi \mathcal{O} e^{-S[\phi(x)]}$$

$$= \left[\prod_x \int d\phi(x) \right] \mathcal{O} e^{-S[\phi(x)]}$$

(integration variable) =
(spacetime points) × (dof of fields)

Multiple Integral
in **Ultra-high** Dimensions!!

Monte-Carlo Integral

integral space

- Monte-Carlo Integral

  Evaluate integrand randomly in the integral space
  ➔ Take the average

$$\int dx^m F(\vec{x}) \simeq \frac{1}{N} \sum_i F(\vec{x}_i)$$

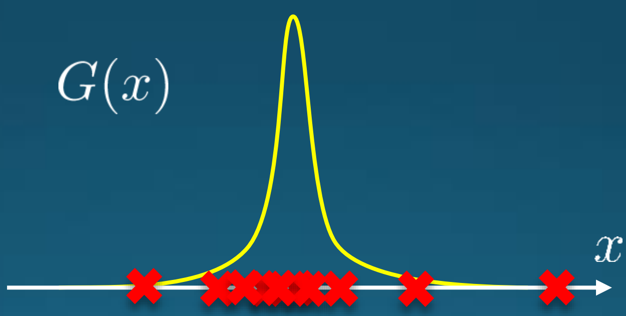
Importance Sampling

- Metropolis Method

If only a part of integral space contribute strongly to the integral:

$$\int dx^m F(\vec{x})G(\vec{x})$$

$G(\vec{x})$:weight func.



Generate the sampling points with the probability $G(x)$

Importance Sampling

integral space

☐ Metropolis Method

If only a part of integral space contribute strongly to the integral:

$$\int dx^m F(\vec{x}) G(\vec{x})$$

$G(\vec{x})$:weight func.

Acceptance/rejection of integrand

$$\begin{cases} G(\vec{x}_{i+1}) \leq G(\vec{x}_i) & \text{accept!} \\ G(\vec{x}_{i+1}) > G(\vec{x}_i) & \text{accept with the probability } G_i/G_{i+1} \end{cases}$$

$$\int dx^m F(\vec{x}) G(\vec{x}) = \frac{1}{N} \sum_{\vec{x}_i} F(\vec{x}_i)$$

Path Integral in QFT

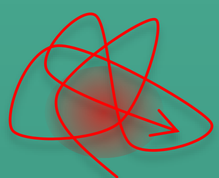
$$\int \mathcal{D}\phi \mathcal{O} e^{-S[\phi(x)]}$$

"Hot spot": Extremely narrow

Acceptance hardly occurs
with the random sampling

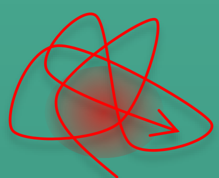
An algorithm that "moves" only around
the hot spot is necessary

Hybrid Monte-Carlo method
(heat-bath method for pure YM)



integral space

Problem in Lattice QCD 1



integral space

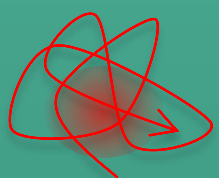
Each step of the HMC need
a matrix inversion of

$$\left(i\gamma_\mu D_\mu - m\right)^{-1}$$

Larger numerical cost when the difference
of the min/max eigenvalues are larger.

Larger numerical cost for smaller quark masses.

Problem in Lattice QCD 2



integral space

$$\int \mathcal{D}\phi \mathcal{O} e^{-S[\phi(x)]}$$

Importance sampling is applicable only when the action S is real and positive.

Complex action cannot be handled.

“Sign Problem”

(complex-phase problem)

- Real-time simulation
- Nonzero density ($\mu \neq 0$)

Sign Problem at $\mu \neq 0$

$$\mathcal{L} = \bar{\psi}(\gamma_\mu D_\mu + m + \mu\gamma_0)\psi = \bar{\psi}\Delta\psi$$

$$\Delta^\dagger(\mu) = -\gamma_\mu D_\mu + m - \mu^*\gamma_0 = \gamma_5\Delta(-\mu^*)\gamma_5$$

$$[\det\Delta(\mu)]^* = \det\Delta(-\mu^*)$$

Quark action becomes complex when $\mu \neq 0$.

☐ Exceptions

- pure imaginary μ
- $\mu_u = -\mu_d$
- $SU(2)_c$

☐ Solutions

- Reweighting, Taylor expansion
- Complex Langevin method
- Lifshitz thimble method
- ...

Reweighting

$$\frac{1}{Z}\int \mathcal{D}\phi \mathcal{O} e^{-S[\phi(x);s]}$$: Action depends on a parameter s

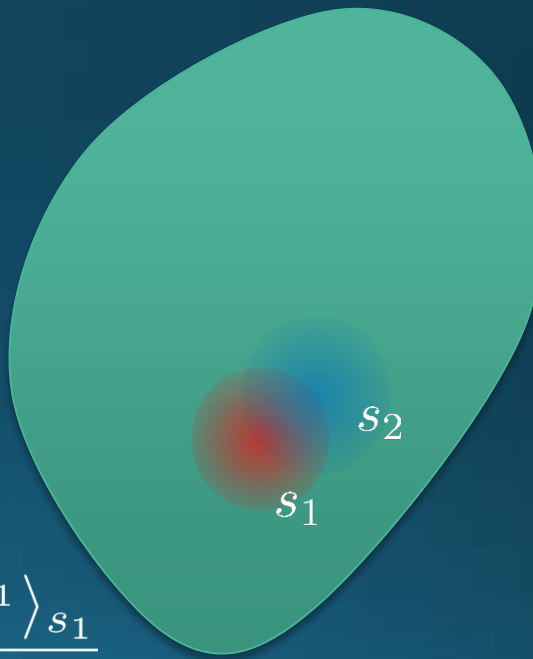
- Monte-Carlo simulation at $s = s_1$

$$\langle \mathcal{O} \rangle_{s_1} = \frac{1}{Z_1}\int \mathcal{D}\phi \mathcal{O} e^{-S[\phi(x);s_1]}$$

- Measurement at $s = s_2$

$$\langle \mathcal{O} \rangle_{s_2} = \frac{1}{Z_2}\int \mathcal{D}\phi \mathcal{O} e^{-S[\phi(x);s_2]}$$

$$= \frac{\int \mathcal{D}\phi \mathcal{O} e^{-S_2+S_1} e^{-S_1}}{\int \mathcal{D}\phi e^{-S_2+S_1} e^{-S_1}} = \frac{\left\langle \mathcal{O} e^{-S_2+S_1} \right\rangle_{s_1}}{\left\langle e^{-S_2+S_1} \right\rangle_{s_1}}$$



→ Measurement at $s = s_2$ from the Monte Carlo simulation at $s = s_1$.

→ Effective when "hot spots" overlaps well

Lattice Spacing a

QCD with zero quark masses

$$\mathcal{L} = \bar{\psi} i \gamma_\mu (\partial_\mu + igA_\mu)\psi + \frac{1}{2}\mathrm{tr}F_{\mu\nu}^2$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu]$$

g is the only parameter. No dimensionful parameters.
Physical scale arises from quantum effects.

Relation b/w g and the lattice spacing a must be determined through the measurement of physical observables.

Contents

Quantum Statistical Mechanics

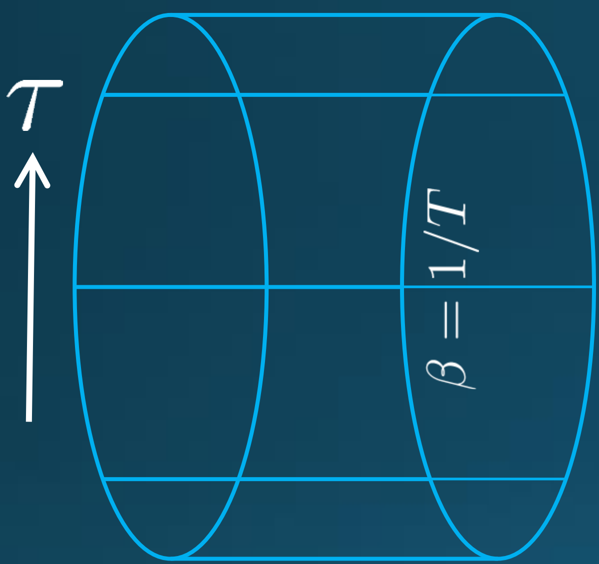
The most important formulae in QSM

$$\rho = \frac{1}{Z} e^{-\beta(H-\mu N)}$$: density matrix

$$Z = \mathrm{Tr} e^{-\beta(H-\mu N)}$$: partition function

$$\langle O \rangle = \mathrm{Tr}[O\rho]$$

QFT @ Nonzero T



$$Z = \mathrm{Tr} e^{-\beta H} = \sum_n \langle n | e^{-\beta H} | n \rangle$$
$$= \int \mathcal{D}\phi e^{-S_T}$$

(Anti-)periodic BC along τ direction
= Nonzero *T* system

$$\langle \mathcal{O} \rangle_T = \frac{1}{Z} \int \mathcal{D}\phi \mathcal{O} e^{-S_T}$$

Thermodynamics

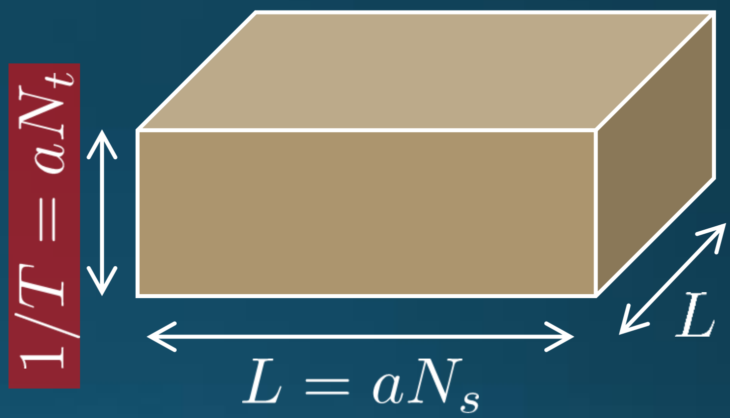
$$\begin{cases} \text{Energy density: } \langle T_{00} \rangle_T \\ \text{Pressure: } \langle T_{11} \rangle_T \end{cases}$$

Suzuki,2013; FlowQCD, 2014

Thermodynamics

Thermodynamic Relations

$$\varepsilon = \frac{T^2}{V}\frac{\partial \ln Z}{\partial T} \qquad p = T\frac{\partial \ln Z}{\partial V}$$



ε and p are obtained from
T, V derivatives of $\ln Z$.

Derivative w.r.t. lattice spacing a with fixed $N_s^3 \times N_t$
➔ Simultaneous variations of V and $1/T$.

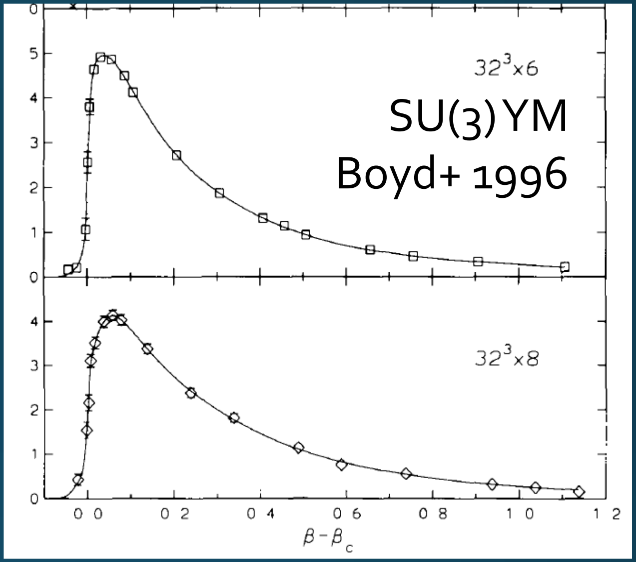
$$a\frac{\partial \ln Z}{\partial a} \sim \frac{V}{T}(\varepsilon - 3p)$$

Integral Method

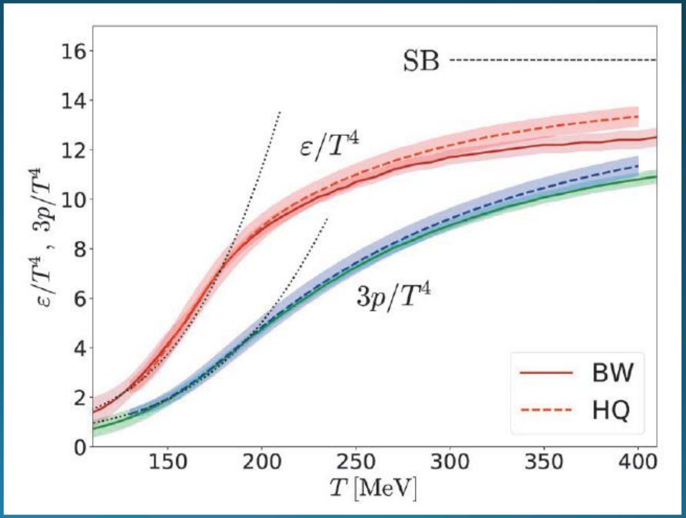
$$\frac{\partial \ln Z}{\partial a} = \frac{\partial \beta}{\partial a}\frac{\partial \ln Z}{\partial \beta} \sim \frac{\partial \beta}{\partial a}\langle S \rangle$$

$$T\frac{\partial (p/T^4)}{\partial T} = \frac{\varepsilon - 3p}{T^4}$$

$$\frac{p}{T^4} = \int_{T_0}^{T} dT \frac{\varepsilon - 3p}{T^5}$$



QCD Thermodynamics



Thermodynamics of SU(3) YM

- **Integral method**
  - Most conventional / established
  - Use themodynamic relations
    Boyd+ 1995; Borsanyi, 2012

$$p = \frac{T}{V} \ln Z$$

$$T\frac{\partial(p/T^4)}{\partial T} = \frac{\varepsilon - 3p}{T^4}$$

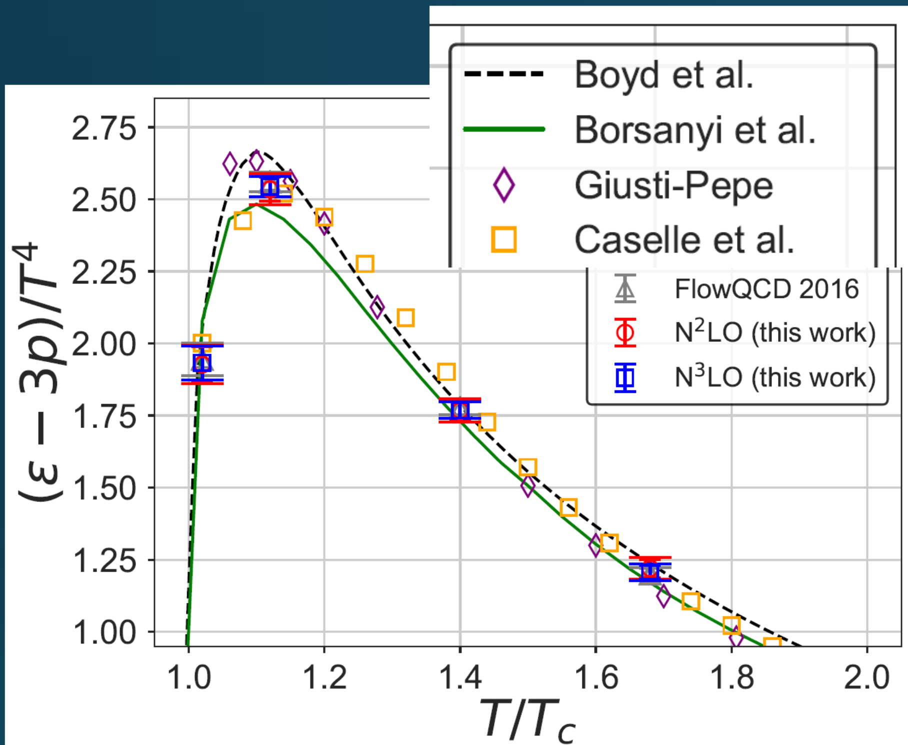
- Gradient-flow method
  - Take expectation values of EMT
    FlowQCD, 2014, 2016

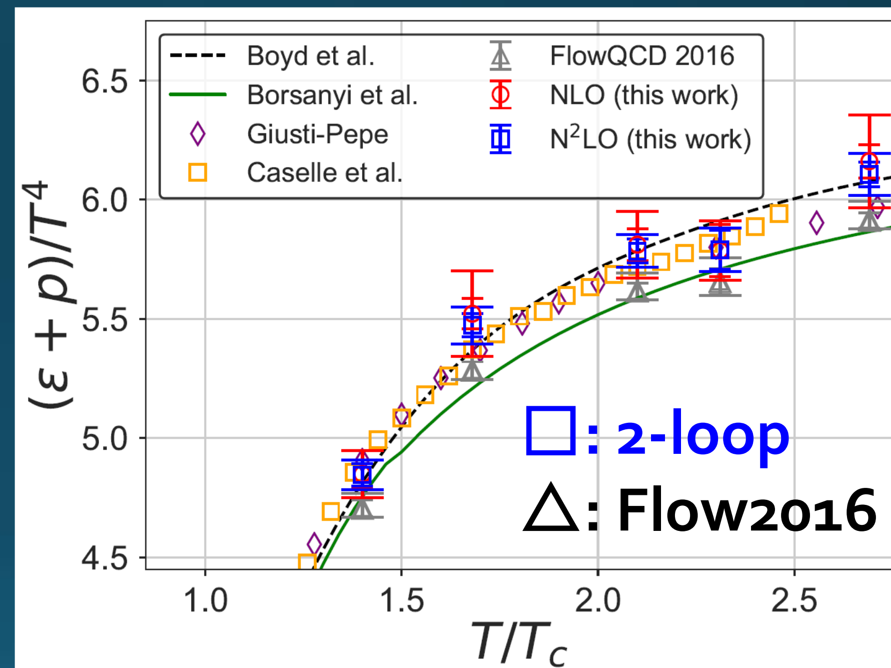
$$\begin{cases} \varepsilon = \langle T_{00} \rangle \\ p = \langle T_{11} \rangle \end{cases}$$

- Moving-frame method
  Giusti, Pepe, 2014~
- Non-equilibrium method
  - Use Jarzynski's equality Caselle+, 2016;2018
- Differential method
  Shirogane+(WHOT-QCD), 2016~

SU(3) Thermodynamics: Comparison

Iritani, MK, Suzuki, Takaura, 2019





Boyd+:1996 / Borsanyi+: 2012

- All results agree well.
- But, the results of integral method has a discrepancy.
  (Older result looks better…)

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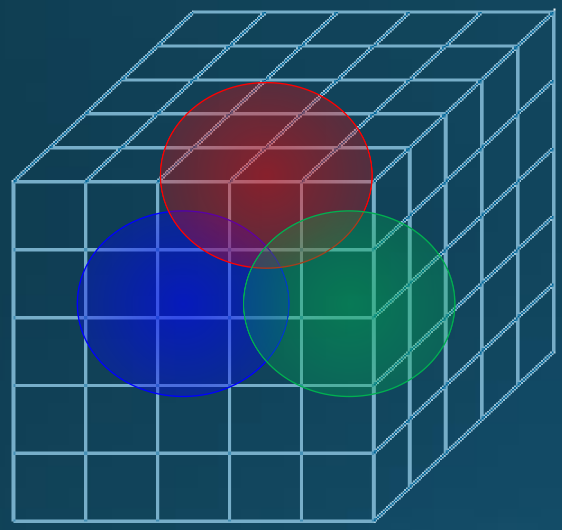
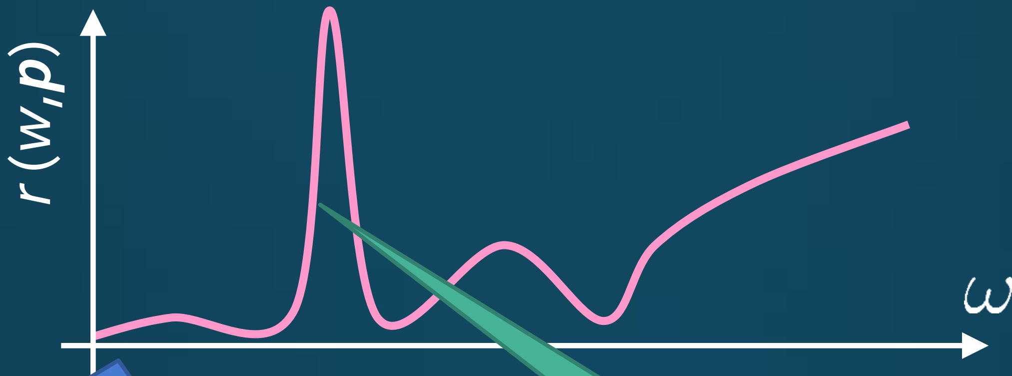
Analytic Continuation

- **Lattice:** imaginary time
- **Dynamics:** real time



Real-time info. have to be extracted from the correlation funcs. in imaginary time.

