YITP-RIKEN iTHEMS International Molecule-type Workshop “Condensed Matter Physics of QCD 2024”, 2024/03/15, YITP, Kyoto

Lee-Yang Zeros
around critical point of heavy-quark QCD

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with
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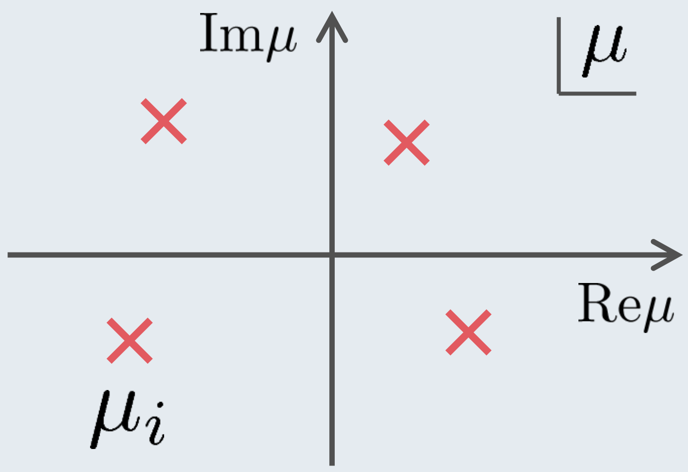
Lee-Yang Zero

Yang, Lee; Lee, Yang ('52)

Partition Function $Z(T,\mu)$

Finite V ➤ Polynomial of μ (or T)

$$Z(T,\mu) = \prod_i (\mu - \mu_i)$$



➤ **zeros** on the complex plane

=**Lee-Yang Zeros**

Lee-Yang Zero

Yang, Lee; Lee, Yang ('52)

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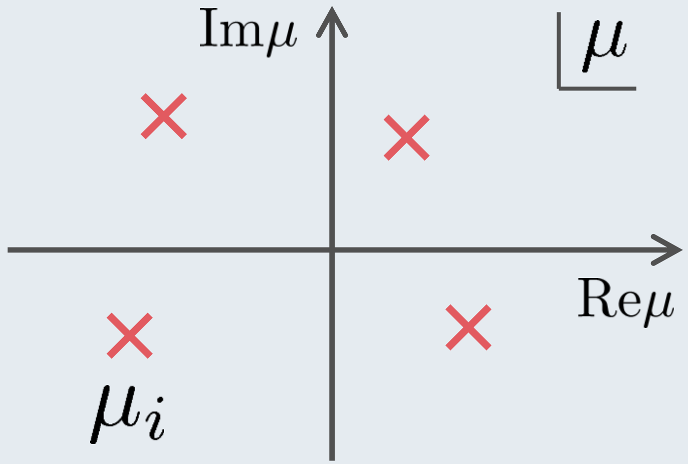


