Energy-momentum tensor around the kink in 1+1d field theories

Hiroaki Ito, Masakiyo Kitazawa
in preparation

KEK Journal club, online, 2022/10/24

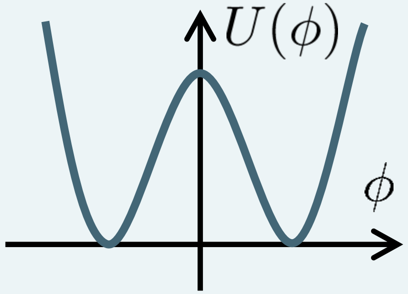
Main Conclusion

ϕ^4 Model & Kink

ϕ^4 model

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - U(\phi)$$

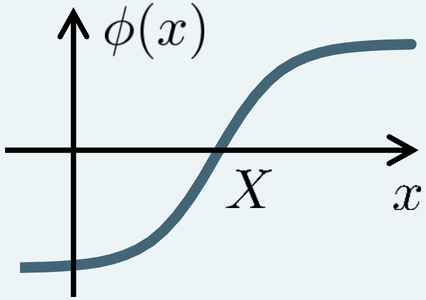
$$U(\phi) = \frac{\lambda}{4}\left(\phi^2 - \frac{m^2}{\lambda}\right)^2$$



Kink (Soliton)

$$\phi_{\text{kink}} = \pm \frac{m}{\sqrt{\lambda}} \tanh \frac{m(x-X)}{\sqrt{2}}$$

- classical solution
- stable
- local



Energy-Momentum Tensor

Classical $\mathcal{O}(\lambda^{-1})$

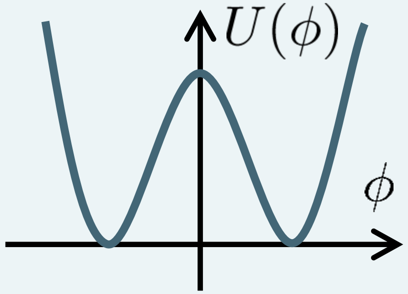
$$\begin{cases} T_{00} = \dfrac{m^4}{\lambda} \cosh^{-4} \dfrac{m(x-X)}{\sqrt{2}} \\ T_{01} = T_{11} = 0 \end{cases}$$

ϕ^4 Model & Kink

ϕ^4 model

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - U(\phi)$$

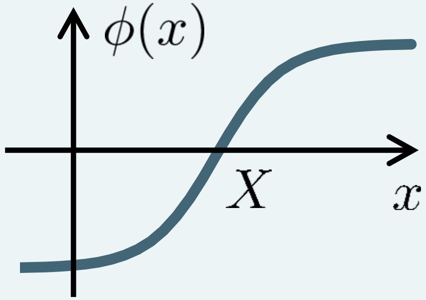
$$U(\phi) = \frac{\lambda}{4}\left(\phi^2 - \frac{m^2}{\lambda}\right)^2$$



Kink (Soliton)

$$\phi_{\text{kink}} = \pm \frac{m}{\sqrt{\lambda}} \tanh \frac{m(x-X)}{\sqrt{2}}$$

- classical solution
- stable
- local



Energy-Momentum Tensor

Classical $\mathcal{O}(\lambda^{-1})$

$$\begin{cases} T_{00} = \dfrac{m^4}{\lambda} \cosh^{-4} \dfrac{m(x-X)}{\sqrt{2}} \\ T_{01} = T_{11} = 0 \end{cases}$$

This study

- **Quantum correction at $\mathcal{O}(\lambda^0)$**
- finite result
- satisfy conservation law

Previous Studies:

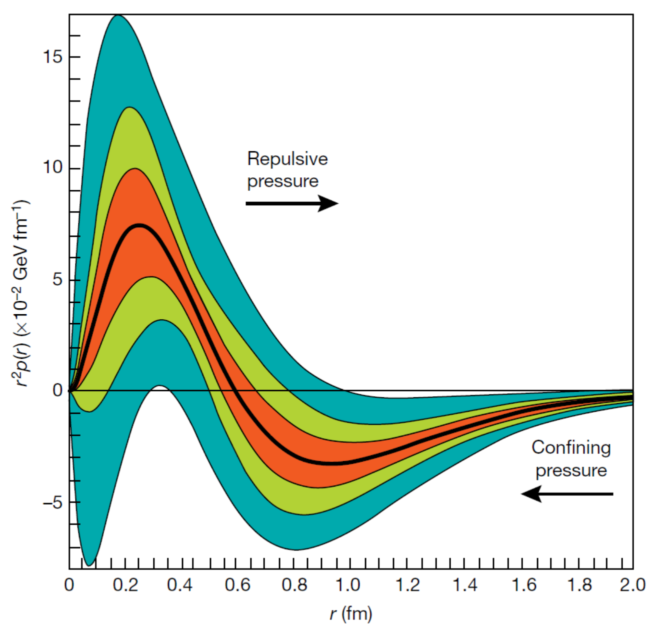
- Total energy: Dashen+ ('74)
- Energy density (?): Goldhaber+ ('03)

Motivations

Gravitational Form Factors & EMT Distribution inside Hadrons

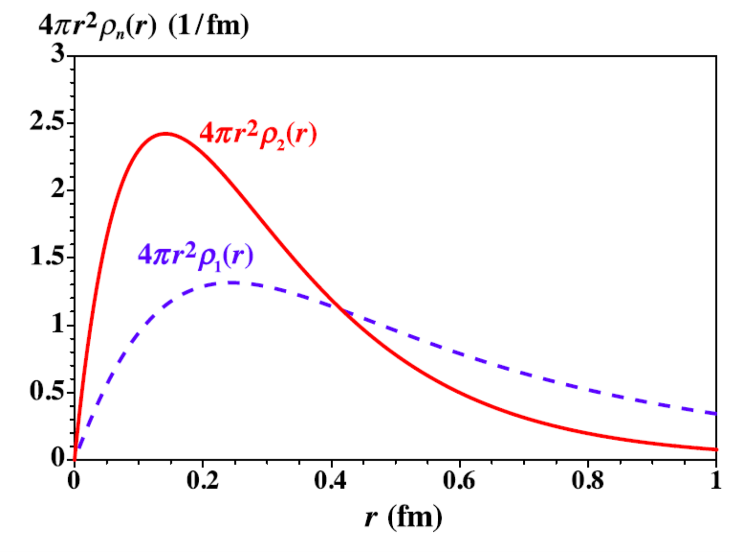
Pressure distribution inside hadrons

Proton



Burket+, Nature 557, 396 ('18)

Pion



Kumano+, PRD 97, 014020 ('18)

$$\langle \mathrm{hadron}|T_{\mu\nu}(x)|\mathrm{hadron}\rangle$$

- EMT → mechanical structure of hadrons.
- Measurement will be refined at the EIC.

