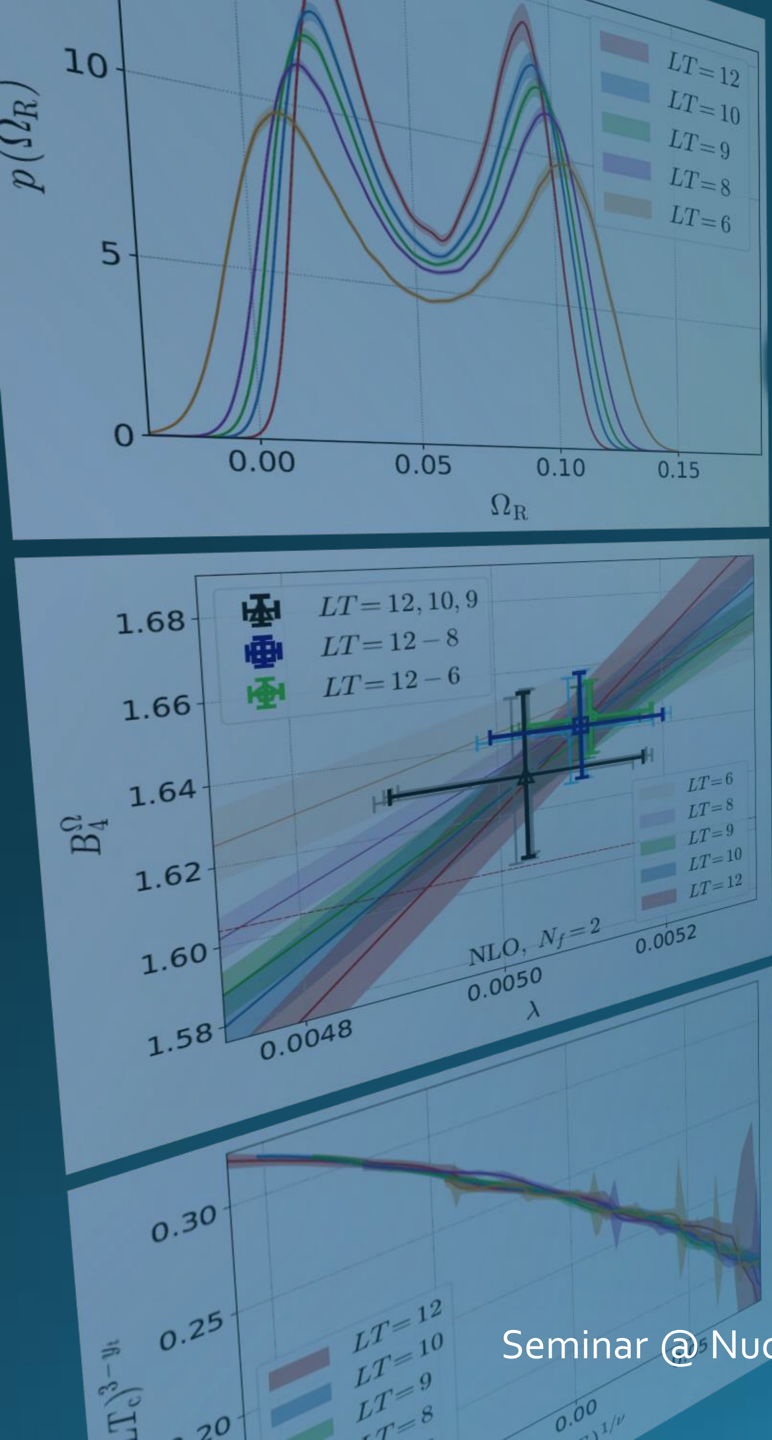


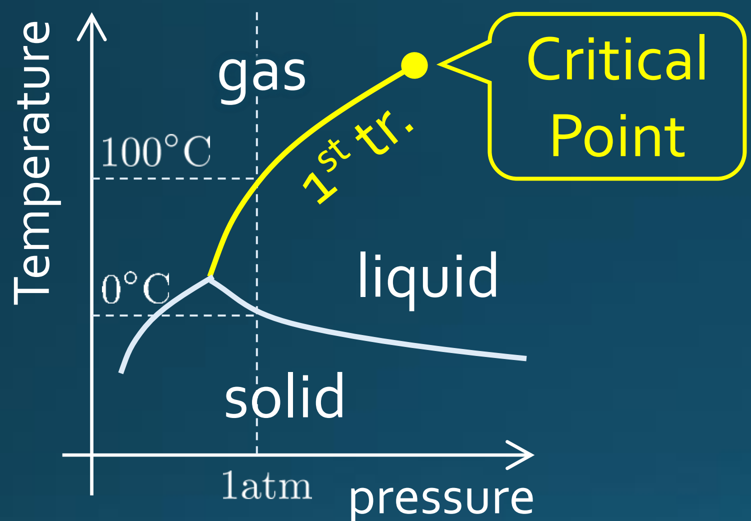
Critical Points in QCD

Masakiyo Kitazawa
(YITP, Kyoto)

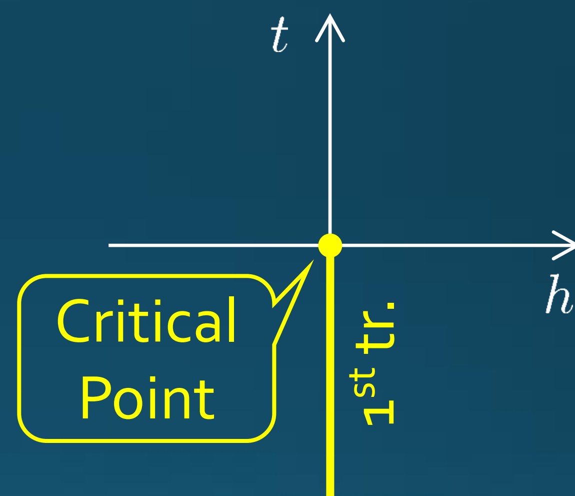
Seminar @ Nuclear Theory Group, Kyoto U., 2022/10/19

Critical Points

- **Water**



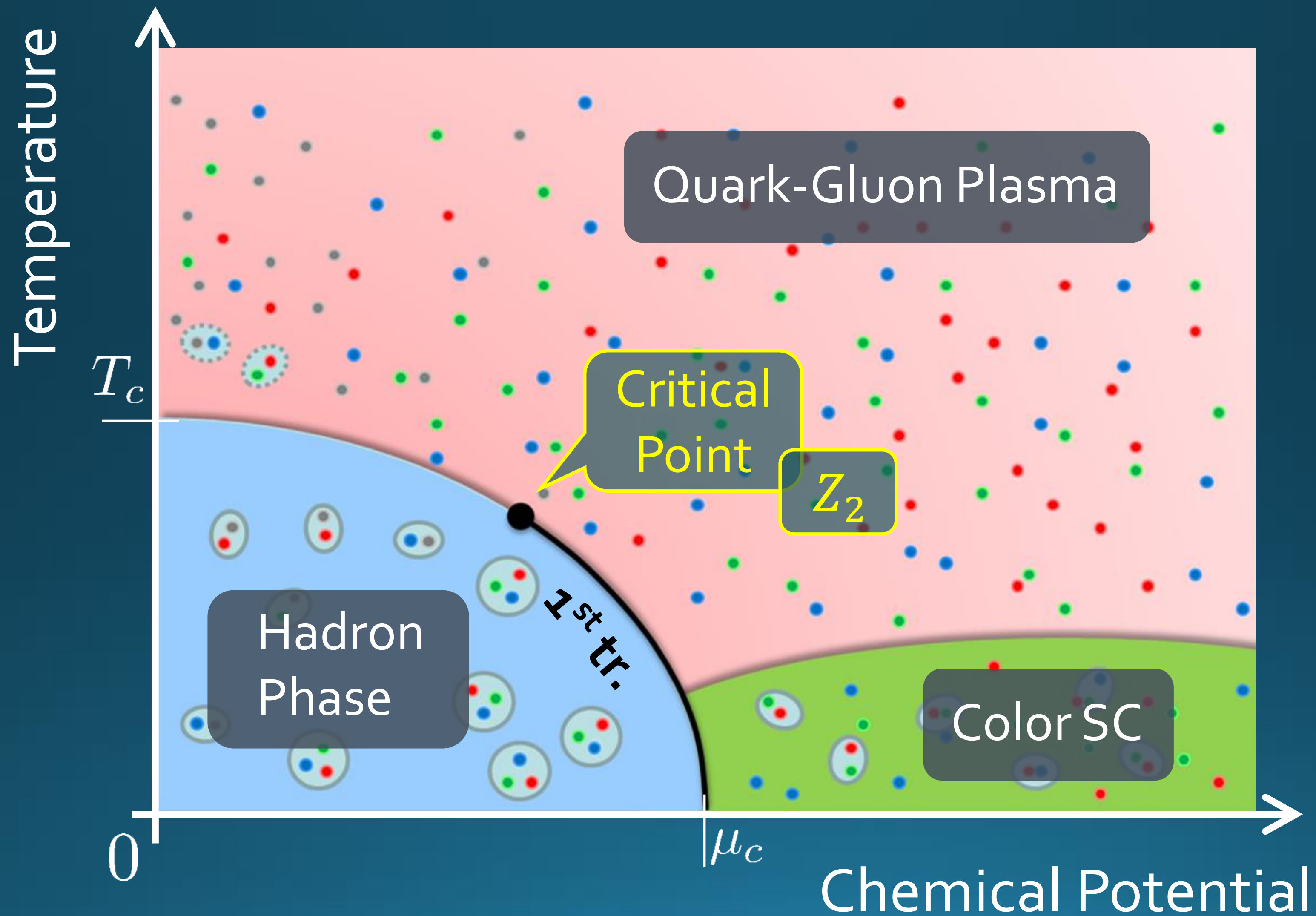
- **Ising Model**



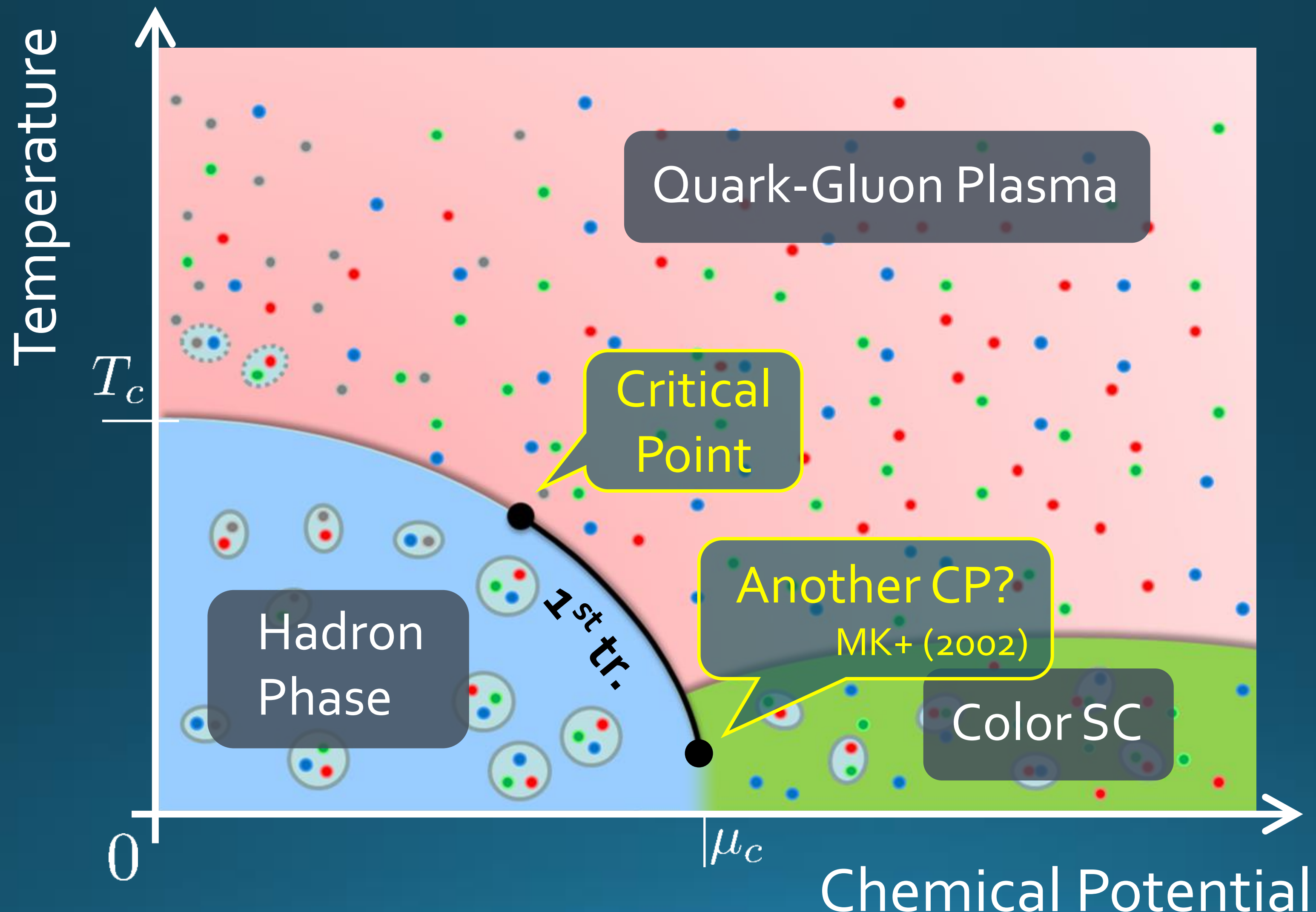
- CP: Second-order transition point.
- Singularities in thermodynamic quantities.
- These CPs belong to the same universality class (Z_2).

