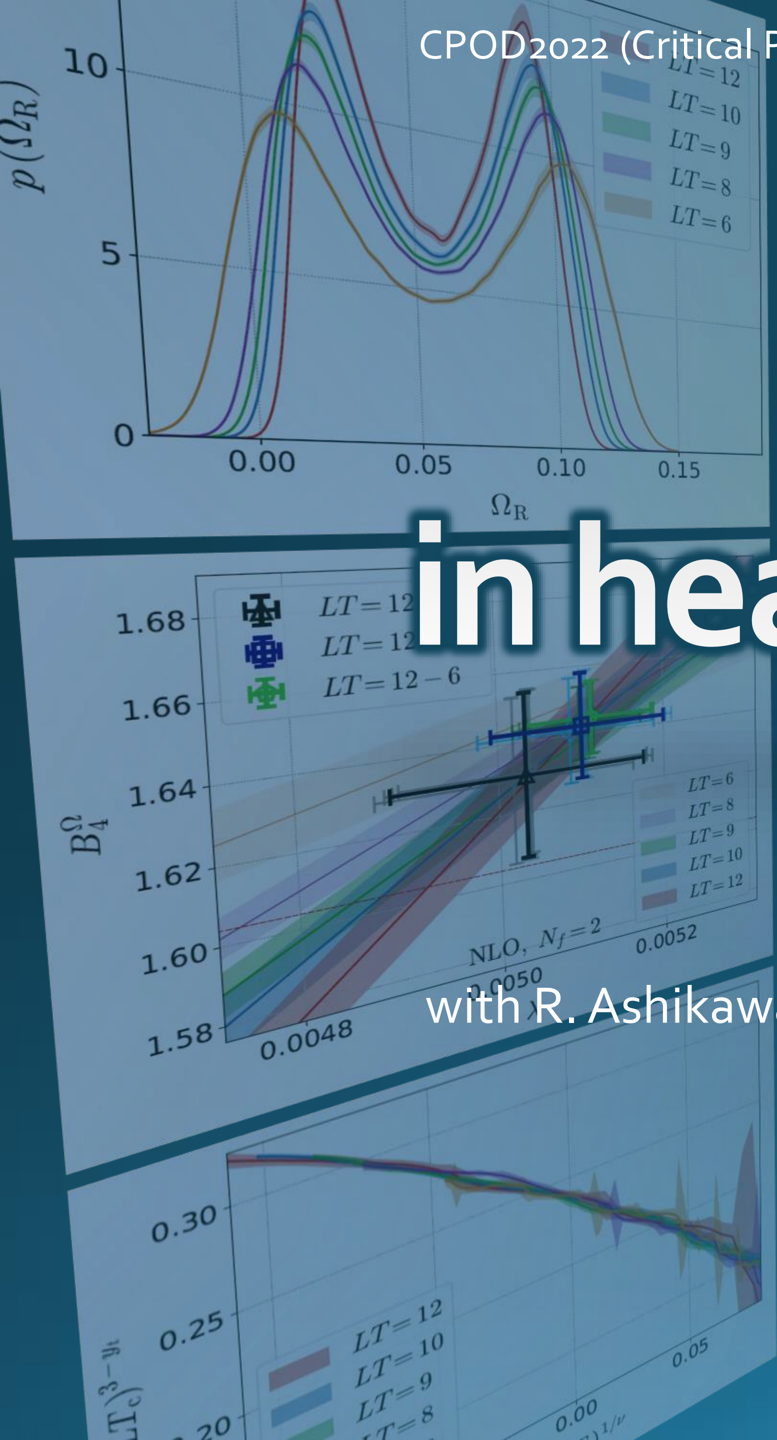
CPOD2022 (Critical Point and Onset of Deconfinement), Dec. 1, 2022

Lattice study of the Critical point in heavy-quark QCD

Masakiyo Kitazawa

(YITP, Kyoto)

with R. Ashikawa, S. Ejiri, K. Kanaya, H. Suzuki, N. Wakabayashi



Ashikawa+, in preparation

Wakabayashi+, PTEP 2022, 033B05 (2022) [2112.06340]

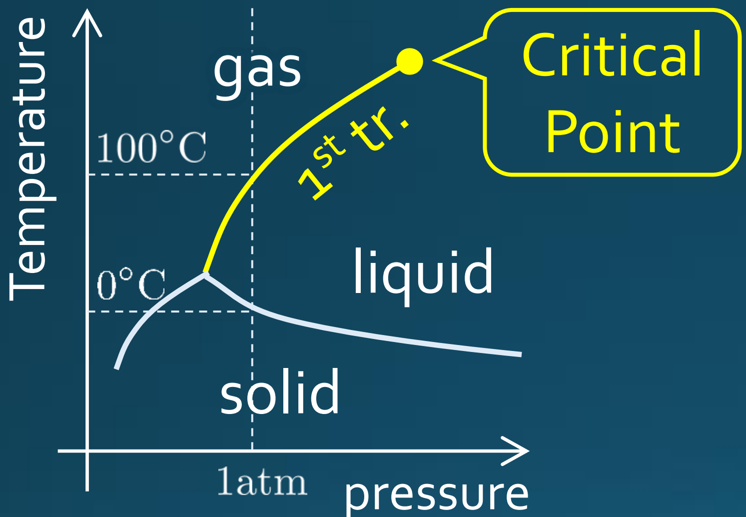
Kiyohara+, Phys. Rev. D104, 114509 (2021) [2108.00118]

Shirogane+, PTEP 2021, 013B08 (2021) [2011.10292]

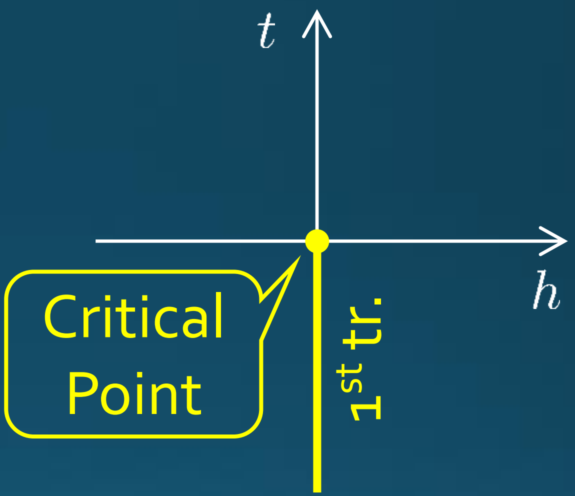
Ejiri+, Phys. Rev. D101, 054505 (2020) [1912.10500]