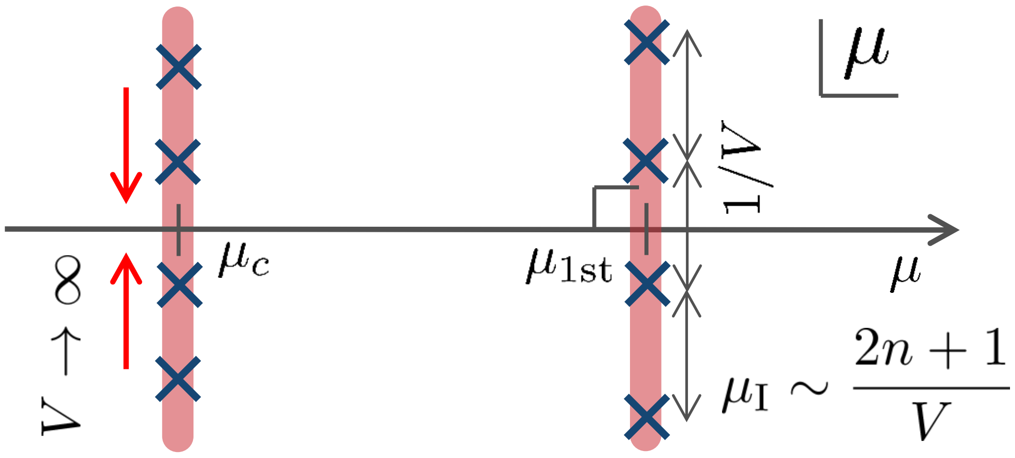
➤ **zeros** on the complex plane

=Lee-Yang Zeros

Phase Transition & LYZ

transition at $\mu = \mu_c$

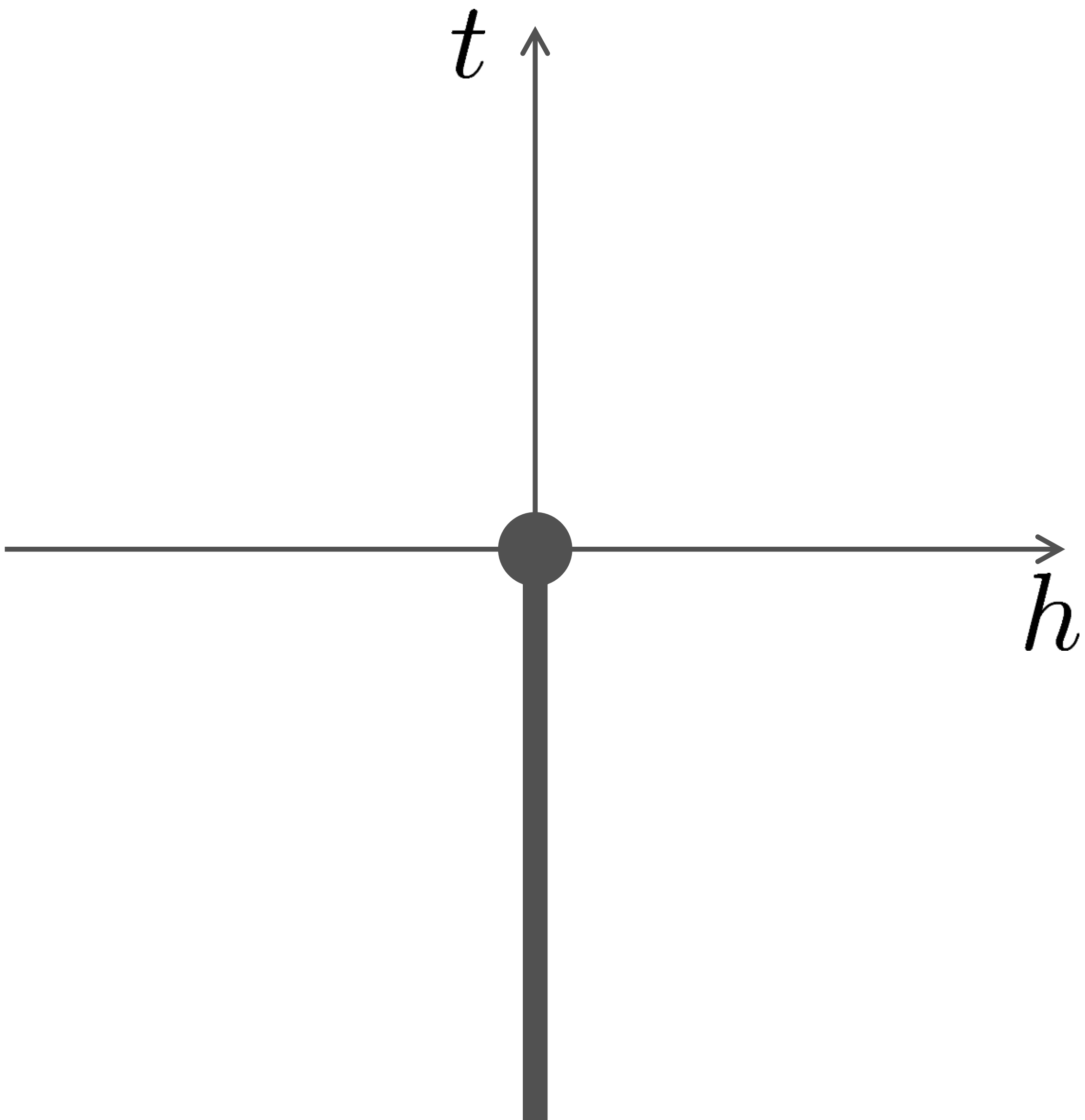
first order transition



— For $V \to \infty$, LYZs are accumulated on the line crossing the real axis at $\mu = \mu_c$.