- **EMT distribution around local structures in QFT.**

EMT on the Lattice QCD

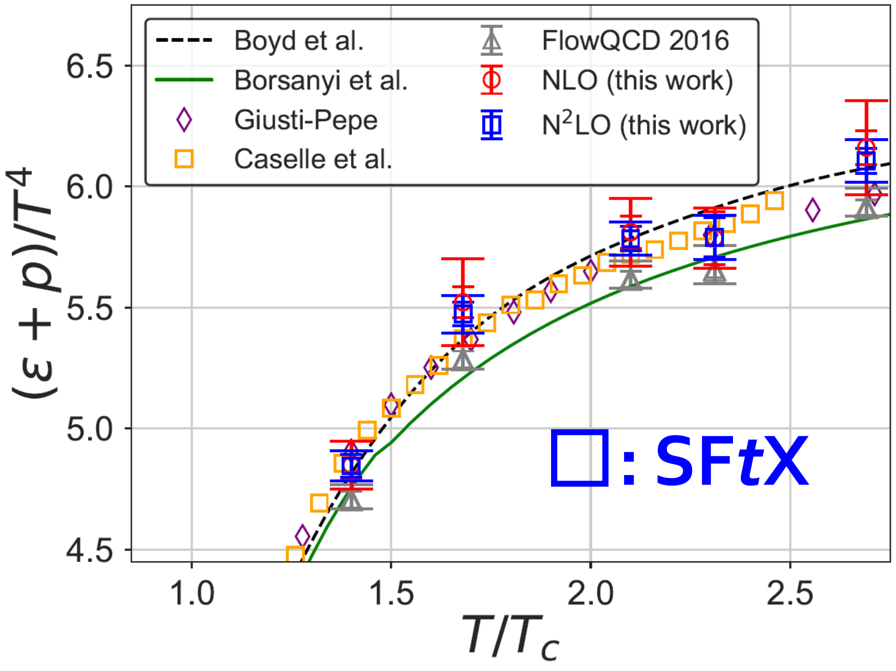
- Measurement of EMT on the lattice QCD simulations had been difficult.
    - Violation of translational sym.
    - Large statistical noise.

- New method based on the gradient flow
- SF*t*X (**S**mall **F**low-***t***ime e**X**pansion) Suzuki ('13)

- Numerical simulations show
    - Valid definition of EMT.
    - Suppression of statistical error.

FlowQCD ('14~); WHOT-QCD ('15~)

Thermodynamics



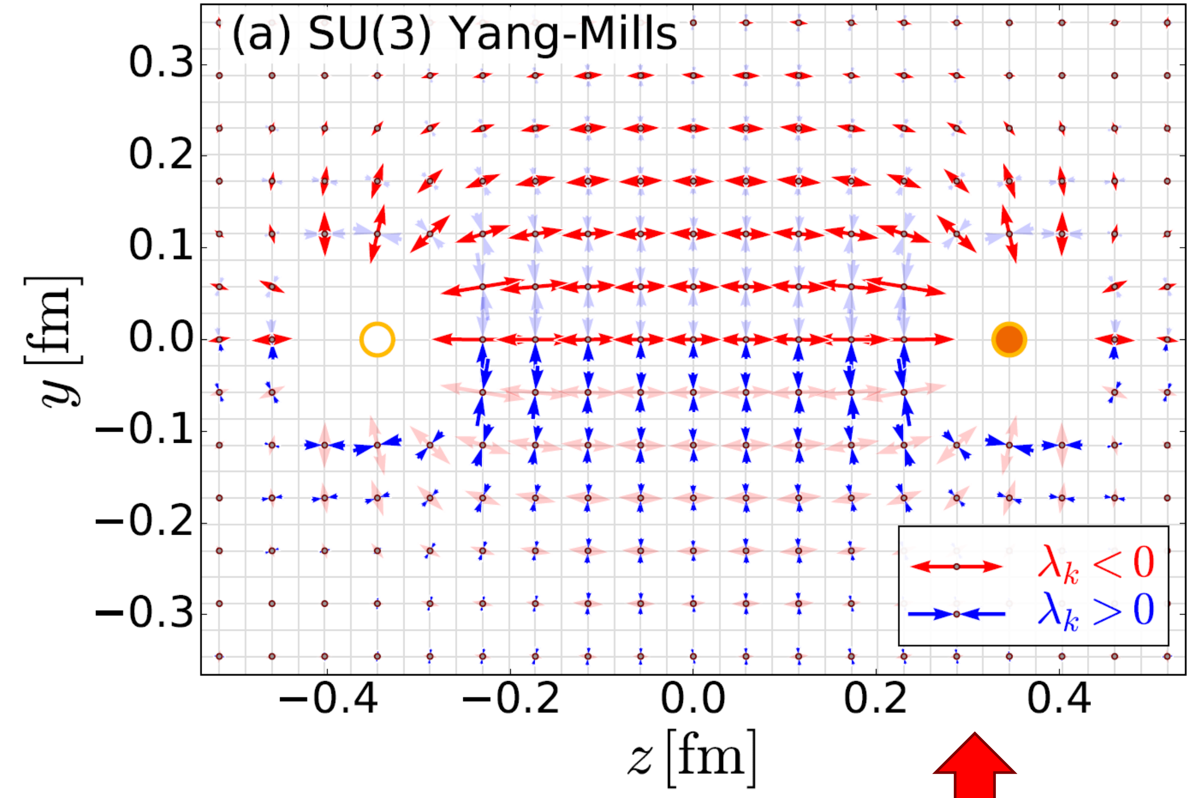
Iritani, MK, Takaura, Suzuki ('19)