Critical Points

- **Water**



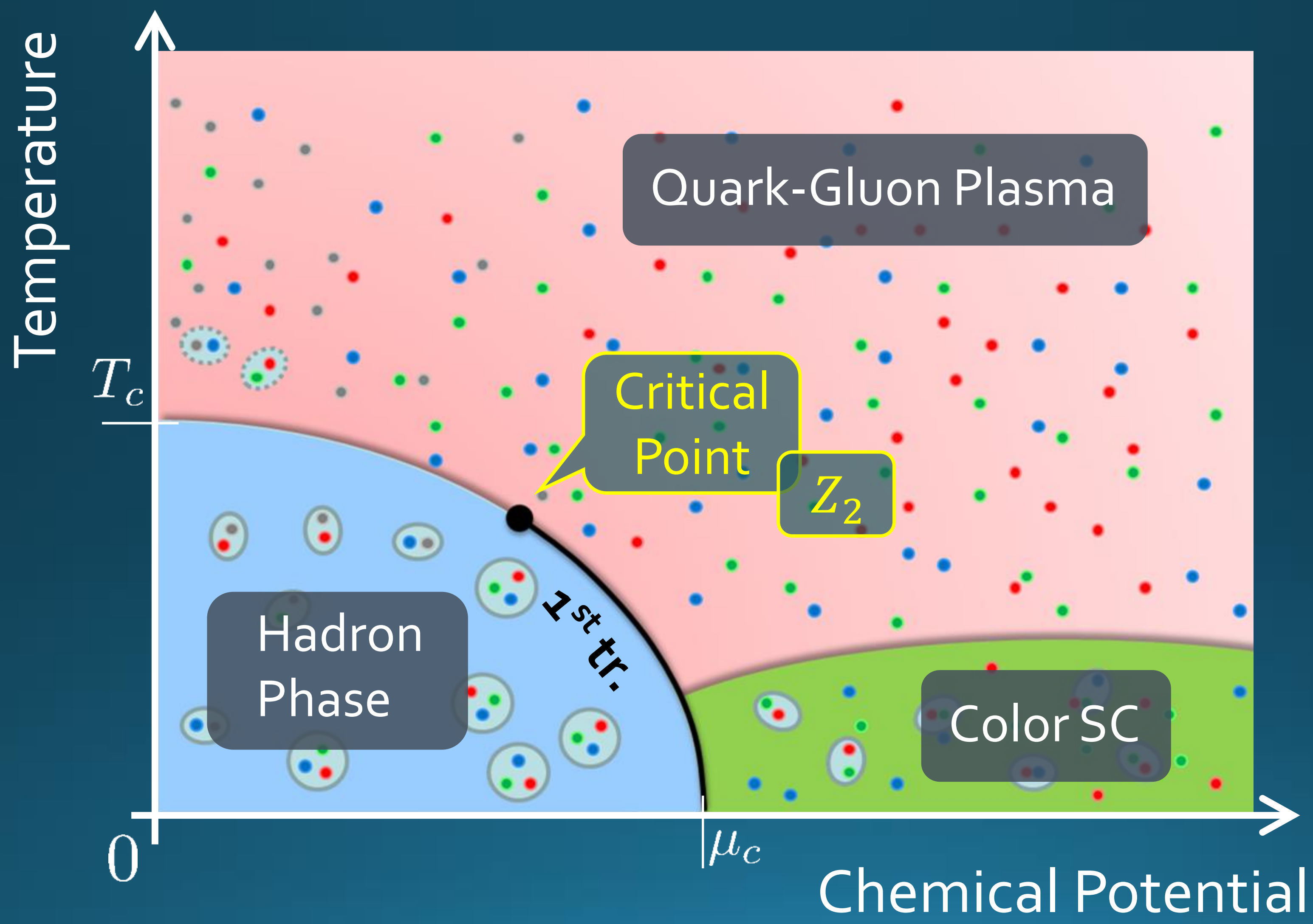
- **Ising Model**



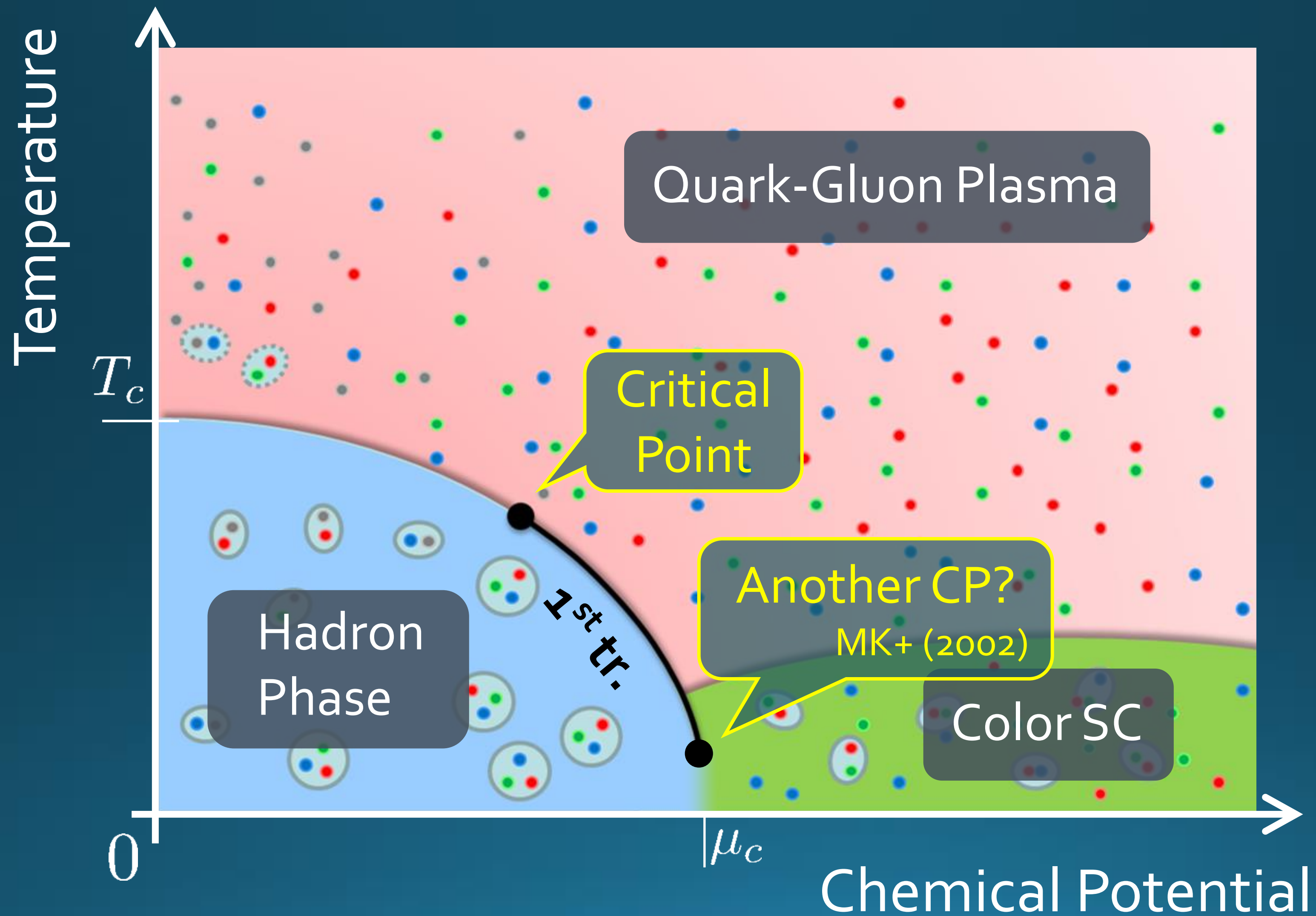
- CP: Second-order transition point.
- Singularities in thermodynamic quantities.
- These CPs belong to the same universality class (Z_2).

➡ Common critical exponents. Ex. $C \sim (T - T_c)^{-\alpha}$

QCD Phase Diagram



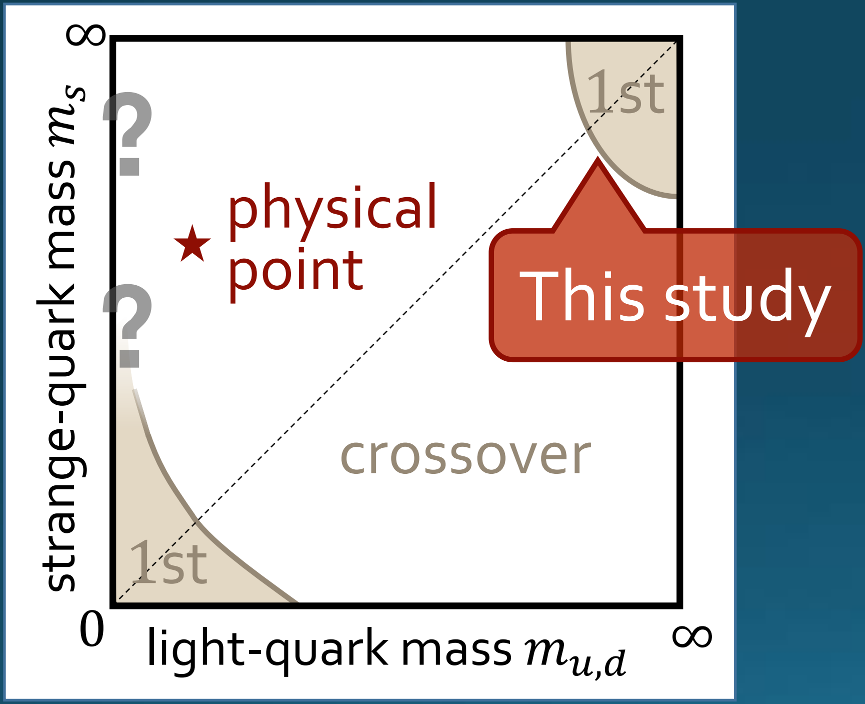
QCD Phase Diagram



Varying Quark Masses @ $\mu_q = 0$

☐ Columbia plot

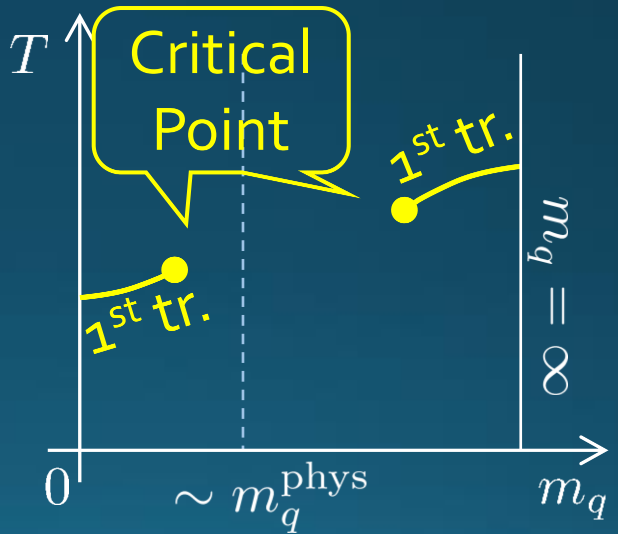
= order of phase tr. at $\mu_q = 0$



☐ Example

Phase diagram in $T - m_q$ plane

$N_f = 3$



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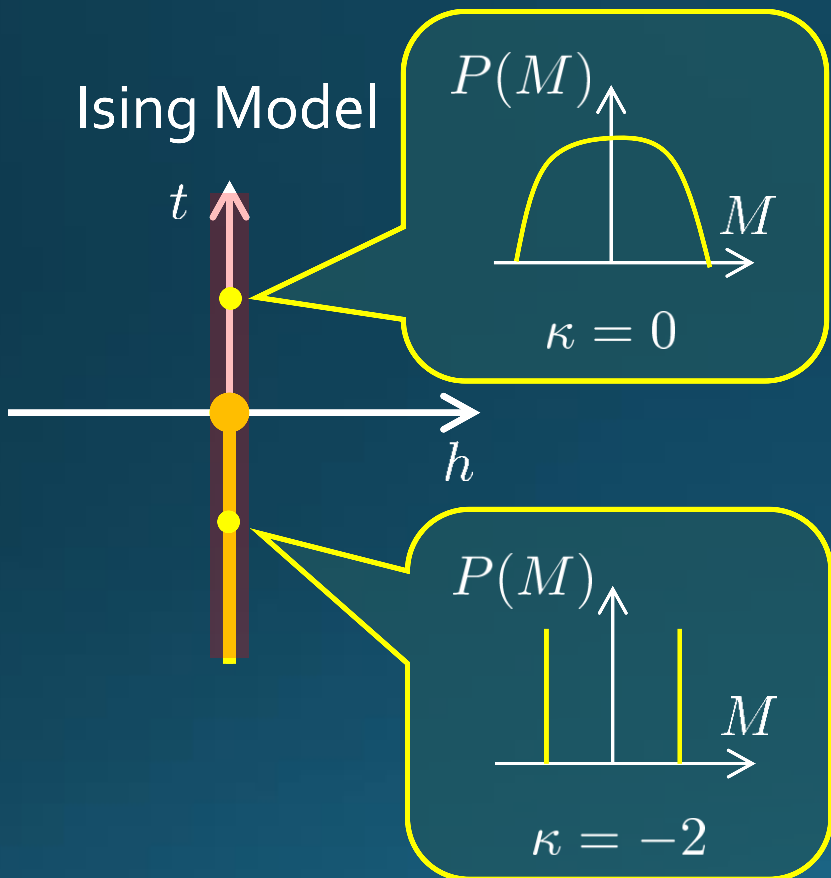
Various orders of phase transition with a variation of m_q.