Spectral Function



slope at the origin

→ transport coefficients

Kubo formulae $\eta \sim \lim_{\omega \to 0} \frac{1}{\omega} \rho(\omega)$

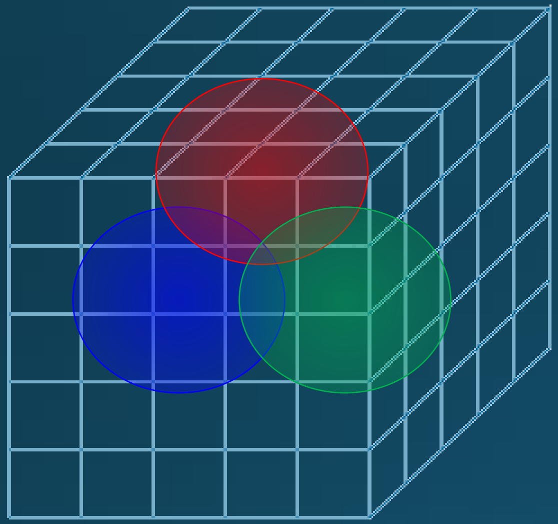
- shear viscosity : T_{12}
- bulk viscosity : T_{mm}
- electric conductivity : J_{ii}

peaks

quasi-particle excitation
width ~ decay rate

Analytic Continuation

- **Lattice:** imaginary time



$\tilde{G}(\tau, \boldsymbol{k})$

discrete and noisy

- **Dynamics:** real time

$\rho(\omega, \boldsymbol{k})$

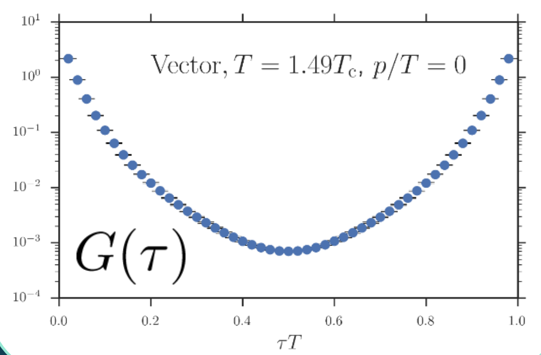
continuous

$$\tilde{G}(\tau) = \int d\omega \, \frac{e^{(\beta/2-\tau)\omega}}{e^{\beta\omega/2} + e^{-\beta\omega/2}} \rho(\omega)$$

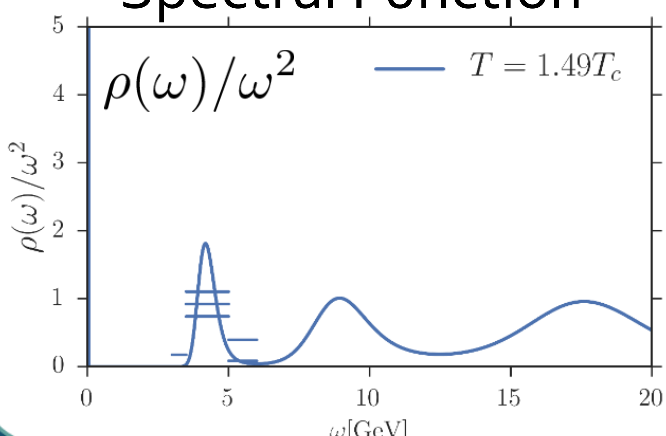
Maximum Entropy Method

Asakawa, Nakahara
Hatsuda, 2001

Lattice data



Spectral Function



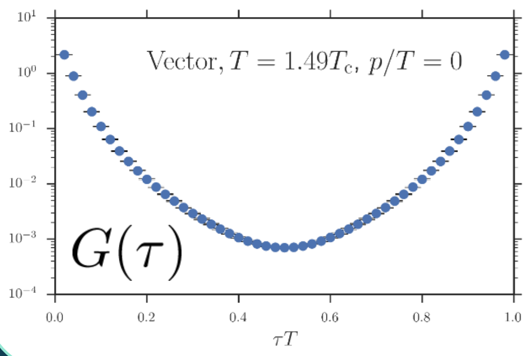
$$G(\tau) = \int_0^\infty d\omega \frac{\cosh(1/2T - \tau)\omega}{\sinh(\omega/2T)} \rho(\omega)$$

"ill-posed problem"

Maximum Entropy Method

Asakawa, Nakahara
Hatsuda, 2001

Lattice data



Bayes theorem ✖

Prior probability

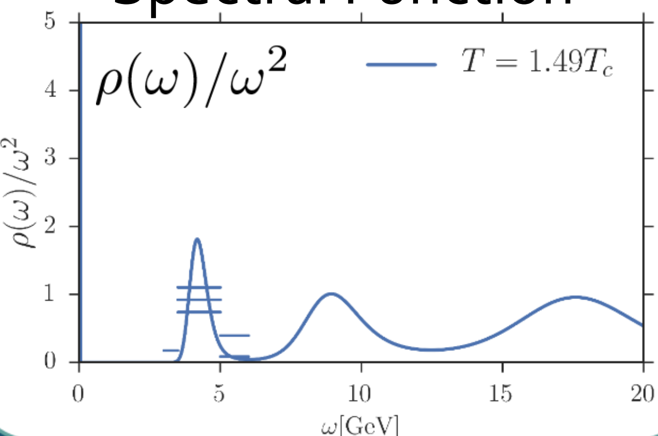
- Shannon-Jaynes entropy
- default model $m(\omega)$

=

Probability of $\rho(\omega)$

$P[\rho(\omega), \alpha]$

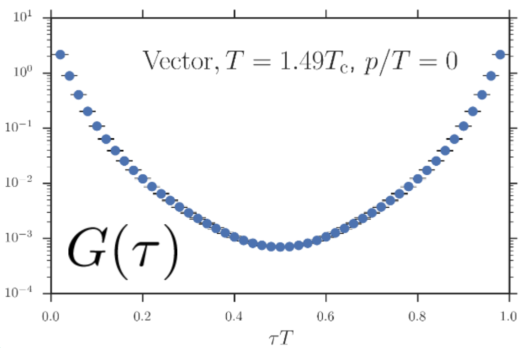
Spectral Function



Maximum Entropy Method

Asakawa, Nakahara
Hatsuda, 2001

Lattice data



Bayes theorem ✖

Prior probability

- Shannon-Jaynes entropy
- default model $m(\omega)$

=

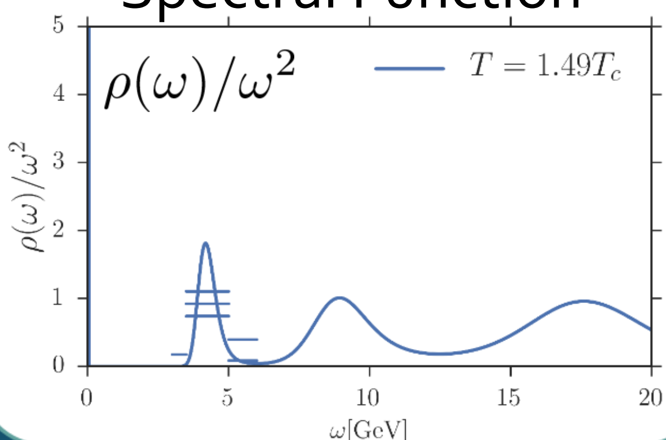
Probability of $\rho(\omega)$

$P[\rho(\omega), \alpha]$

expectation value
$\langle \rho(\omega) \rangle_P$

$$\langle \mathcal{O} \rangle_P = \int d\alpha \int [d\rho] P[\rho, \alpha] \mathcal{O}$$

Spectral Function

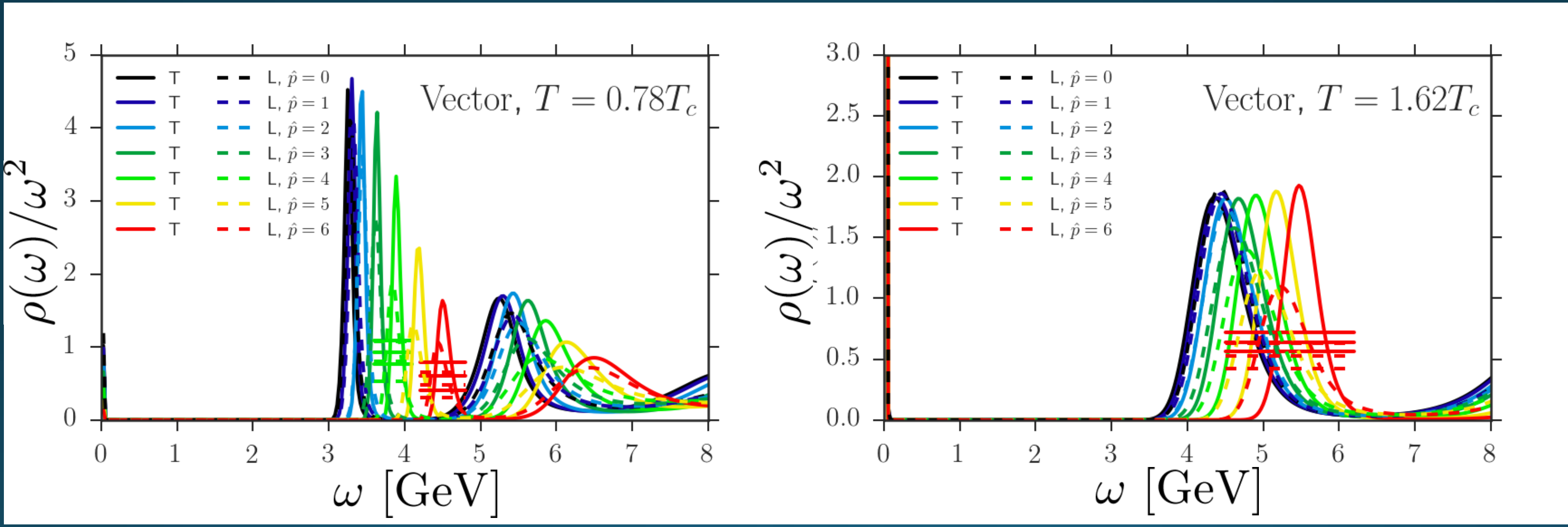


- ☐ Output of MEM is jus an expectation value.
- ☐ Error analysis is necessary!!!

Charmonium SPC

Ikeda, Asakawa, MK
PRD 2017

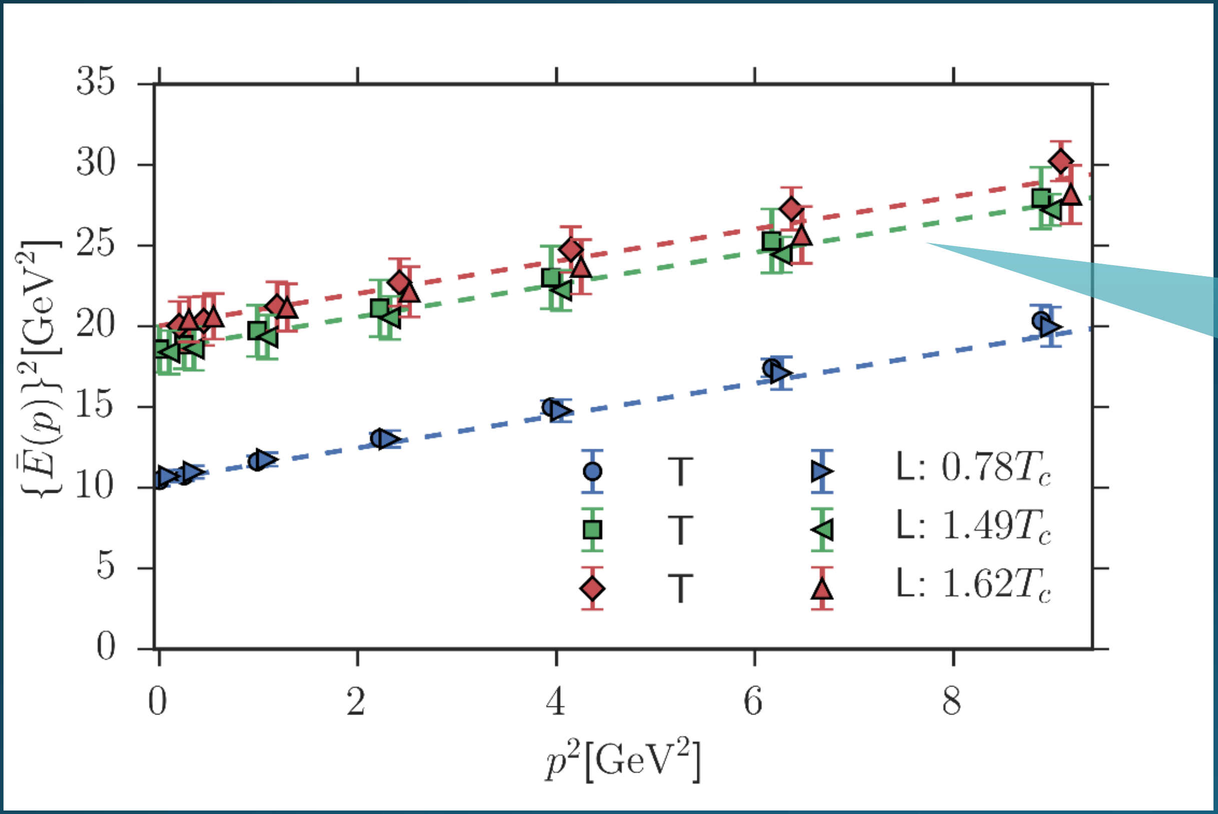
Spectral function of *J/ψ*



- Transverse/longitudinal decomposed
- Mass enhancement in medium?

Dispersion Relation of Charmonia

Ikeda, Asakawa, MK
PRD 2017



Disp. Rel. in vacuum

$$E = \sqrt{p^2 + m^2}$$

- Large mass enhancement at nonzero T.
- Disp. Rel. of J/ψ is unchanged from the vacuum one.

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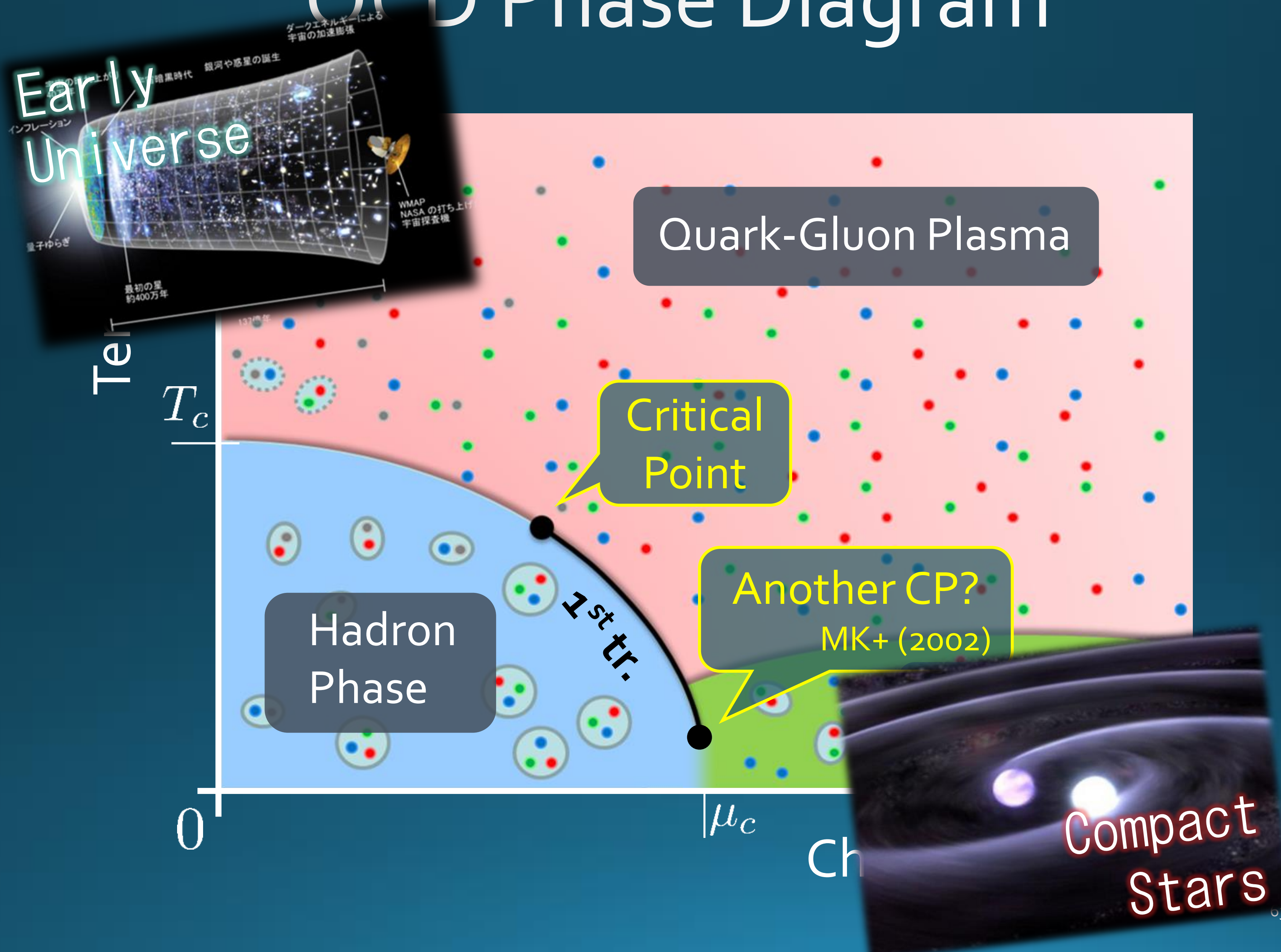
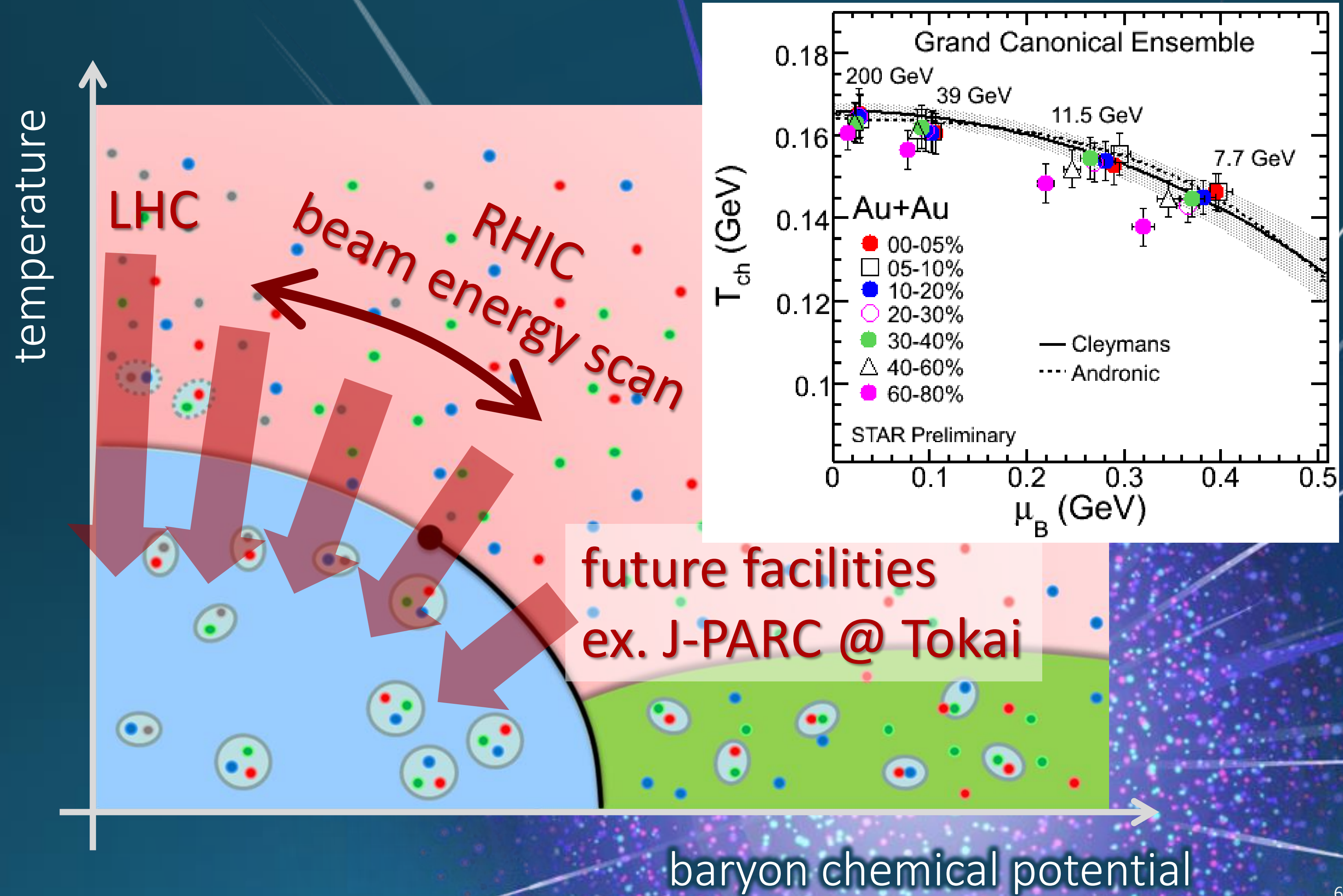
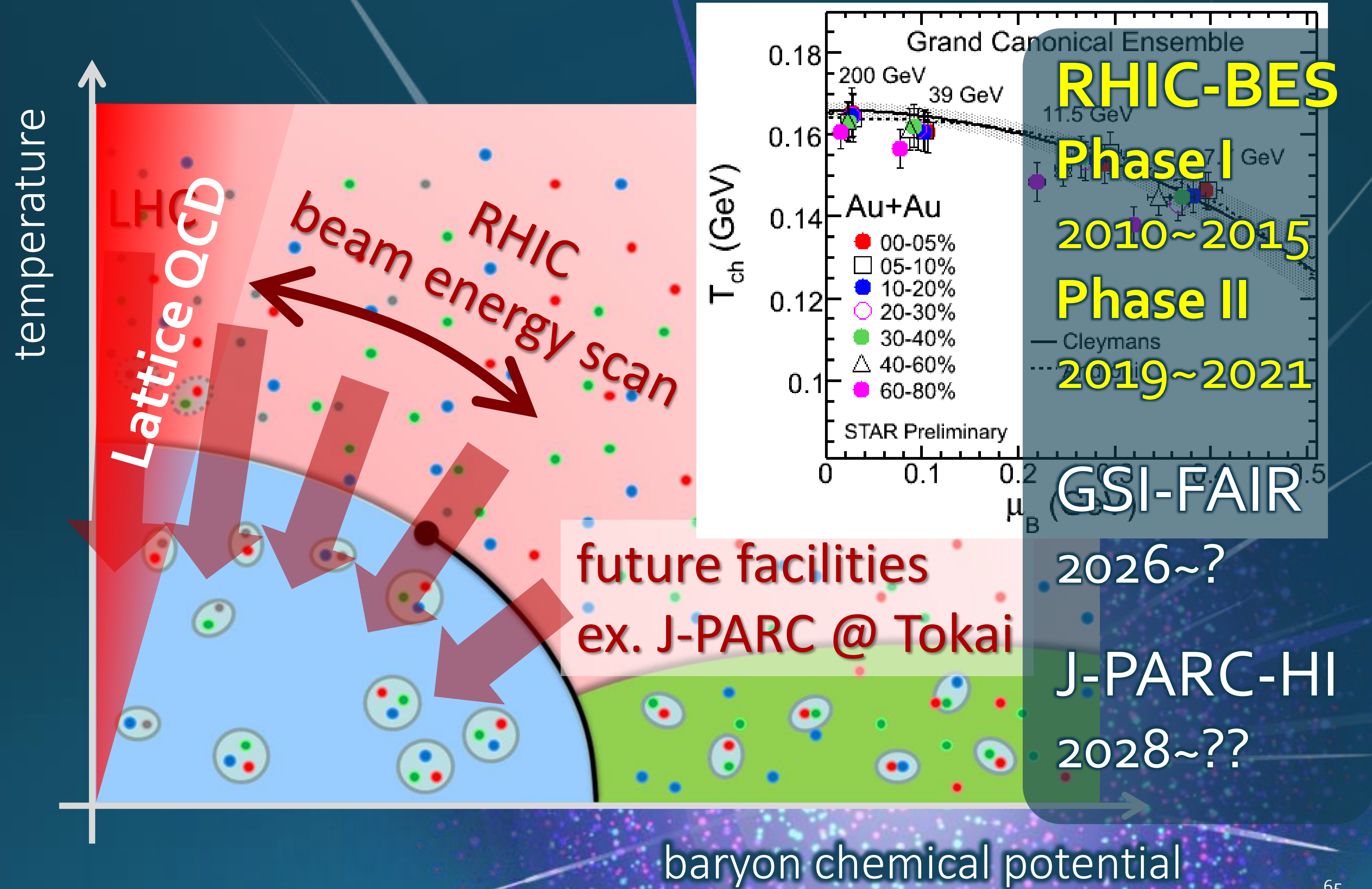
QCD Phase Diagram