$$\epsilon = \langle T_{00} \rangle, \quad p = \langle T_{11} \rangle$$

EMT Distribution in $\bar{Q}Q$ System

Eigenvectors of Stress Tensor

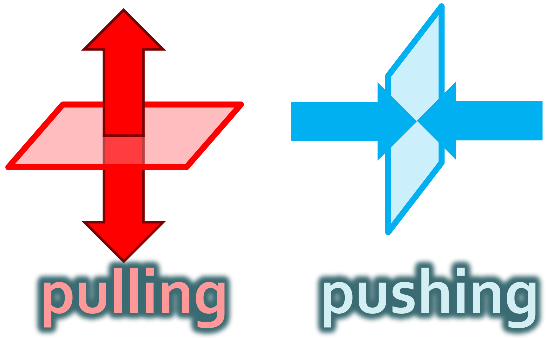
Yanagihara, MK+ (FlowQCD Col.), PLB ('19)



- **$\bar{Q}Q$ system: Flux tube**
- Visualization of the flux tube by a gauge-invariant observable.
- Mechanical structure of the flux tube.
- Force mediation via local interaction.

SU(3) YM

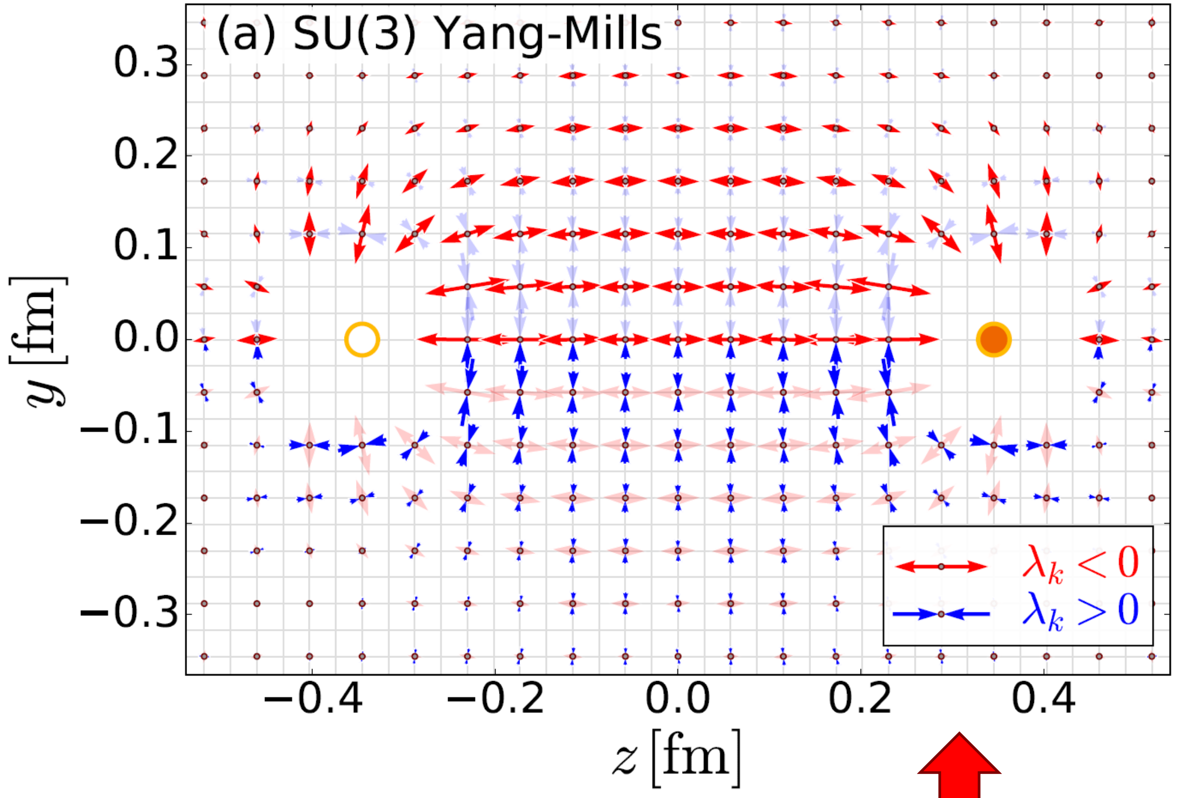
$a = 0.029\text{fm}, R = 0.69\text{fm}$

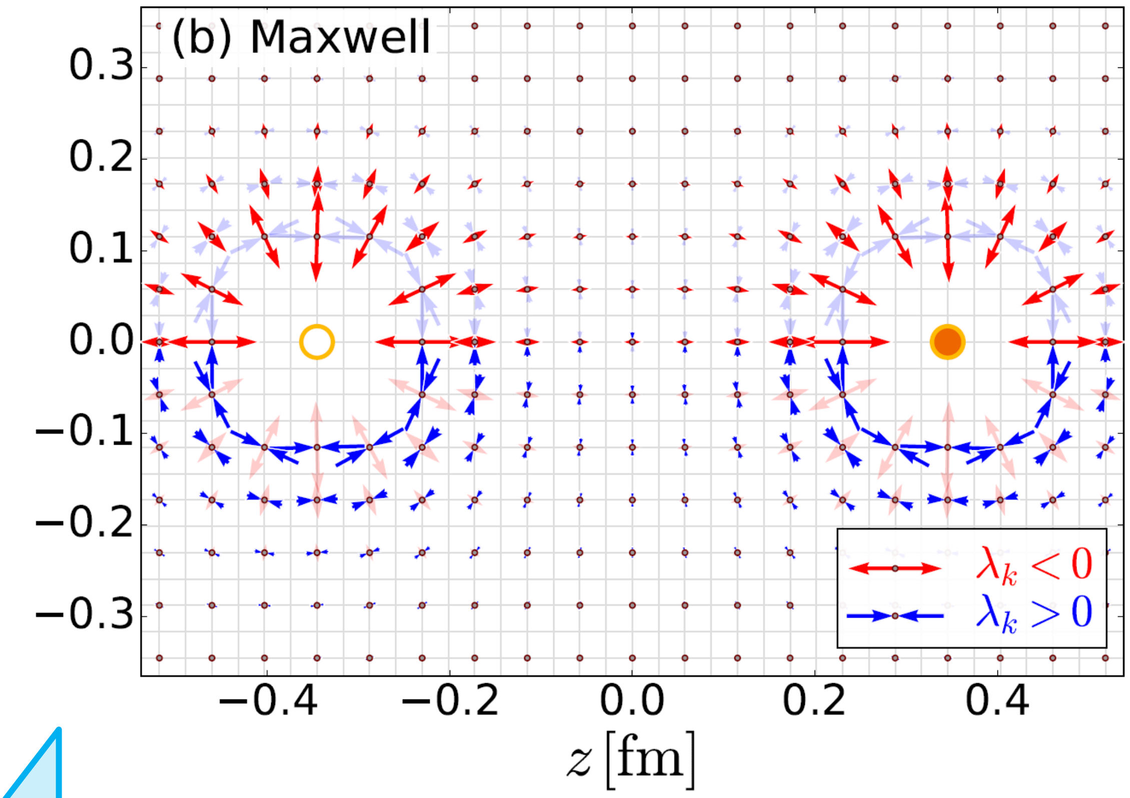


EMT Distribution in $\bar{Q}Q$ System

Eigenvectors of Stress Tensor

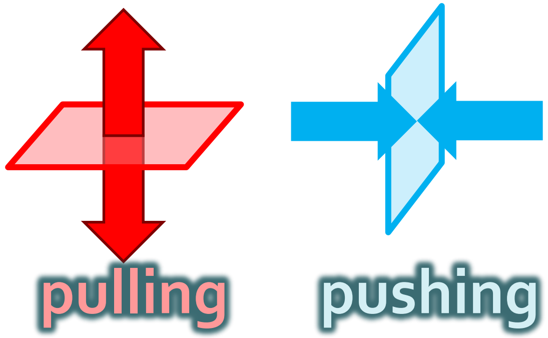