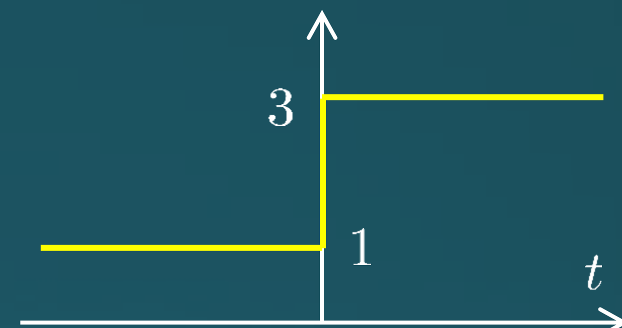
Binder Cumulant B_4

Ising Model



Binder Cumulant

$$B_4 = \frac{\langle M^4 \rangle_c}{\langle M^2 \rangle_c^2} + 3 = \kappa + 3$$



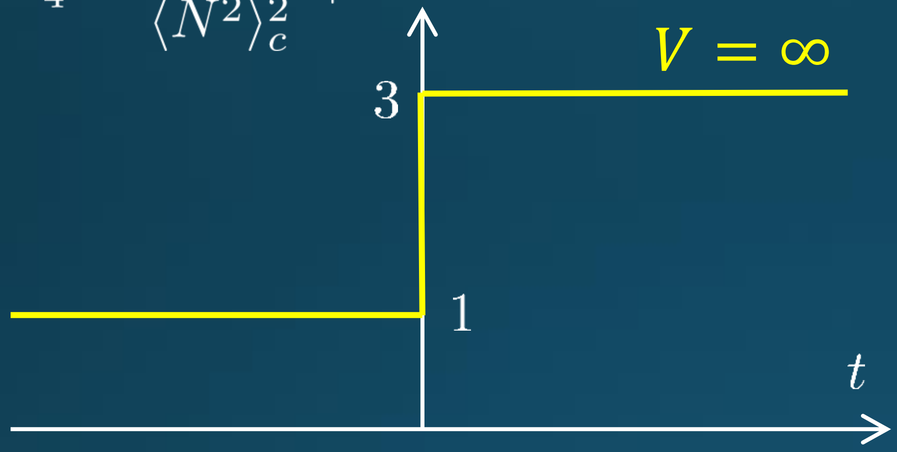
Kurtosis: $\kappa = \dfrac{\langle N^4 \rangle_c}{\langle N^2 \rangle_c^2}$

- $\langle M^4 \rangle_{c,h=0}$ changes discontinuously at the CP.

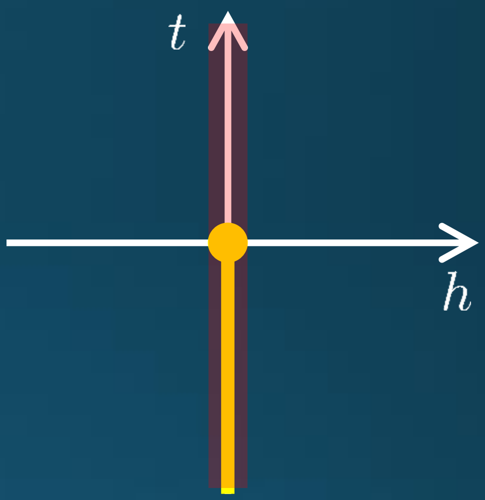
Finite-Volume Effects

Binder Cumulant

$$B_4 = \frac{\langle N^4 \rangle_c}{\langle N^2 \rangle_c^2} + 3$$



Ising Model

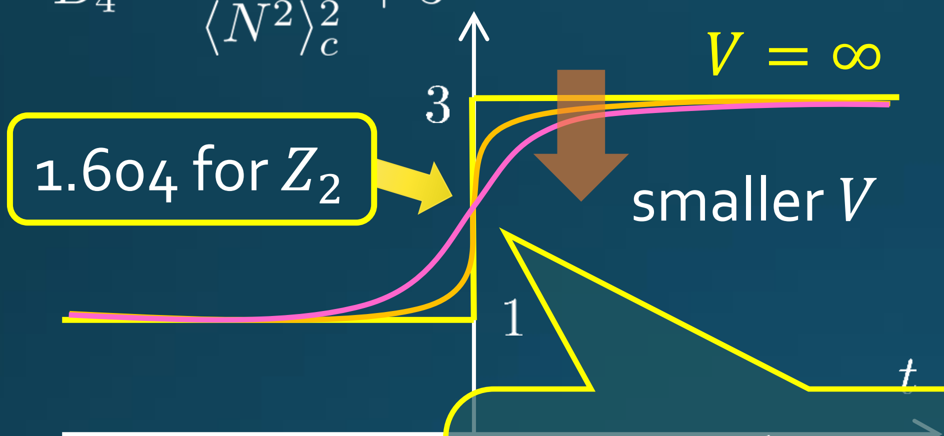


- Discontinuity of B_4 at the CP is smeared on finite V.
- B_4 obtained at various V have crossing at $t = 0$.

Finite-Volume Effects

Binder Cumulant

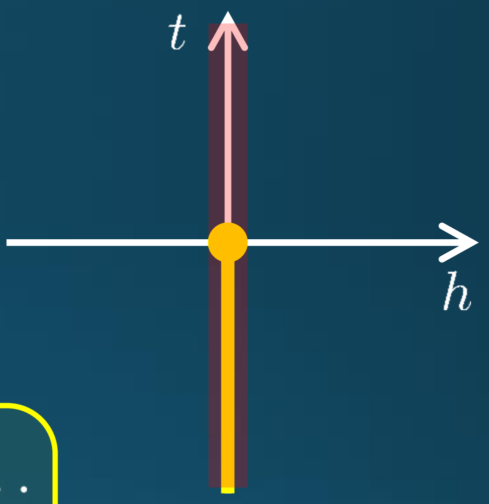
$$B_4 = \frac{\langle N^4 \rangle_c}{\langle N^2 \rangle_c^2} + 3$$



$$B_4(t, 0, L^{-1}) = b_4 + ctL^{y_t} + \cdots$$

Z(2) CP $\begin{cases} b_4 = 1.604 \\ \nu = 1/y_t = 0.630 \end{cases}$

Ising Model

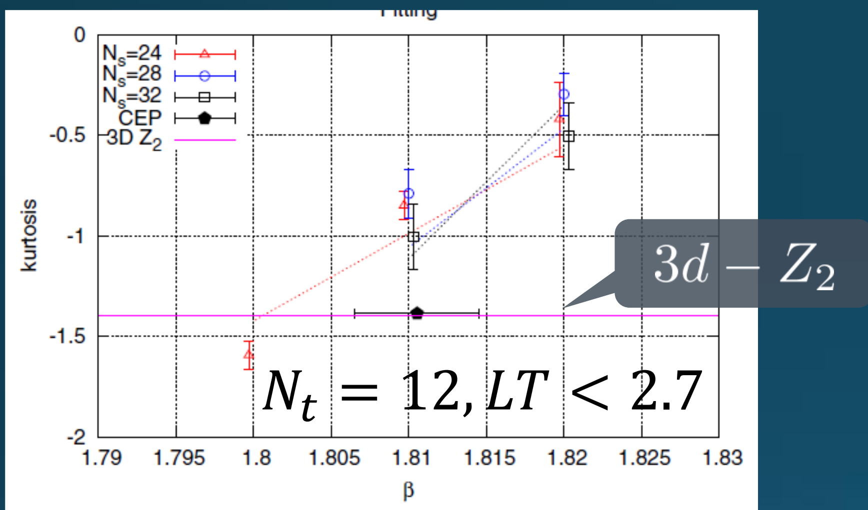


- Discontinuity of B_4 at the CP is smeared on finite V.
- B_4 obtained at various V have crossing at $t = 0$.

Lattice Studies of Binder-Cumulant

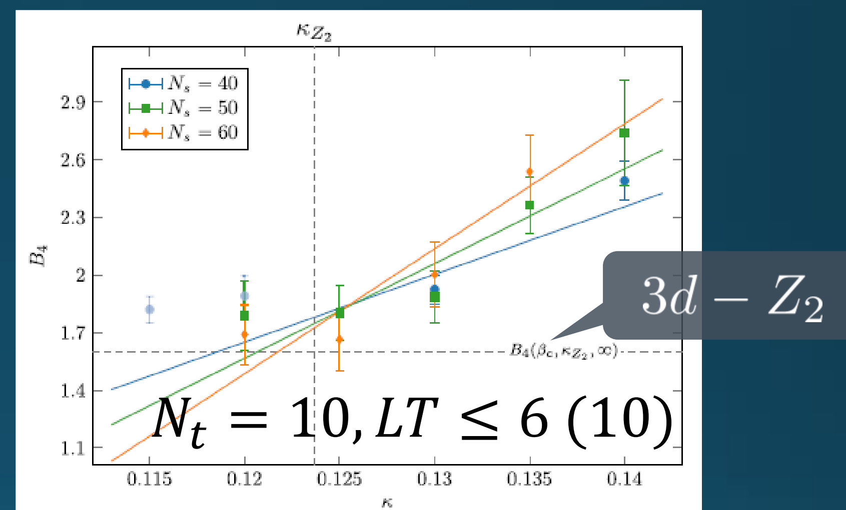
☐ Light-quark region

Kuramashi, Nakamura, Ohno, Takeda, ’20



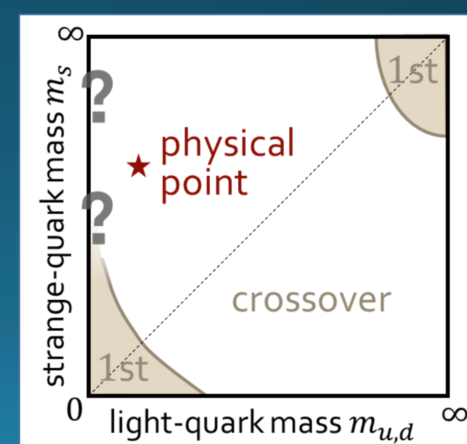
☐ Heavy-quark region

Cuteri , Philipsen , Schön, Sciarra, ’21



- ☐ Statistically-significant deviation of the crossing point from the 3d-Ising value.