→ Common critical exponents. Ex. $C \sim (T - T_c)^{-\alpha}$

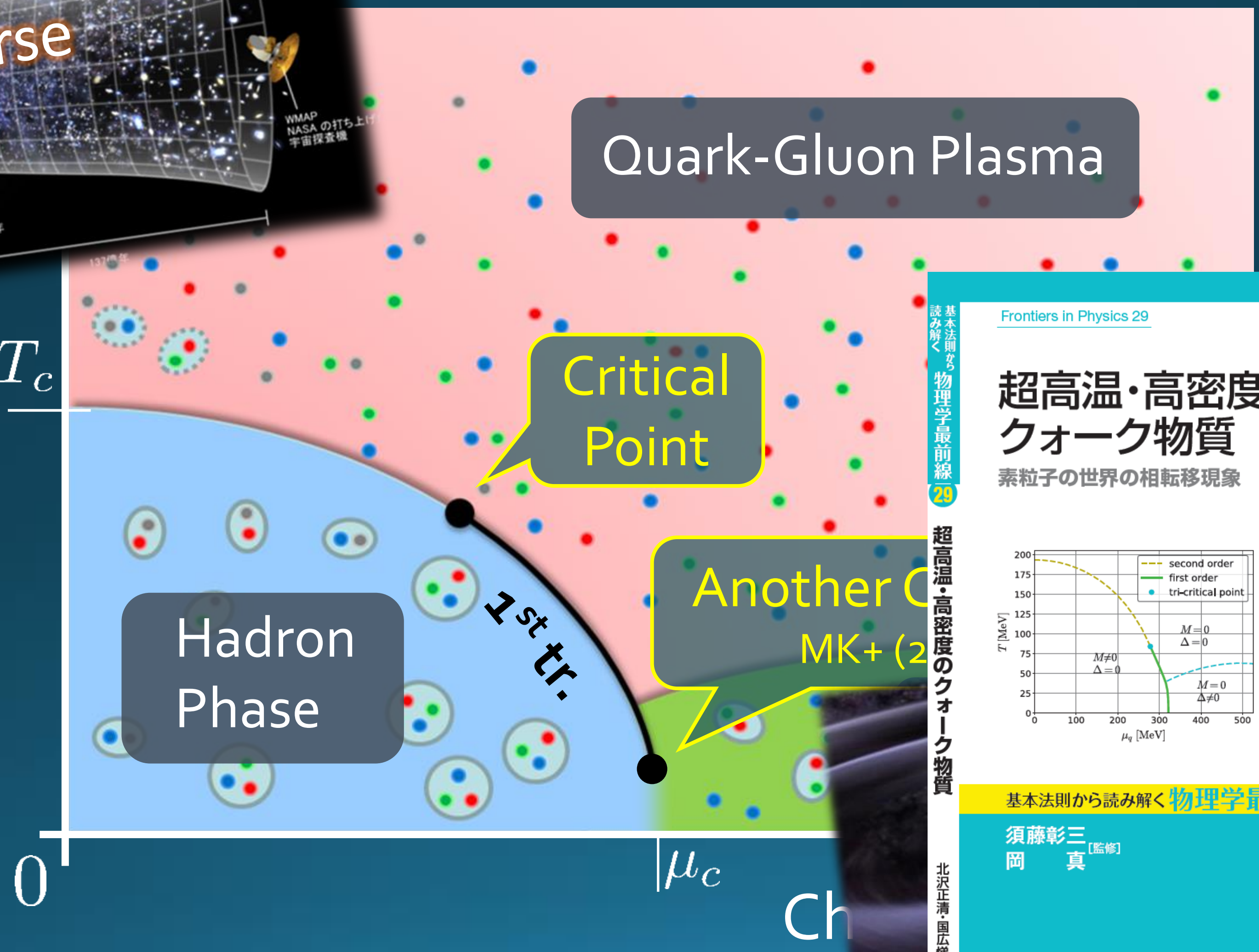
QCD Phase Diagram

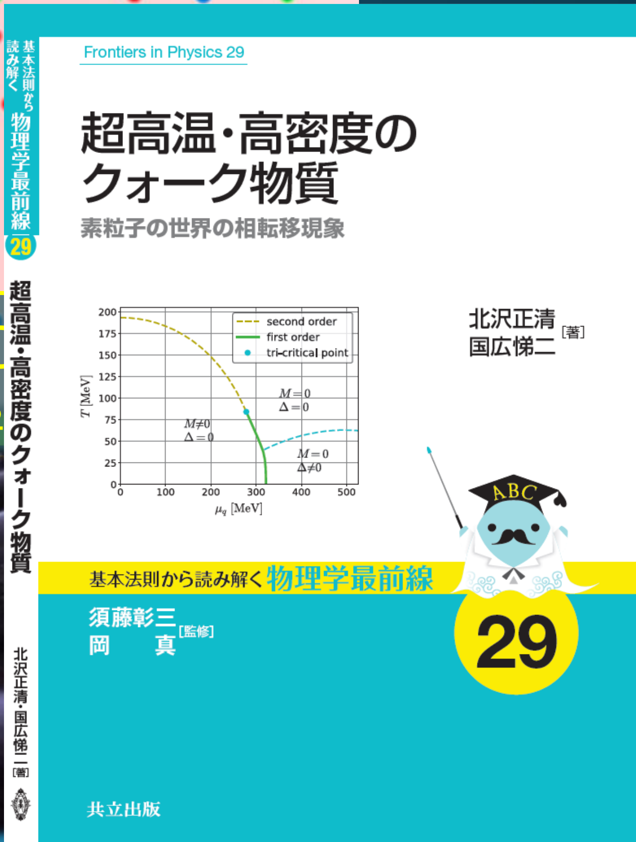


QCD Phase Diagram

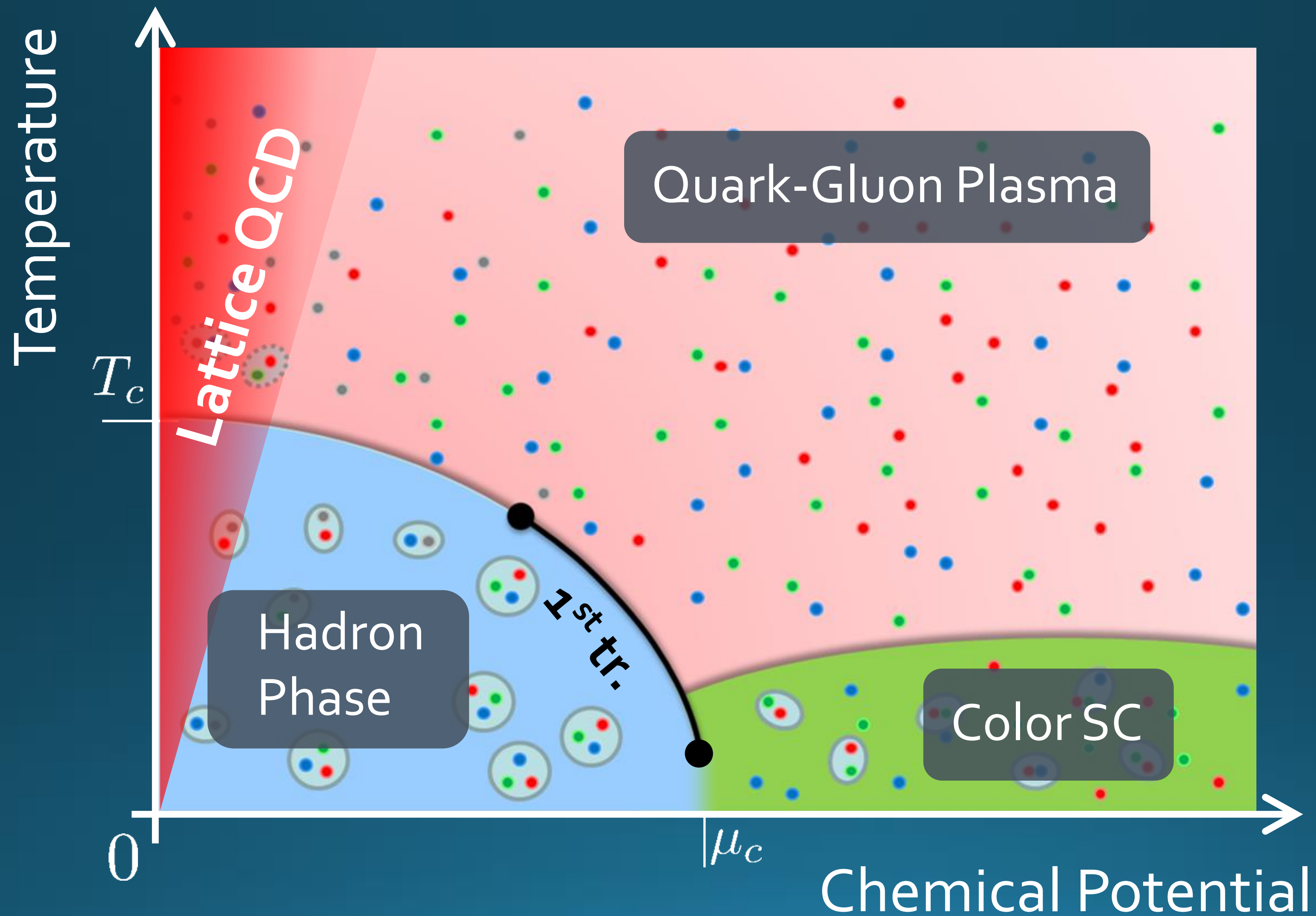


QCD Phase Diagram

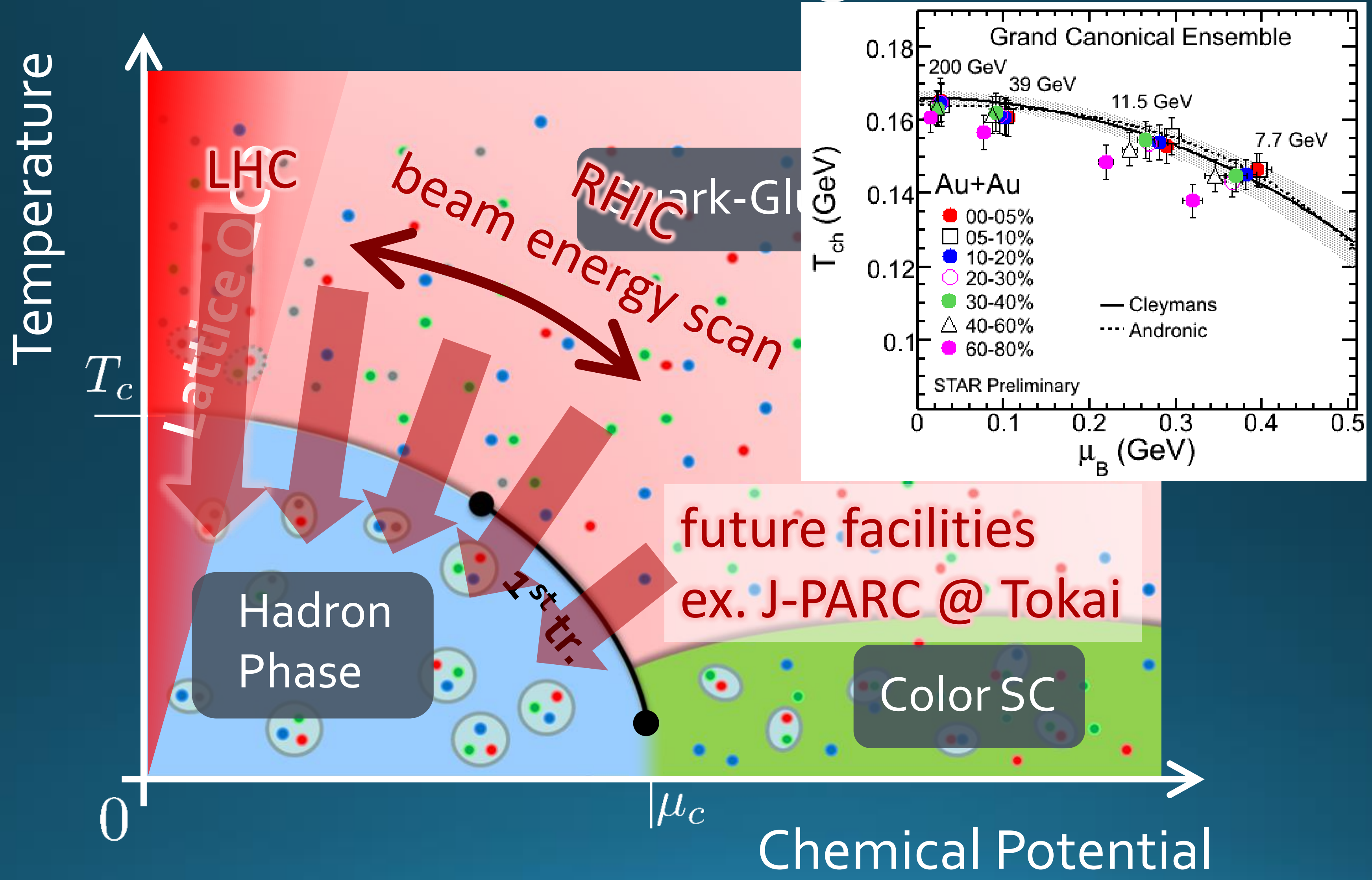




QCD Phase Diagram



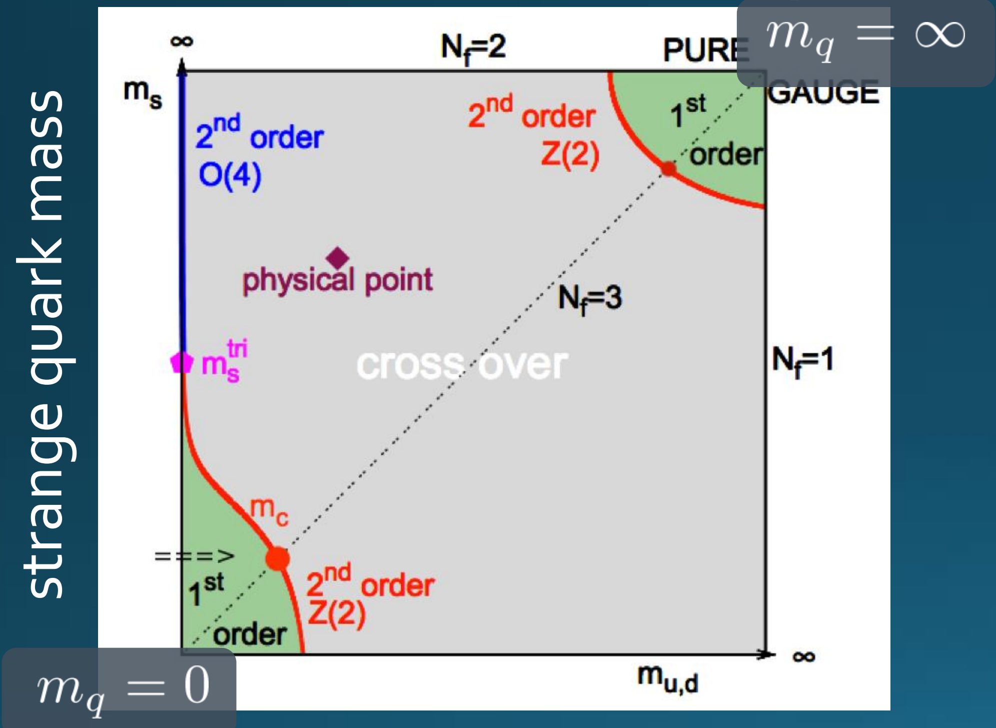
QCD Phase Diagram



Varying Quark Masses @ $\mu_q = 0$

- **Columbia plot**

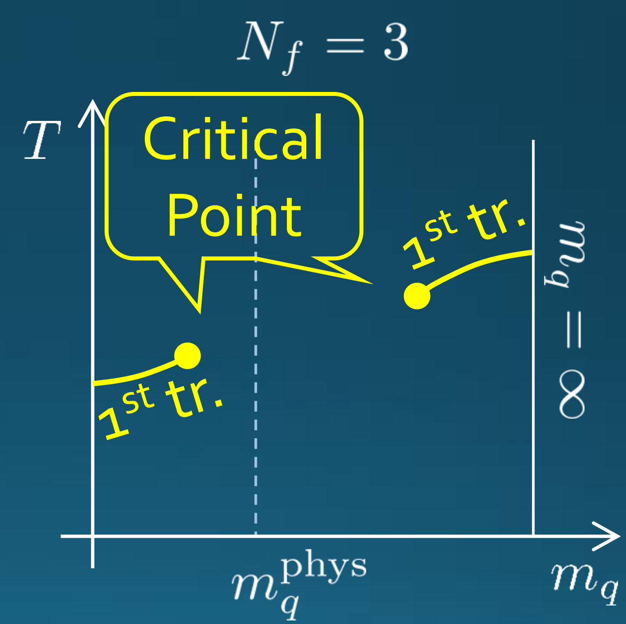
  = order of phase tr. at $\mu_q = 0$



light (ud) quark masses

- **Example**

  Phase diagram in $T - m_q$ plane



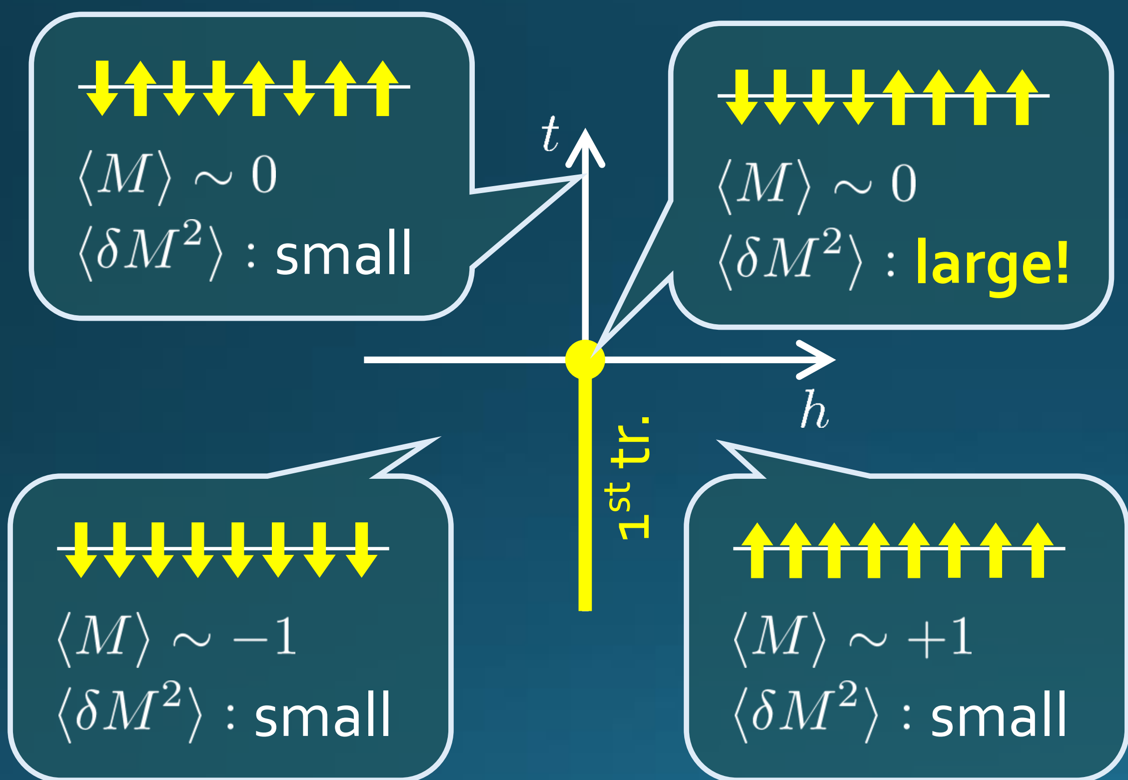
Various orders of phase transition with a variation of m_q.

Fluctuations and Scaling near CP

CP in Ising Model

$$H = -J\sum_{\langle ij\rangle} S_i S_j - H\sum_i S_i$$

$$M = \frac{1}{N}\sum_{i=1}^{N} S_i$$



Scaling of free energy

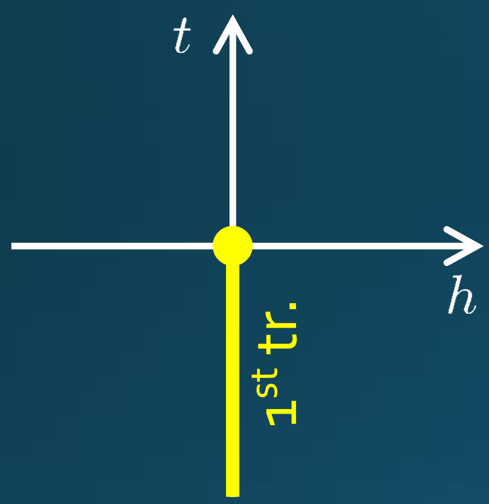
$$F(t,h) = F(b^{y_t} t, b^{y_h} h)$$

$$M = \frac{1}{V}\frac{\partial F}{\partial h}$$

Critical exponents

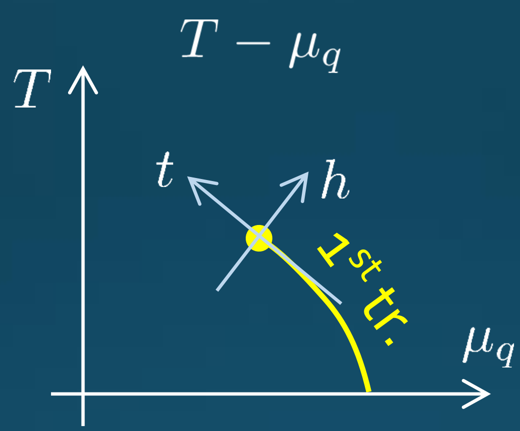
Mapping b/w Ising & QCD

- **Ising Model**

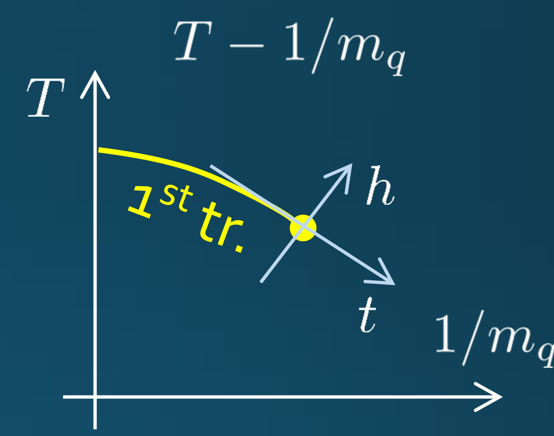


$$F(t,h) = F(b^{y_t} t, b^{y_h} h)$$

- **QCD**



$$\begin{pmatrix} t \\ h \end{pmatrix} = M \begin{pmatrix} T \\ \mu_q \end{pmatrix}$$