Yanagihara, MK+ (FlowQCD Col.), PLB ('19)





SU(3) YM

$a = 0.029\text{fm}, R = 0.69\text{fm}$



Flux Tube in Dual SC Model

Yanagihara, MK, PTEP ('19)

Dual Superconductor Picture

Nambu ('70); Nielsen, Olesen ('73);
t 'Hooft ('81); …

QCD vacuum	⟺	E/M dual of SC
Flux tube	⟺	Magnetic vortex

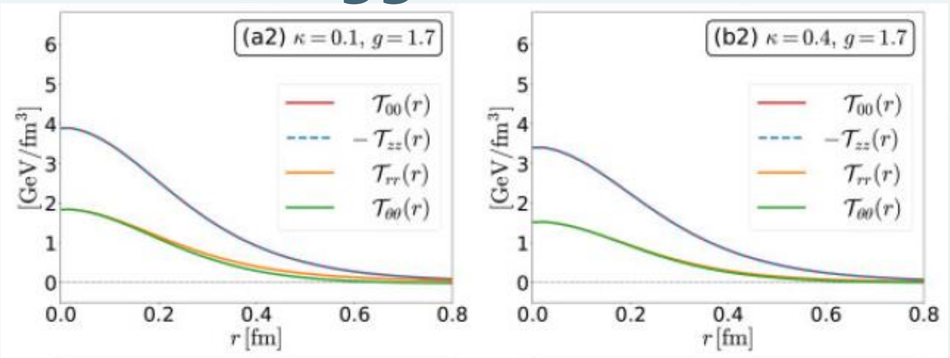
ex) Abelian-Higgs Model

$$\mathcal{L}_{\rm AH} = -\frac{1}{4}F_{\mu\nu}^2 + |(\partial_\mu + igA_\mu)\phi|^2 - \lambda(\phi^2 - v^2)^2$$

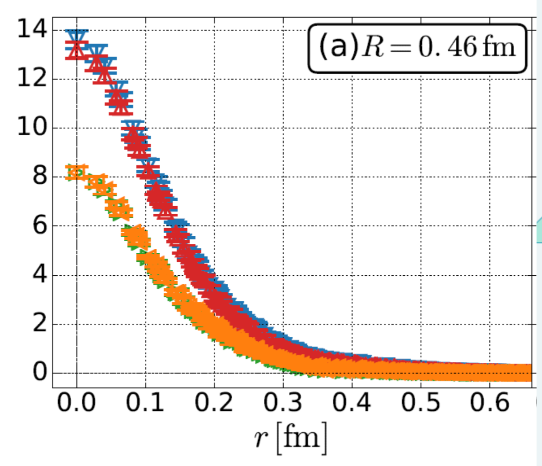
- Classical solution in the AH model
- Lattice EMT can be reproduced by the AH model qualitatively.
- Quantitative reproduction is not possible.