  ➡ V may not be large enough?



Our Strategy

Kiyohara+, PRD104 (2021); Ashikawa+, in prep.

Numerical simulations on large spatial volume up to $LT = 15$

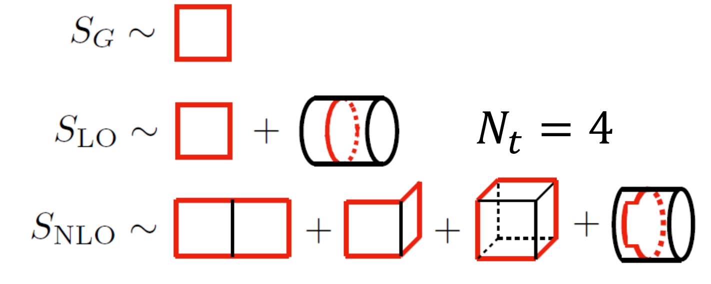
To realize it:

- ☐ CP in the heavy-quark region
- ☐ Simulations on coarse lattices ($N_t = 1/aT = 4 \rightarrow 6$)
- ☐ Hopping parameter ($\kappa \sim 1/m_q$) expansion (HPE)

$$\kappa \sim \frac{1}{2m_q a}$$

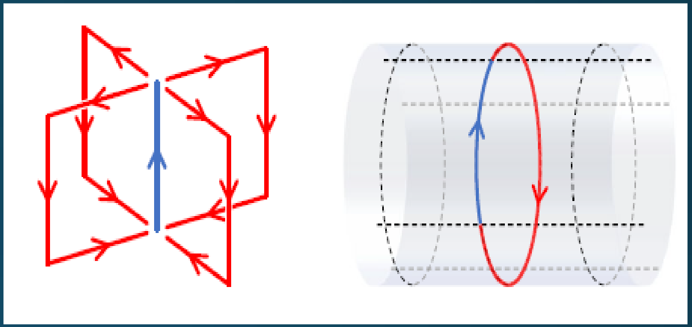
$$\ln\det M(\kappa) = -\sum_{n=1}^{\infty} \frac{1}{n} \mathrm{tr}[B^n] \kappa^n$$

$\mathrm{tr}[B_n]$ is given by the closed trajectories of length n.



Hopping Parameter Expansion

- **Monte Carlo Simulation @ LO**
  - heat bath & over relaxation with modified staple
  - ➡ Numerical cost is almost the same as the pure YM!

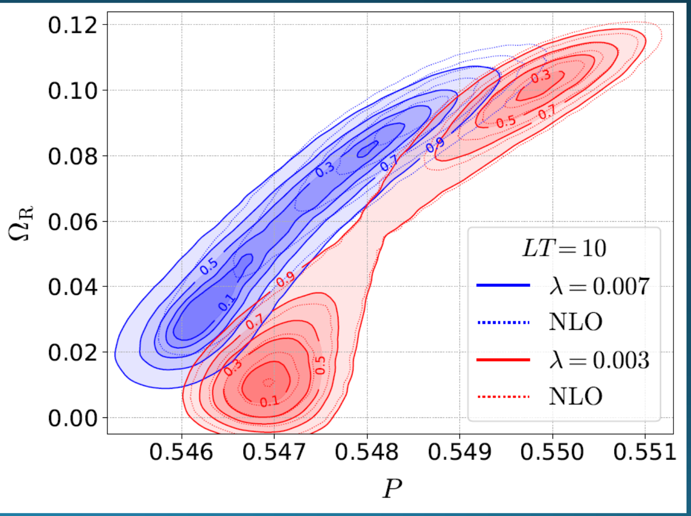


- **NLO by Reweighting**

$$\langle \mathcal{O} \rangle_{\mathrm{NLO}} = \frac{\langle \hat{O} e^{-S_{\mathrm{NLO}}} \rangle_{\mathrm{LO}}}{\langle e^{-S_{\mathrm{NLO}}} \rangle_{\mathrm{LO}}}$$

  - Overlapping problem is well suppressed due to the LO confs.
  - ➡ Realize high statistical analysis

$$\lambda = 64 N_c N_f \kappa^4$$



Numerical Simulation @ $N_t = 4$

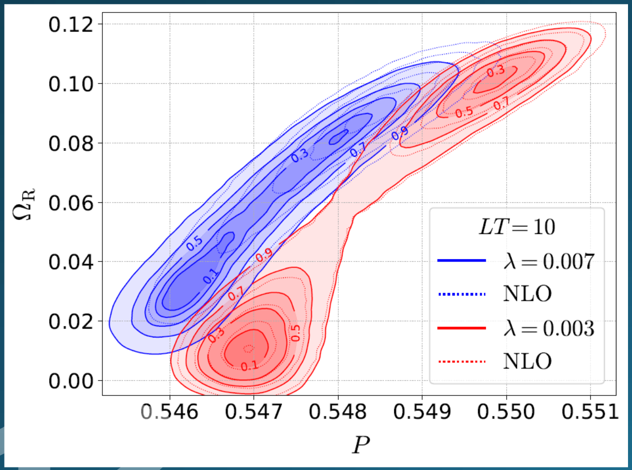
Kiyohara+, PRD104 (2021) 114509

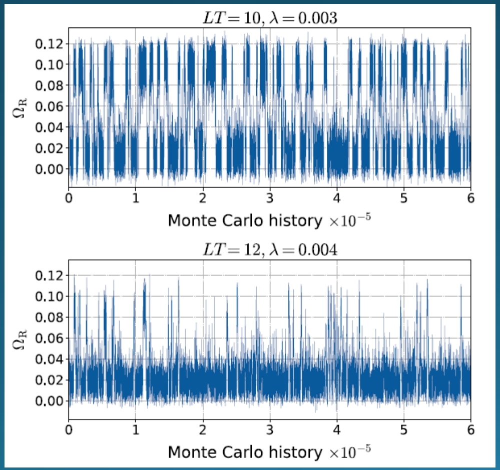
- Coarse lattice: $N_t = 4$
- But **large spatial volume**:
  $LT = N_s \,/\, N_t \le 12$

- Hopping-param. (~$1/m_q$) expansion
- Monte-Calro with LO action
- High statistical analysis (6×10^5 meas.)

Simulation params.

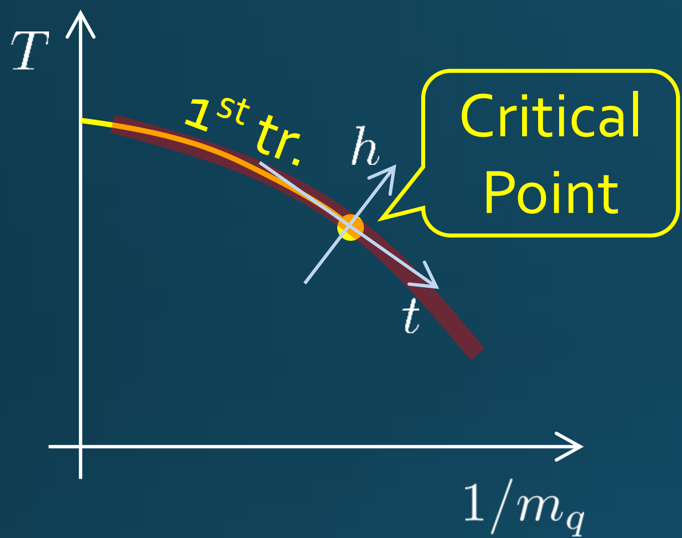
lattice size	β^*	λ	$\kappa^{N_f=2}$
$48^3 \times 4$	5.6869	0.004	0.0568
	5.6861	0.005	0.0601
	5.6849	0.006	0.0629
$40^3 \times 4$, $36^3 \times 4$	5.6885	0.003	0.0529
	5.6869	0.004	0.0568
	5.6861	0.005	0.0601
	5.6849	0.006	0.0629
	5.6837	0.007	0.0653
$32^3 \times 4$	5.6885	0.003	0.0529
	5.6865	0.004	0.0568
	5.6861	0.005	0.0601
	5.6845	0.006	0.0629
	5.6837	0.007	0.0653
$24^3 \times 4$	5.6870	0.0038	0.0561
	5.6820	0.0077	0.0669
	5.6780	0.0115	0.0740