Beam-Energy Scan



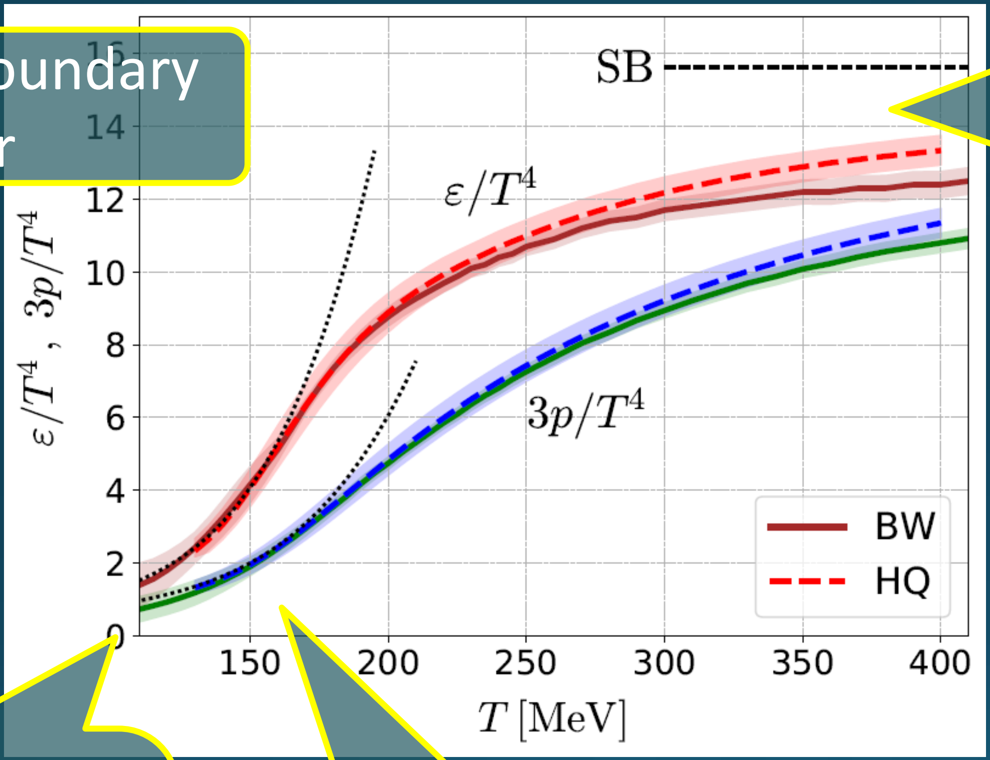
Beam-Energy Scan



Contents

QCD Thermodynamics

No phase boundary
➔Crossover



High T: approach to Stefan-Boltzmann limit

Budapest-Wuppertal '14, HotQCD '14

Low T: consistent w/ hadron resonance gas model

Sudden increase around $T \simeq 160$MeV

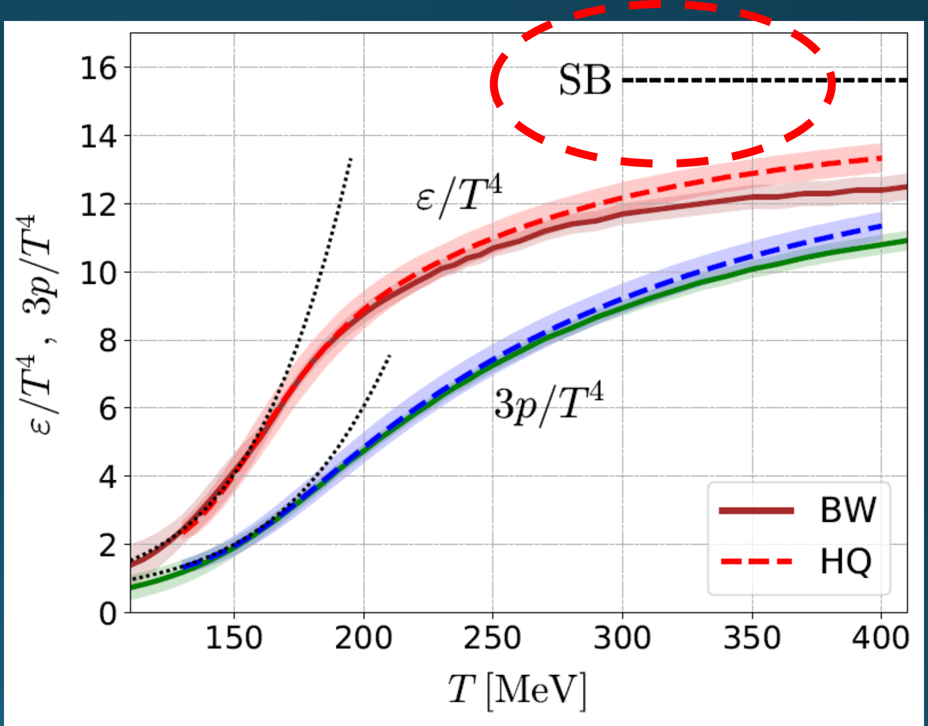
$1\text{eV} \simeq 10^4\text{K}$
$1\text{MeV} \simeq 10^{10}\text{K}$
$100\text{MeV} \simeq 1$兆K

Stefan-Boltzmann Limit

SB limit = Free gas of massless quarks & gluons

$$\epsilon = \left(16 + \frac{21}{2}N_f\right)\frac{\pi^2}{30}T^4$$

$$\epsilon = 3p$$



$$\epsilon_{\text{free}} = g\int \frac{d^3p}{(2\pi)^3}\frac{p}{e^{p/T}\pm 1} = \frac{\pi^2}{30}T^4$$

Hadron Resonance Gas (HRG) Model

= Free gas composed of all known hadrons

$$\epsilon = \sum_{i=\text{hadrons}} \epsilon_i$$

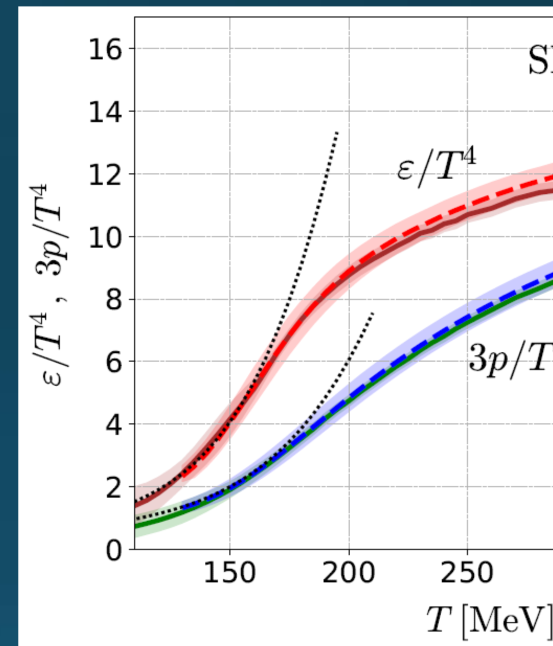
$$\epsilon_i = \int \frac{d^3p}{(2\pi)^3} \frac{E_p^{(i)}}{e^{E_p^{(i)}/T} \pm 1}$$

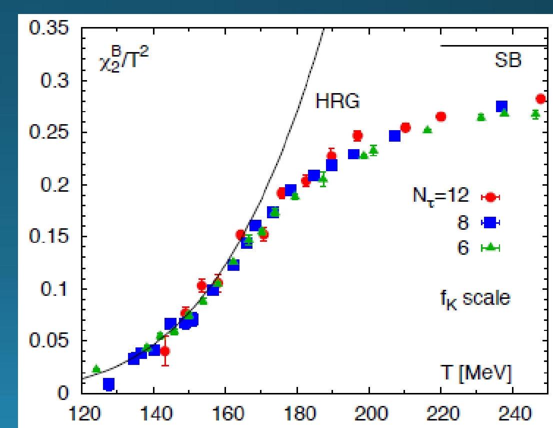
$$E_p = \sqrt{m^2 + p^2}$$

HRG reproduces QCD thermodynamics for $T < 160$MeV quite well

Particle data group

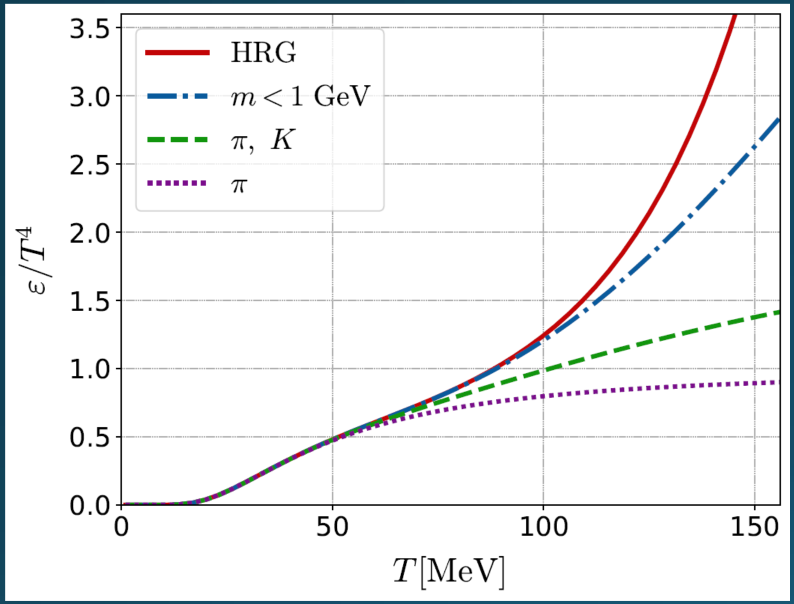
• $\pi^\pm$	$1^-(0^-)$
• π^0	$1^-(0^{-+})$
• η	$0^+(0^{-+})$
• $f_0(500)$	$0^+(0^{++})$
• $\rho(770)$	$1^+(1^{--})$
• $\omega(782)$	$0^-(1^{--})$
• $\eta'(958)$	$0^+(0^{-+})$
• $f_0(980)$	$0^+(0^{++})$
• $a_0(980)$	$1^-(0^{++})$
• $\phi(1020)$	$0^-(1^{--})$
• $h_1(1170)$	$0^-(1^{+-})$
• $b_1(1235)$	$1^+(1^{+-})$
• $a_1(1260)$	$1^-(1^{++})$
• $f_2(1270)$	$0^+(2^{++})$
• $f_1(1285)$	$0^+(1^{++})$
• $\eta(1295)$	$0^+(0^{-+})$
• $\pi(1300)$	$1^-(0^{-+})$
• $a_2(1320)$	$1^-(2^{++})$
• $f_0(1370)$	$0^+(0^{++})$
$h_1(1380)$	$?^-(1^{+-})$
• $\pi_1(1400)$	$1^-(1^{-+})$



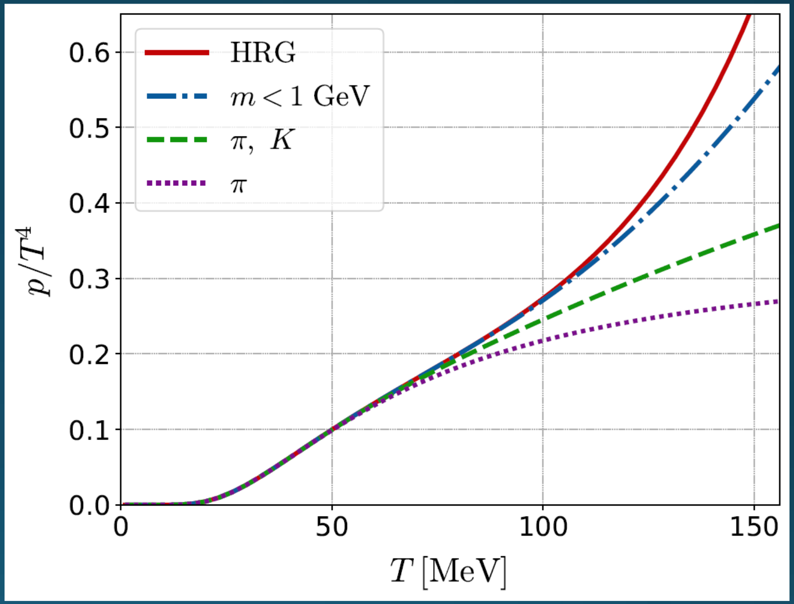


HRG Model 2: Exercise in Phthon3

ϵ/T^4



p/T^4



sample codes: https://www.dropbox.com/sh/tojgef5khp5cb7h/AABiBSFtP8j
code: https://github.com/MasakiyoK/Saizensen/Chap3/

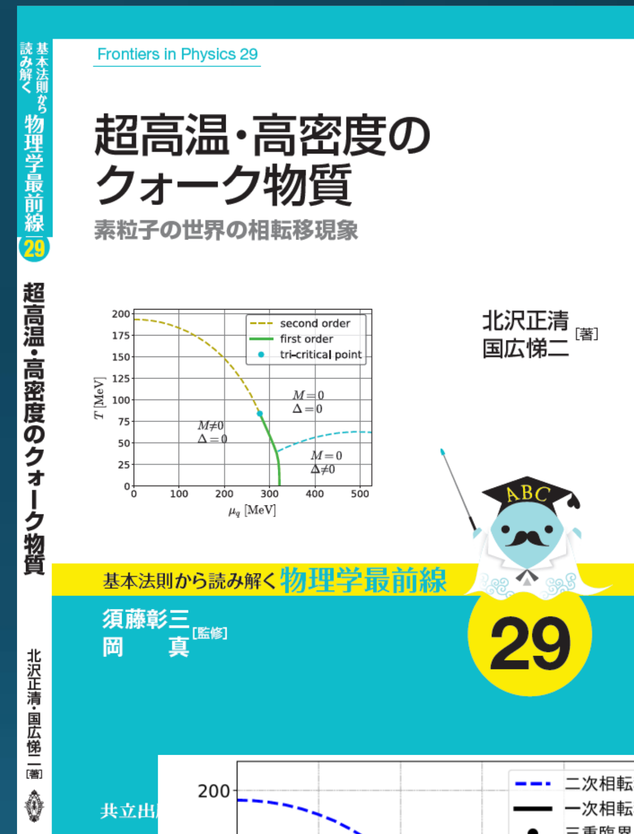
List of hadrons: Bollweg+, PRD104, 7 ('21) https://arxiv.org/abs/2107.10011

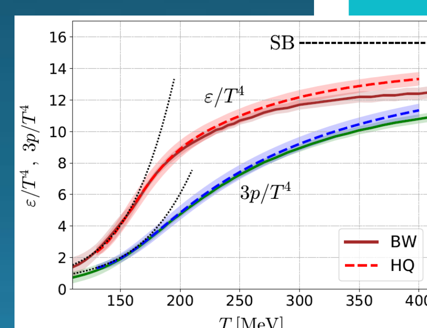
Advertising

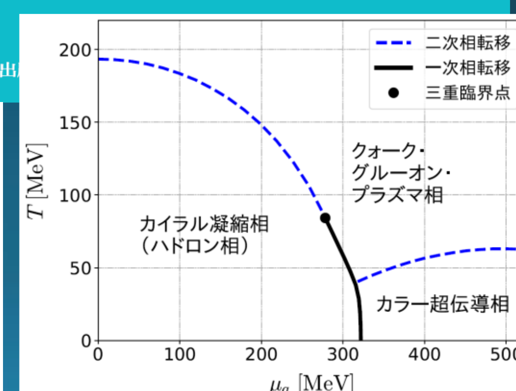
A book **"Quark matter at extreme conditions: Phase transitions in the world of elementary particles"** will come soon (**end of August**)!

- ☐ Intro. to hot and dense QCD
- ☐ Relativistic heavy-ion collisions
- ☐ BCS theory
- ☐ Phase diagram in NJL model
- ☐ Linear response, collective modes
- ☐ Color superconductivity
- ☐ **Numerical codes in Python**

Codes at:
https://github.com/MasakiyoK/Saizensen

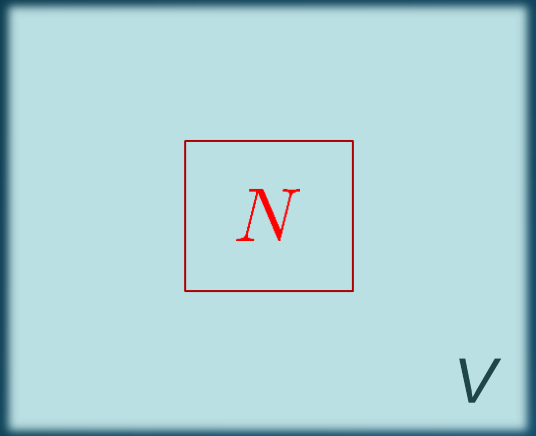


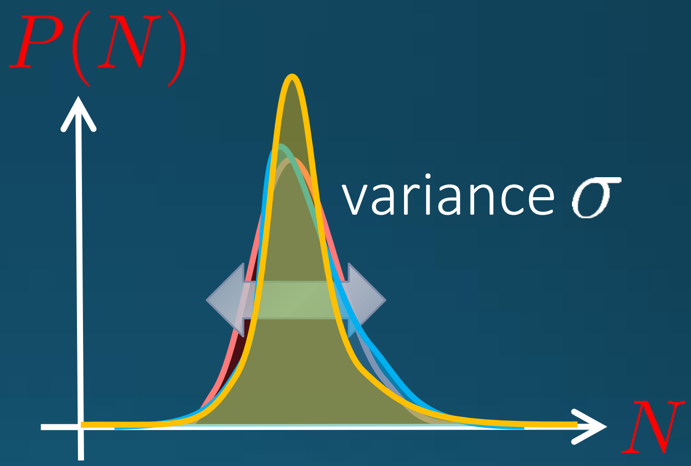




Thermal Fluctuations

Observables in equilibrium are fluctuating!





Enhancement & sign change of higher order cumulants will be used for the signal of the QCD critical point.