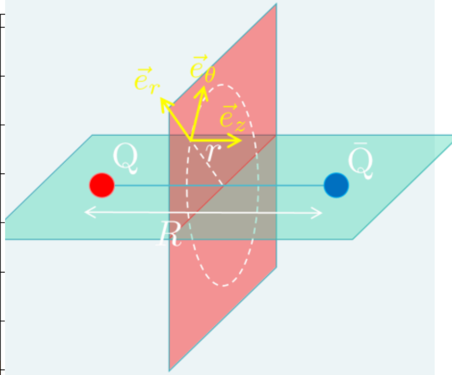
EMT on the Midplane

Abelian-Higgs model



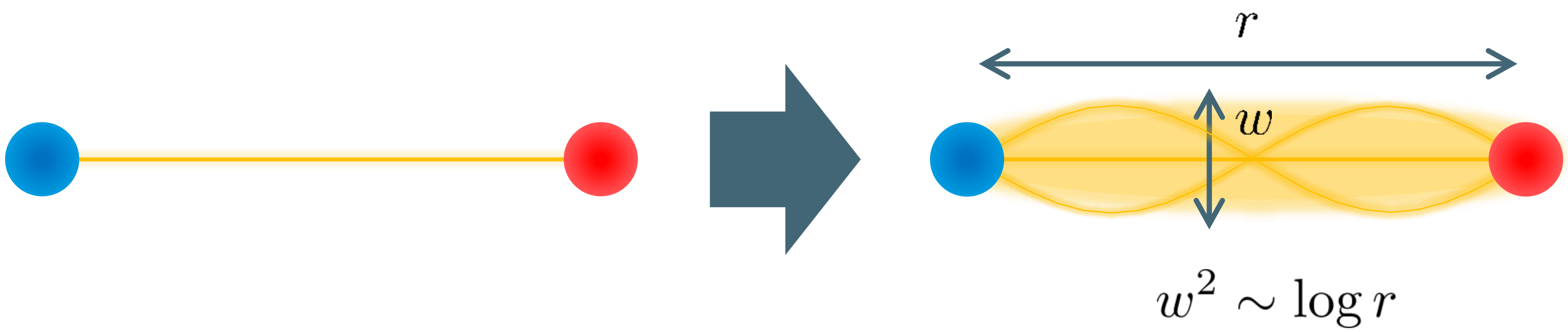
Lattice





Quantum Effect on Flux Tube

- Classical vortex is unstable against quantum fluctuations Nielsen, Olesen ('73)
- Quantum effects give rise to
  - Luscher term in potential Luscher (1981)
  - Fattening of the tube Luscher, Munster, Weisz (1981)



- How does quantum vibration of the flux tube modify the EMT distribution?
- Discrimination of "intrinsic" and "vibration" via EMT?

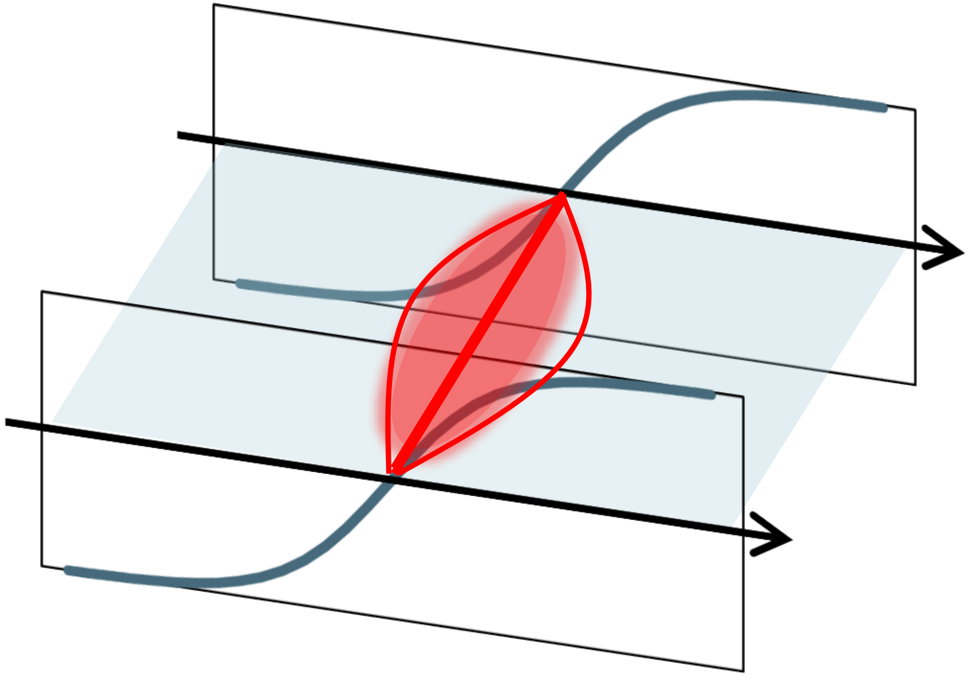
Quantum Effect in 2+1d ϕ^4 Model

$$\phi_{\text{kink}}(x,y) = \pm\frac{m}{\sqrt{\lambda}}\tanh\frac{m(x-X)}{\sqrt{2}}$$

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{\lambda}{4}\left(\phi - \frac{m^2}{\lambda}\right)^2$$

The kink in 2+1d forms a stable surface at the **classical level**.

How do the **quantum effects** blur the surface?



Analysis of ϕ^4 Model

Fluctuations around the Kink

Model: ϕ^4 theory (1+1d)

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{\lambda}{4}\left(\phi - \frac{m^2}{\lambda}\right)^2$$

$\phi(x)$: real scalar

Kink

$$\phi_{\mathrm{kink}} = \frac{m}{\sqrt{\lambda}} \tanh \frac{m(x-X)}{\sqrt{2}}$$

Expansion around $\phi_{\mathrm{kink}}(x)$

$$\phi(x) = \phi_{\mathrm{kink}}(x) + \eta(x)$$

$$S[\eta] = S_{\mathrm{cl}} + \int d^2x \left\{ \underbrace{\frac{1}{2}(\partial_0 \eta)^2 - \frac{1}{2}\eta\left(-\partial_x^2 - m^2 + 3\lambda\phi_{\mathrm{kink}}^2\right)\eta}_{\text{quadratic} \Rightarrow \text{diagonalize}} - \underbrace{\lambda\phi_{\mathrm{kink}}\eta^3}_{\mathcal{O}(\lambda^{1/2})} - \frac{\lambda}{4}\eta^4 \right\}$$

quadratic ➡ diagonalize

$\mathcal{O}(\lambda^{1/2})$

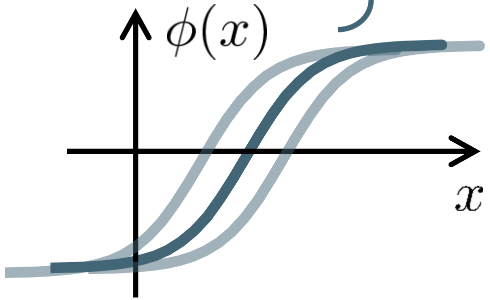
- First-order terms can be eliminated by the partial integral & EoM.