$$\begin{pmatrix} t \\ h \end{pmatrix} = M \begin{pmatrix} T \\ m_q^{-1} \end{pmatrix}$$

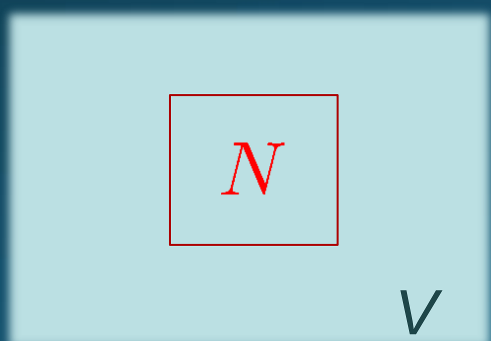
- **Singular part:**

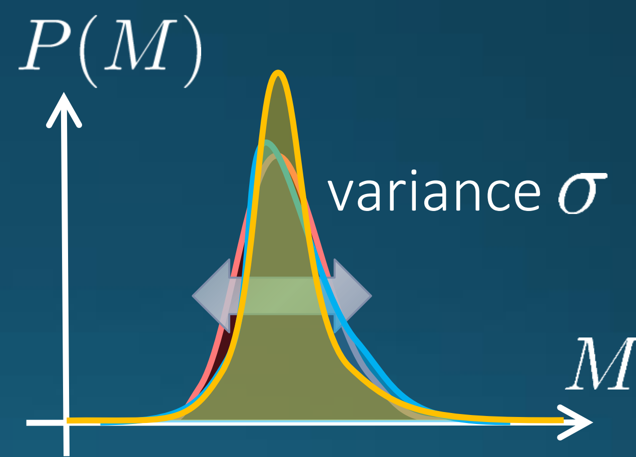
$$F_{\mathrm{QCD}}(T,\mu_q) = F_{\mathrm{Ising}}(M(T,\mu_q))$$

Fluctuations = Prob. Distr.

Observables are fluctuating even in equilibrium!

$$M = \sum_{i=1}^{N} S_i$$





Cumulants

Binder Cumulant

Cumulants

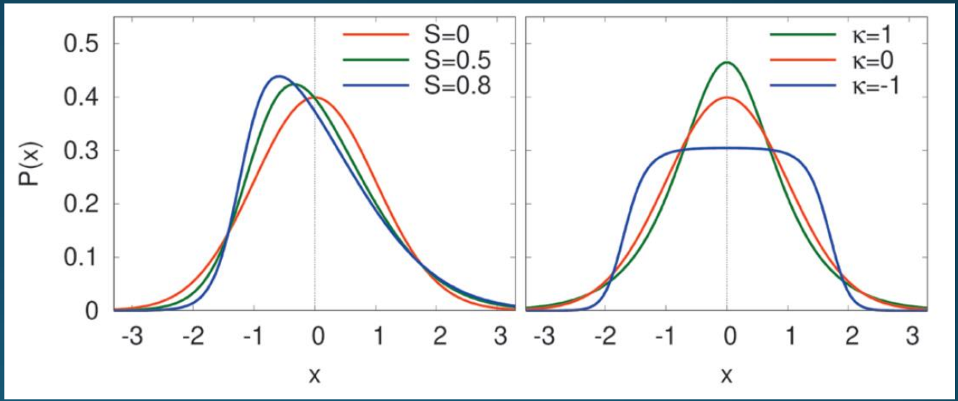
$$\begin{cases} \langle N \rangle_c = \langle N \rangle & \text{average} \\ \langle N^2 \rangle_c = \langle \delta N^2 \rangle & \text{variance} \\ \langle N^3 \rangle_c = \langle \delta N^3 \rangle \\ \langle N^4 \rangle_c = \langle \delta N^4 \rangle - 3\langle \delta N^2 \rangle^2 \end{cases}$$

- skewness

$$S = \frac{\langle N^3 \rangle_c}{\langle N^2 \rangle_c^{3/2}}$$

- kurtosis

$$\kappa = \frac{\langle N^4 \rangle_c}{\langle N^2 \rangle_c^2}$$

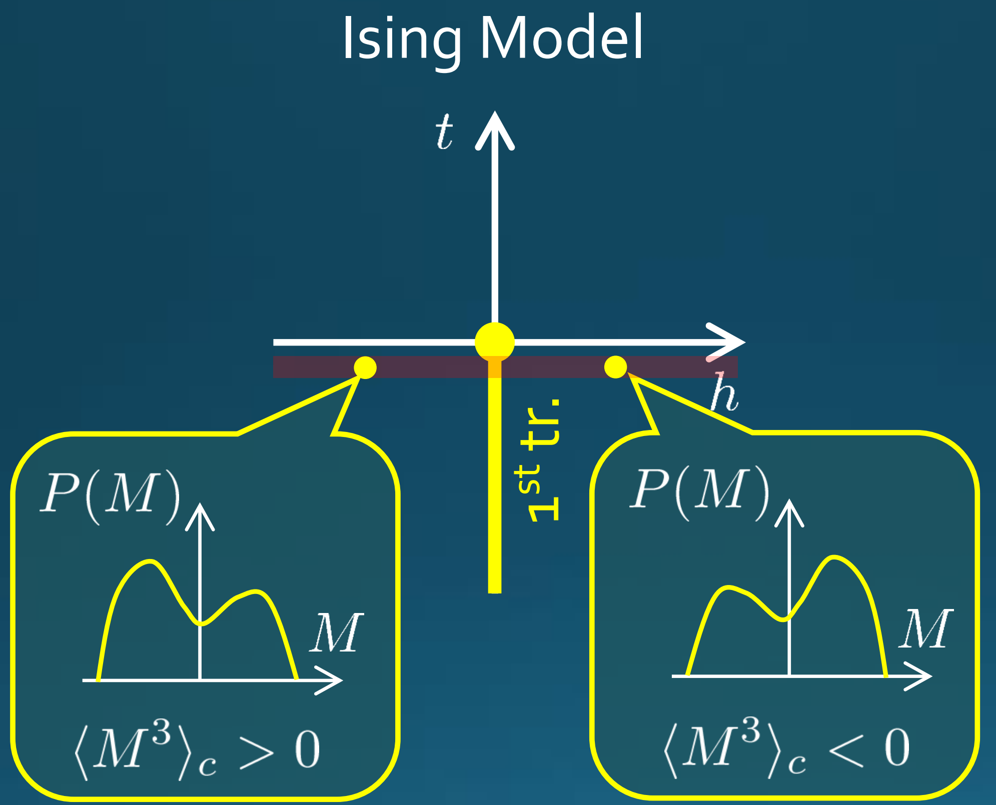


NOTE

- Gauss distribution: $\langle N^3 \rangle_c = \langle N^4 \rangle_c = \cdots = 0$
- Poisson distribution: $\langle N^2 \rangle_c = \langle N^3 \rangle_c = \langle N^4 \rangle_c = \cdots = \langle N \rangle$

Review: Asakawa, MK, PPNP **90** (2016)

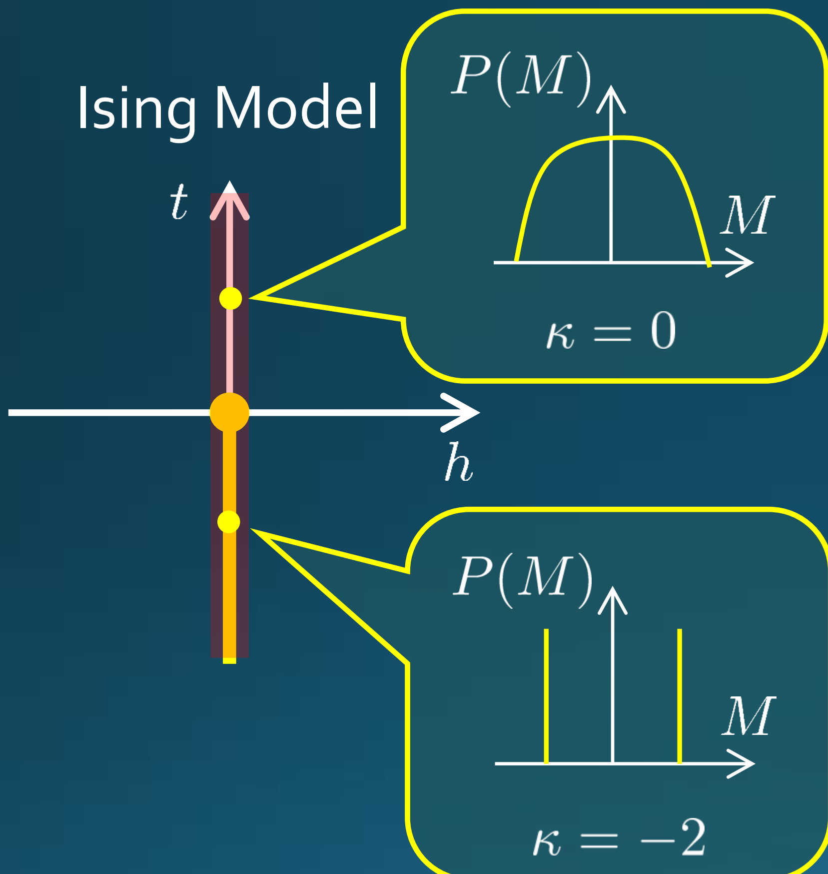
Cumulants around Critical Point



- Sign of $\langle M^3 \rangle_c$ is flipped at $h = 0$.