Stephanov, '09; Asakawa, Ejiri, MK, '09

Cumulants

Cumulants

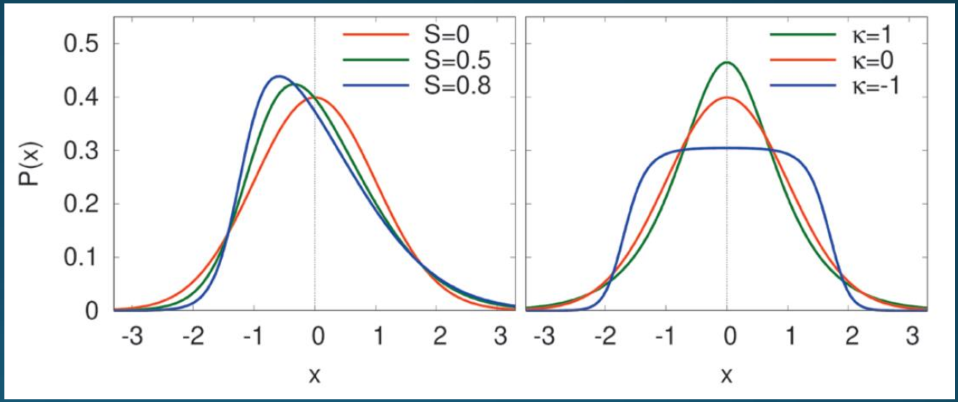
$$\begin{cases} \langle N \rangle_c = \langle N \rangle & \text{average} \\ \langle N^2 \rangle_c = \langle \delta N^2 \rangle & \text{variance} \\ \langle N^3 \rangle_c = \langle \delta N^3 \rangle & \\ \langle N^4 \rangle_c = \langle \delta N^4 \rangle - 3\langle \delta N^2 \rangle^2 & \end{cases}$$

□ skewness

$$S = \frac{\langle N^3 \rangle_c}{\langle N^2 \rangle_c^{3/2}}$$

□ kurtosis

$$\kappa = \frac{\langle N^4 \rangle_c}{\langle N^2 \rangle_c^2}$$



□ NOTE

- Gauss distribution: $\langle N^3 \rangle_c = \langle N^4 \rangle_c = \cdots = 0$
- Poisson distribution: $\langle N^2 \rangle_c = \langle N^3 \rangle_c = \langle N^4 \rangle_c = \cdots = \langle N \rangle$

Review: Asakawa, MK, PPNP **90** (2016) Sec. 2

Cumulants of Conserved Charges
=Observable on the Lattice

- Fluctuation-Response Relations

$$\langle N_B^m \rangle_c = V \chi_m^B$$

Thermal Fluctuation

Susceptibility

$$\chi_m^B \sim \frac{\partial^m p}{\partial \mu_B^m}$$

$$p(T,\mu) = p(T,0) + \frac{\chi_2}{2}\left(\frac{\mu}{T}\right)^2 + \cdots$$

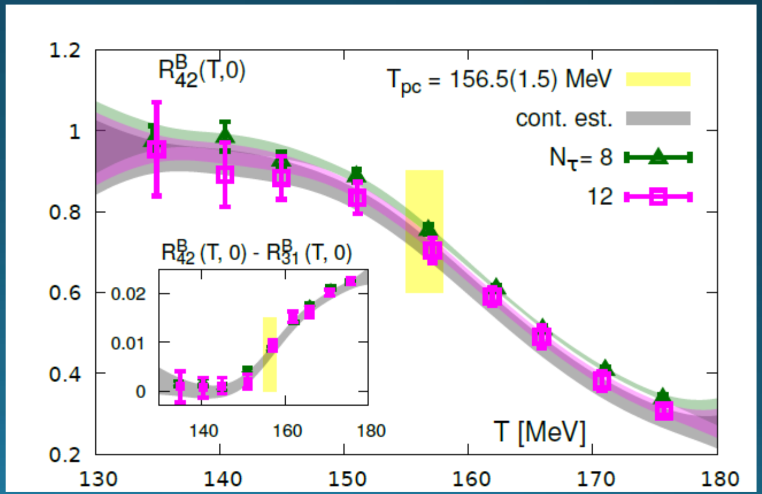
- Volume dependence canceled out in ratios

Ejiri, Karsch, Redlich, '05

➡ useful for comparison w/ HIC

Review: Asakawa, MK, PPNP **90** (2016)

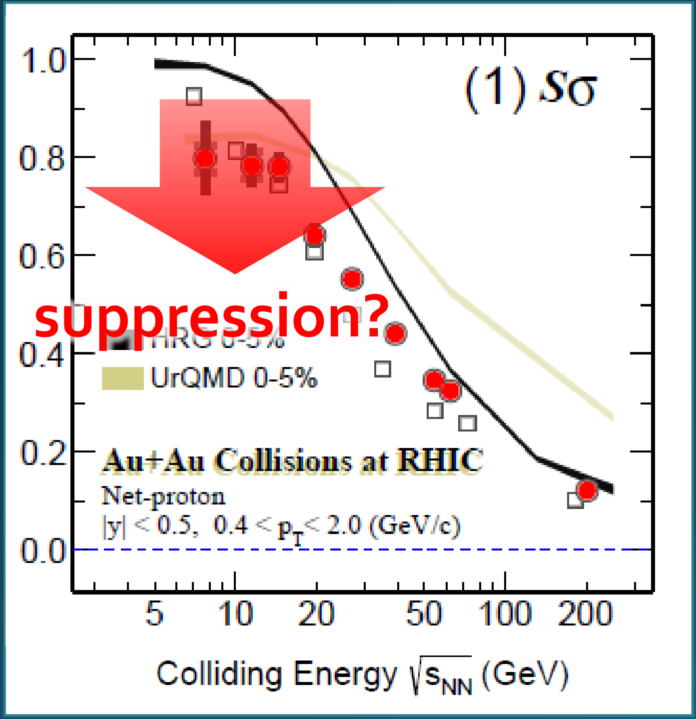
$\langle N_B^4 \rangle_c / \langle N_B^2 \rangle_c$



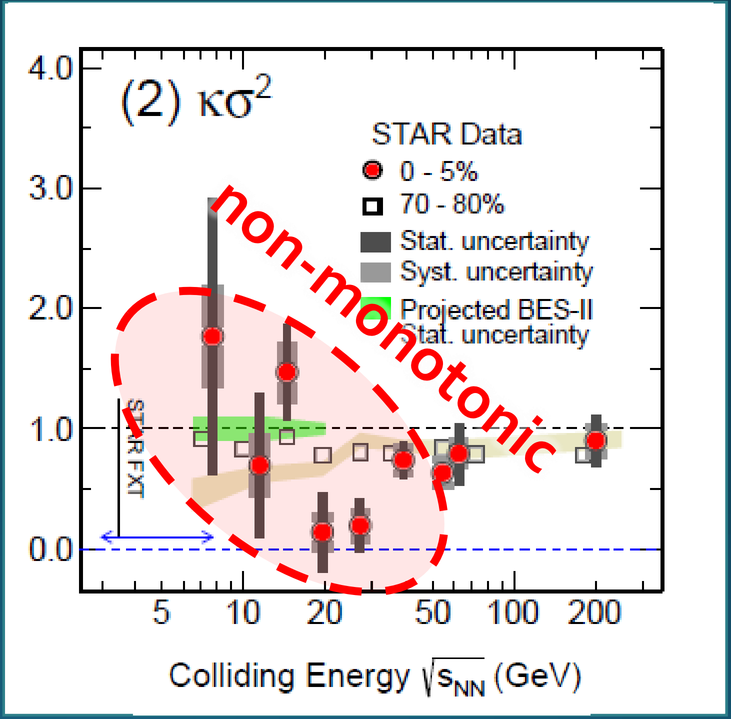
HotQCD, PRD**101** (2020)

Proton Number Cumulants in HIC

$\langle N_p^3 \rangle_c / \langle N_p^2 \rangle_c$



$\langle N_p^4 \rangle_c / \langle N_p^2 \rangle_c$



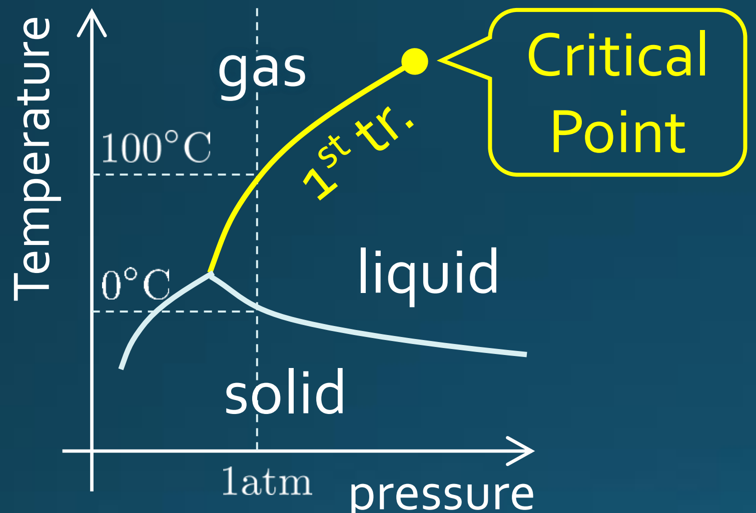
STAR, PRC 2020 [2001.06419]

- **Nonzero and non-Poissonian cumulants are experimentally established.**

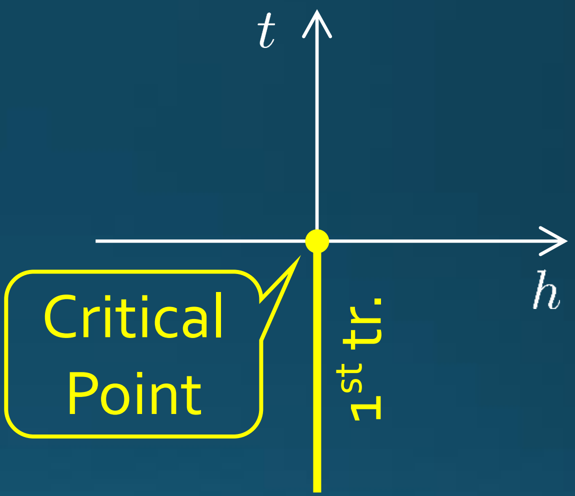
Contents

Critical Points

☐ Water



☐ Ising Model

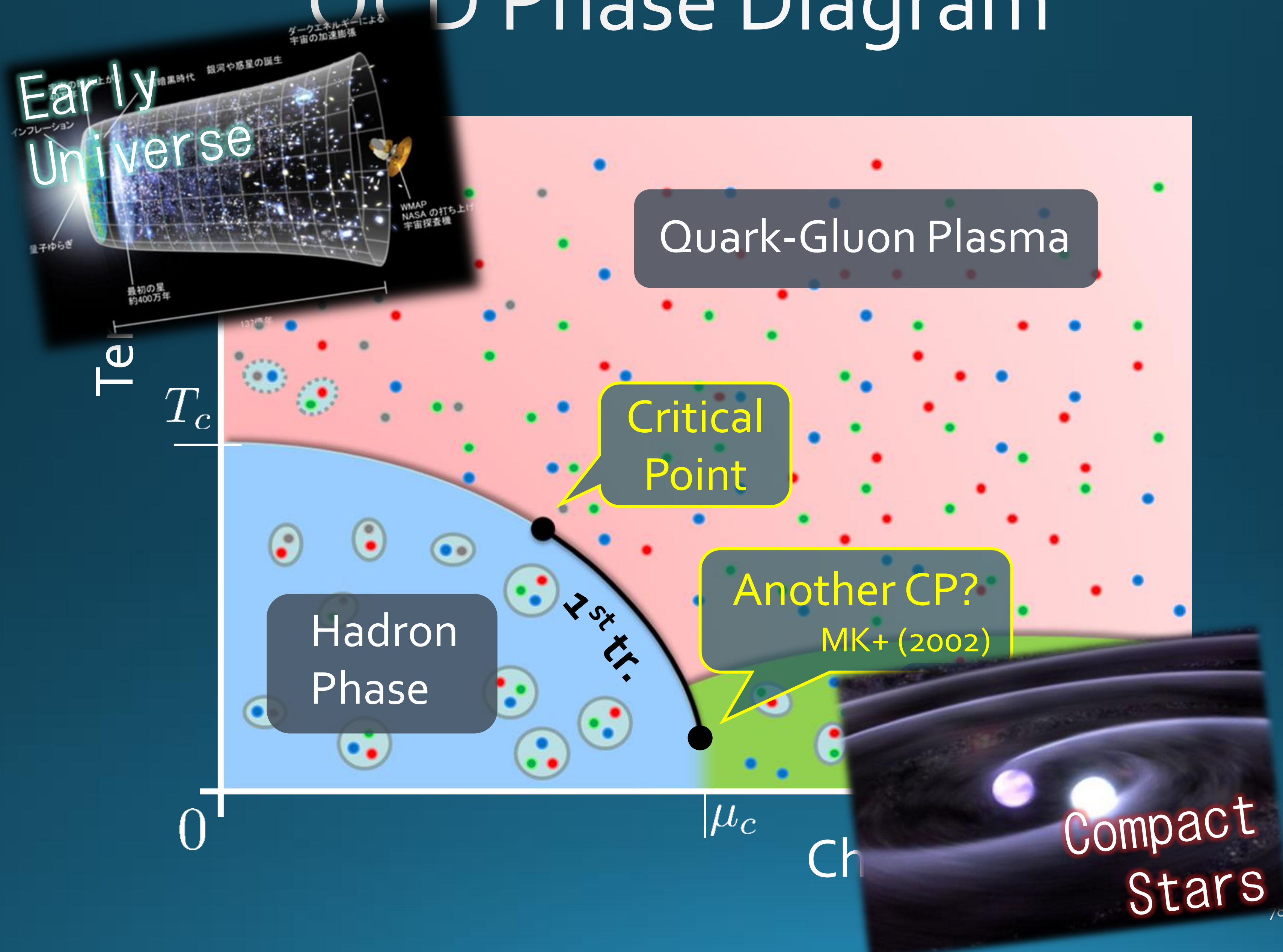


These CPs belong to the same universality class (Z_2).

➡ Common critical exponents.

ex. $C \sim (T - T_c)^{-\alpha}$

QCD Phase Diagram



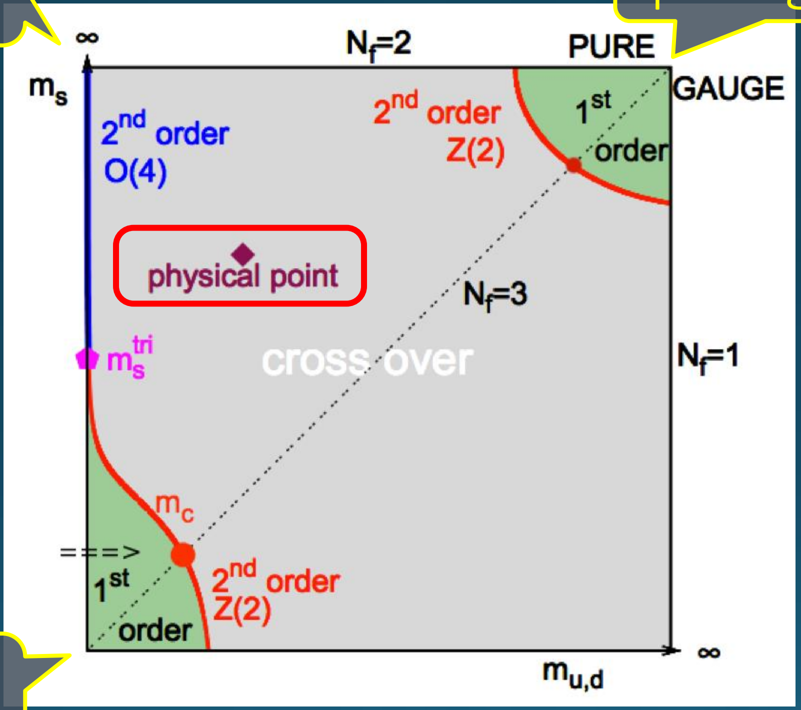
Columbia Plot

= order of phase tr. at $\mu = 0$

massless 2-flavor

pure gauge

s quark mass



massless 3-flavor

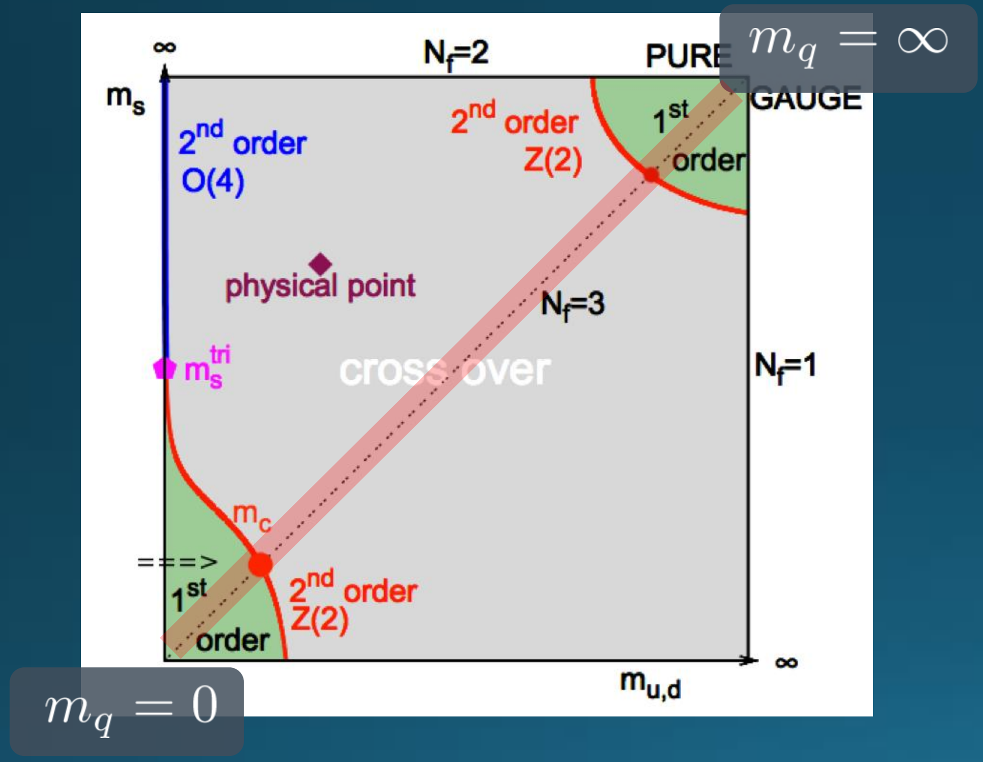
u,d (degenerate) quark mass

Various orders of phase transition with variation of m_q.

Varying Quark Masses

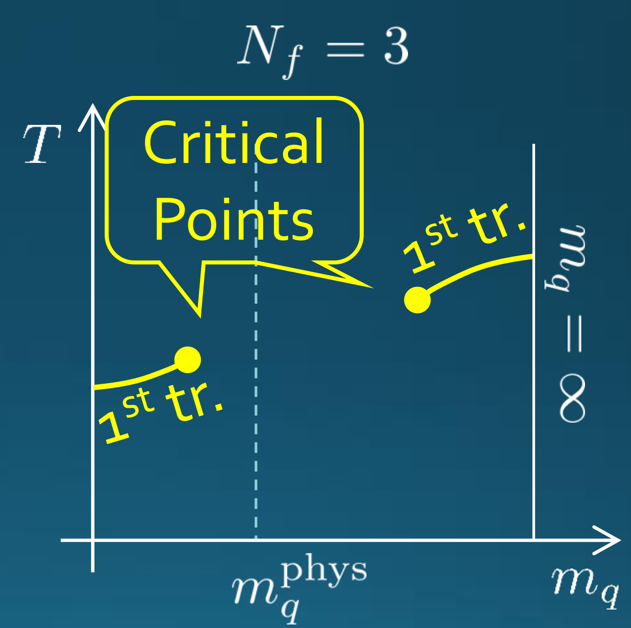
☐ Columbia plot

= order of phase tr. at $\mu = 0$

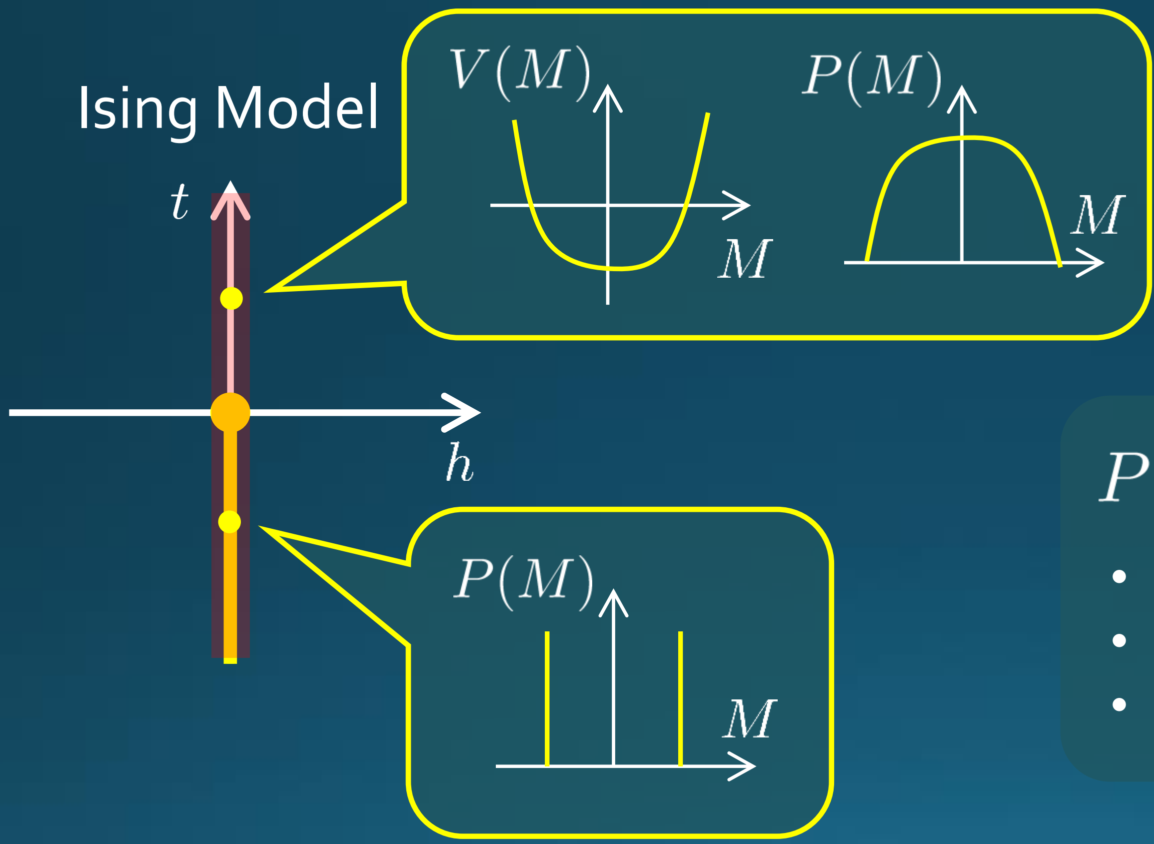


☐ Phase Diagram

on the $T - m_q$ plane



Cumulants around Critical Point



$$P(M) \sim e^{-V(M)}$$

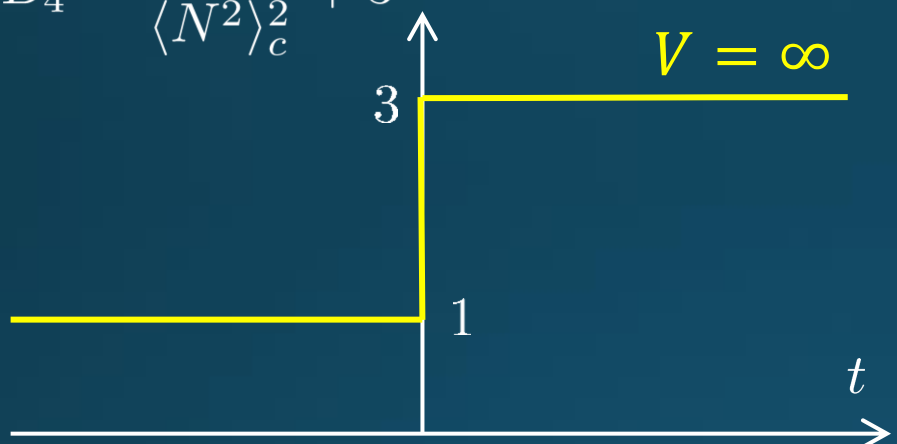
- $P(M)$: probability distr.
- $V(M)$: effective potential
- M : order parameter

- $\langle N^4 \rangle_c$ changes discontinuously at the CP.

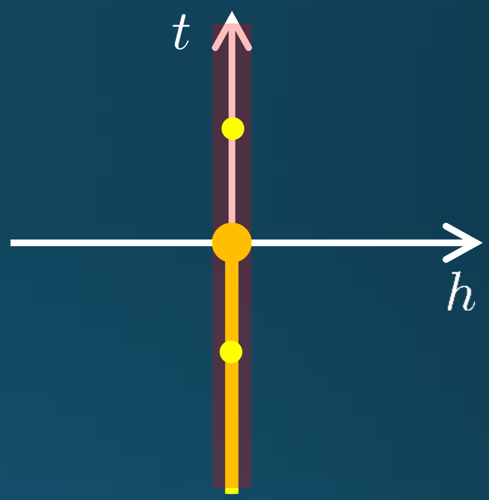
Finite-Volume Effects

Binder Cumulant

$$B_4 = \frac{\langle N^4 \rangle_c}{\langle N^2 \rangle_c^2} + 3$$



Ising Model



- Sudden change of B_4 at the CP is smeared by finite V effect.
- B_4 obtained for various V has crossing at $t = 0$.
- At the crossing point, $B_4 = 1.604$ in Z_2 universality class.