Diagonalization

$$\left(-\partial_x^2 - m^2 + 3\lambda\phi_{\text{kink}}^2 \right)\psi_n = \omega_n^2 \psi_n$$

ex)
Rajaraman,
"Solitons & Instantons"

Eigenvals	**Eigenfuncs**		
$\omega_q^2 = q^2 + 2m^2$	$\psi_q(x) \to \exp\left[i\left(qx \pm \frac{1}{2}\delta\right)\right]$	δ: phase shift	scattering states
$\omega_1^2 = \frac{3}{2}m^2$	$\psi_1(x)$	: vibrational mode	bound states
$\omega_0^2 = 0$	$\psi_0(x) \sim \partial_x \phi_{\text{kink}}(x)$	: translational mode (zero mode)	bound states



Total Energy at $O(\lambda^0)$ (Dashen+, '74)

Total Energy

Sum of zero-point energy of all modes

$$E = E_{\rm cl} + \frac{1}{2}\sum_n \omega_n$$

➡ **Divergent**

Two Steps to Remove the Divergence

1) Vacuum Subtraction $E = E_{\rm kink} - E_{\rm vac}$

2) Mass Renormalization

mass counter term in vacuum sector

Vacuum Subtraction (Dashen+, ’74)

Rebhan and Nieuwenhuizen (’97)
Rajaraman, “solitons & instantons”

Mode Number Cutoff

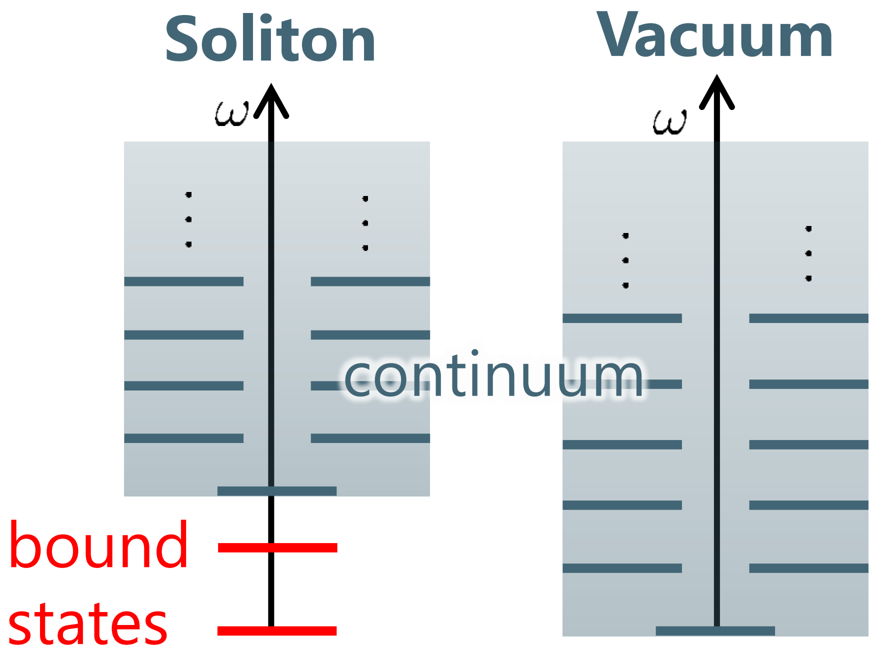
- System with finite length L
  → Scattering modes become discrete.
- Perform the vacuum subtraction with the same mode number N.
- Take $N \to \infty$, and then $L \to \infty$.

Scattering modes $\psi_q(x)$ satisfy

- Soliton sector: $q_n + 2\delta(q_n) = 2n\pi/L$
- Vacuum sector: $q_n = 2n\pi/L$

$$E_{\rm kink} = \frac{2\sqrt{2}}{3}\frac{m^3}{\lambda} + \left(\frac{\sqrt{3}}{6\sqrt{2}} - \frac{3}{\sqrt{2}\pi}\right)m$$

Spectra for periodic BC



Energy Momentum Tensor

EMT as a Noether Current

$$T_{\mu\nu}(x) = \frac{1}{2}(\partial_\mu \phi)(\partial_\nu \phi) - \frac{1}{4}\delta_{\mu\nu}(\partial_\rho \phi)^2 + U(\phi)$$

$$\phi(x) = \phi_{\mathrm{kink}}(x) + \eta(x)$$

$$\begin{aligned} T_{00} =& T_{00}^{\mathrm{kink}} + \frac{1}{2}(\partial_0 \eta)^2 + \frac{1}{2}(\partial_1 \eta)^2 + (\partial_1 \phi_{\mathrm{kink}})(\partial_1 \eta) + \lambda \phi_{\mathrm{kink}}\left(\phi_{\mathrm{cl}}^2 - \frac{m^2}{\lambda}\right)\eta \\ &+ \frac{\lambda}{2}\left(3\phi_{\mathrm{kink}}^2 - \frac{m^2}{\lambda}\right)\eta^2 + \mathcal{O}(\lambda^1) \end{aligned}$$

- Calculate expectation value of each term.
- First-order terms do exist.
- Zero mode gives rise to divergences in perturbative expansion.

Collective Coordinate Method

Zero mode gives rise to IR divergence!

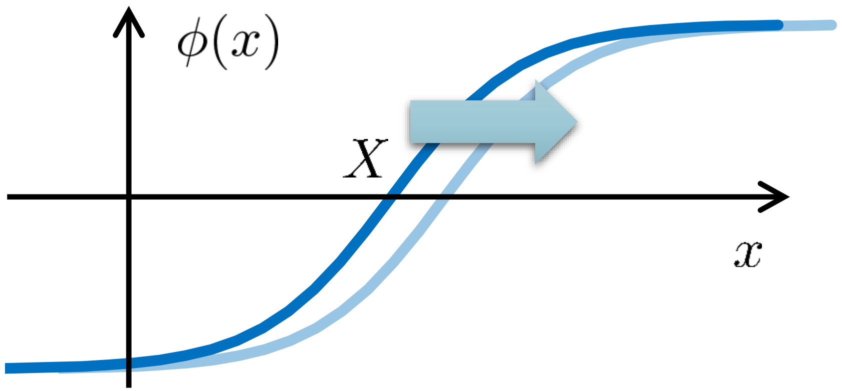
Gervais, Sakita '74
Gervais, Jevicki, Sakita, '75
Tomboulis, '75; Christ, Lee, '75