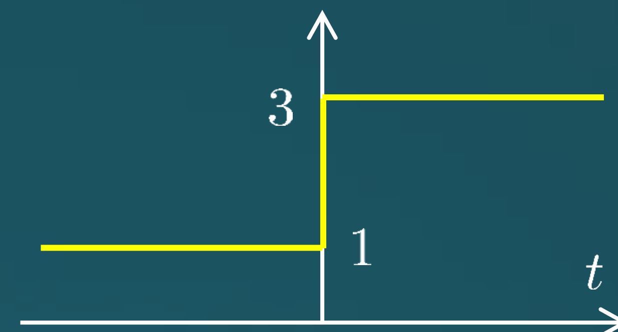
Cumulants around Critical Point

Ising Model



Binder Cumulant

$$B_4 = \frac{\langle M^4 \rangle_c}{\langle M^2 \rangle_c^2} + 3 = \kappa + 3$$



Kurtosis: $\kappa = \frac{\langle N^4 \rangle_c}{\langle N^2 \rangle_c^2}$

- $\langle M^4 \rangle_{c,h=0}$ changes discontinuously at the CP.

Experimental Search for QCD Critical Point in Heavy-Ion Collisions

Reviews:

Asakawa, MK, PPNP ('16)

Bluhm, MK+, NPA 1003 ('20)

MK, Esumi, Nonaka, JPS journal, 2021/8


解説

非ガウスゆらぎで探る宇宙最高密度の

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esumi.shinichi.gn@u.tsukuba.ac.jp

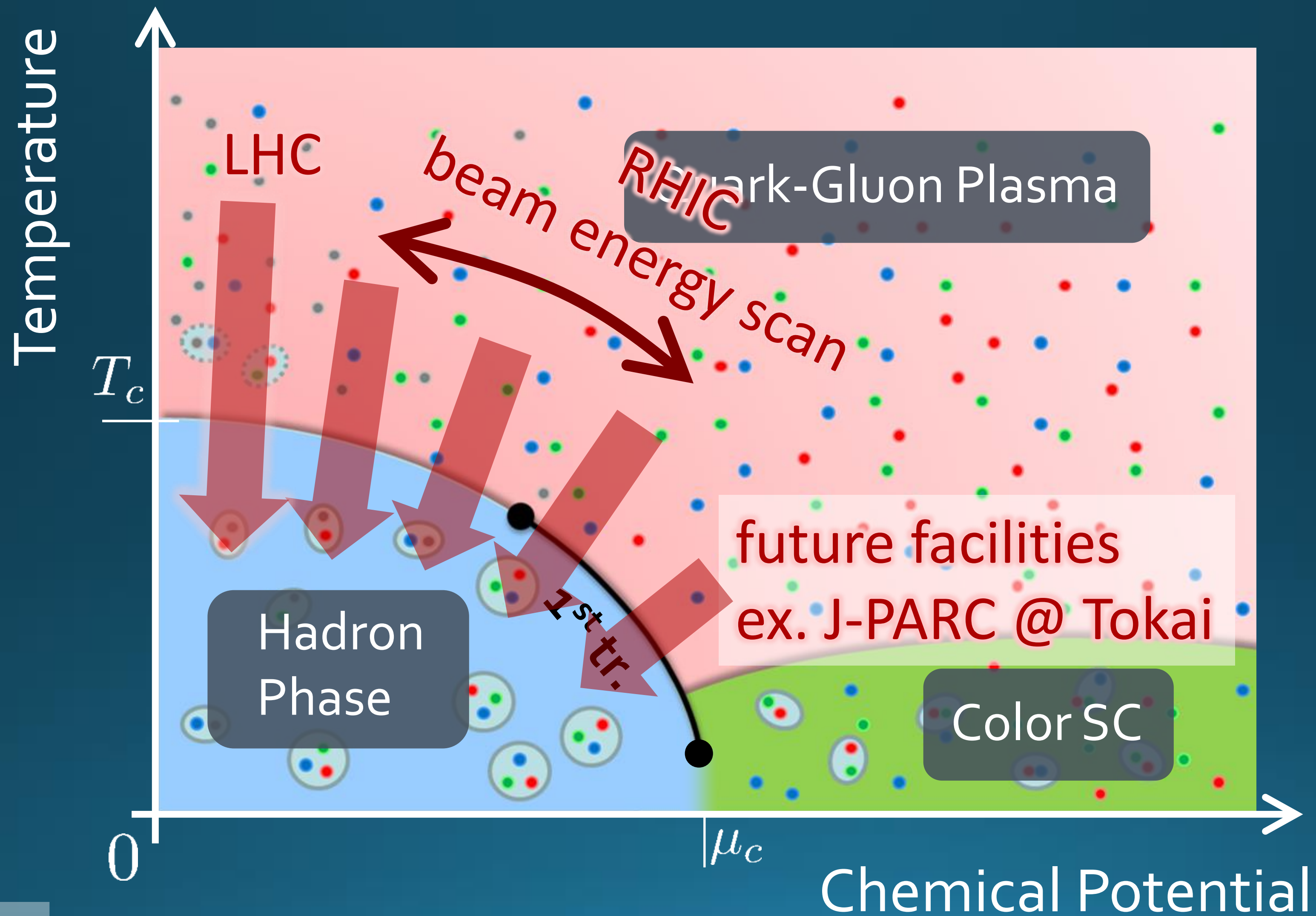
現在，およそ10^{15} g/cm^{3}という超高密度で実現するとされる相転移の実験的探索が世界各地の実験施設で行われているのをご

これら一連の実験が目指す最重要課題が，ビームエネルギー走査による高密度領域の相構造探索である．

用語解説

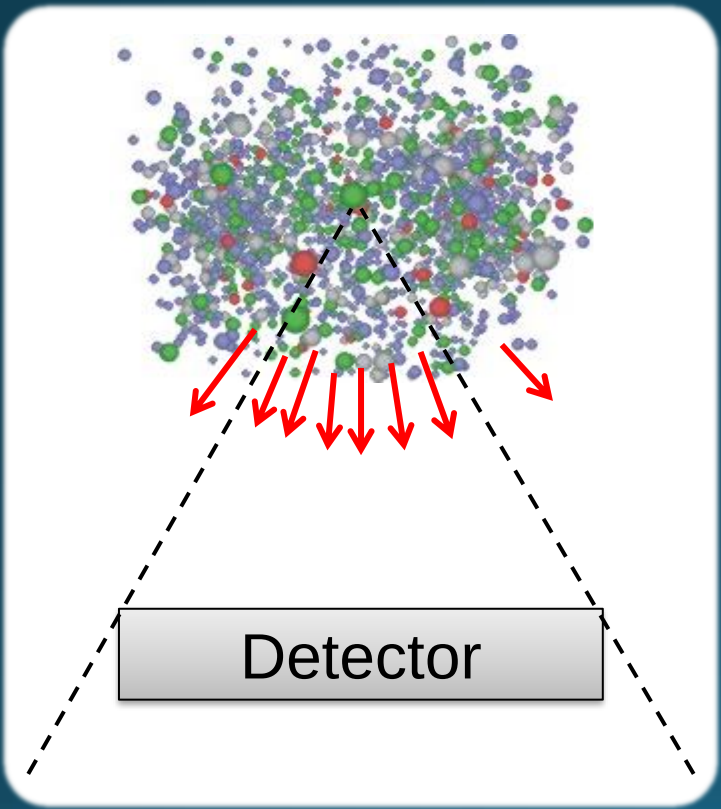
QCD臨界点：
QCD真空にクォーク数密度を印加していくと，10^{15} g/cm^3

QCD Phase Diagram

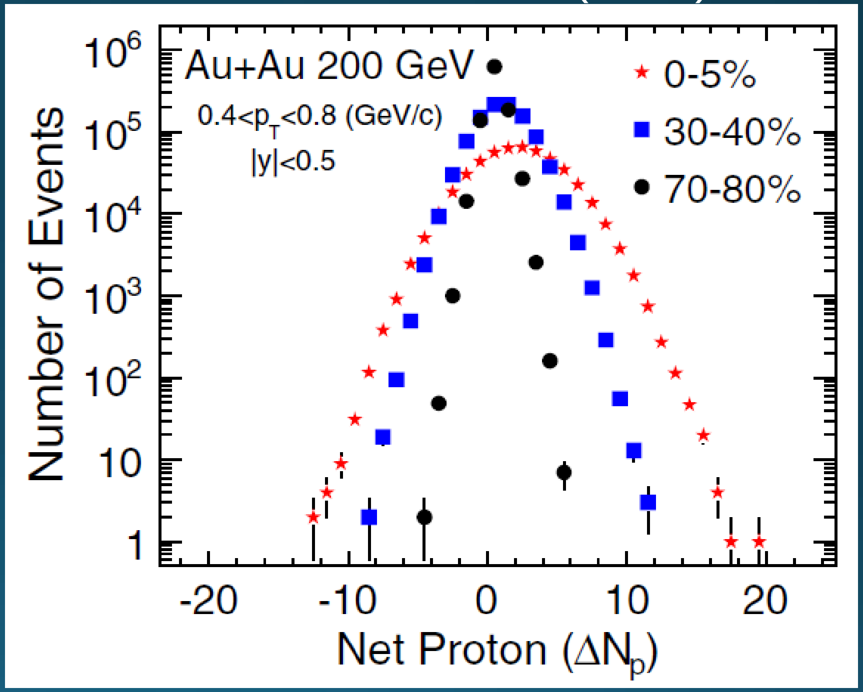


Event-by-event Fluctuations

Review: Asakawa, MK, PPNP **90** (2016)



STAR, PRL**105** (2010)



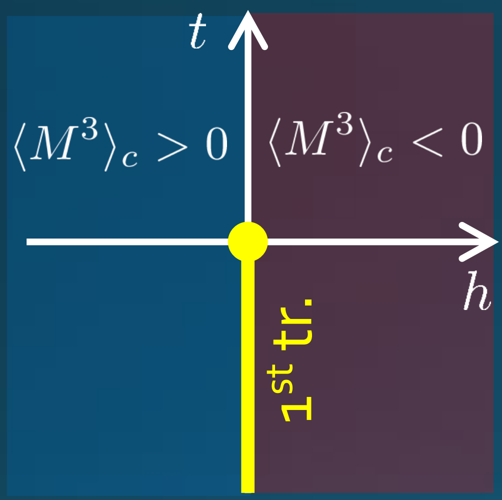
Cumulants

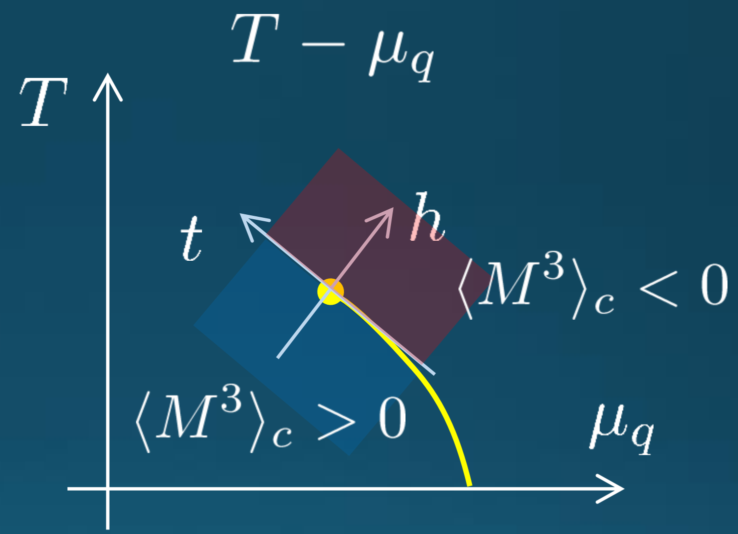
$$\langle \delta N_p^2 \rangle, \langle \delta N_p^3 \rangle, \langle \delta N_p^4 \rangle_c$$

Sign of non-Gaussian Cumulants

Stephanov, '11

- Ising Model
- QCD





- Sign of $\langle N^3 \rangle_c$ flips on the phase boundary of QCD.
- Same idea is also applicable to higher order cumulants.

Sign Change of Cumulant

Asakawa, Ejiri, MK, '09

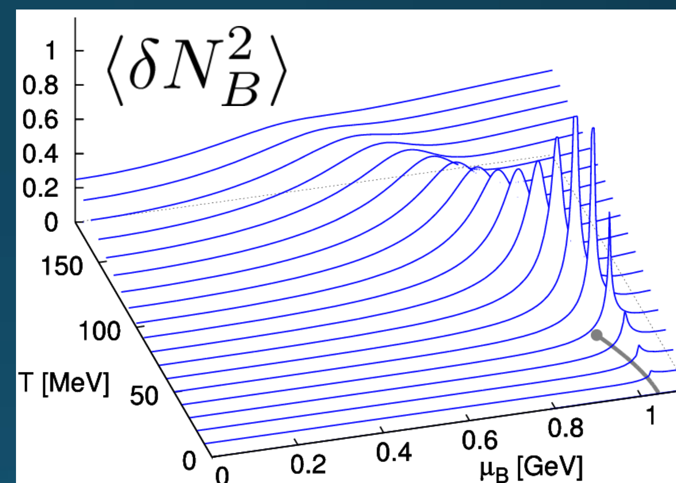
☐ Geometric interpretation on the signs

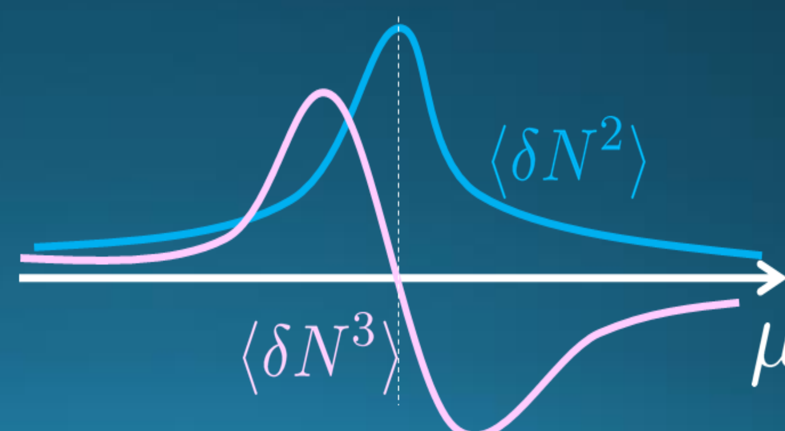
Fluctuations $\langle N_B^2 \rangle_c$ diverge at the QCD-CP.