Finite-Volume Effects

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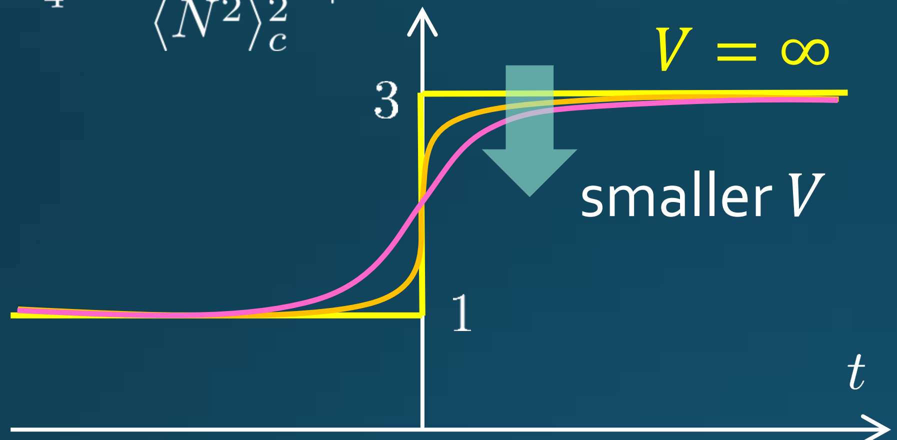
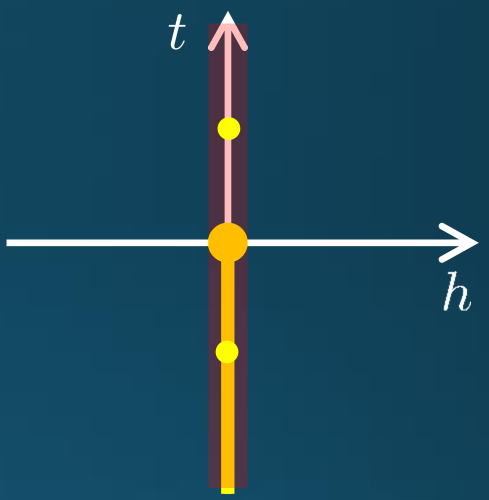


Ising Model

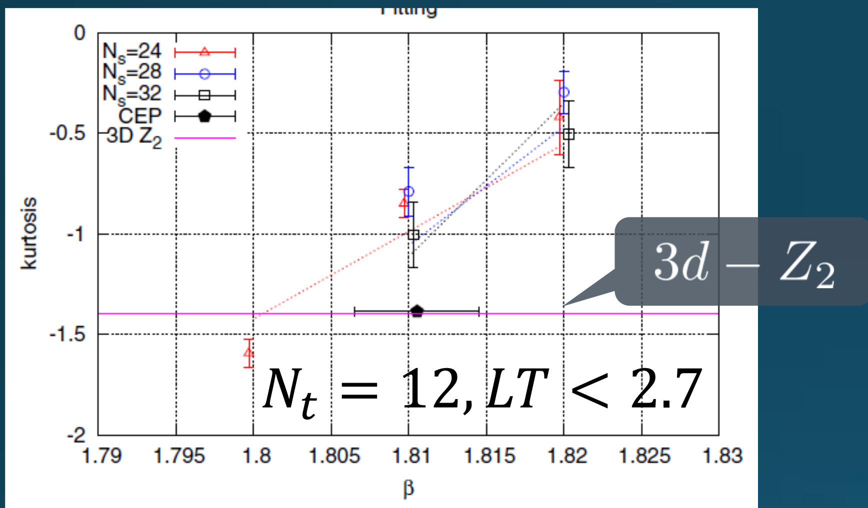


- ☐ Sudden change of B_4 at the CP is smeared by finite V effect.
- ☐ B_4 obtained for various V has crossing at $t = 0$.
- ☐ At the crossing point, $B_4 = 1.604$ in Z_2 universality class.

Binder-Cumulant Analysis

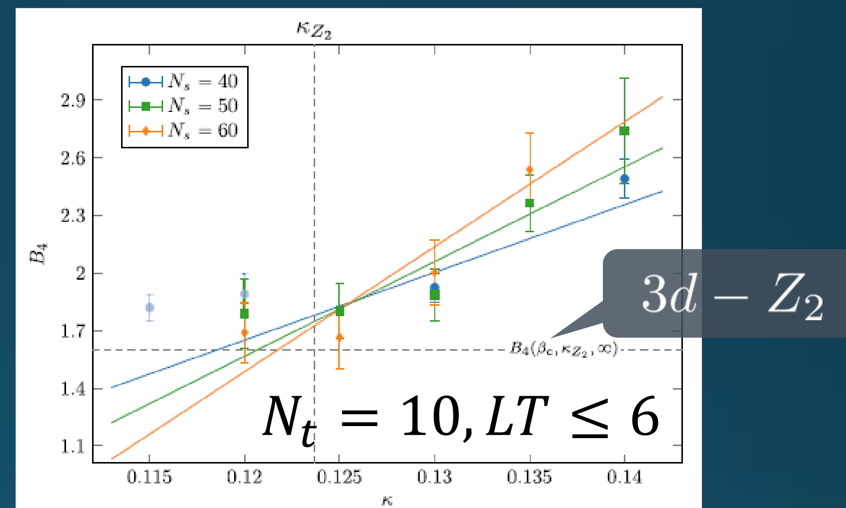
☐ Light-quark region

Kuramashi, Nakamura, Ohno, Takeda, '20



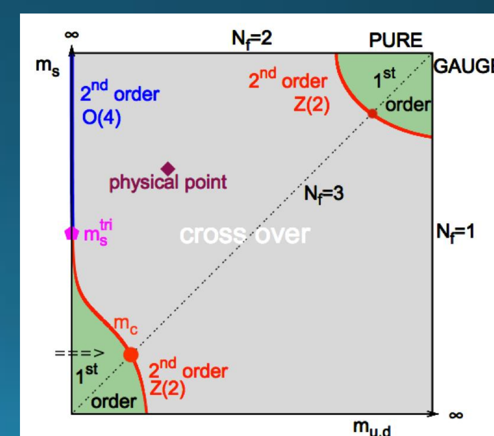
☐ Heavy-quark region

Cuteri , Philipsen , Schön, Sciarra, '21



☐ Statistically-significant deviation of the crossing point from the 3d-Ising value.

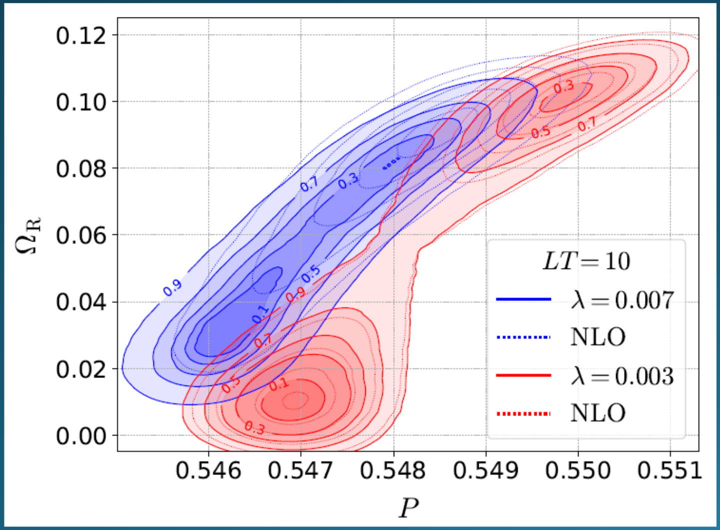
➡ Too large finite-V effects?



Numerical Simulation

- **Coarse lattice: $N_t = 4$**
- But **large spatial volume**:
  $LT = N_s \,/\, N_t \leq 12$

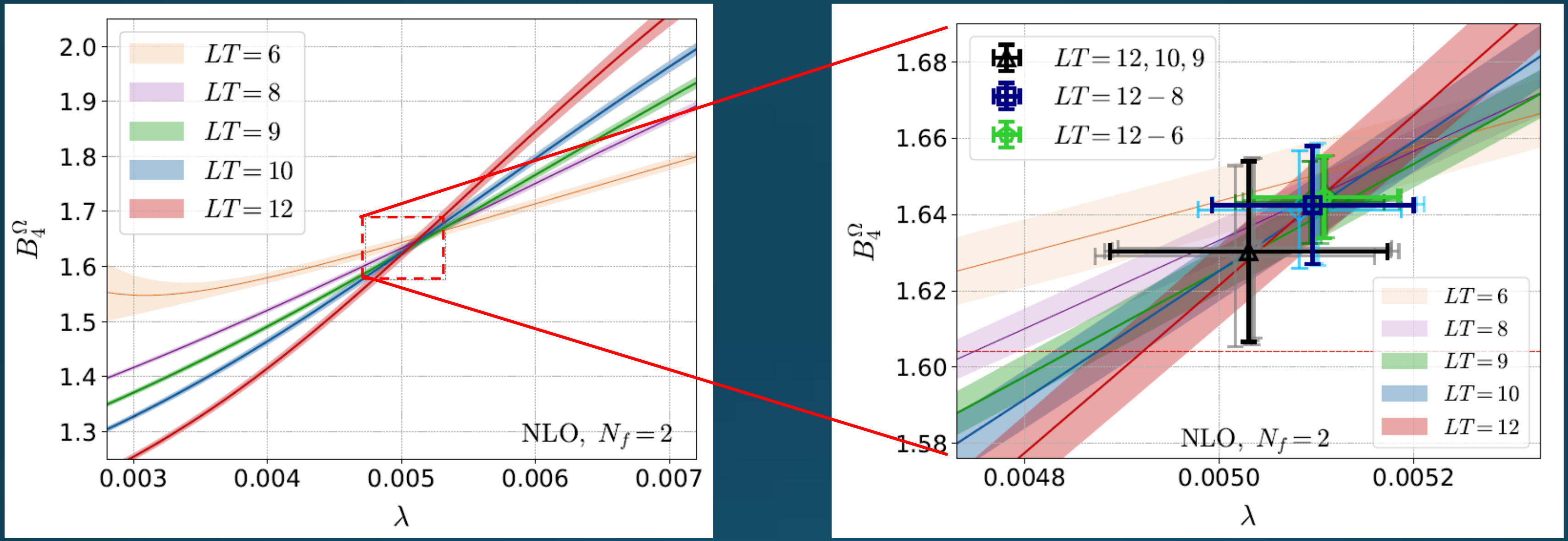
- Hopping-param. ($\sim 1/m_q$) expansion
- Monte-Calro with LO action
- High statistical analysis



Simulation params.

lattice size	β^*	λ	$\kappa^{N_f=2}$
$48^3 \times 4$	5.6869	0.004	0.0568
	5.6861	0.005	0.0601
	5.6849	0.006	0.0629
$40^3 \times 4$, $36^3 \times 4$	5.6885	0.003	0.0529
	5.6869	0.004	0.0568
	5.6861	0.005	0.0601
	5.6849	0.006	0.0629
	5.6837	0.007	0.0653
$32^3 \times 4$	5.6885	0.003	0.0529
	5.6865	0.004	0.0568
	5.6861	0.005	0.0601
	5.6845	0.006	0.0629
	5.6837	0.007	0.0653
$24^3 \times 4$	5.6870	0.0038	0.0561
	5.6820	0.0077	0.0669
	5.6780	0.0115	0.0740

Binder-Cumulant Analysis

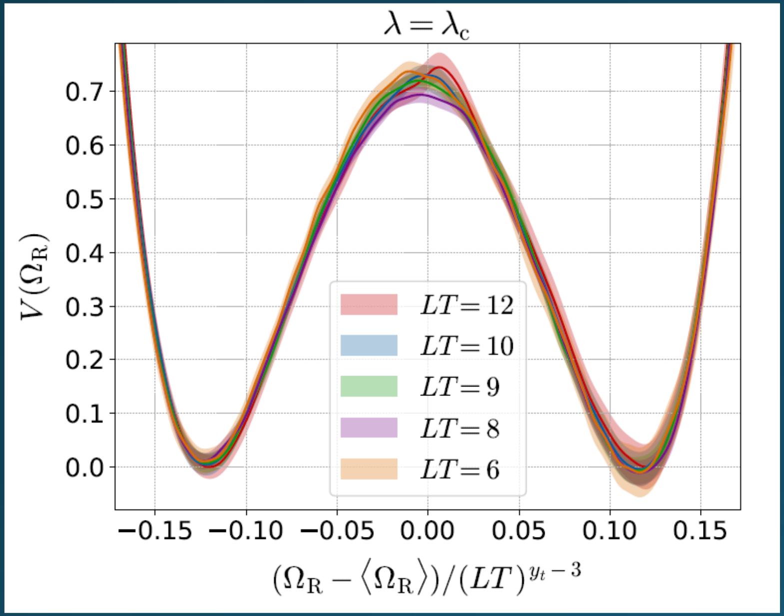


Z2	$B_4 = 1.604$	$\nu = 0.630$
$LT \geq 9$	$B_4 = 1.630(24)(2),$	$\nu = 0.614(48)(3)$
$LT \geq 8$	$B_4 = 1.643(15)(2),$	$\nu = 0.614(29)(3)$

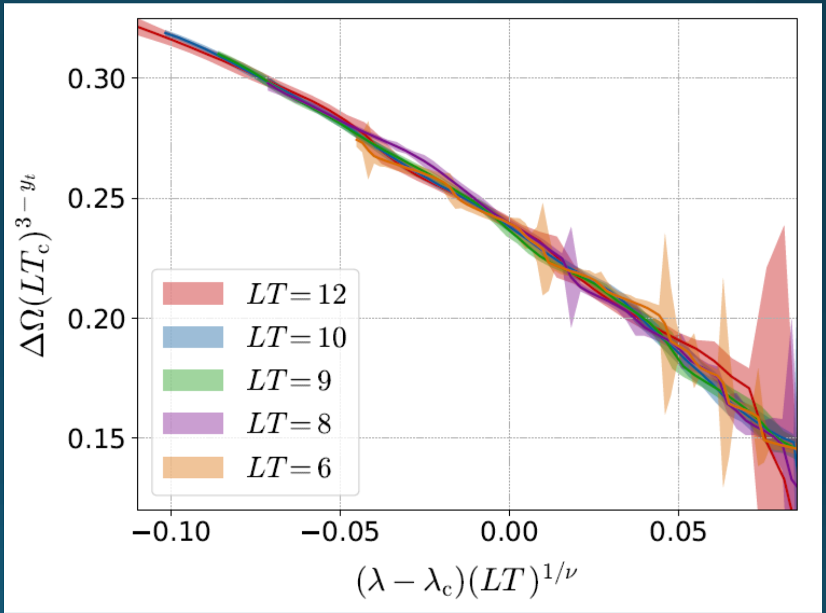
- ☐ B_4 and ν are consistent with Z2 universality class only when $LT \geq 9$ data are used for the analysis.

Further Check of Finite-V Scaling

- Effective potential at the CP



- Scaling of order parameter



Z2 scaling is well established

Contents

Energy-Momentum Tensor

energy momentum

$$T_{\mu\nu} = \begin{bmatrix} T_{00} & T_{01} & T_{02} & T_{03} \\ T_{10} & T_{11} & T_{12} & T_{13} \\ T_{20} & T_{21} & T_{22} & T_{23} \\ T_{30} & T_{31} & T_{32} & T_{33} \end{bmatrix}$$

pressure

stress

All components are important physical observables!

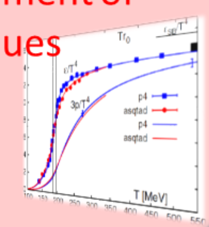
EMT with Gradient Flow
"SFtE Method"

New measurement of the renormalized EMT on the lattice.
Suzuki 2013; FlowQCD 2014~; WHOT-QCD 2017~

Thermodynamics

direct measurement of expectation values

$\langle T_{00} \rangle, \langle T_{ii} \rangle$



Fluctuations and Correlations

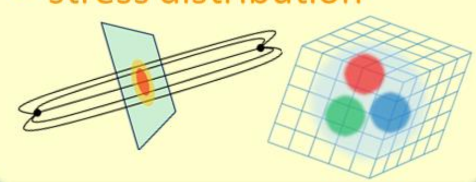
viscosity, specific heat, …

$\eta = \int_0^\infty dt \langle T_{12} ; T_{12} \rangle$

$c_V \sim \langle \delta T_{00}^2 \rangle$

Hadron Structure

- flux tube / hadrons
- stress distribution



Yang-Mills Gradient Flow

$$\frac{\partial}{\partial t} A_\mu(t,x) = -\frac{\partial S_{\rm YM}}{\partial A_\mu}$$

t: "flow time"
dim:[length²]

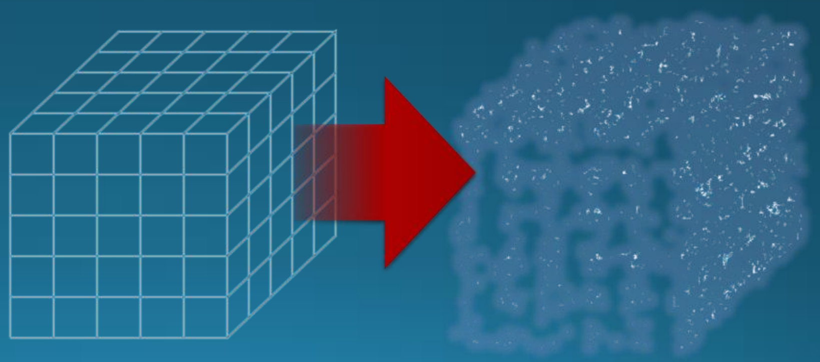
Luscher 2010
Narayanan, Neuberger, 2006
Luscher, Weiss, 2011

$$A_\mu(0,x) = A_\mu(x)$$

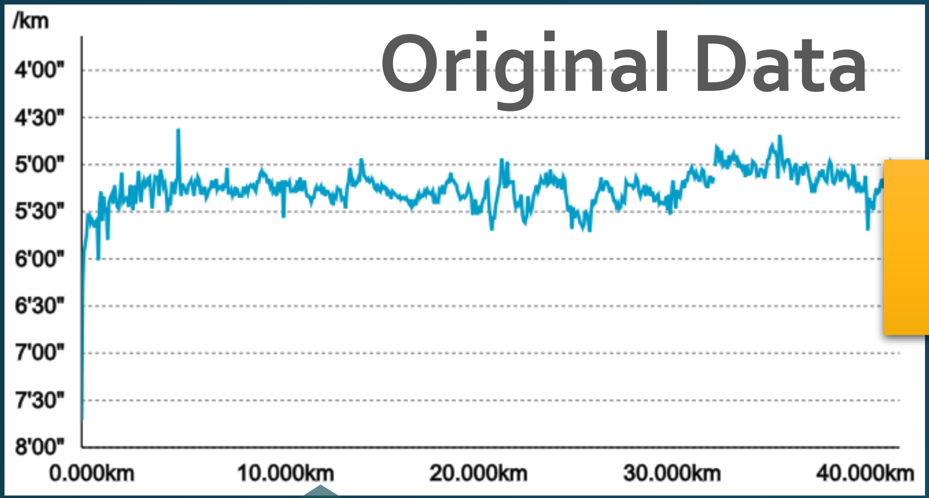
leading

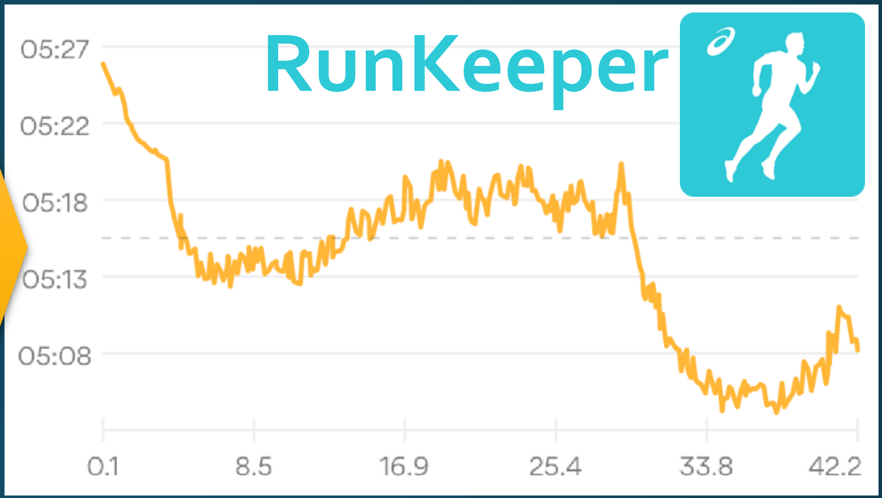
$$\boxed{\partial_t A_\mu} = D_\nu G_{\mu\nu} = \boxed{\partial_\nu \partial_\nu A_\mu} + \cdots$$

- ☐ diffusion equation in 4-dim space
- ☐ diffusion distance $d \sim \sqrt{8t}$
- ☐ "continuous" cooling/smearing
- ☐ No UV divergence at t>0



Gradient Flow = Smearing



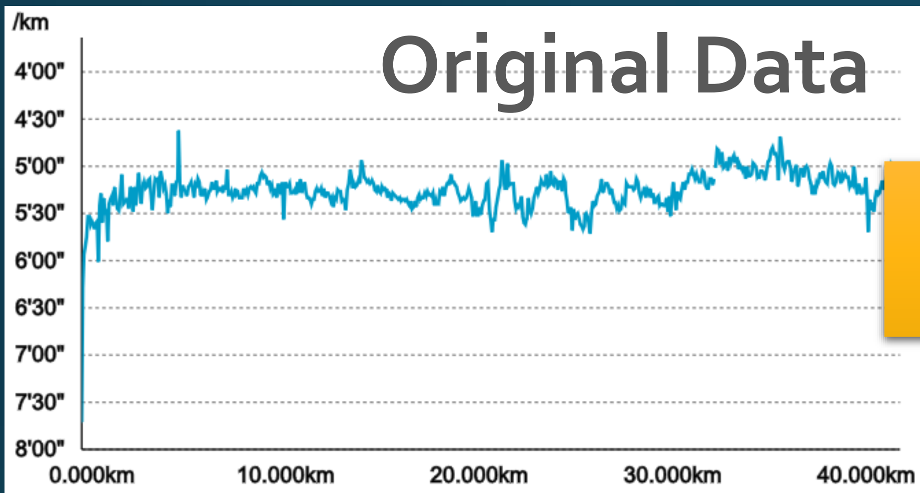


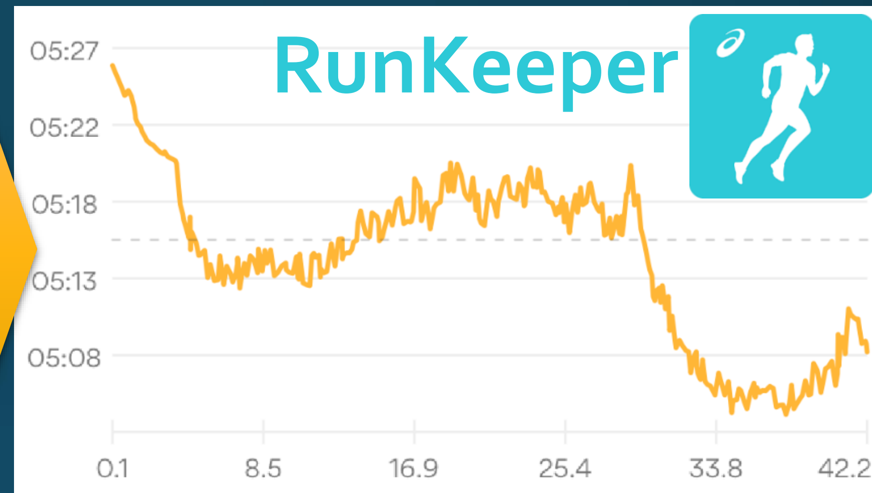
Sasayama Marathon

2019/3/3 (Sun.)