$\lambda = 64 N_c N_f \kappa^4$

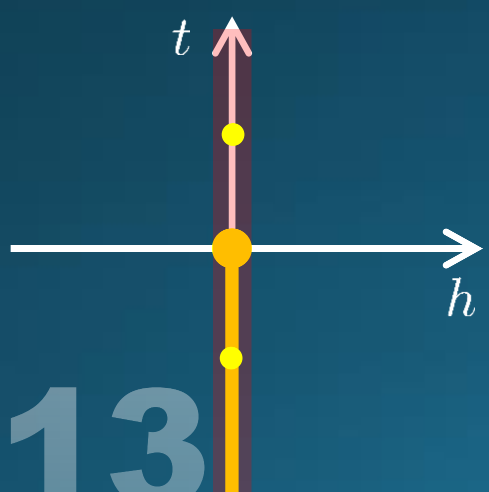
Transition Line

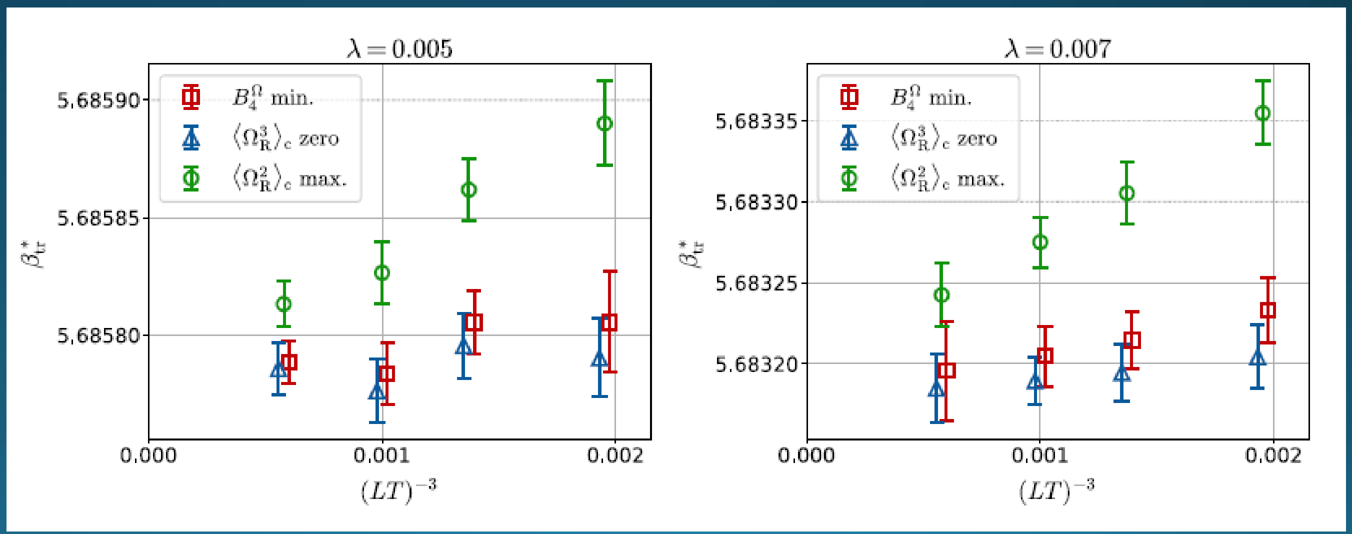


Definitions of transition line

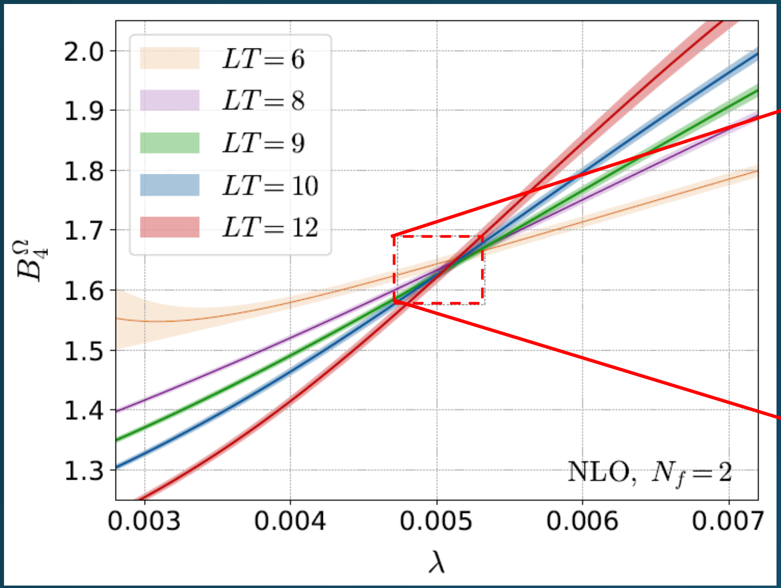
- ☐ Maximum of $\langle \Omega_R^2 \rangle$
- ☐ Zero of $\langle \Omega_R^2 \rangle$
- ☐ Minimum of B_4