CCM: Basic idea

zero mode = translational mode

$$\psi_0(x) \sim \partial_x \phi_{\text{kink}}(x)$$

- Eliminate the zero mode
- Promote X to a dynamical val.



$$\phi(x,t) = \phi_{\text{kink}}(x - X(t)) + \tilde{\eta}(x - X(t))$$

$X(t)$: dynamical

constraint: $\int dx \tilde{\eta}(x)\psi_0(x) = 0$

Collective Coordinate Method

$$\phi(x,t) = \phi_{\rm kink}(x - X(t)) + \tilde{\eta}(x - X(t))$$

Gervais, Sakita '75
Gervais, Jevicki, Sakita, '75
Tomboulis, '75; Christ, Lee, '75

Dynamical variables: $X, \tilde{\eta}(x)$

Conjugate momenta: $P, \tilde{\pi}(x)$

Constraints:

$$\int dx \tilde{\eta}(x)\psi_0(x) = 0, \quad \int dx \tilde{\pi}(x)\psi_0(x) = 0$$

New Hamiltonian

$$\mathcal{H}[X, P, \eta, \pi] = E_{\rm cl} + \frac{P^2}{2E_{\rm cl}} + \tilde{\mathcal{H}}[\tilde{\eta}, \tilde{\pi}] + \mathcal{O}(\lambda)$$

Hamiltonian in the center of mass frame

- Non-trivial coupling between $X, P, \tilde{\eta}, \tilde{\pi}$ in higher order terms.

Expectation Values

$$T_{00} = T_{00}^{\text{kink}} \qquad \mathcal{O}(\lambda^{-1})$$

$$+(\partial_1 \phi_{\text{kink}})(\partial_1 \eta) + \lambda \phi_{\text{kink}} \left(\phi_{\text{kink}}^2 - \frac{m^2}{\lambda} \right) \eta \qquad \mathcal{O}(\lambda^{-1/2})$$

$$+\frac{1}{2}(\partial_0 \eta)^2 + \frac{1}{2}(\partial_1 \eta)^2 + \frac{\lambda}{2} \left(3\phi_{\text{kink}}^2 - \frac{m^2}{\lambda} \right) \eta^2 \qquad \mathcal{O}(\lambda^0)$$

$$+\mathcal{O}(\lambda^1)$$

Mass Counter Term

$$\delta \mathcal{L} = \delta m^2 \phi(x)^2$$

$O(\lambda^0)$ Terms

$$\langle \eta(x)^2 \rangle = G(x, x)$$

$$\langle (\partial_0 \eta(x))^2 \rangle \neq \partial_0 \partial_0' G(x, x)$$

Green Function

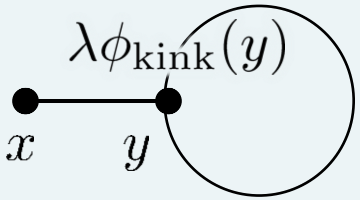
$$G(x, y) = \int d\omega {\sum_n}' \psi_n(x) \frac{i}{\omega^2 - \omega_n^2 + i\epsilon} \psi_n^*(y)$$

Tadpole Diagram

$$T_{00} = (\partial_1 \phi_{\rm kink})(\partial_1 \eta) + \lambda \phi_{\rm kink}\left(\phi_{\rm kink}^2 - \frac{m^2}{\lambda}\right)\eta + \cdots \qquad \mathcal{O}(\lambda^{-1/2})$$

$$\mathcal{L}_I = -\lambda \phi_{\rm kink} \eta^3 \qquad \mathcal{O}(\lambda^{1/2})$$

$\langle \eta(x) \rangle =$



$$= \int dy \lambda G(x,y) \phi_{\rm kink}(y) G(y,y) \qquad \mathcal{O}(\lambda^0)$$

analytically calculable

EoM $\quad \left(-\partial_x^2 - m^2 + \lambda \phi_{\rm kink}^2\right)\phi_{\rm kink} = 0$

Eigen eq. $\quad \left(-\partial_x^2 - m^2 + 3\lambda \phi_{\rm kink}^2\right)\psi_n = \omega_n^2 \psi_n$

Note:

Anti-periodic BC

$\eta(x+L) = -\eta(x)$

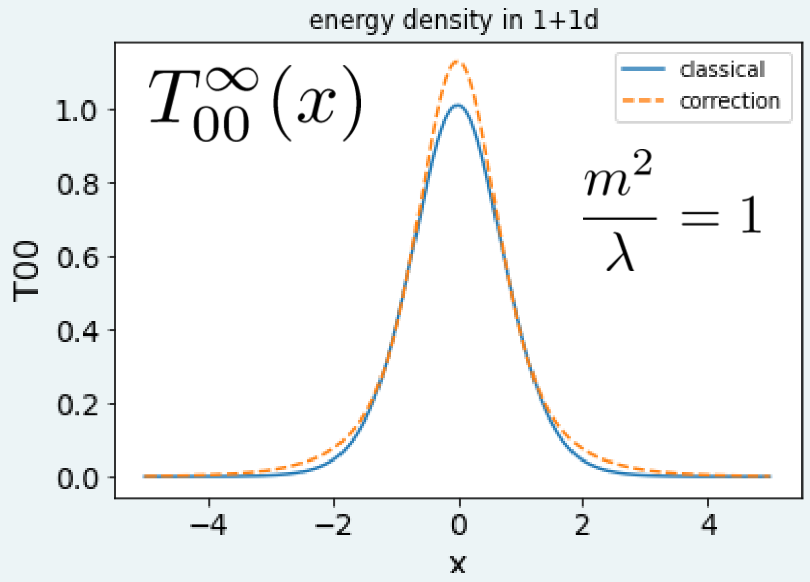
must be imposed.