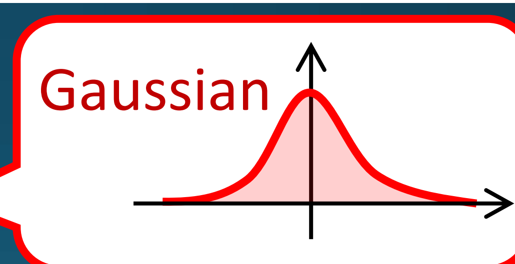
record: 3:42.45

Gradient Flow = Smearing





① $x(t) \to x'(t) \sim \int dt' \exp\left[-\frac{(t-t')^2}{2\sigma^2}\right] x(t')$



$\sigma = \sqrt{2s}$

② $\frac{d}{ds} x(t;s) = \frac{d^2}{dt^2} x(t,s) \qquad x(t;0) = x(t)$

Gradient Flow

$\partial_t A_\mu = \partial_\nu \partial_\nu A_\mu + \cdots$

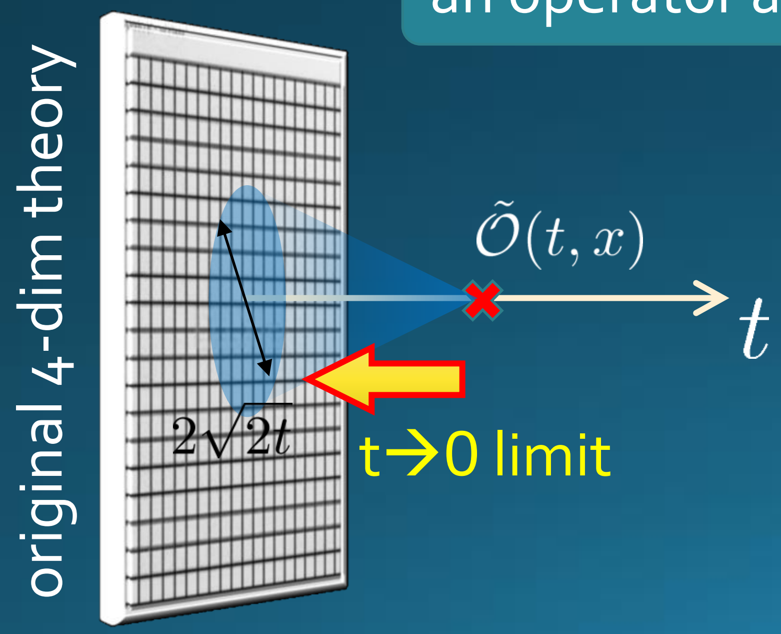
Small Flow-Time Expansion

Luescher, Weisz, 2011
Suzuki, 2013

$$\tilde{\mathcal{O}}(t,x) \xrightarrow[t\to 0]{} \sum_i c_i(t)\mathcal{O}_i^R(x)$$

an operator at t>0

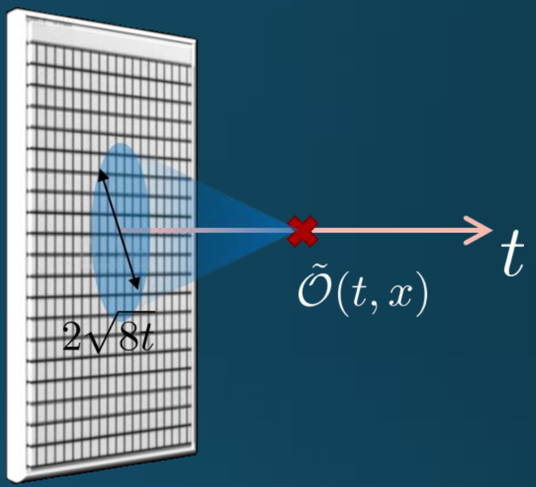
remormalized operators of original theory



Constructing EMT 1

Suzuki, 2013

$$\tilde{\mathcal{O}}(t,x) \xrightarrow[t \to 0]{} \sum_i c_i(t)\mathcal{O}_i^R(x)$$



- **Gauge-invariant dimension 4 operators**

$$\begin{cases} U_{\mu\nu}(t,x) = G_{\mu\rho}(t,x)G_{\nu\rho}(t,x) - \frac{1}{4}\delta_{\mu\nu}G_{\mu\nu}(t,x)G_{\mu\nu}(t,x) \\ E(t,x) = \frac{1}{4}\delta_{\mu\nu}G_{\mu\nu}(t,x)G_{\mu\nu}(t,x) \end{cases}$$

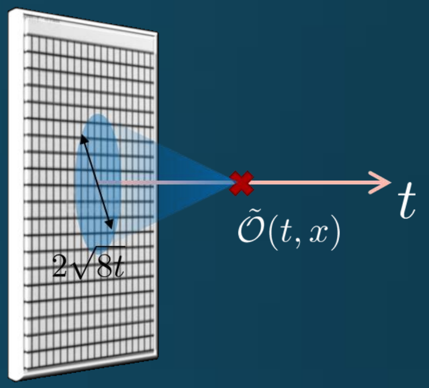
Constructing EMT

Suzuki, 2013

$$U_{\mu\nu}(t,x) = \alpha_U(t)\left[\boxed{T^R_{\mu\nu}(x) - \frac{1}{4}\delta_{\mu\nu}T^R_{\rho\rho}(x)}\right] + \mathcal{O}(t)$$

$$E(t,x) = \langle E(t,x)\rangle + \alpha_E(t)\boxed{T^R_{\rho\rho}(x)} + \mathcal{O}(t)$$

vacuum subtr.



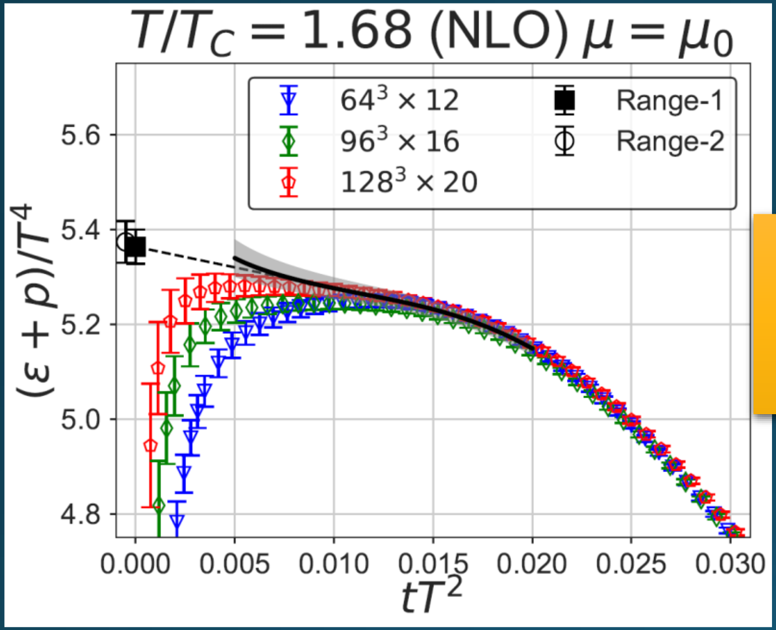
Remormalized EMT

$$T^R_{\mu\nu}(x) = \lim_{t\to 0}\left[c_1(t)U_{\mu\nu}(t,x) + \delta_{\mu\nu}c_2(t)E(t,x)_{\mathrm{subt.}}\right]$$

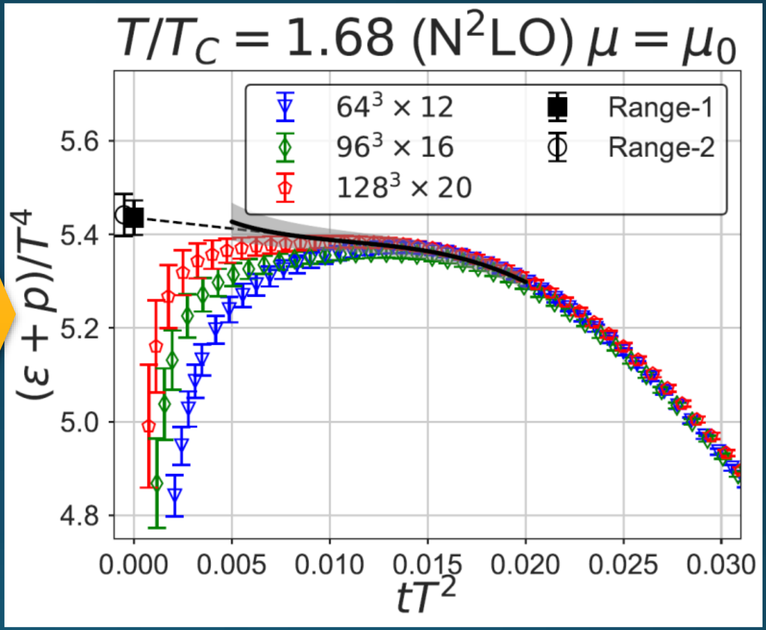
Perturbative coefficient:
Suzuki (2013); Makino, Suzuki (2014); Harlander+ (2018); Iritani, MK, Suzuki, Takaura (2019)

Higher Order Coefficient: ε+p

NLO (1-loop)



N²LO (2-loop)



Iritani, MK, Suzuki, Takaura, PTEP 2019

- ☐ t dependence becomes milder with higher order coeff.
- ☐ Better t→0 extrapolation

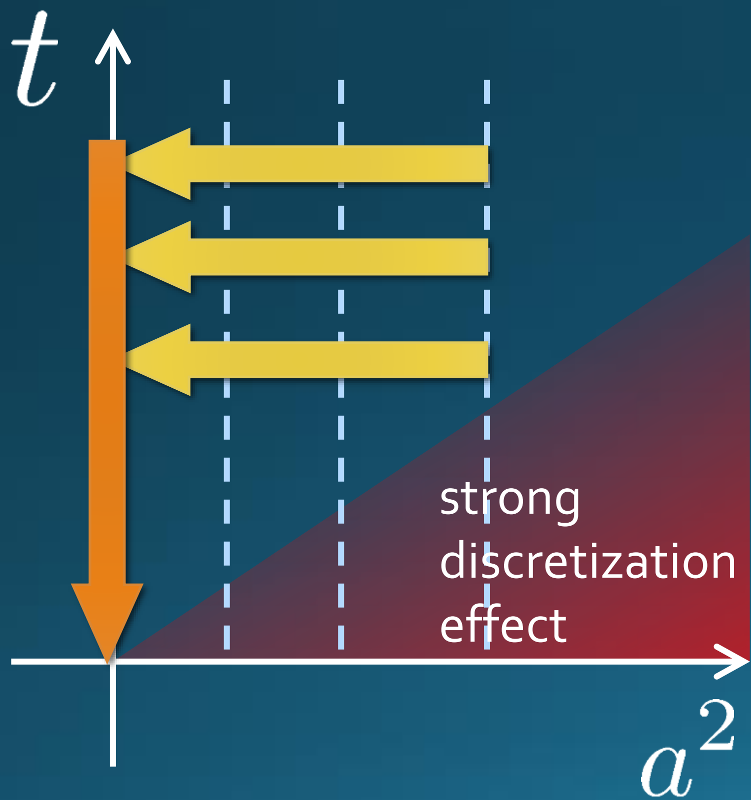
- ☐ Systematic error: μ_0 or μ_d, uncertaintyof Λ (±3%), fit range
- ☐ Extrapolation func: linear, higher order term in c_1 ($\sim g^6$)

Double Extrapolation

$t \to 0,\ a \to 0$

$$\langle T_{\mu\nu}(t)\rangle_{\rm latt} = \langle T_{\mu\nu}(t)\rangle_{\rm phys} + \boxed{C_{\mu\nu} t} + \boxed{D_{\mu\nu}(t)\frac{a^2}{t}}$$

O(t) terms in SFTE lattice discretization



Continuum extrapolation

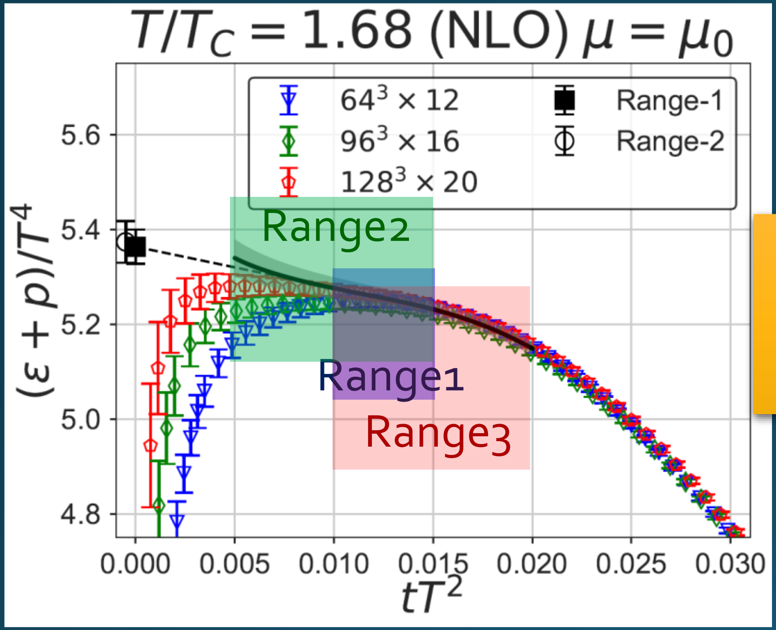
$$\langle T_{\mu\nu}(t)\rangle_{\rm cont} = \langle T_{\mu\nu}(t)\rangle_{\rm lat} + C(t)a^2$$

Small t extrapolation

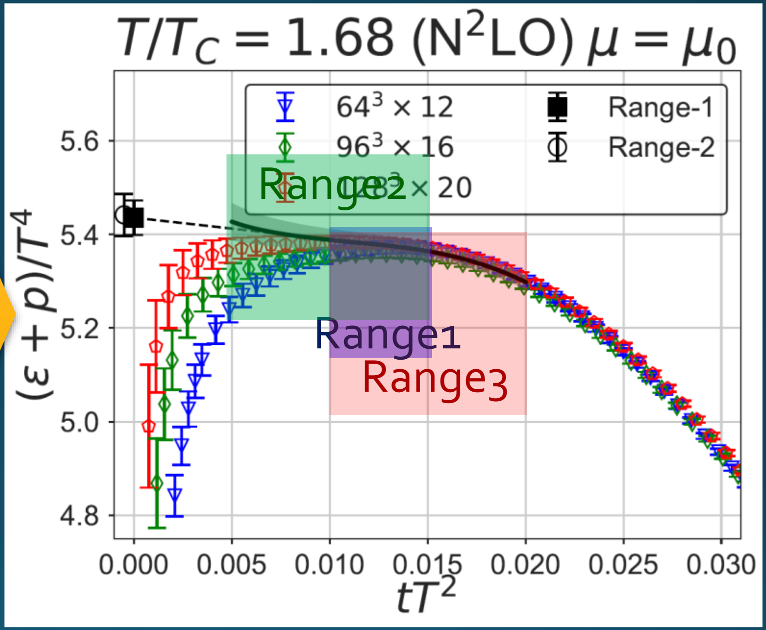
$$\langle T_{\mu\nu}\rangle = \langle T_{\mu\nu}(t)\rangle + C' t$$

Higher Order Coefficient: ε+p

NLO (1-loop)



N²LO (2-loop)



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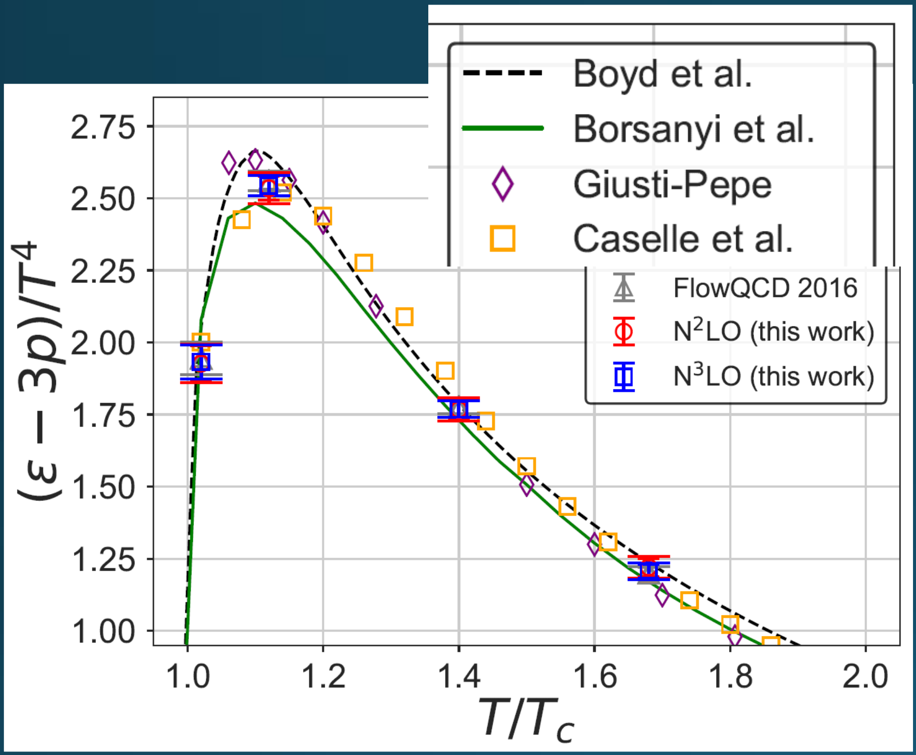
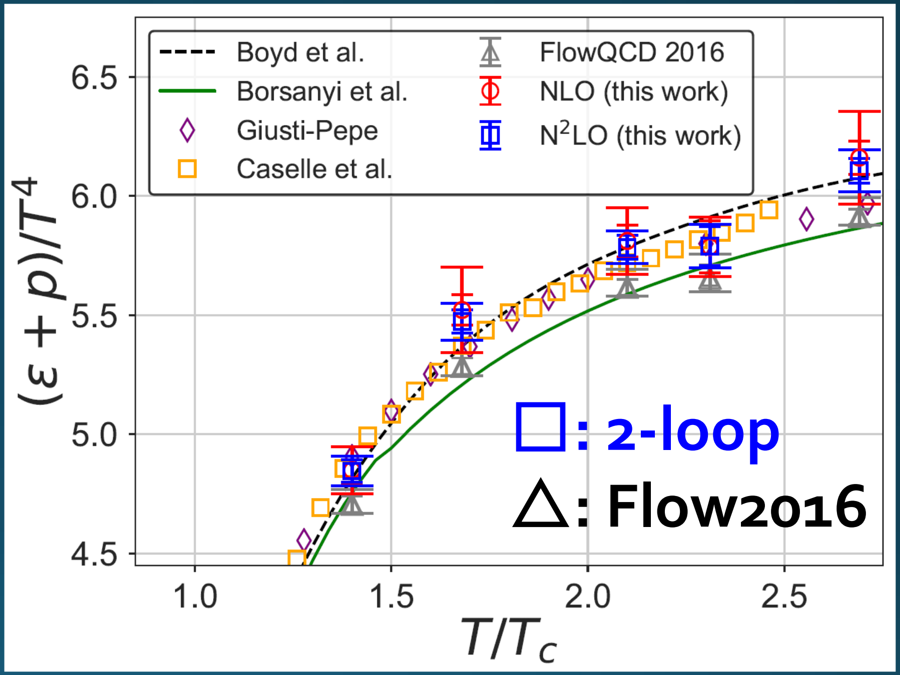
Effect of Higher-Order Coeffs.