Themodynamic Relation

$$\langle N_{\rm B}^{m+1} \rangle_c = T \frac{\partial \langle N_{\rm B}^m \rangle_c}{\partial \mu_{\rm B}}$$

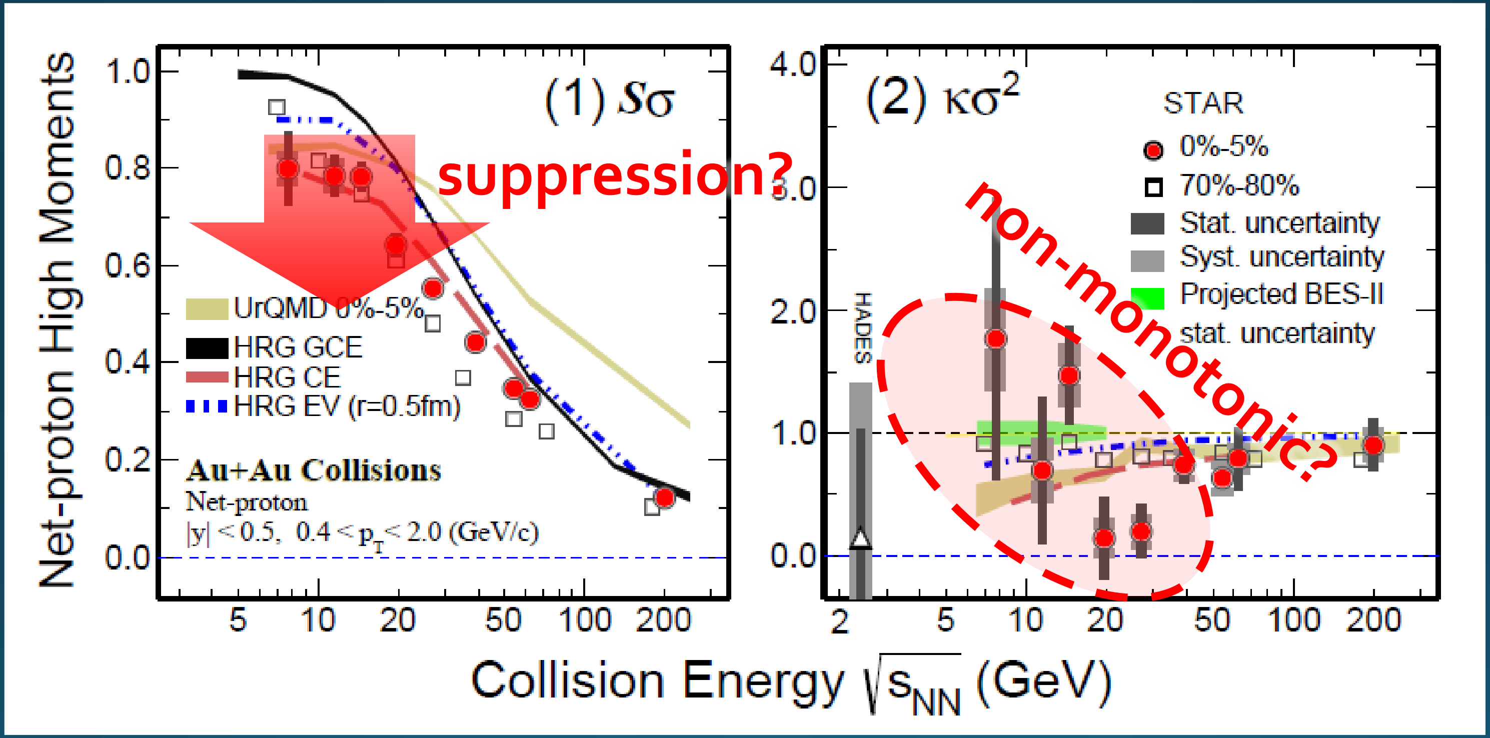
Sign of $\langle N_B^3 \rangle_c$ can distinguish near and away sides!





Proton Number Cumulants

$\langle N_p^3 \rangle_c / \langle N_p^2 \rangle_c$ $\langle N_p^4 \rangle_c / \langle N_p^2 \rangle_c$



STAR, PRL**126** ('21)

- Nonzero and non-Poissonian cumulants are experimentally established.

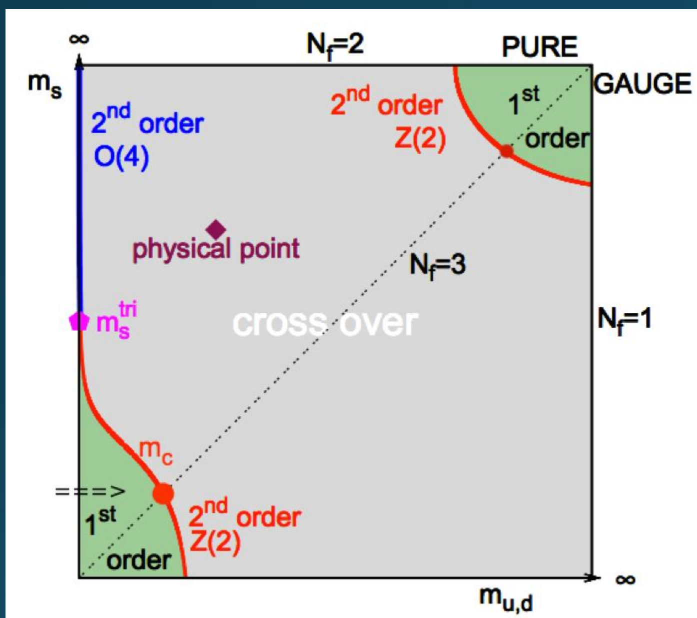
Lattice Simulations of QCD-CP in Heavy-Quark Region

Ejiri+, Phys.Rev.D 101 (2020) 054505
Kiyohara+, Phys.Rev.D 104 (2021) 114509
Wakabayashi+, PTEP 2022 (2022) 033B05
Ashikawa+, in prep.

Varying Quark Masses

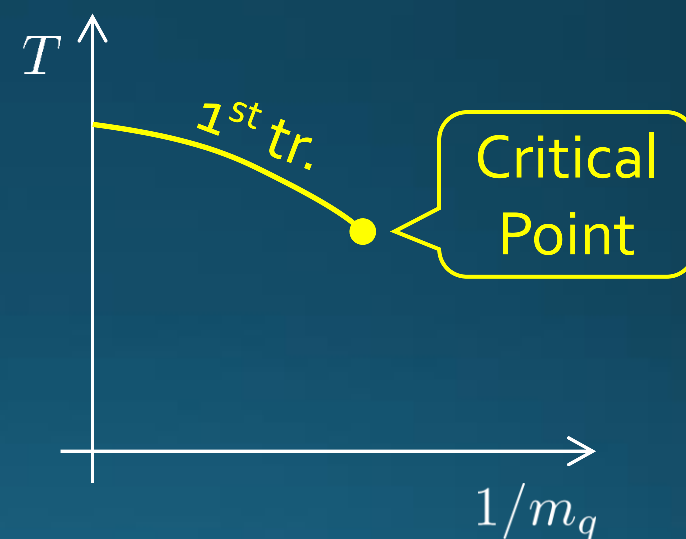
- **Columbia plot**

  = order of phase tr. at $\mu = 0$



- **Example**

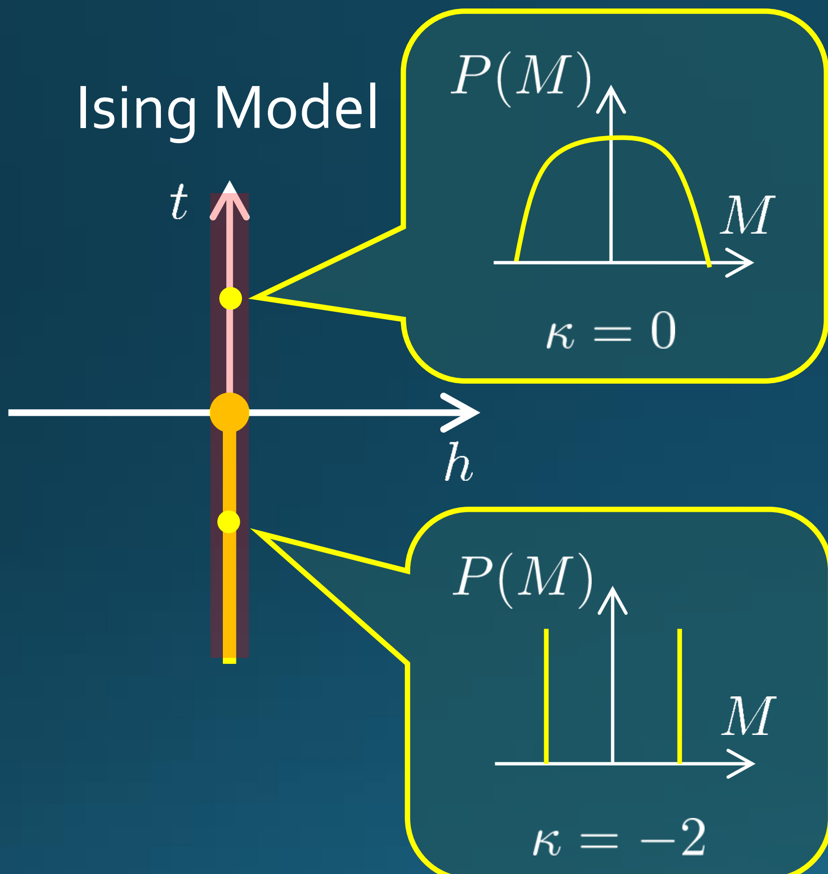
  Phase diagram in heavy-quark region



Lattice QCD numerical simulations are possible!

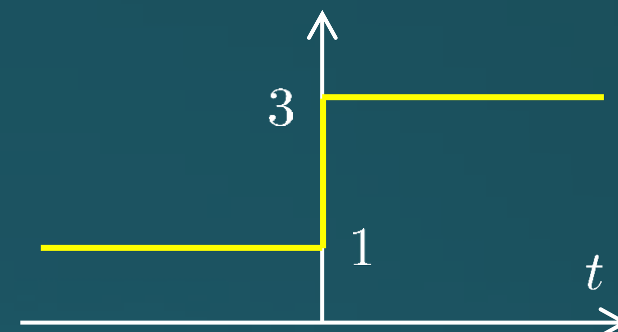
- Where are the CPs?
- Correct scaling behavior?

Cumulants around Critical Point



Binder Cumulant

$$B_4 = \frac{\langle N^4 \rangle_c}{\langle N^2 \rangle_c^2} + 3 = \kappa + 3$$



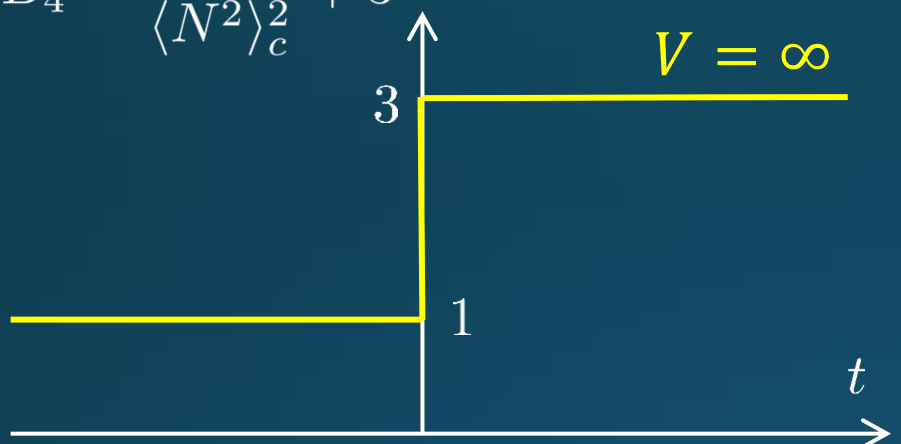
Kurtosis: $\kappa = \frac{\langle N^4 \rangle_c}{\langle N^2 \rangle_c^2}$

- $\langle M^4 \rangle_c$ changes discontinuously at the CP.

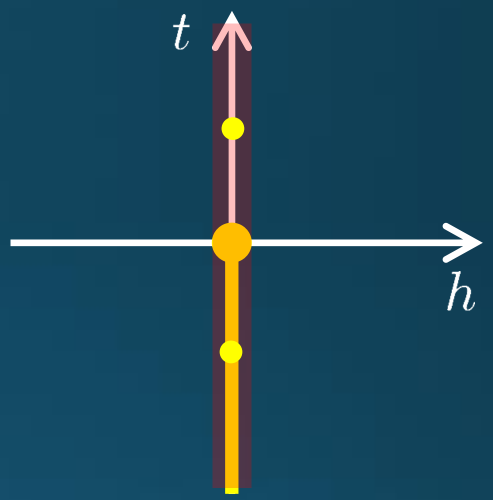
Finite-Volume Effects

Binder Cumulant

$$B_4 = \frac{\langle N^4 \rangle_c}{\langle N^2 \rangle_c^2} + 3$$



Ising Model

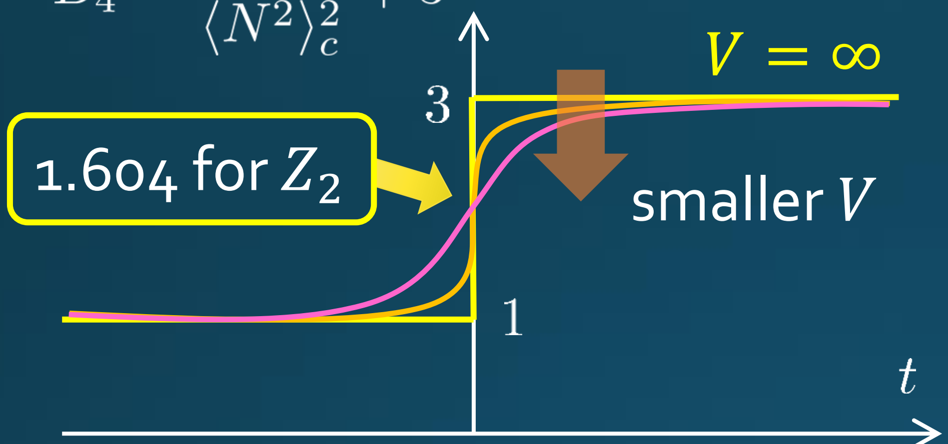


- Sudden change of B_4 at the CP is smeared by finite V effect.
- B_4 obtained for various V has crossing at $t = 0$.
- At the crossing point, $B_4 = 1.604$ in Z_2 universality class.