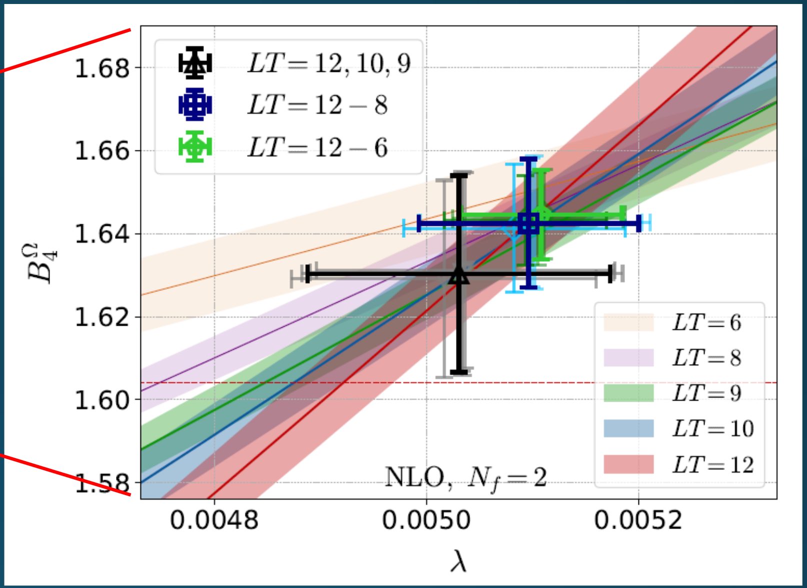
Ising Model





Binder-Cumulant Analysis





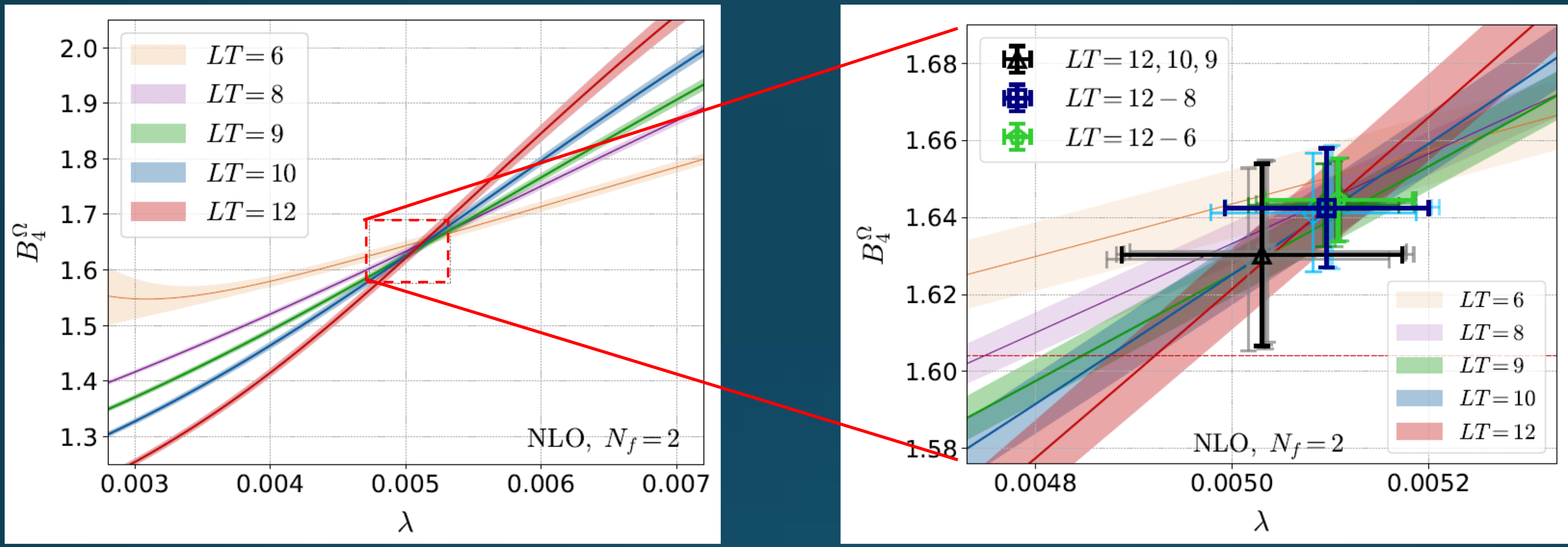
$$\lambda = 64 N_f N_c \kappa^4$$

Fitting function

$$B_4(\lambda, LT) = b_4 + c(\lambda - \lambda_c)(LT)^{1/\nu}$$

params: b_4, c, λ_c, ν

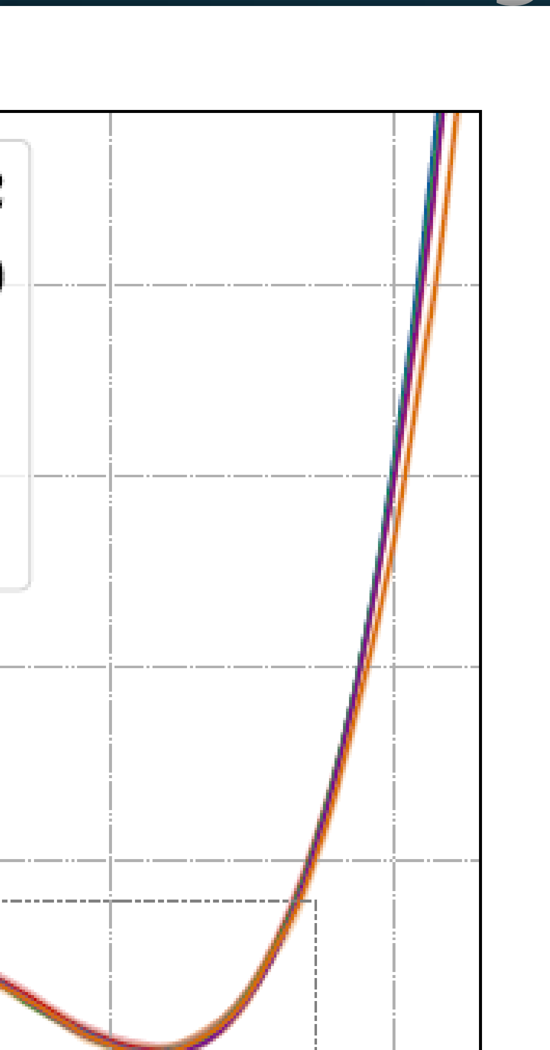
Binder-Cumulant Analysis



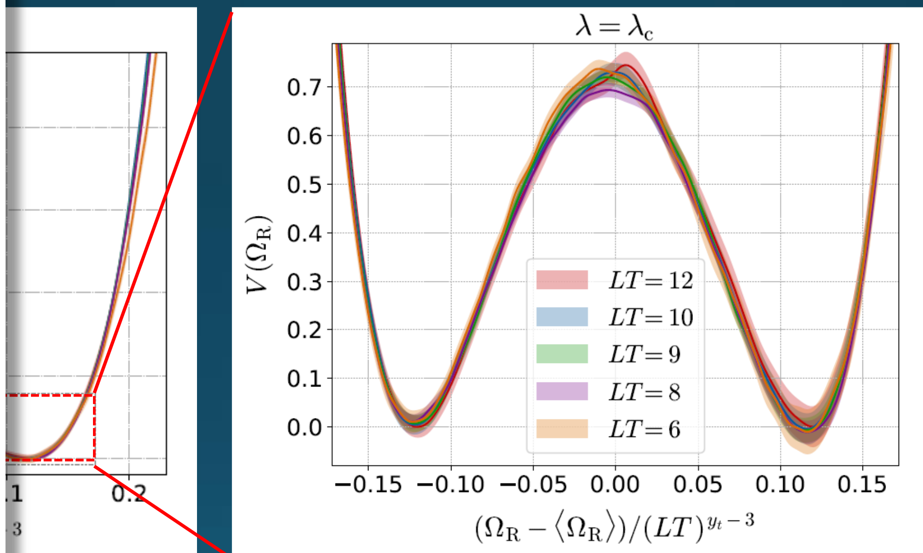
$LT \geq 9$	$B_4 = 1.630(24)(2),$	$\nu = 0.614(48)(3)$
$LT \geq 8$	$B_4 = 1.643(15)(2),$	$\nu = 0.614(29)(3)$
Z2	$B_4 = 1.604$	$\nu = 0.630$

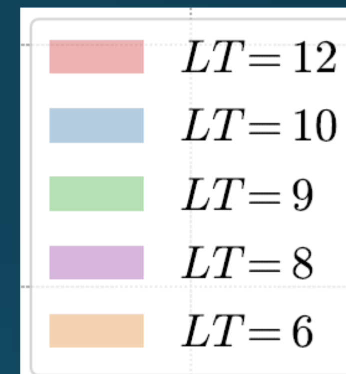
- B_4 and ν are consistent with Z2 universality class only when $LT \geq 9$ data are used for the analysis.

Scaling of Distribution Function



e Potential: $V(\Omega_{\mathrm{R}}) = -\log P(\Omega_{\mathrm{R}})$



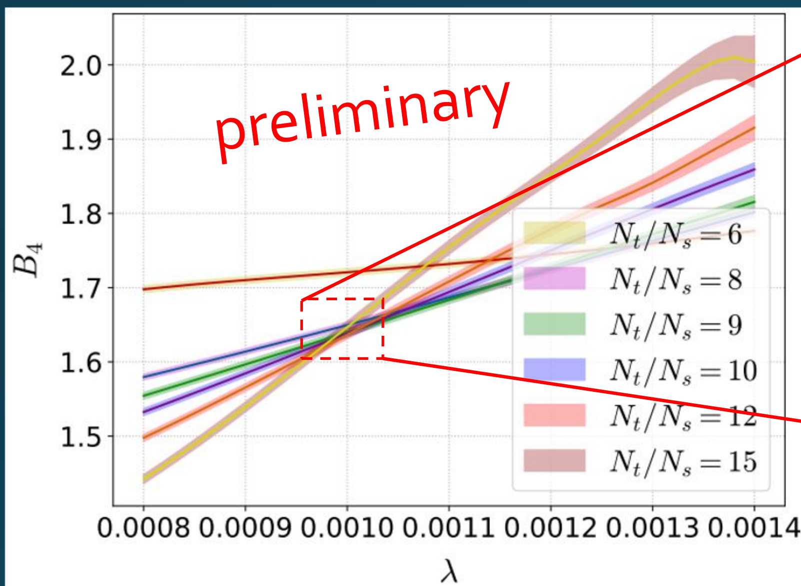


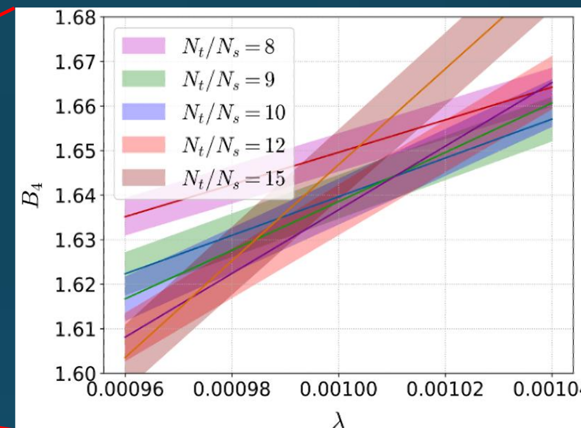
Result at $\lambda = \lambda_c$

- Z_2-FSS is well applied near the peaks
- Scaling violation around the edges
  - ➡ lead to scaling violation of B_4

Numerical Simulation @ $N_t = 6$

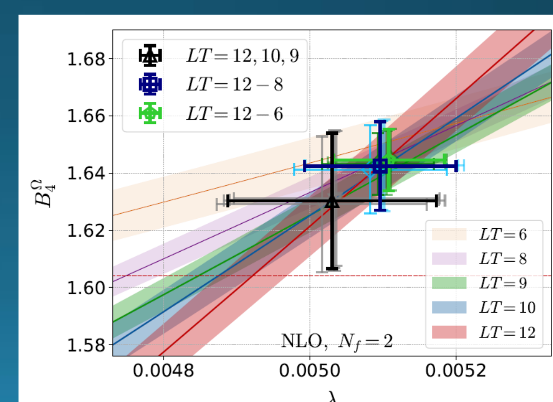
Ashikawa+, in prep.





$$a = \frac{1}{N_t T}$$

$N_t = 4$

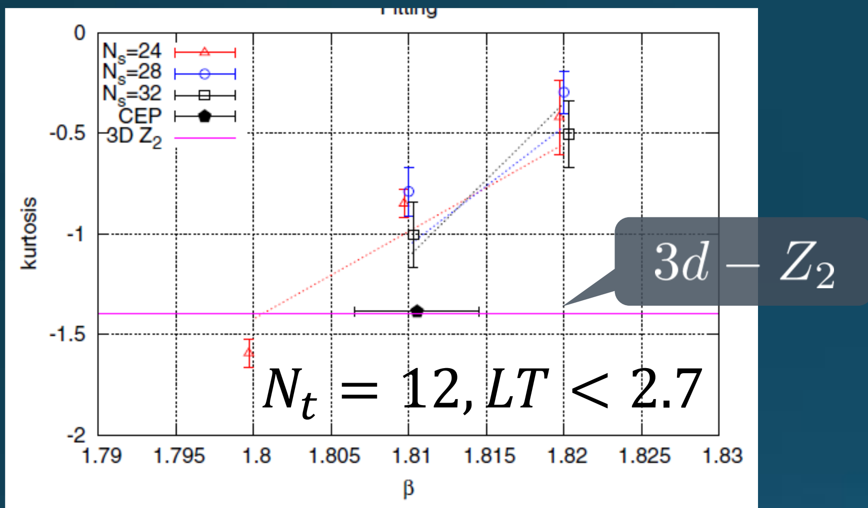


- For $N_t = 6$, deviation from the finite-size scaling is more significant with the same V.

Lattice Studies of Binder-Cumulant

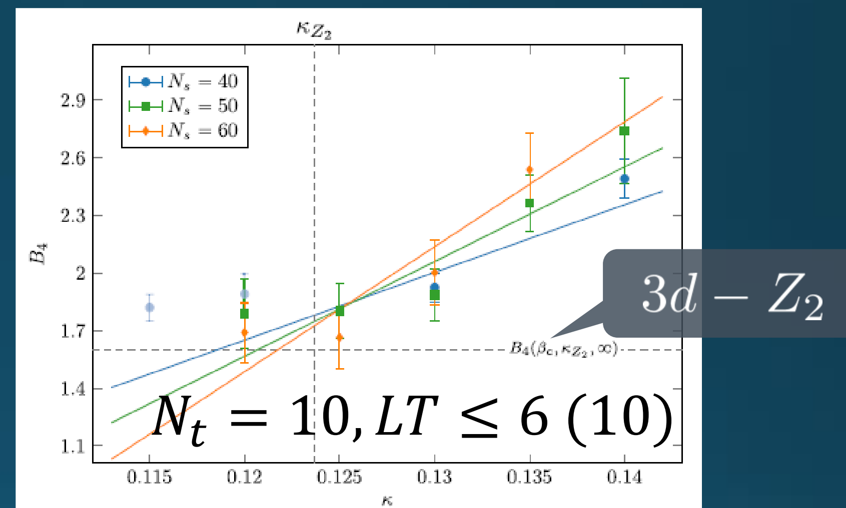
☐ Light-quark region

Kuramashi, Nakamura, Ohno, Takeda, '20



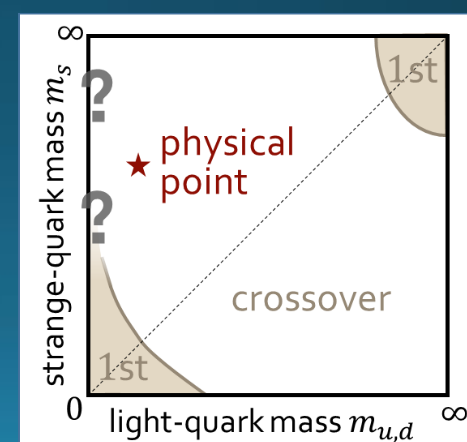
☐ Heavy-quark region

Cuteri , Philipsen , Schön, Sciarra, '21



☐ Statistically-significant deviation of the crossing point from the 3d-Ising value.