Iritani, MK, Suzuki, Takaura, 2019





Systematic error: μ_0 or μ_d, Λ, t→0 function, fit range

More stable extrapolation with higher order c_1 & c_2 (pure gauge)

Energy-Momentum Tensor

$$\begin{array}{c} \qquad\qquad\qquad \text{energy} \qquad\qquad\qquad \text{momentum} \\ T_{\mu\nu} = \begin{bmatrix} \boxed{T_{00}} & T_{01} & T_{02} & T_{03} \\ T_{10} & T_{11} & T_{12} & T_{13} \\ T_{20} & T_{21} & T_{22} & T_{23} \\ T_{30} & T_{31} & T_{32} & T_{33} \end{bmatrix} \\ \qquad\qquad\qquad\qquad\qquad \text{stress} \end{array}$$

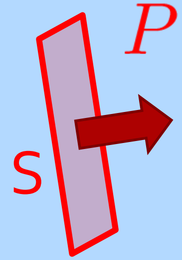
pressure

Spatial components of EMT: Stress Tensor

Stress = Force per Unit Area

Stress = Force per Unit Area

Pressure

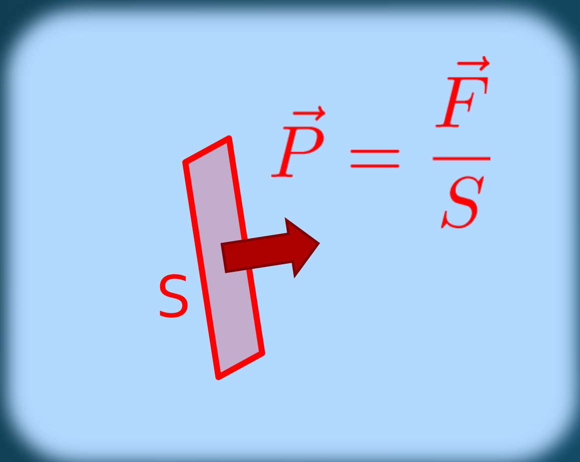


$$\vec{P} = \frac{\vec{F}}{S}$$

$$\vec{P} = P\vec{n}$$

Stress = Force per Unit Area

Pressure

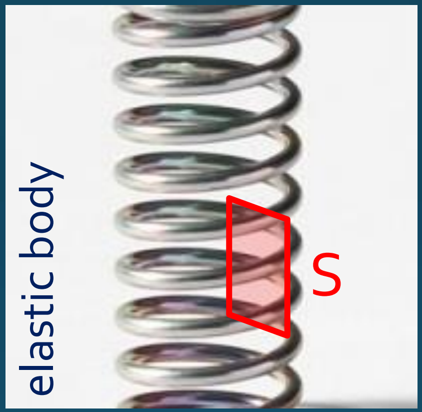


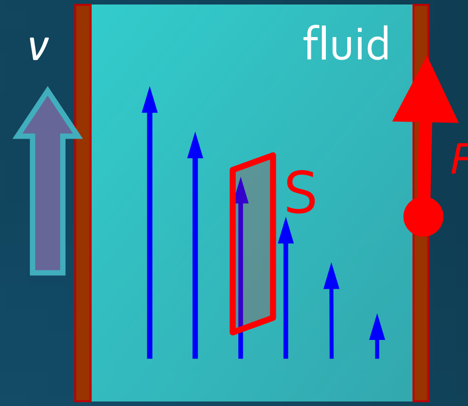
$$\vec{P} = P\vec{n}$$

In thermal medium

$$T_{ij} = P\delta_{ij}$$

Generally, F and n are not parallel





$$\frac{F_i}{S} = \sigma_{ij} n_j$$

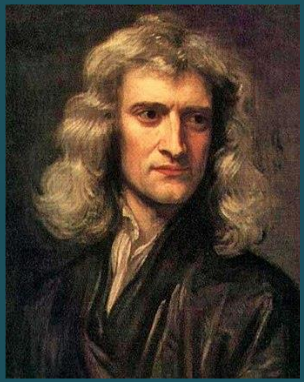
Stress Tensor

$$\sigma_{ij} = -T_{ij}$$

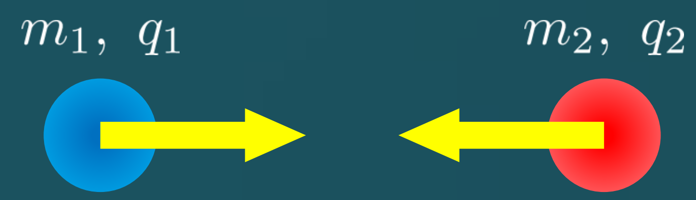
Landau
Lifshitz

Force

Action-at-a-distance



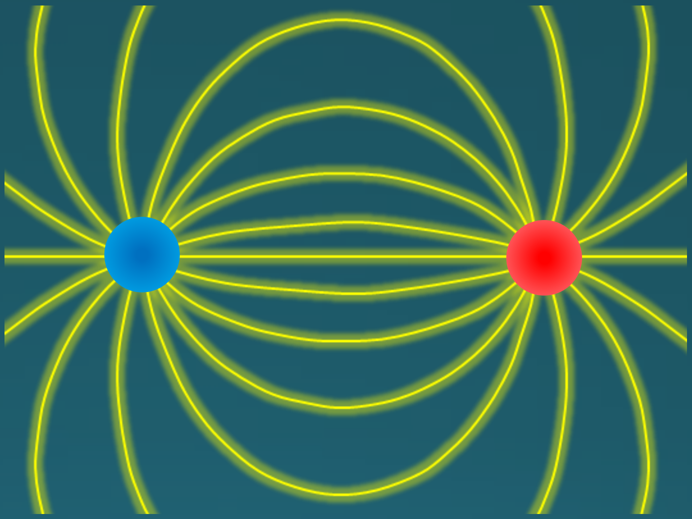
Newton
1687



$$F = -G\frac{m_1 m_2}{r^2} \qquad F = -\frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r^2}$$

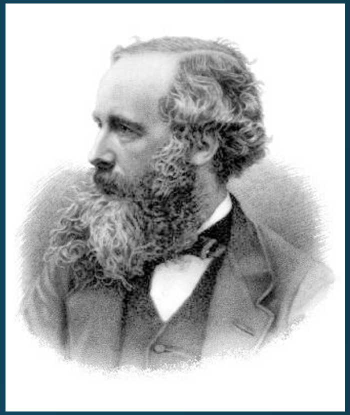
Local interaction

Faraday
1839



Maxwell Stress
(in Maxwell Theory)

$$\sigma_{ij} = \varepsilon_0 E_i E_j + \frac{1}{\mu_0} B_i B_j - \frac{1}{2}\delta_{ij}\left(\varepsilon_0 E^2 + \frac{1}{\mu_0} B^2\right)$$

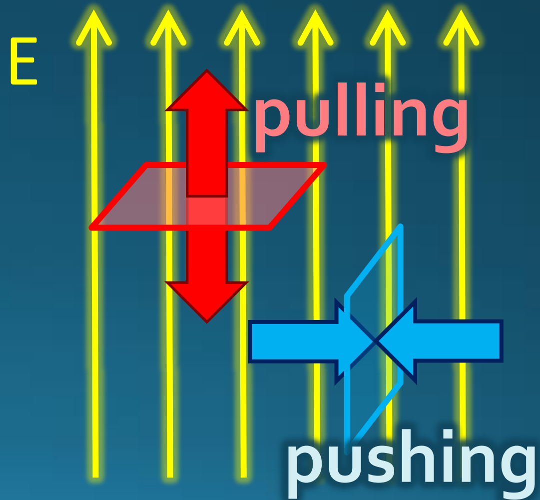


Maxwell

$$\vec{E} = (E, 0, 0)$$

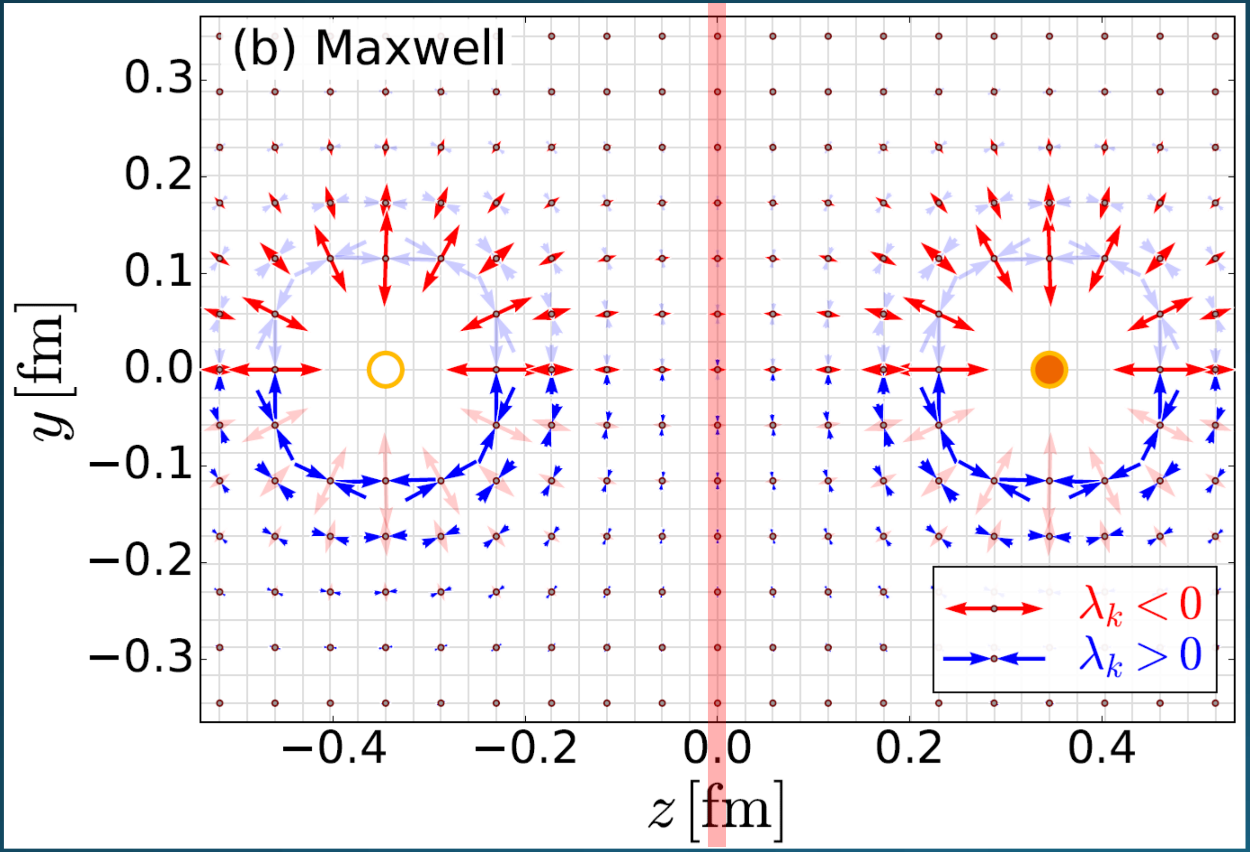
$$T_{ij} = \begin{pmatrix} -E^2 & 0 & 0 \\ 0 & E^2 & 0 \\ 0 & 0 & E^2 \end{pmatrix}$$

- Parallel to field: **Pulling**
- Vertical to field: **Pushing**



Maxwell Stress

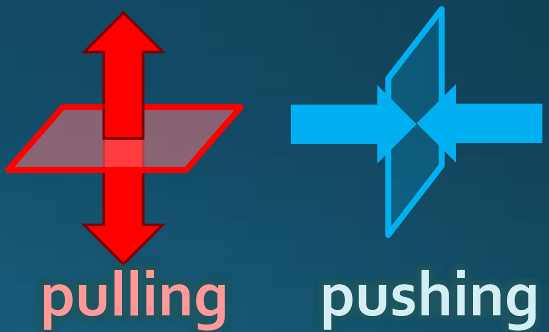
(in Maxwell Theory)



$$T_{ij} v_j^{(k)} = \lambda_k v_i^{(k)}$$

$$(k = 1, 2, 3)$$

length: $\sqrt{|\lambda_k|}$

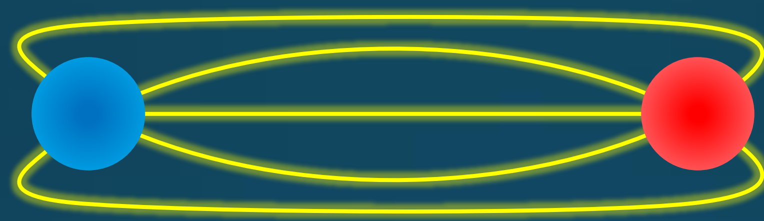


Definite physical meaning

- ☐ Distortion of field, line of the field
- ☐ Propagation of the force as local interaction

Quark-Anti-quark system

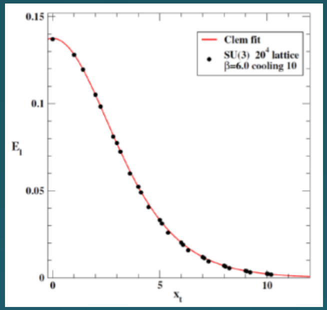
Formation of the flux tube → confinement



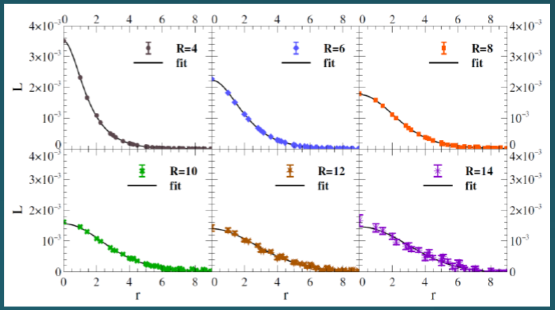
Previous Studies on Flux Tube

- ☐ Potential
- ☐ Action density
- ☐ Color-electric field

so many studies…

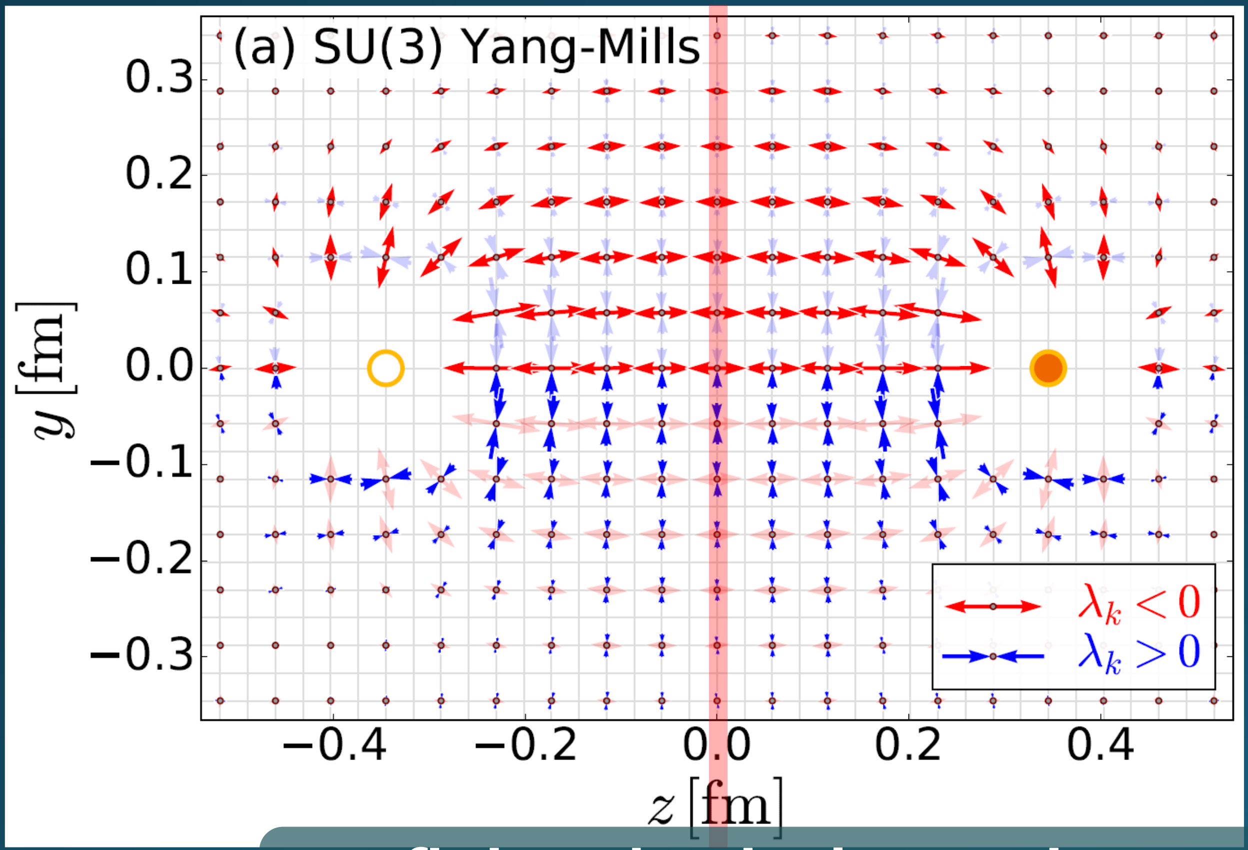


Cea+ (2012)



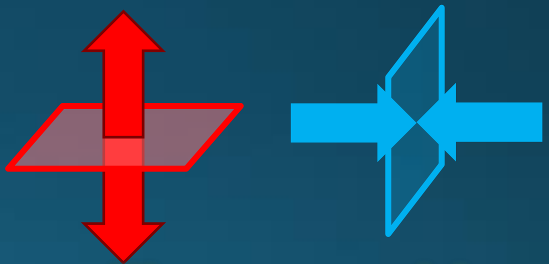
Cardoso+ (2013)

Stress Tensor in $Q\bar{Q}$ System



Yanagihara+, 1803.05656
PLB, in press

Lattice simulation
SU(3) Yang-Mills
a=0.029 fm
R=0.69 fm
t/a²=2.0

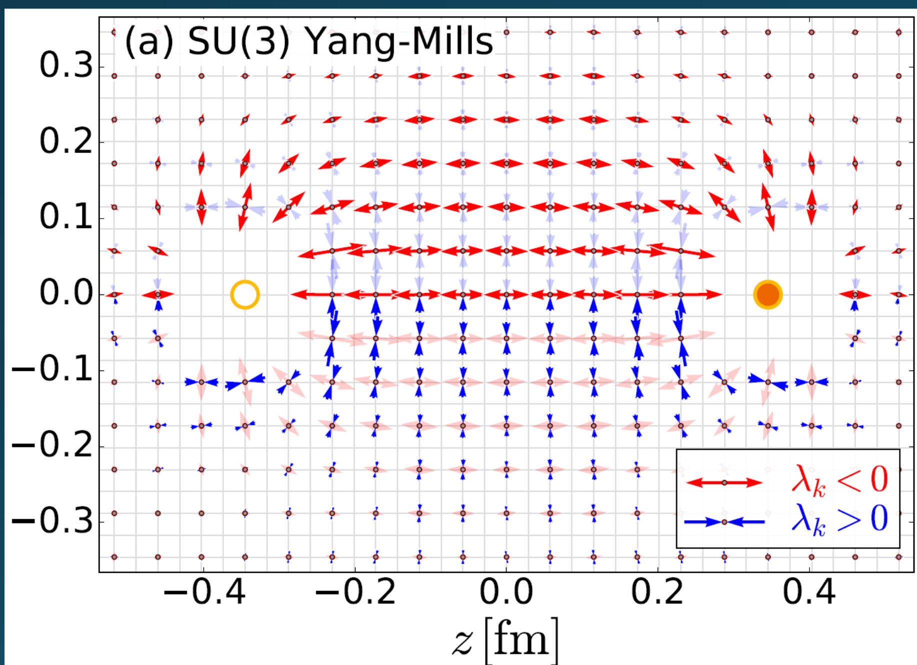


Definite physical meaning

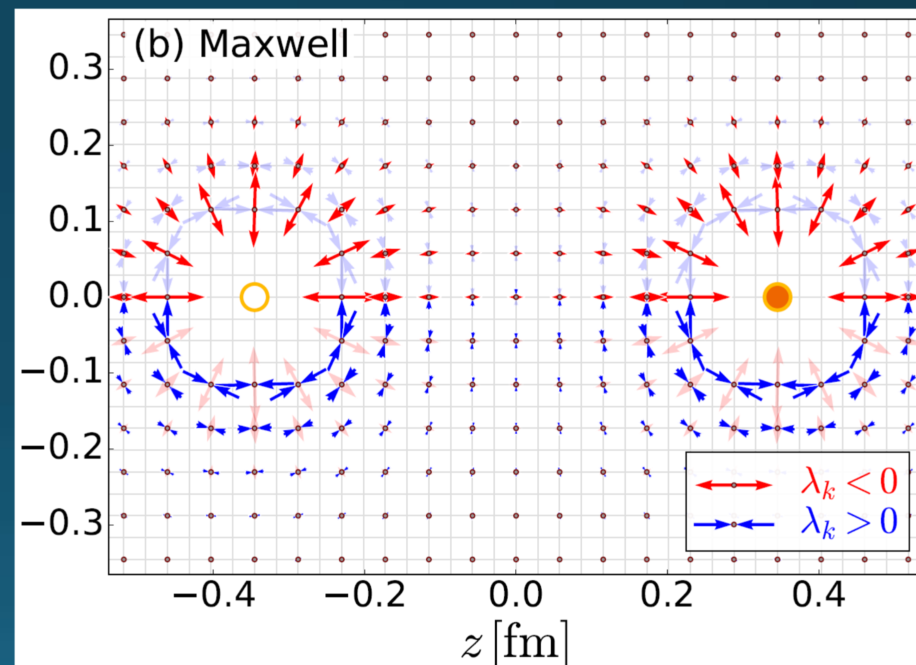
- ☐ Distortion of field, line of the field
- ☐ Propagation of the force as local interaction
- ☐ Manifestly gauge invariant

SU(3) YM vs Maxwell

SU(3) Yang-Mills
(quantum)



Maxwell
(classical)



Propagation of the force is clearly different
in YM and Maxwell theories!