Results

$$T_{00}(x) = T_{00}^{\infty}(x) - \frac{3\sqrt{2}m}{2\pi}\frac{1}{L}$$

$$T_{11}(x) = \frac{3\sqrt{2}m}{2\pi}\frac{1}{L}$$

- T_{00}^{∞}: finite & local function
- L: spatial length



Note:

- $T_{\mu\nu}(x)$ has a constant term $\sim 1/L$ that vanishes at $L \to \infty$.
- $E = \int_{-L/2}^{L/2} dx\, T_{00}(x)$ reproduces Dashen+ ('74), but $\int dx\, T_{00}^{\infty}(x)$ does not.
- Is the result in Dashen+ the total energy of the soliton?
- $T_{11}(x)$ does not have x dependence. ➔ consistent with EM conservation

$$\partial_0 T_{01} - \partial_1 T_{11} = 0$$

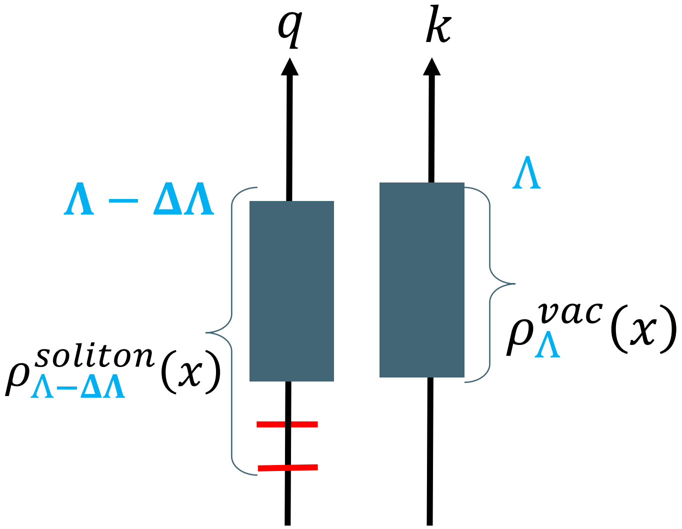
Another Scheme (Goldhaber+, ’03)

Local Mode Regularization

- Infinite system
- Subtraction with “local density of states” for each sector.

$$\rho^{\mathrm{soliton}}_{\Lambda-\Delta\Lambda}(x) = \rho^{\mathrm{vac}}_{\Lambda}(x)$$

$$\rho_N(x) \equiv \sum_{n=0}^{N} |\eta_n(x)|^2$$



Results in LMR

$$\begin{cases} T^{\mathrm{LMR}}_{00}(x) = T^{\mathrm{ours}}_{00} - \dfrac{3m}{4\pi}\cosh^{-2}\dfrac{mx}{\sqrt{2}} \\ T^{\mathrm{LMR}}_{11}(x) = T^{\mathrm{ours}}_{11} - \dfrac{3m}{4\pi}\cosh^{-2}\dfrac{mx}{\sqrt{2}} \end{cases}$$

- $E = \int_{-L/2}^{L/2} dx\, T^{\mathrm{LMR}}_{00}(x)$ reproduces Dashen+ (’74).
- $T_{11}(x)$ has x dependence.
  ➔ violates EM conservation

Kink in Sine-Gordon Model

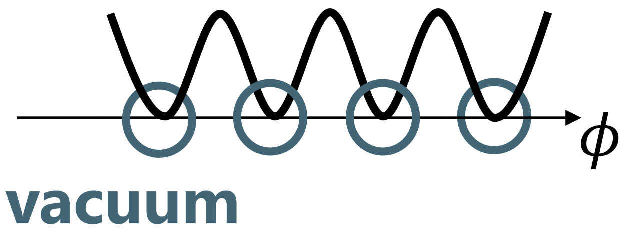
Ito, JPS meeting 2022fall

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{m^4}{\lambda}\left(\cos\frac{\sqrt{\lambda}\phi}{m} - 1\right)$$

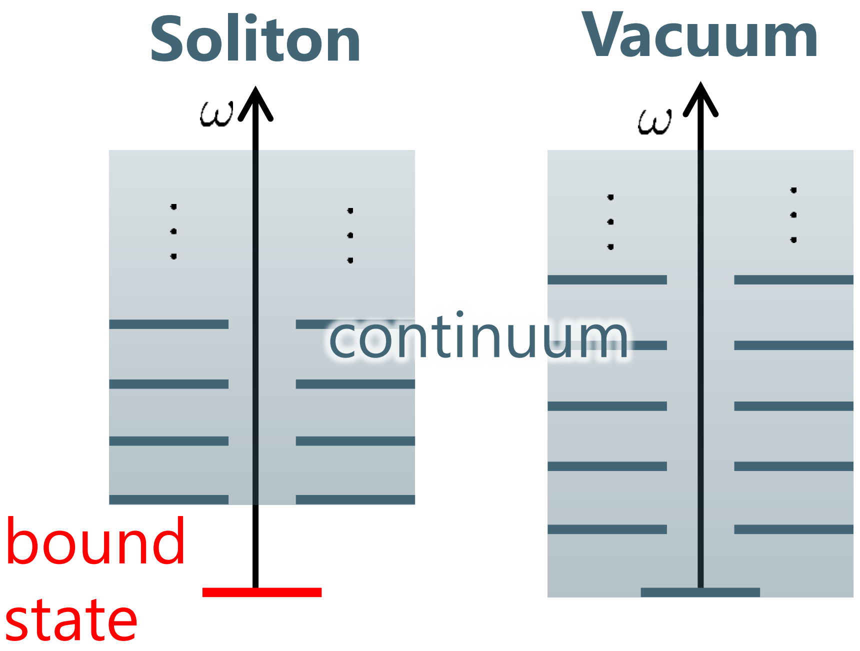
$$\phi_{\text{kink}}(x) = \frac{4m}{\sqrt{\lambda}}\arctan(e^{m(x-X)})$$

- Same calculation is feasible.
- Obtained EMT satisfies the momentum conservation.

Difference in the spectra:
SG model has only a bound state.



Spectra for periodic BC



Summary & Outlook

- Calculation of EMT distribution around a kink at the one-loop order.
- EMT distr. at the quantum level around a localized structure in QFT.
- Our result satisfies the momentum conservation.
- Conservation laws can be used for discriminating the correct scheme.

Future Studies

- Interpretation of $\sim 1/L$ term in the EMT.
- Breather state in Sine-Gordon model.
- Similar problem in 2+1dimensional systems → fattening of the surface