➡ V may not be large enough?



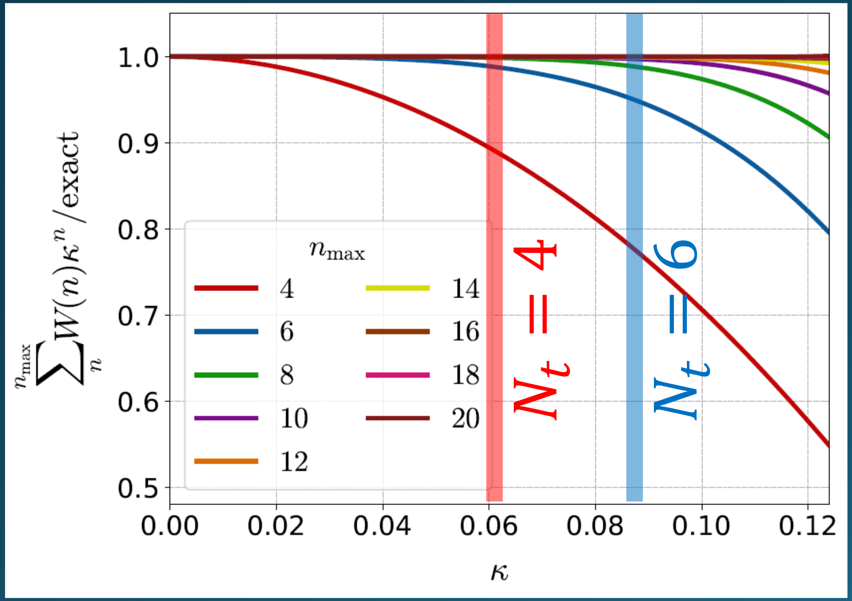
Convergence of HPE

Wakabayashi+ ('22)

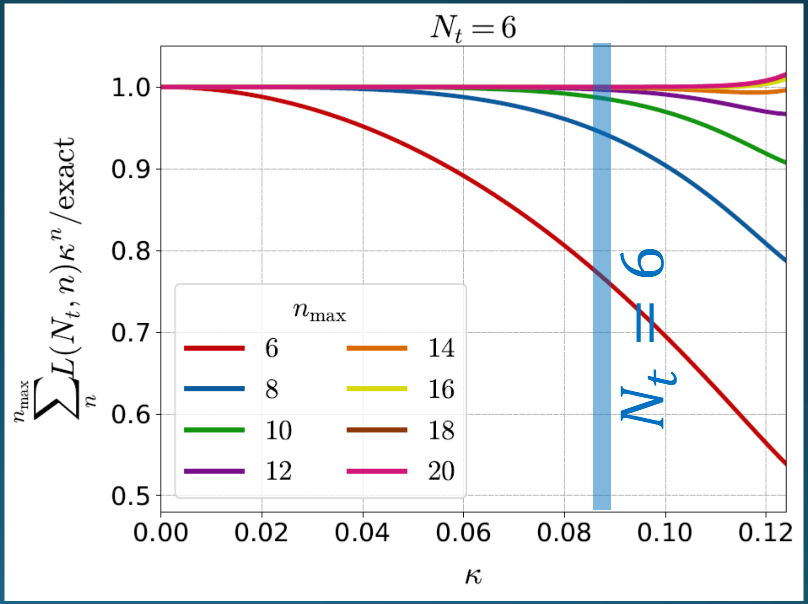
HPE for free lattice field (U=1)

=The worst convergence case

Wilson-loop-type



Polyakov-loop-type



$N_t = 4$ $\kappa_c = 0.0602(4)$ Kiyohara+,'21

$N_t = 6$ $\kappa_c = 0.0877(9)$ Cuteri+, '21

NLO of HPE is reliable for $N_t = 6$.

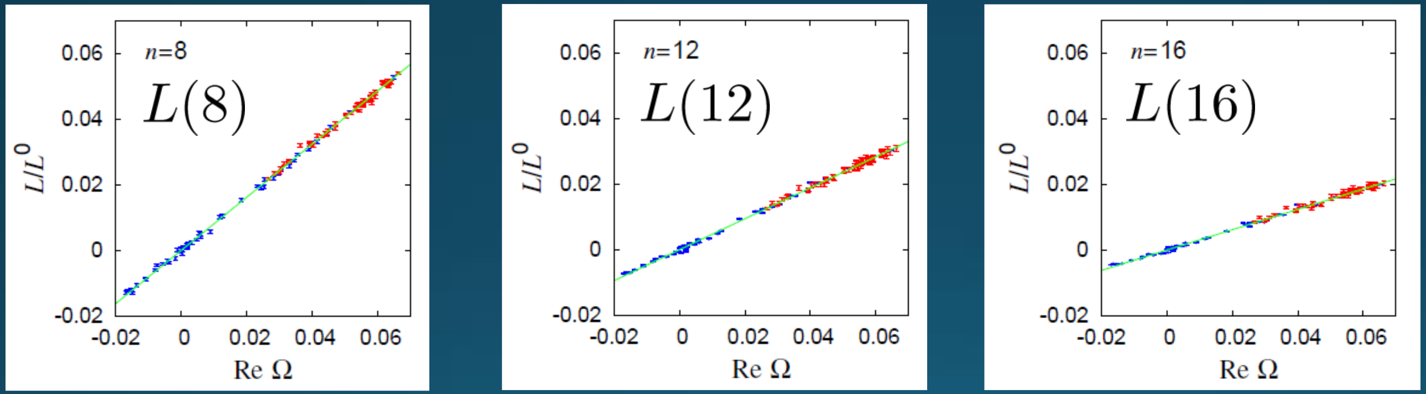
Effective Inclusion of Higher-order Terms

Ejiri+ ('20)
Wakabayashi+ ('22)

$$\ln \det M(\kappa) = \sum_n W(n)\kappa^n + \sum_n L(n)\kappa^n$$

Wilson-loop　Polyakov-loop

$32^3 \times 6$



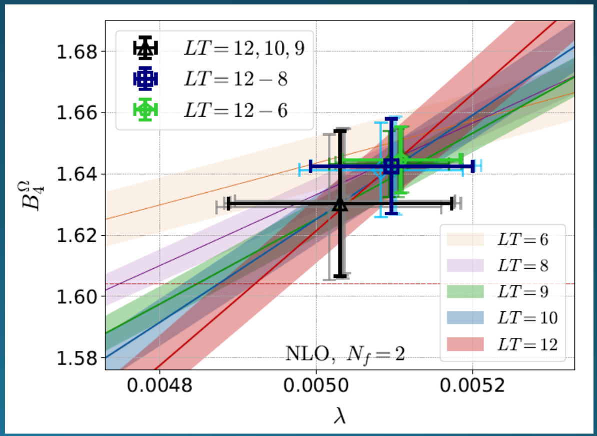
$\mathrm{Re}\Omega = L(6)$

- $L(n)$ are strongly correlated with each other.
- Higher order terms: $L(n) \simeq c_n L(N_t)$
- incorporated by determining c_n.

Summary

- CP in the light- & heavy-quark QCD can be investigated by the lattice QCD simulations.
- The hopping-parameter expansion allows us to study the CP in the heavy-quark region quite efficiently.
  - Our method: Monte-Carlo@LO + NLO by Reweighting
  - Good convergence at the NLO for $N_t = 4, 6$.
  - Effective method will extend the applicability of the HPE further.

- Confirmation of Z(2) scaling behavior.
- But, large spatial volume is necessary.
- Precise determination of κ_c

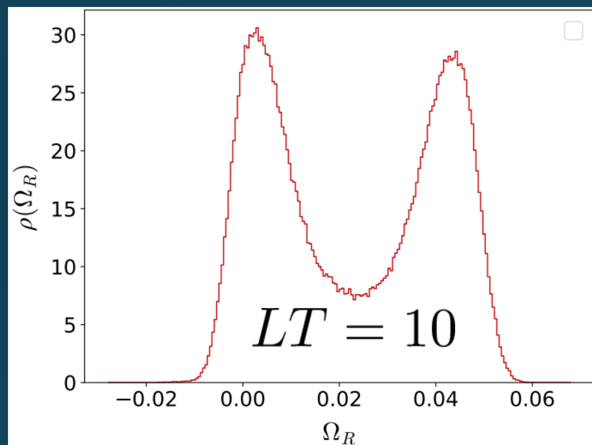


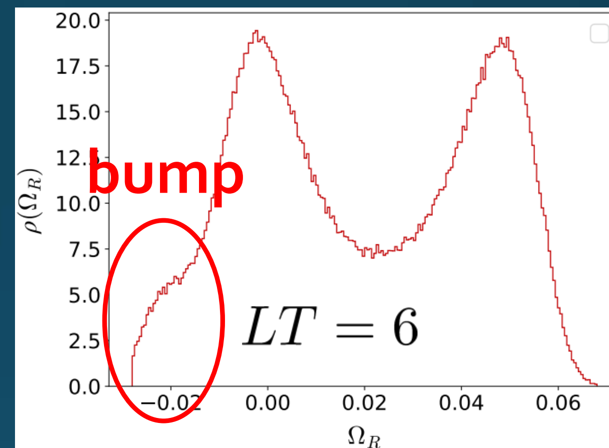
Future: $N_t = 8, 10, \cdots$

Violation of FSS & Remnant of Z(3)