Stress Distribution on Mid-Plane

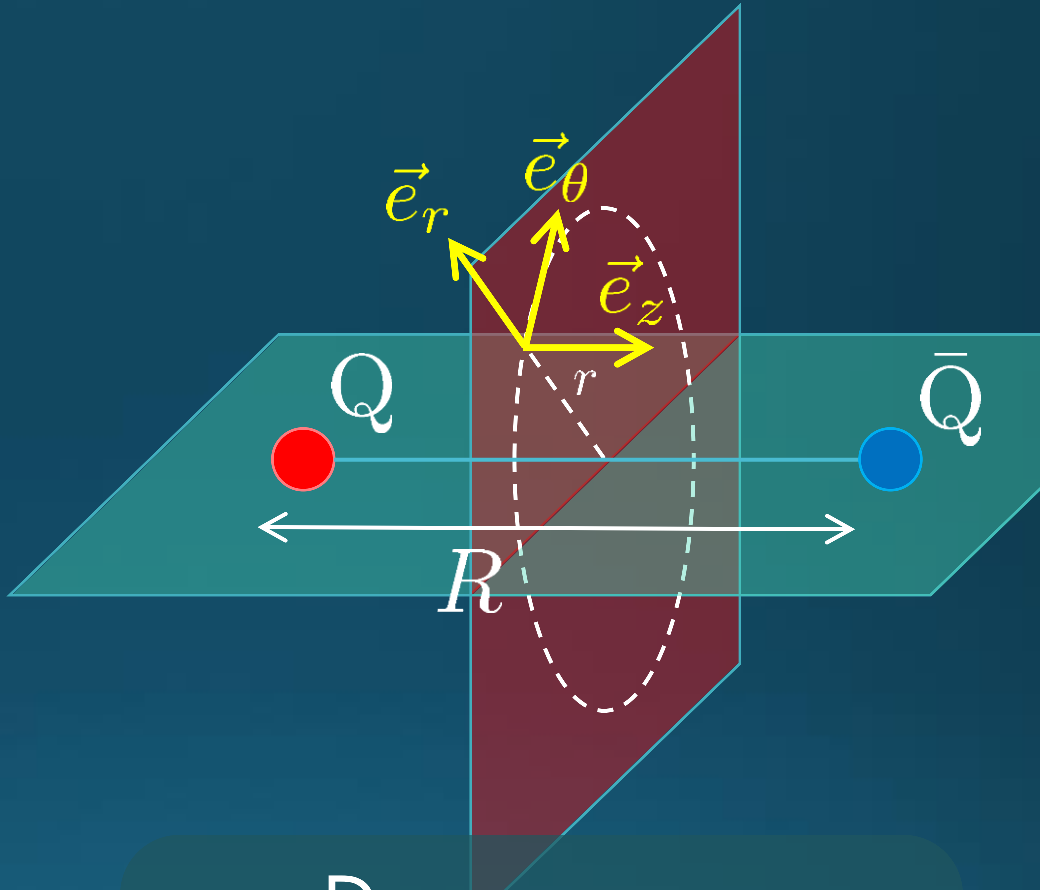
From rotational symm. & parity

EMT is diagonalized
in Cylindrical Coordinates

$$T_{cc'}(r) = \begin{pmatrix} T_{rr} & & & \\ & T_{\theta\theta} & & \\ & & T_{zz} & \\ & & & T_{44} \end{pmatrix}$$

$$T_{rr} = \vec{e}_r^{\,T} T \vec{e}_r$$

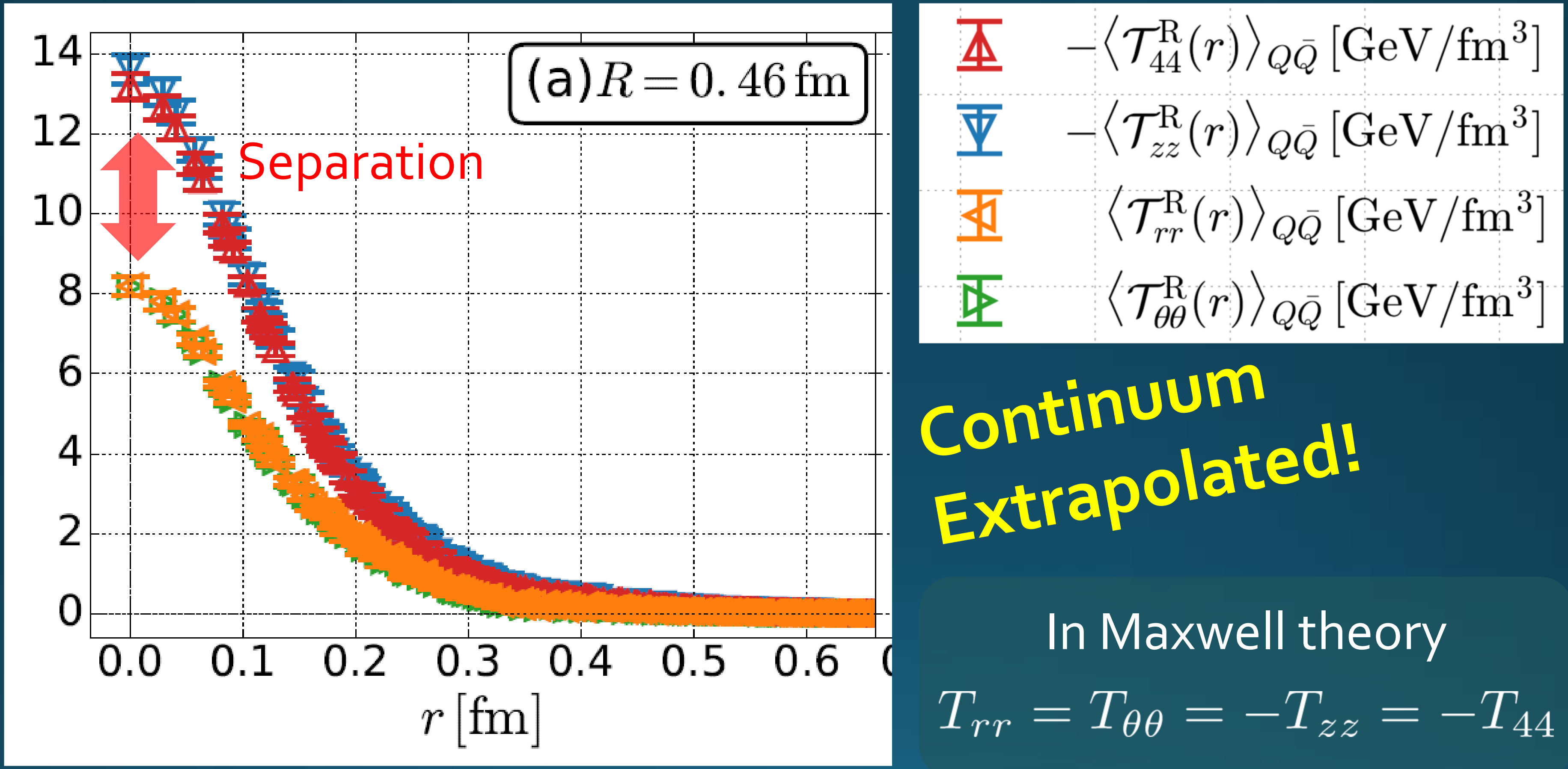
$$T_{\theta\theta} = \vec{e}_\theta^{\,T} T \vec{e}_\theta$$



Degeneracy
in Maxwell theory

$$T_{rr} = T_{\theta\theta} = -T_{zz} = -T_{44}$$

Mid-Plane



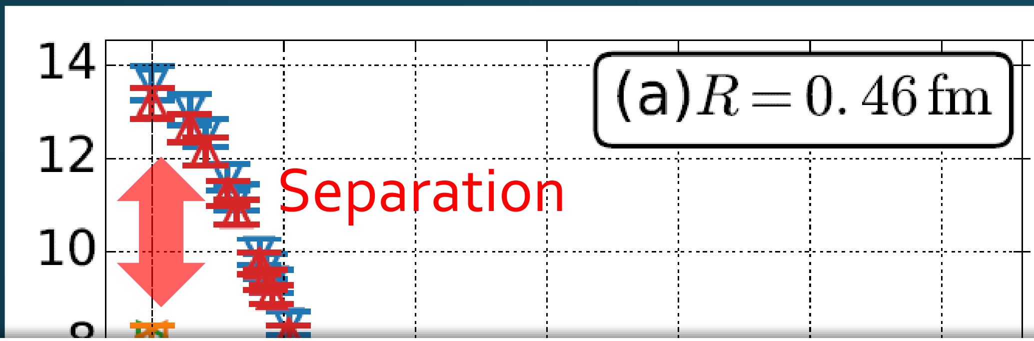
Continuum Extrapolated!

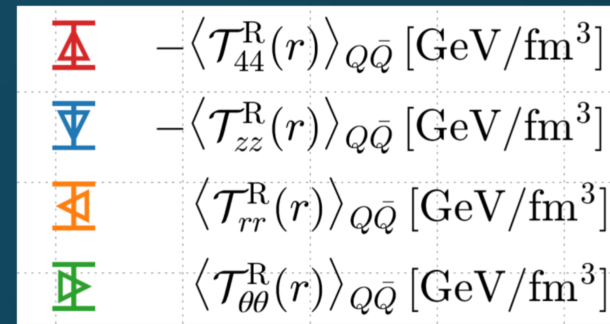
In Maxwell theory

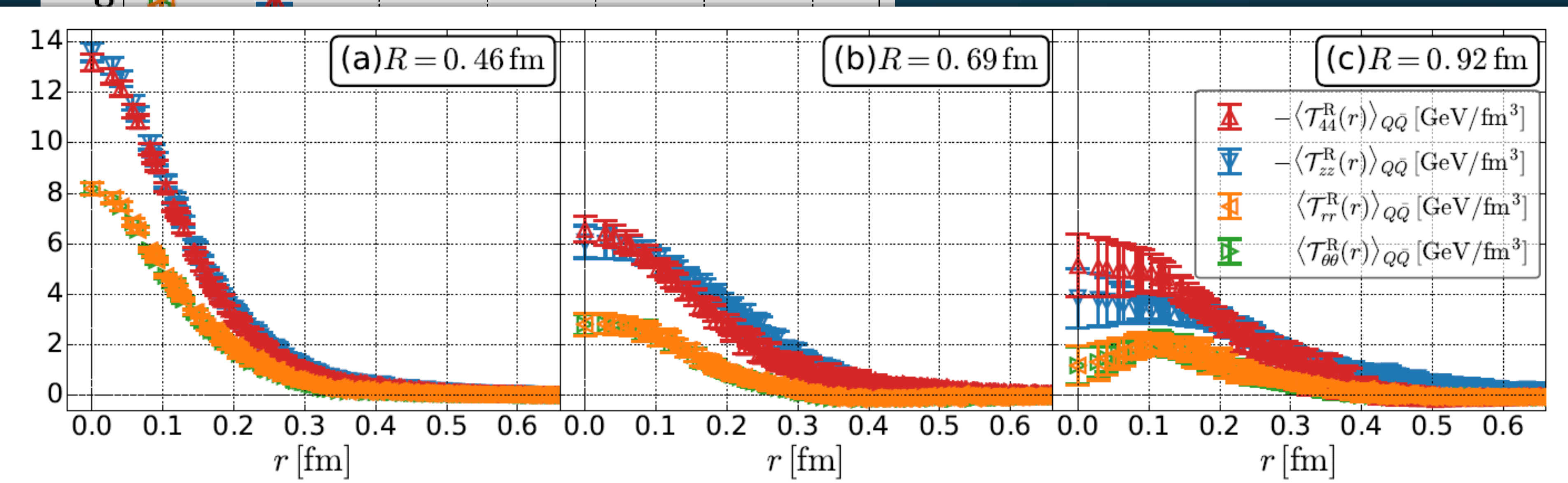
$$T_{rr} = T_{\theta\theta} = -T_{zz} = -T_{44}$$

- ☐ Degeneracy: $T_{44} \simeq T_{zz}, \quad T_{rr} \simeq T_{\theta\theta}$
- ☐ Separation: $T_{zz} \neq T_{rr}$
- ☐ Nonzero trace anomaly $\sum T_{cc} \neq 0$

Mid-Plane







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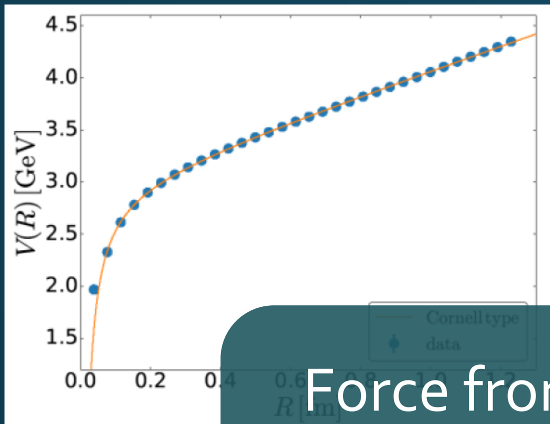
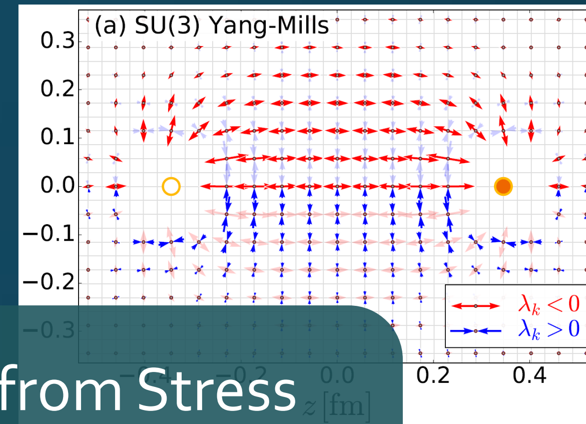
Force





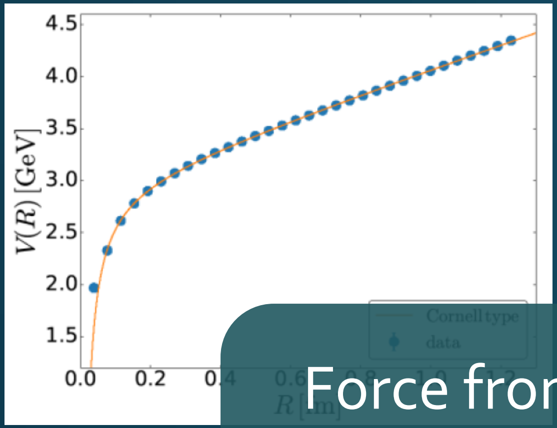
Force from Potential

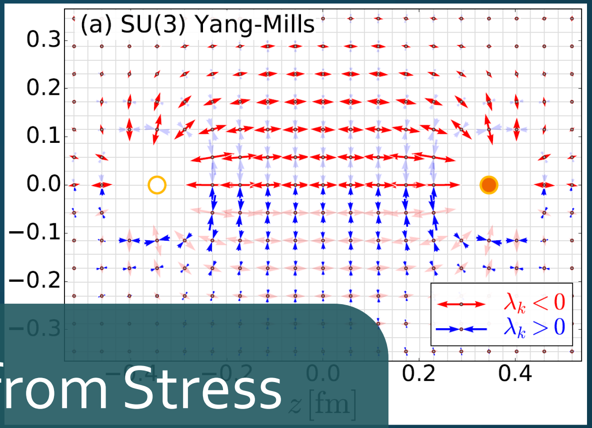
$$F_{\mathrm{pot}} = -\frac{dV}{dR}$$

Force from Stress

$$F_{\mathrm{stress}} = \int_{\mathrm{mid.}} d^2x T_{zz}(x)$$

Force





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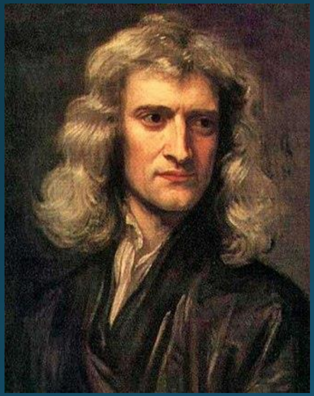
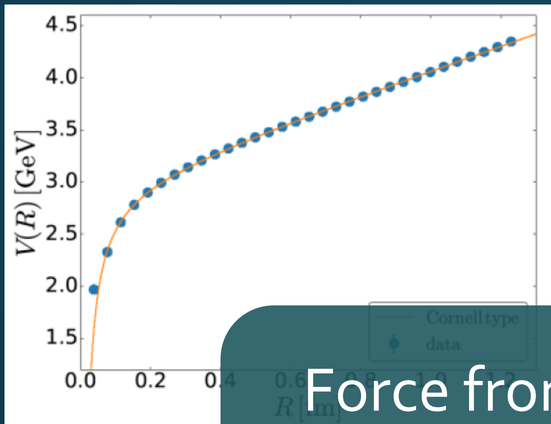
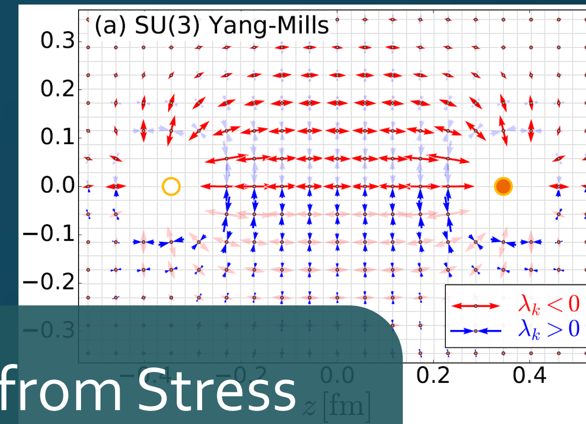


Newton
1687

Faraday
1839

Force





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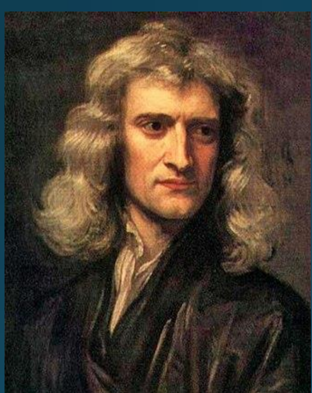

Newton
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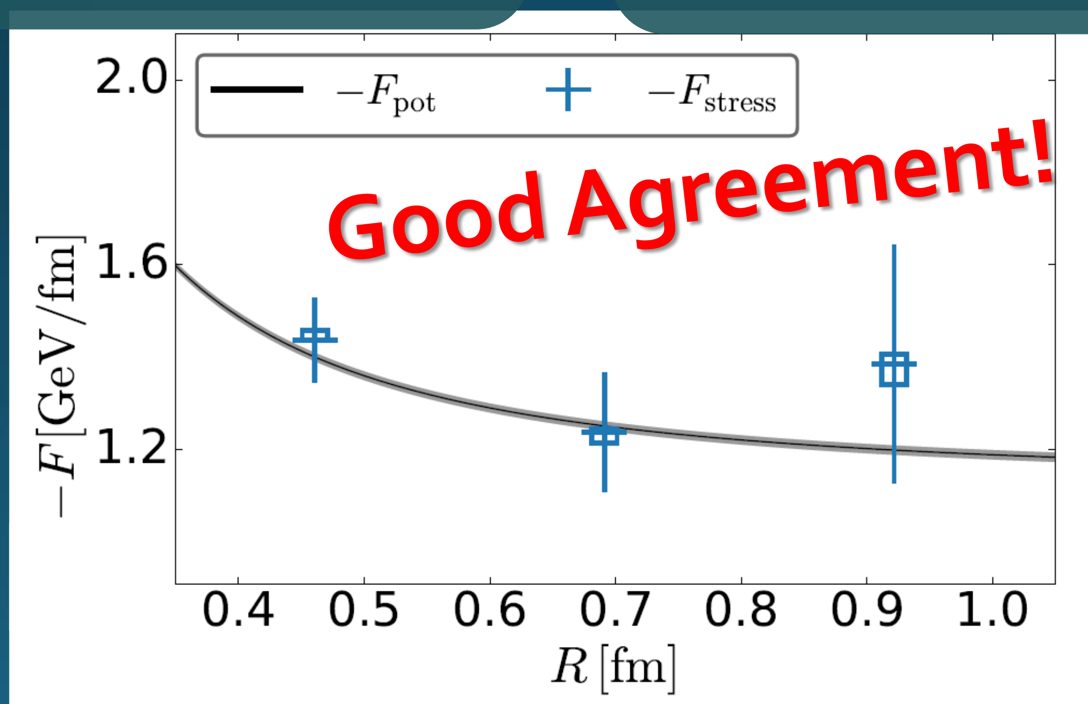


Faraday
1839

Summary

- Lattice QCD numerical simulations are unique tools to investigate non-perturbative aspects of QCD.

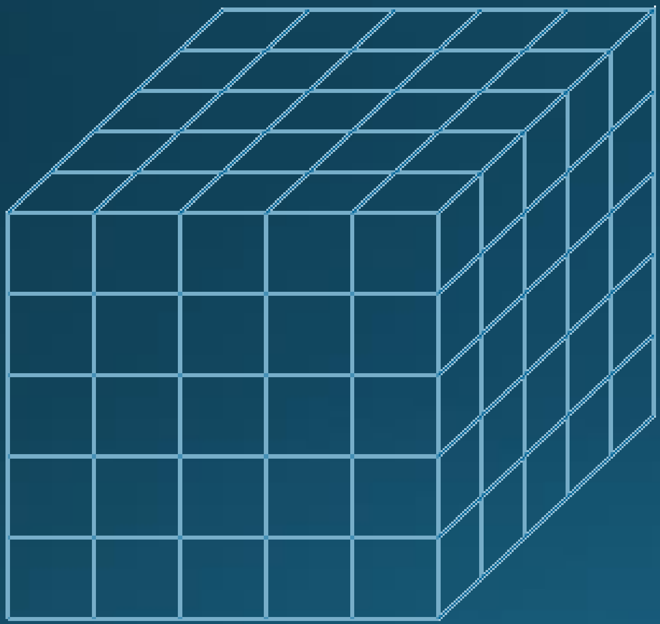
- Observables that can be measured on the lattice are strictly limited due to our ignorance of physical states and Euclidean formulation.
- There still are many things that can be obtained from there.

- **More studies based on novel ideas are awaited!**

Data & Physics

Gauge Configuration

128^4

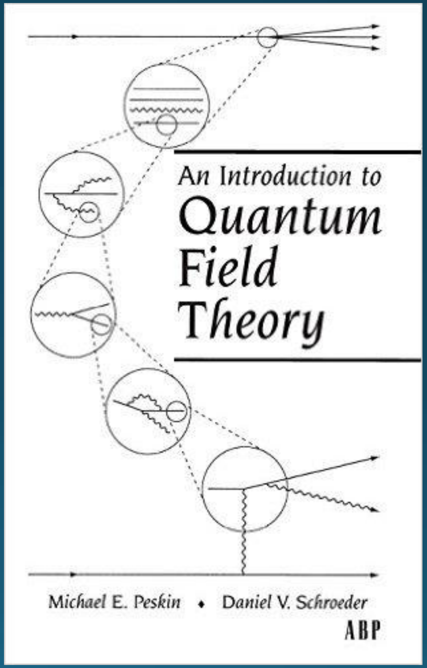


$$128^4 \times 4 \times 9 \times 2 \times 8 \text{ Bytes}$$

$$= 144 \text{ GB}$$

Textbook

Peskin-Schroeder



$$\sim 10\text{MB}$$