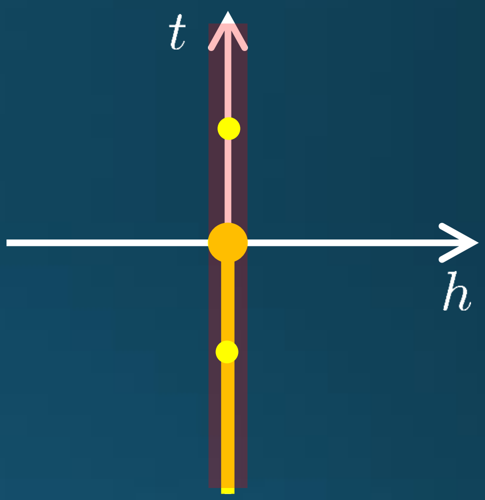
Finite-Volume Effects

Binder Cumulant

$$B_4 = \frac{\langle N^4 \rangle_c}{\langle N^2 \rangle_c^2} + 3$$



Ising Model



- Sudden change of B_4 at the CP is smeared by finite V effect.
- B_4 obtained for various V has crossing at $t = 0$.
- At the crossing point, $B_4 = 1.604$ in Z_2 universality class.

Finite-Size Scaling

Infinite vol.: $F(t,h) = F(b^{y_t} t, b^{y_h} h)$

Finite vol.: $\tilde{F}(t,h,L^{-1}) = \tilde{F}(b^{y_t} t, b^{y_h} h, bL^{-1})$

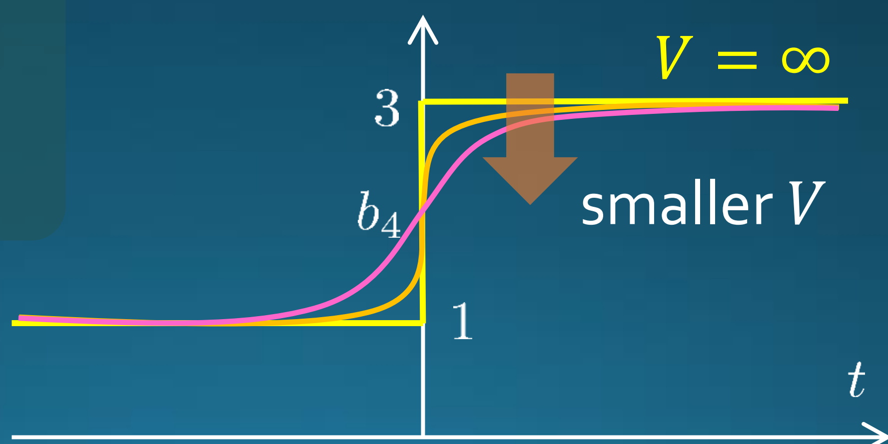
$= \tilde{F}(L^{y_t} t, L^{y_h} h, 1)$ $\quad b = L$

$$\langle M^n \rangle_c = \frac{\partial^n F}{\partial h^n}$$

$$B_4(t,0,L^{-1}) = b_4 + ctL^{y_t} + \cdots$$

$Z(2)$ universality class:

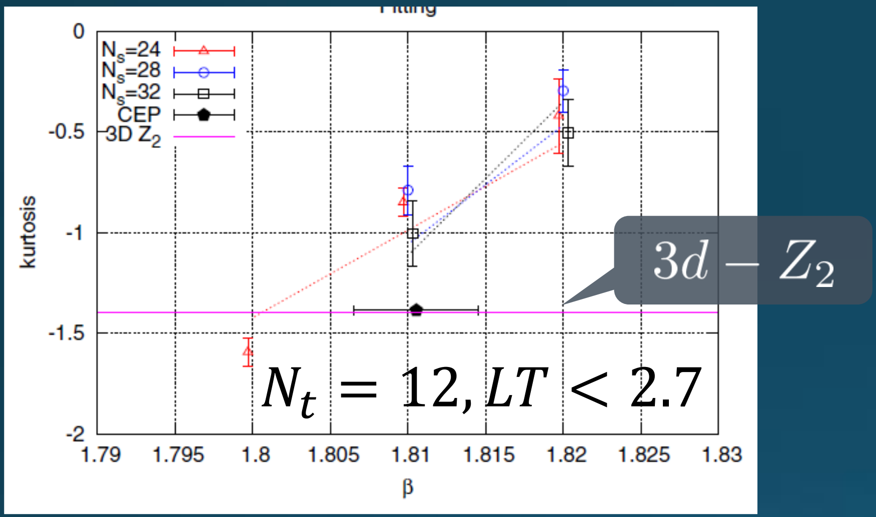
$b_4 = 1.604, \quad \nu = 1/y_t = 0.630$



Lattice Studies of Binder-Cumulant

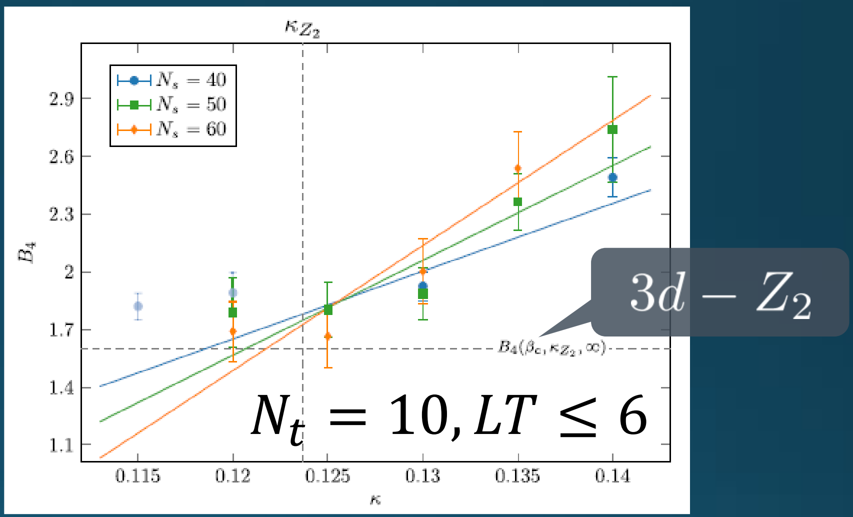
Light-quark region

Kuramashi, Nakamura, Ohno, Takeda, ’20



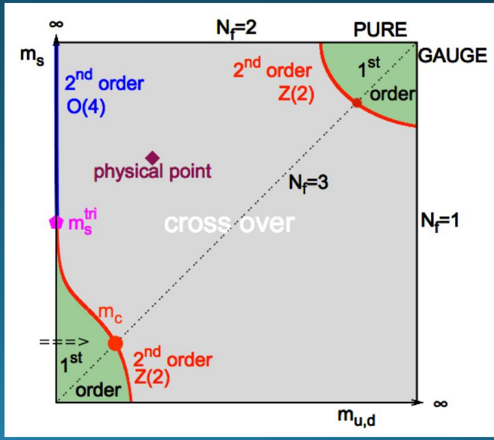
Heavy-quark region

Cuteri , Philipsen , Schön, Sciarra, ’21



- Statistically-significant deviation of the crossing point from the 3d-Ising value.

  ➡ V may not be large enough?



Our Strategy

Kiyohara+, PRD104 (2021); Ashikawa+, in prep.

Numerical simulations with large spatial volume

$$LT \geq 10$$

To realize it:

- ☐ Focus on the CP in the heavy-quark region
- ☐ Simulations on coarse lattices ($N_t = 1/aT = 4 \to 6$)
- ☐ Hopping parameter ($\sim 1/m_q$) expansion (HPE)
  - ☐ Monte Carlo simulations at the LO of HPE
  - ☐ Measurement at the NLO of HPE by reweighting

Hopping Parameter Expansion 1

Wilson fermion

$$S_q = \sum_{x,y} \bar{\psi}_x M_{xy} \psi_y \qquad \kappa \sim \frac{1}{2m_q a}$$

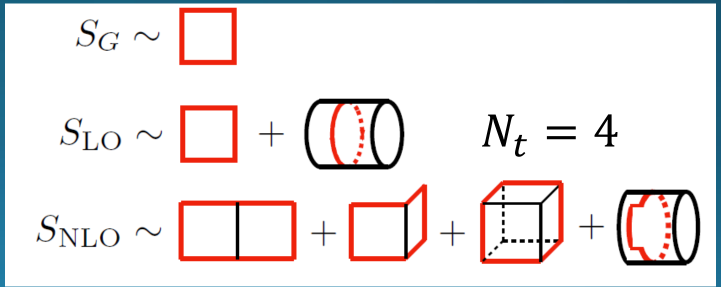
$$M_{xy} = \delta_{xy} - \kappa B_{xy} \qquad B_{xy} = \sum_{\mu=1}^{4} \left[(1-\gamma_\mu) U_{x,\mu} \delta_{y,x+\hat{\mu}} + (1+\gamma_\mu) U^\dagger_{y,\mu} \delta_{y,x-\hat{\mu}} \right]$$

nonzero only for neighboring (x, y)

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int \mathcal{D}U_\mu \mathcal{O} e^{-S_g + \ln \det M(\kappa)}$$

$$\ln \det M(\kappa) = -\sum_{n=1}^{\infty} \frac{1}{n} \mathrm{tr}[B^n] \kappa^n$$

$\mathrm{tr}[B_n]$ is given by the closed trajectories of length n.

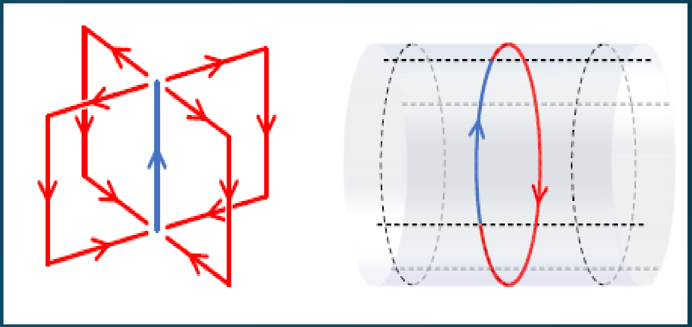


Hopping Parameter Expansion 2

□ Monte Carlo Simulation @ LO

- □ heat bath & over relaxation with modified staple

➡ Numerical cost is almost the same as the pure YM!



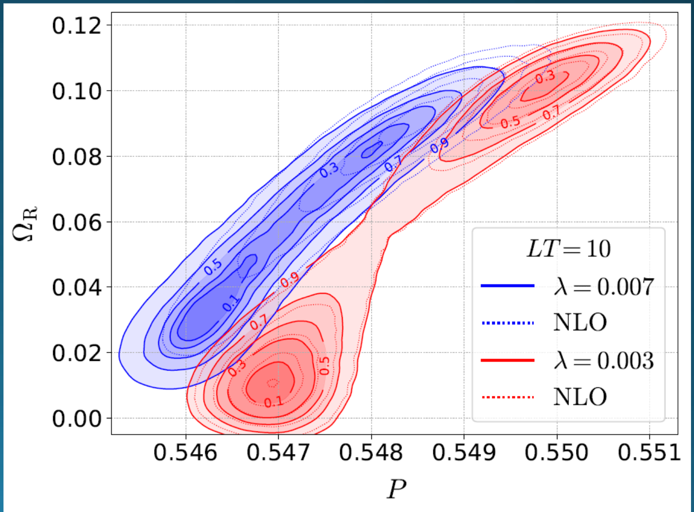
□ NLO by Reweighting

$$\langle \mathcal{O} \rangle_{\rm NLO} = \frac{\langle \hat{O} e^{-S_{\rm NLO}} \rangle_{\rm LO}}{\langle e^{-S_{\rm NLO}} \rangle_{\rm LO}}$$

- □ Overlapping problem is well suppressed due to the LO confs.

➡ Realize high statistical analysis

$$\lambda = 64 N_c N_f \kappa^4$$



Numerical Simulation @ $N_t = 4$

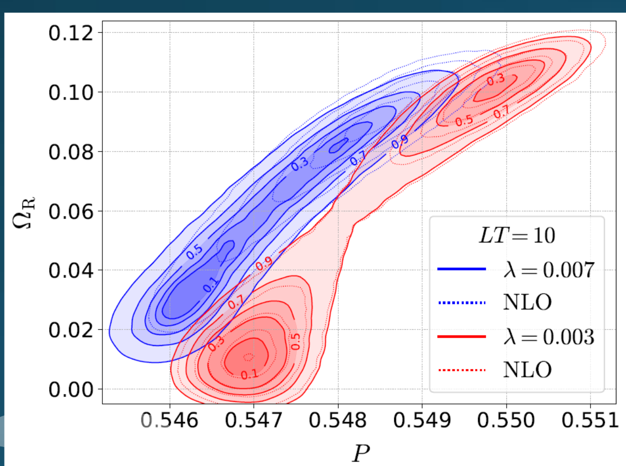
Kiyohara+, PRD104 (2021) 114509

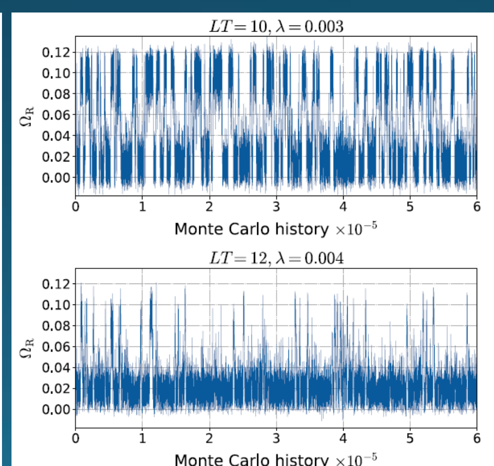
- Coarse lattice: $N_t = 4$
- But **large spatial volume**:
  $LT = N_s \,/\, N_t \le 12$

- Hopping-param. (~$1/m_q$) expansion
- Monte-Calro with LO action
- High statistical analysis (6×10^5 meas.)

Simulation params.