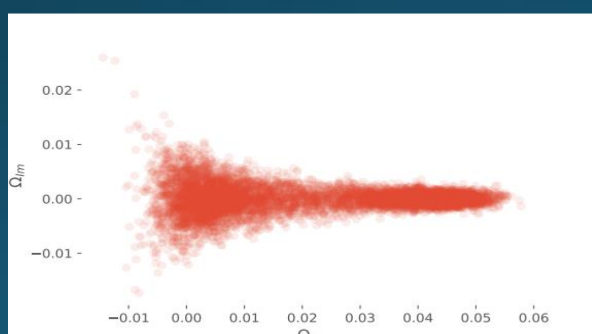
□ Probability Distribution of Polyakov loop

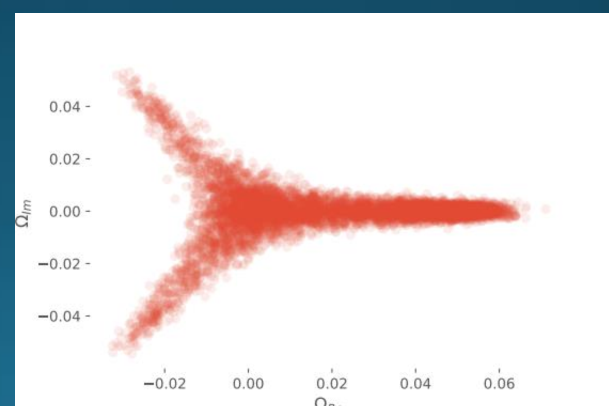
Real Part





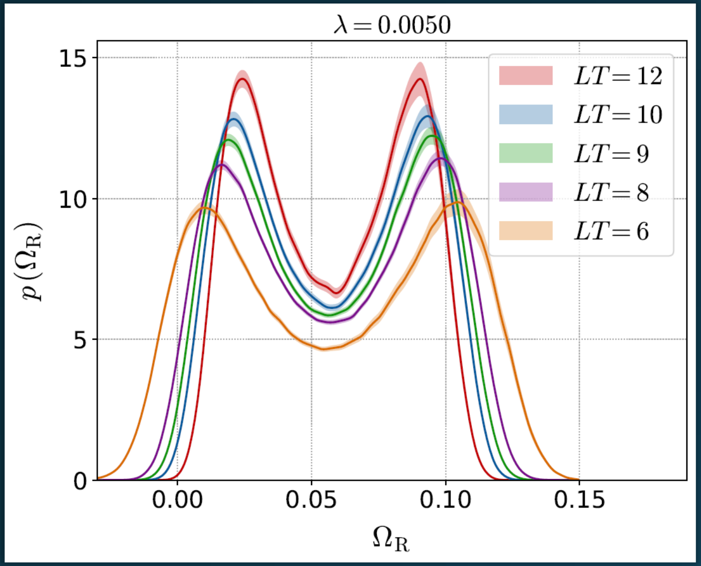
Complex Plane



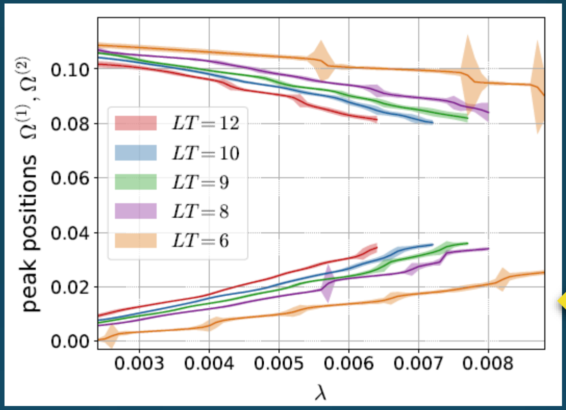


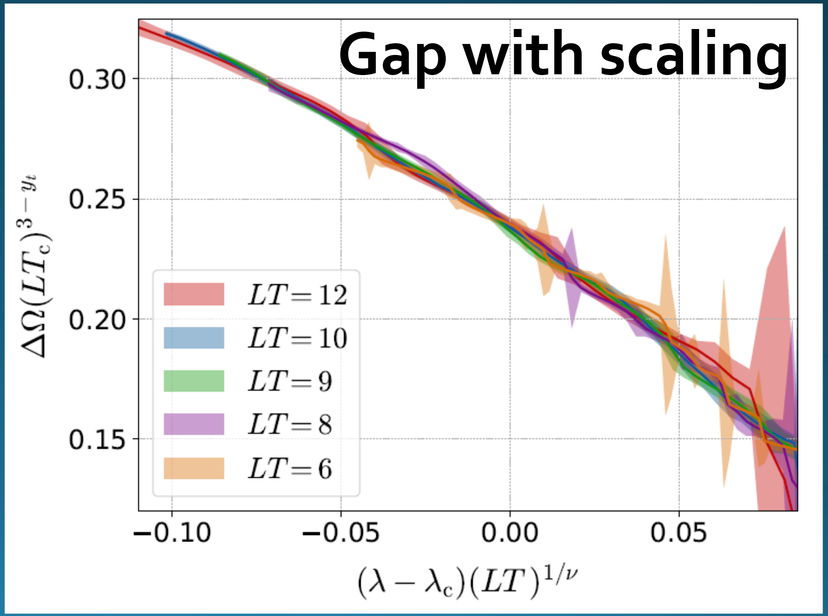
Scaling of Gap of Peaks

Polyakov-loop Distribution



Peak Position

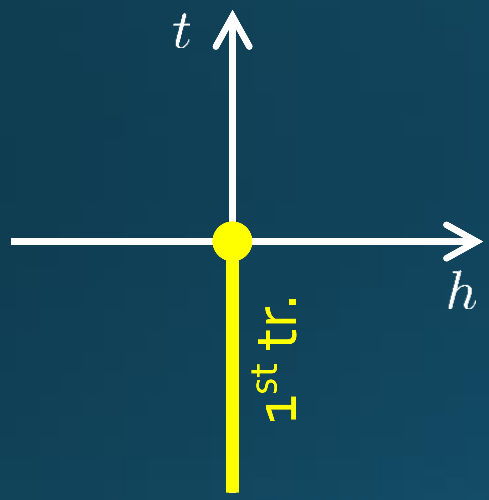




λ dep. of the gap agrees well with Z_2-FSS.

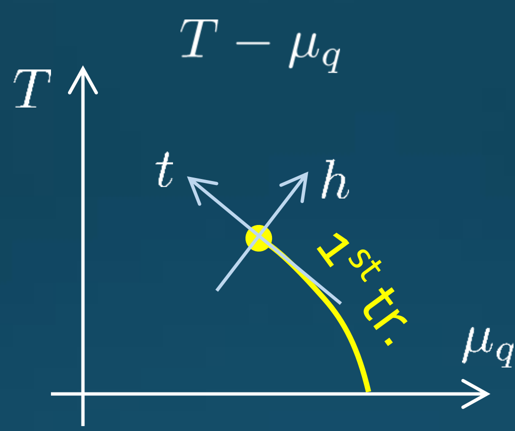
Mapping b/w Ising & QCD

- **Ising Model**

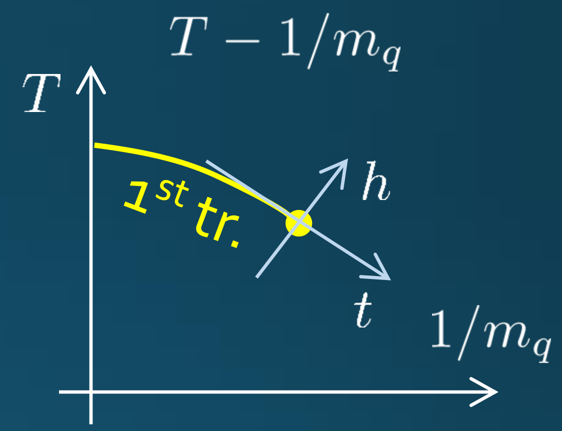


$$F(t,h) = F(b^{y_t} t, b^{y_h} h)$$

- **QCD**



$$\begin{pmatrix} t \\ h \end{pmatrix} = M \begin{pmatrix} T \\ \mu_q \end{pmatrix}$$



$$\begin{pmatrix} t \\ h \end{pmatrix} = M \begin{pmatrix} T \\ m_q^{-1} \end{pmatrix}$$

- **Singular part:**

$$F_{\rm QCD}(T,\mu_q) = F_{\rm Ising}(M(T,\mu_q))$$

Hopping Parameter Expansion 1

Wilson fermion

$$S_q = \sum_{x,y} \bar{\psi}_x M_{xy} \psi_y \qquad \kappa \sim \frac{1}{2m_q a}$$

$$M_{xy} = \delta_{xy} - \kappa B_{xy} \qquad B_{xy} = \sum_{\mu=1}^{4} \Big[(1-\gamma_\mu) U_{x,\mu} \delta_{y,x+\hat{\mu}} + (1+\gamma_\mu) U^\dagger_{y,\mu} \delta_{y,x-\hat{\mu}} \Big]$$

nonzero only for neighboring (x, y)

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int \mathcal{D}U_\mu \mathcal{O} e^{-S_g + \ln \det M(\kappa)}$$

$$\ln \det M(\kappa) = -\sum_{n=1}^{\infty} \frac{1}{n} \mathrm{tr}[B^n] \kappa^n$$

$\mathrm{tr}[B_n]$ is given by the closed trajectories of length n.