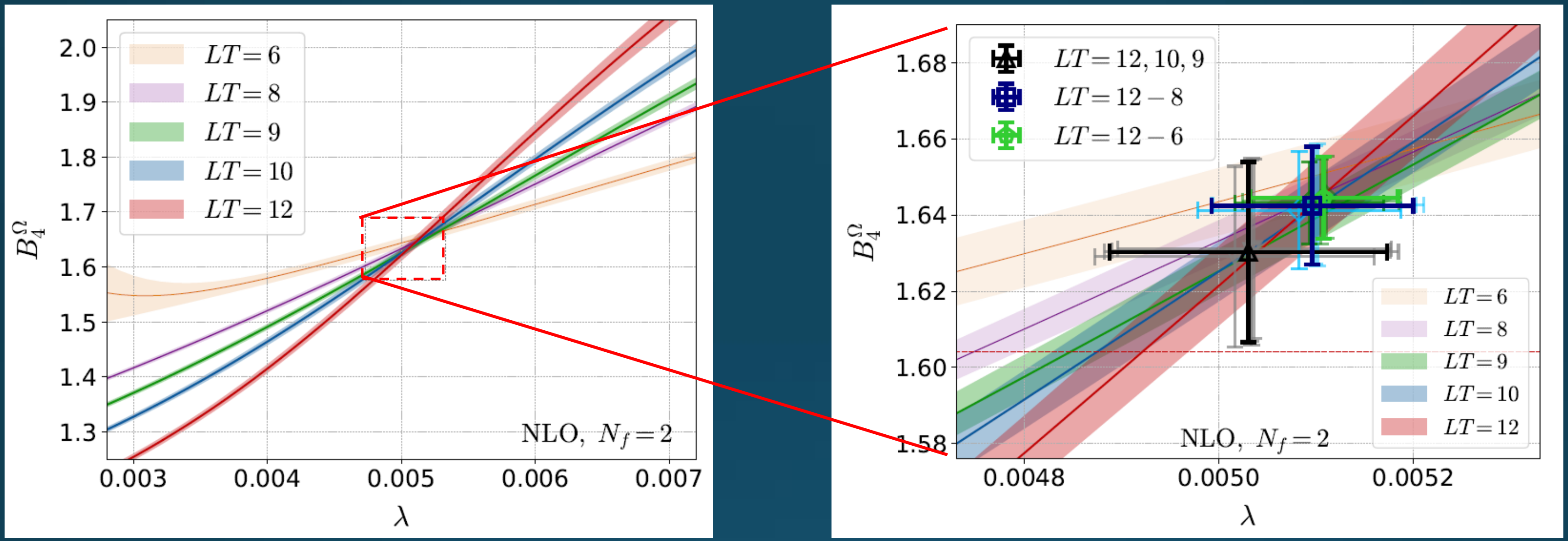
lattice size	β^*	λ	$\kappa^{N_f=2}$
$48^3 \times 4$	5.6869	0.004	0.0568
	5.6861	0.005	0.0601
	5.6849	0.006	0.0629
$40^3 \times 4$, $36^3 \times 4$	5.6885	0.003	0.0529
	5.6869	0.004	0.0568
	5.6861	0.005	0.0601
	5.6849	0.006	0.0629
	5.6837	0.007	0.0653
$32^3 \times 4$	5.6885	0.003	0.0529
	5.6865	0.004	0.0568
	5.6861	0.005	0.0601
	5.6845	0.006	0.0629
	5.6837	0.007	0.0653
$24^3 \times 4$	5.6870	0.0038	0.0561
	5.6820	0.0077	0.0669
	5.6780	0.0115	0.0740





$\lambda = 64 N_c N_f \kappa^4$

Binder-Cumulant Analysis

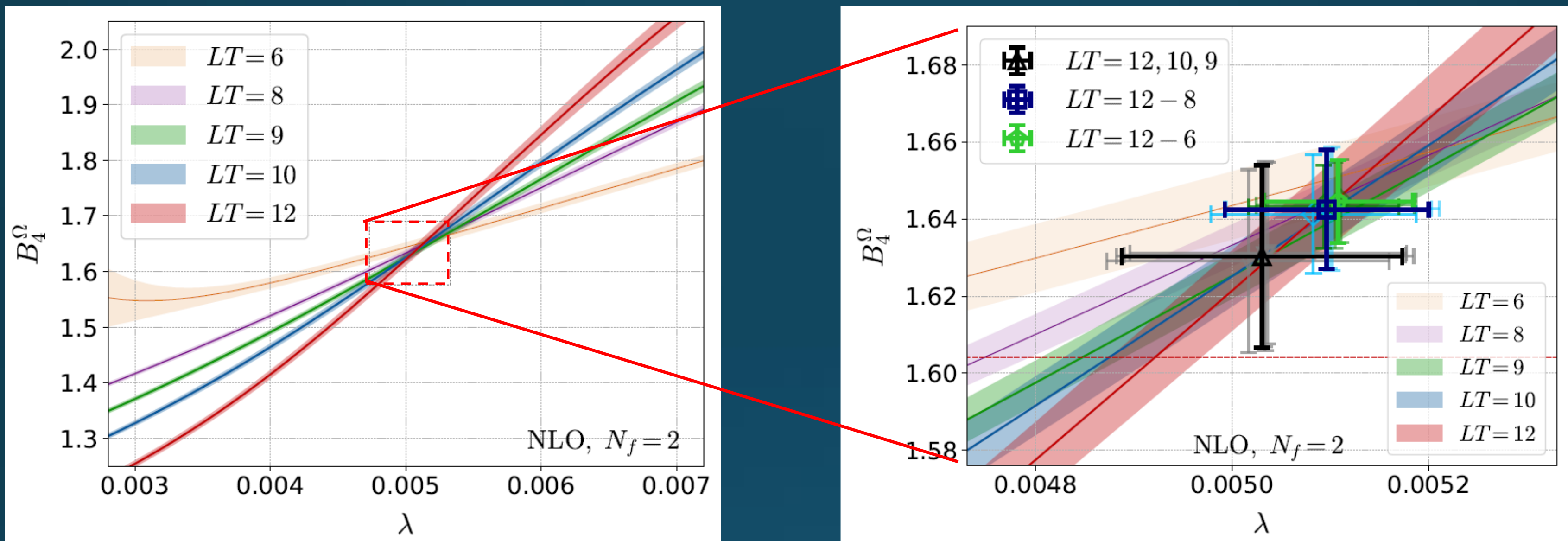


Fitting function

$$B_4(\lambda, LT) = b_4 + c(\lambda - \lambda_c)(LT)^{1/\nu}$$

params: b_4, c, λ_c, ν

Binder-Cumulant Analysis

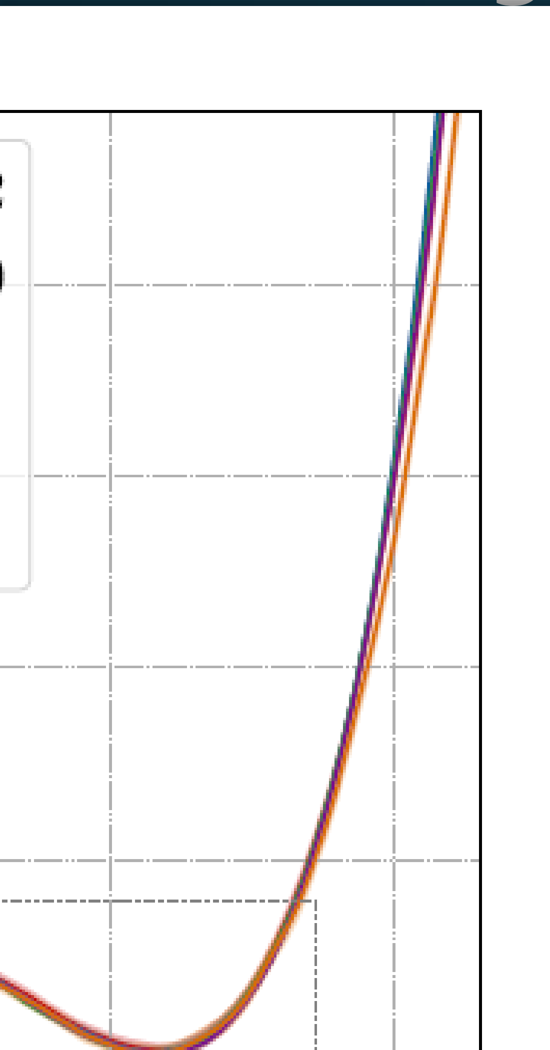


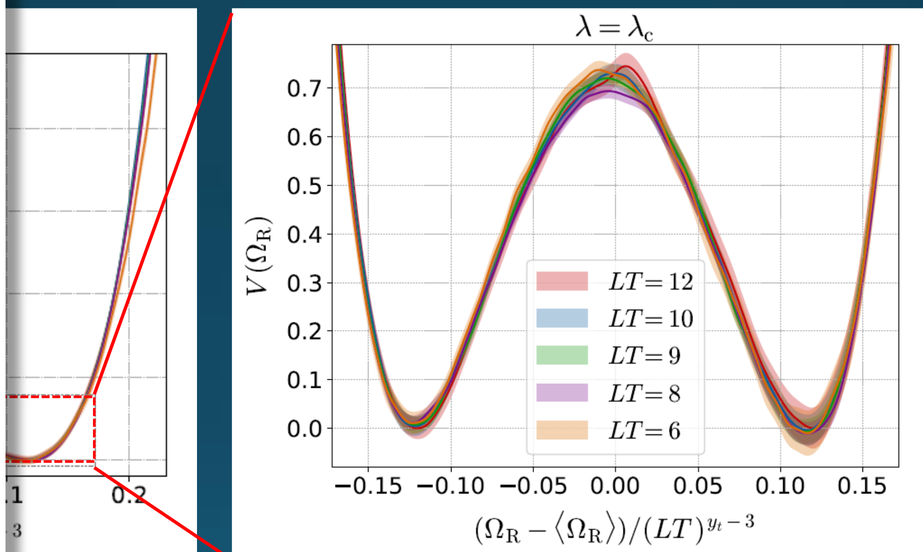
$LT \geq 9$	$B_4 = 1.630(24)(2),$	$\nu = 0.614(48)(3)$
$LT \geq 8$	$B_4 = 1.643(15)(2),$	$\nu = 0.614(29)(3)$
Z2	$B_4 = 1.604$	$\nu = 0.630$

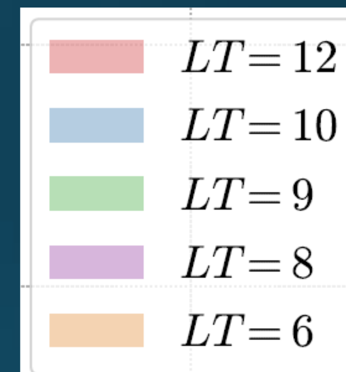
- B_4 and ν are consistent with Z2 universality class only when $LT \geq 9$ data are used for the analysis.

Scaling of Distribution Function

e Potential: $V(\Omega_{\mathrm{R}}) = -\log P(\Omega_{\mathrm{R}})$





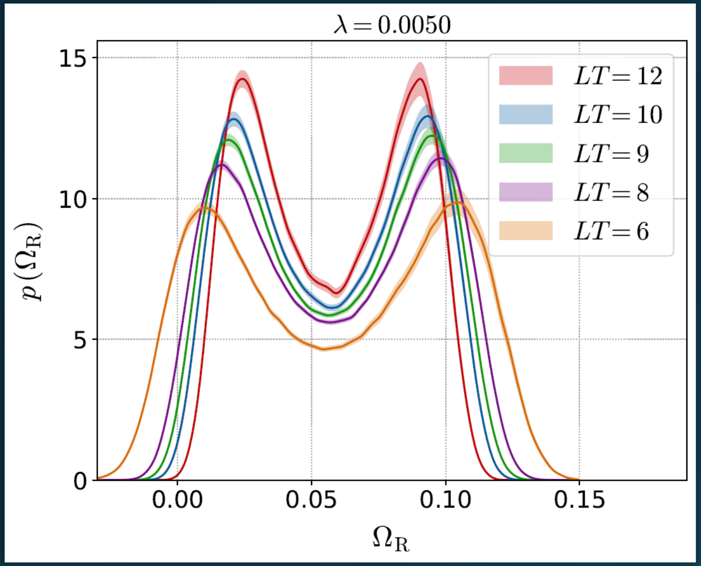


Result at $\lambda = \lambda_c$

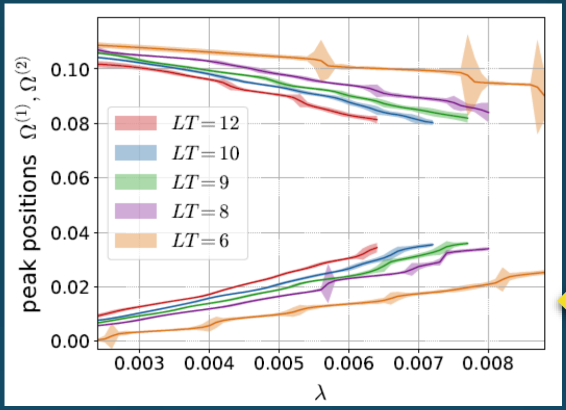
- Z_2-FSS is well applied near the peaks
- Scaling violation around the edges
  - ➡ lead to scaling violation of B_4

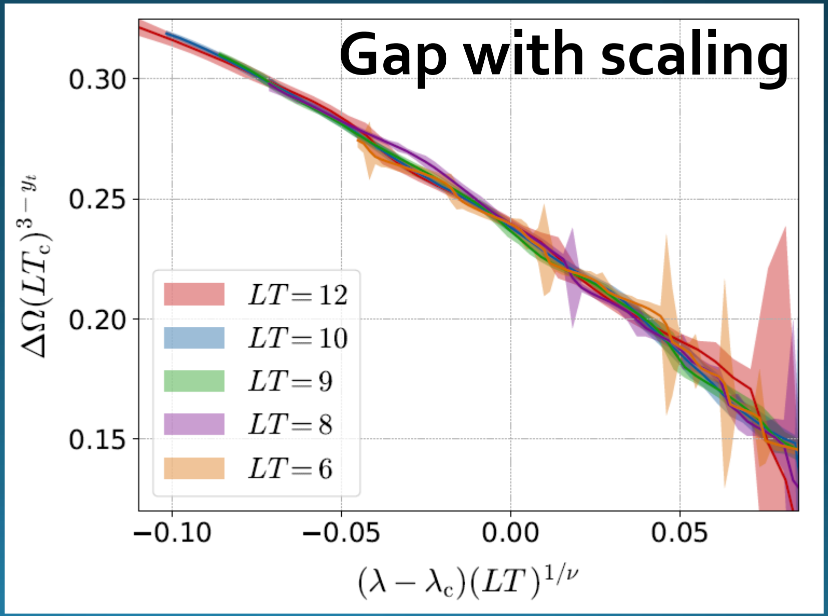
Scaling of Gap of Peaks

Polyakov-loop Distribution



Peak Position





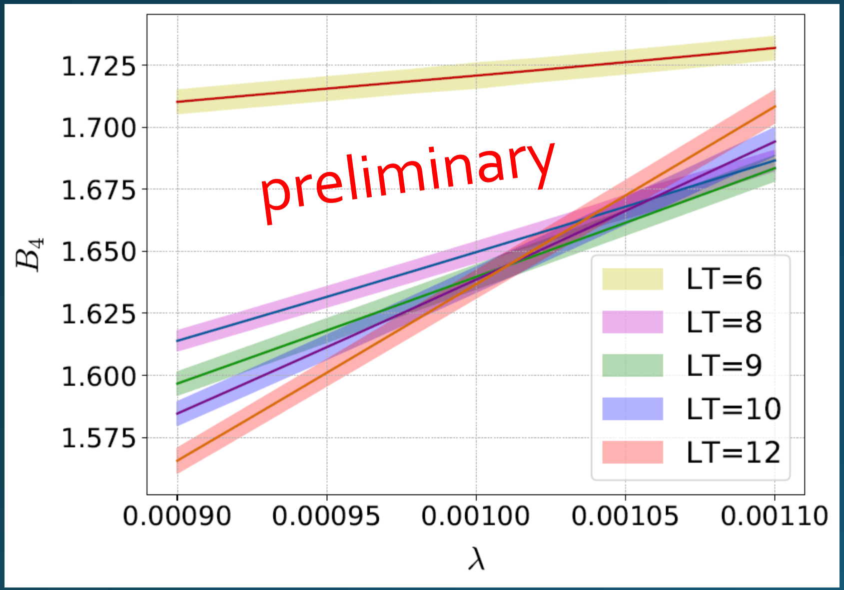
λ dep. of the gap agrees well with Z_2-FSS.

Numerical Simulation @ $N_t = 6$

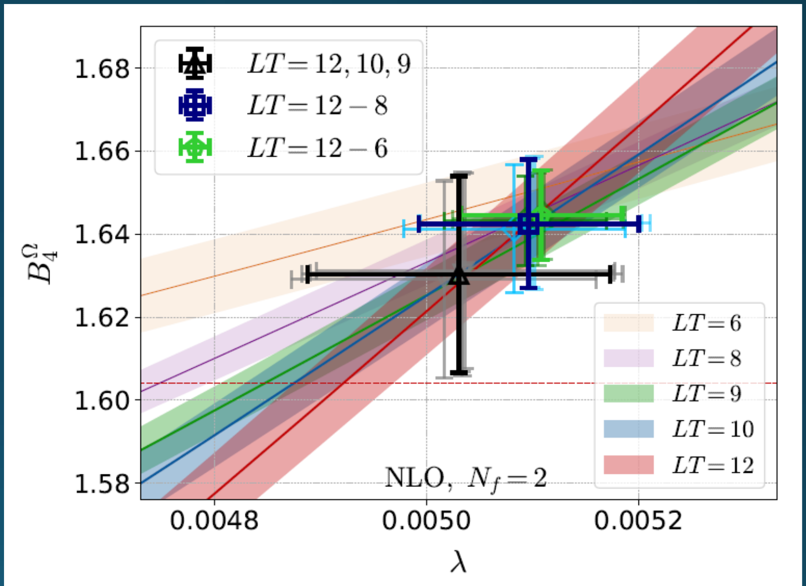
New result for $N_t = 6$ (Ashikawa+, in prep.)

$$a = \frac{1}{N_t T}$$

$N_t = 6$



$N_t = 4$

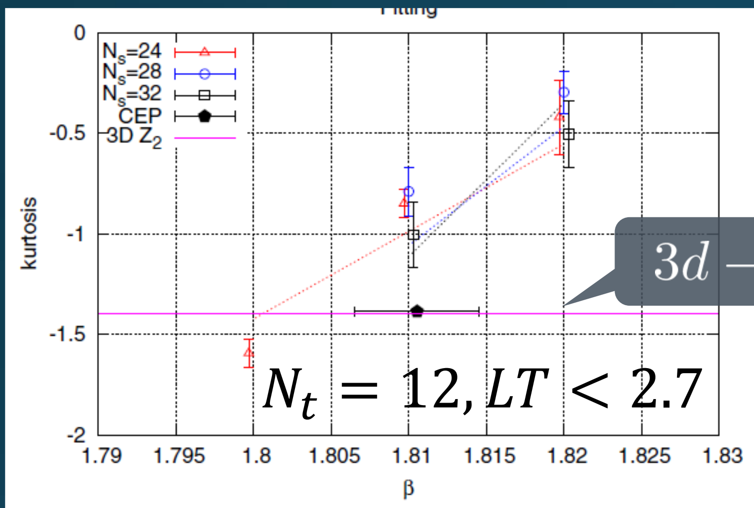


- For $N_t = 6$, deviation from the finite-size scaling is more significant with the same V.

Lattice Studies of Binder-Cumulant

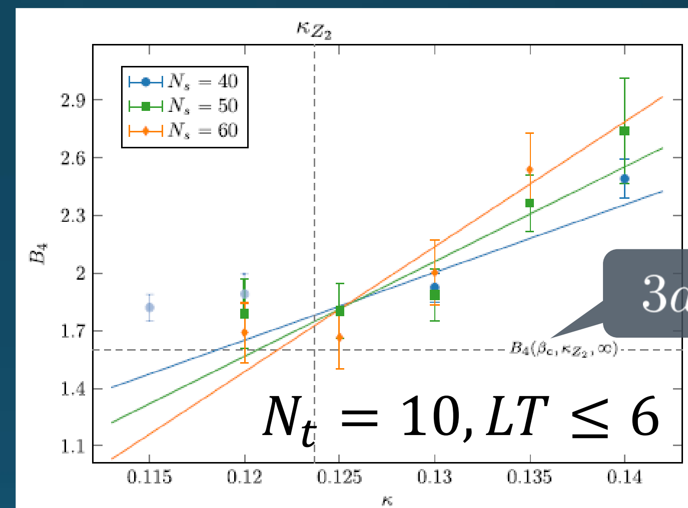
☐ Light-quark region

Kuramashi, Nakamura, Ohno, Takeda, '20



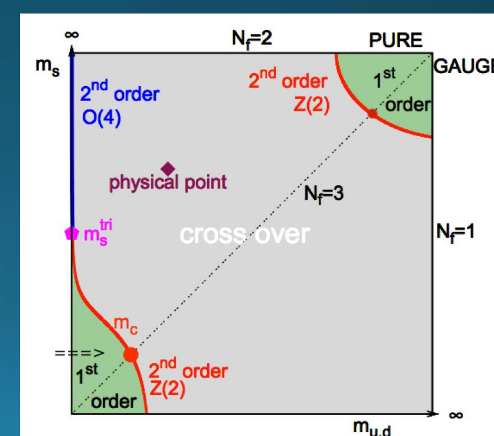
☐ Heavy-quark region

Cuteri , Philipsen , Schön, Sciarra, '21



☐ Statistically-significant deviation of the crossing point from the 3d-Ising value.

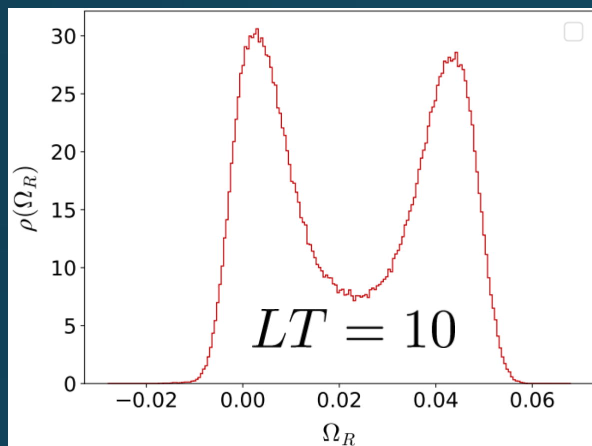
➡ V may not be large enough?

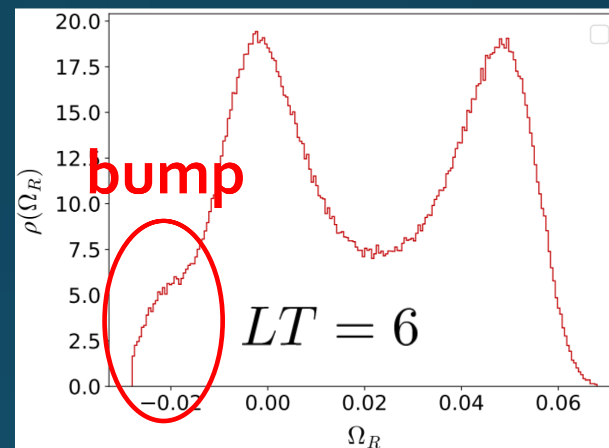


Violation of FSS & Remnant of Z(3)

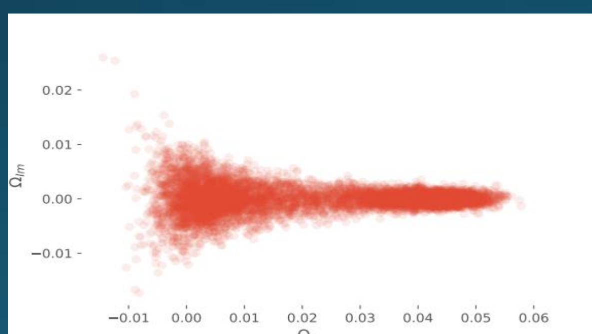
☐ Probability Distribution of Polyakov loop

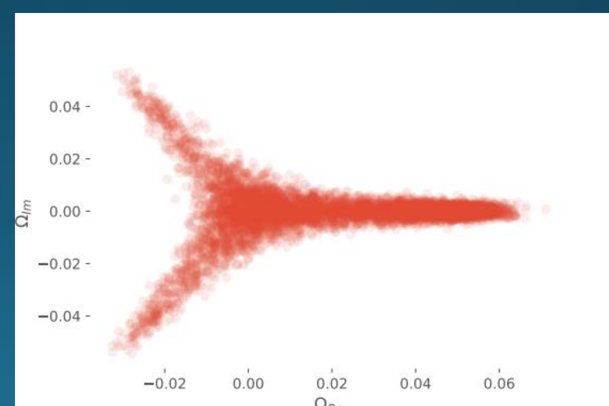
Real Part





Complex Plane



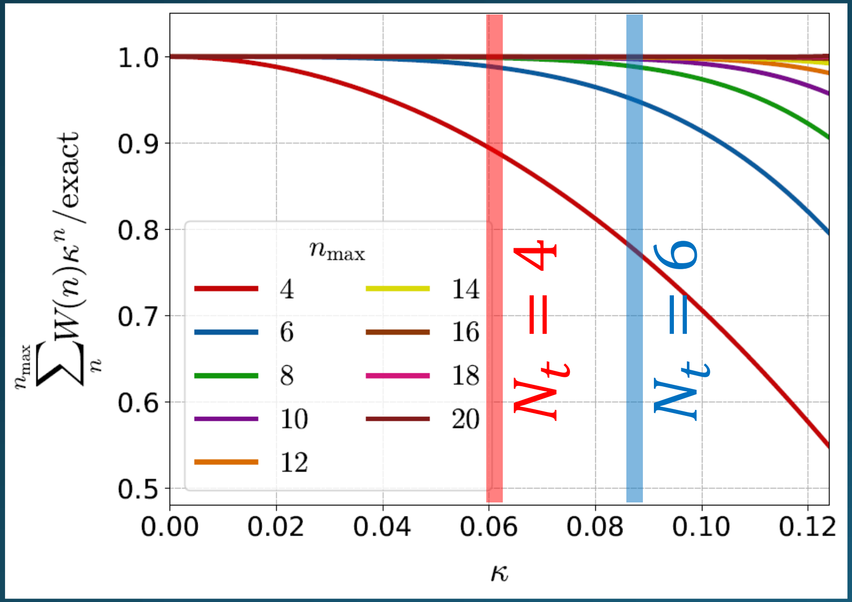


Convergence of HPE

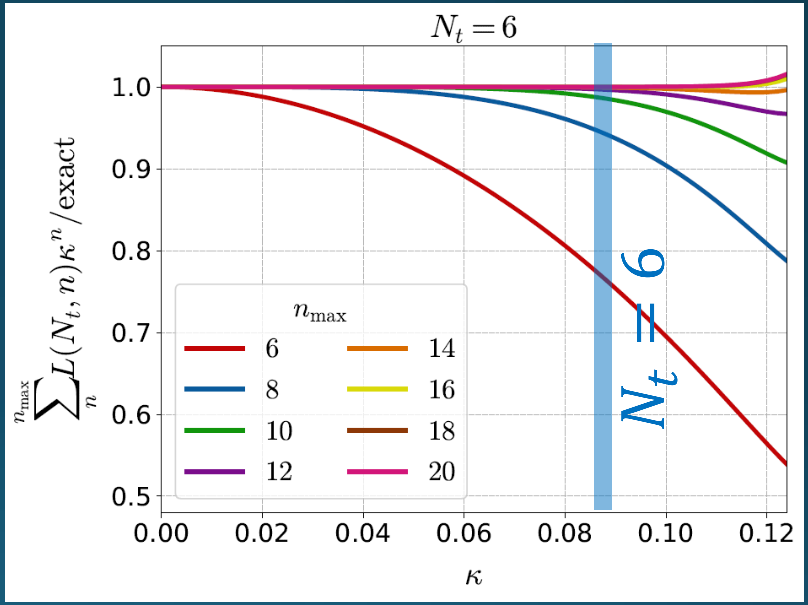
Wakabayashi+ ('22)

- HPE of free lattice field (U=1)

Wilson-loop-type



Polyakov-loop-type



$N_t = 4 \ \ \kappa_c = 0.0602(4)$ Kiyohara+,'21

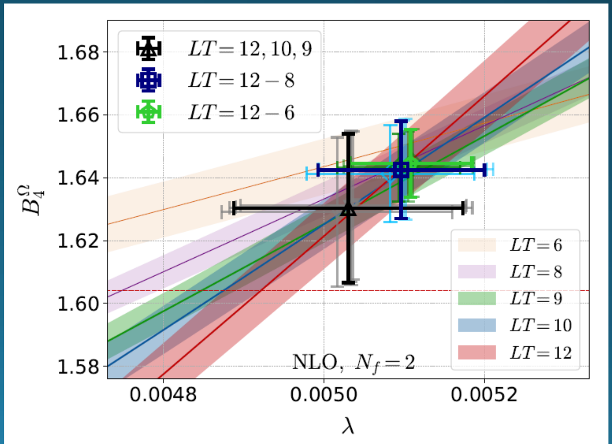
$N_t = 6 \ \ \kappa_c = 0.0877(9)$ Cuteri+, '21

NNLO and higher

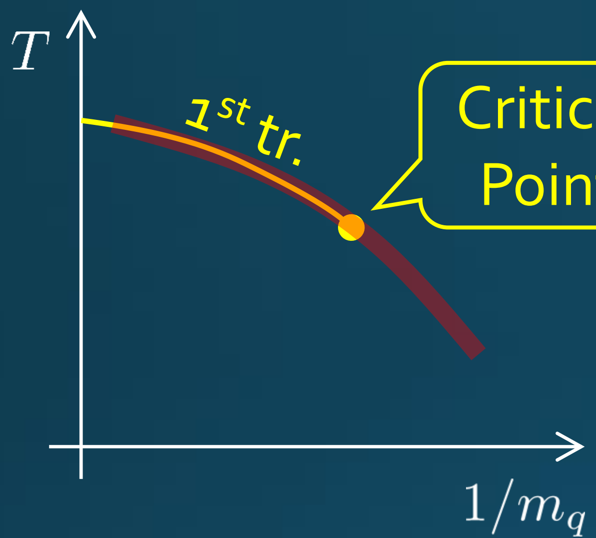
Wakabayashi+ ('22)

Summary

- Critical points appear many places in QCD at nonzero temperature.
- Among them, a search for the CPs
  1) on the $T-\mu_q$ plane by Relativistic heavy-ion collisions
  2) on the $T-m_q$ plane by Lattice QCD simulations
  are interesting subjects that are actively studied.
- Large spatial-volume simulations are mandatory to apply for the FSS for the analysis of the CP on the lattice.



Transition Line



Definitions of transition line

- ☐ Maximum of $\langle \Omega_R^2 \rangle$
- ☐ Zero of $\langle \Omega_R^2 \rangle$
- ☐ Minimum of B_4