— For a 1st transition, LYZs appear at equal distance of length $1/V$.

LY Zeros around a Critical Point



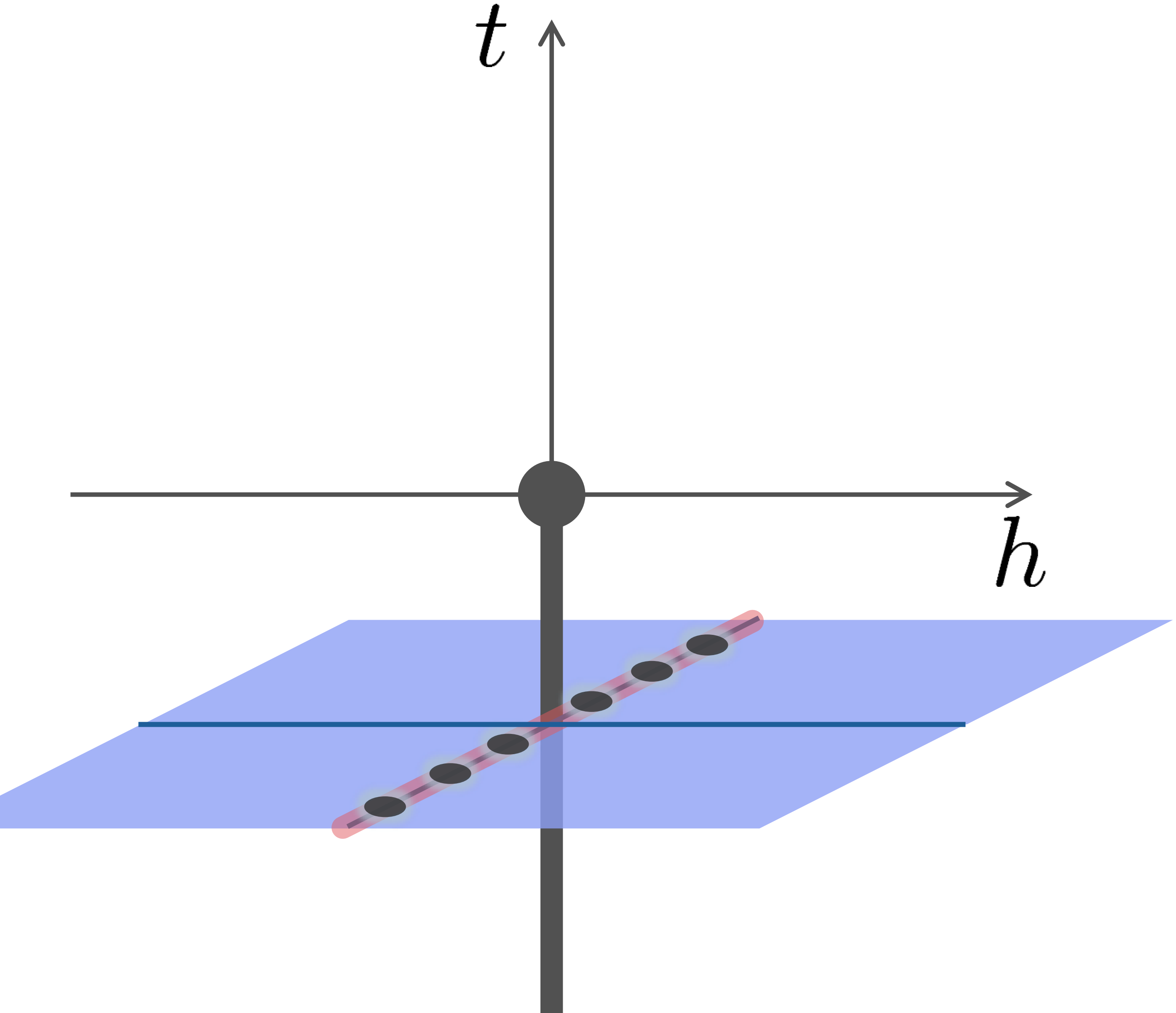
1st-transition

singularity on the real h axis

Crossover

no singularity on the real axis

LY Zeros around a Critical Point



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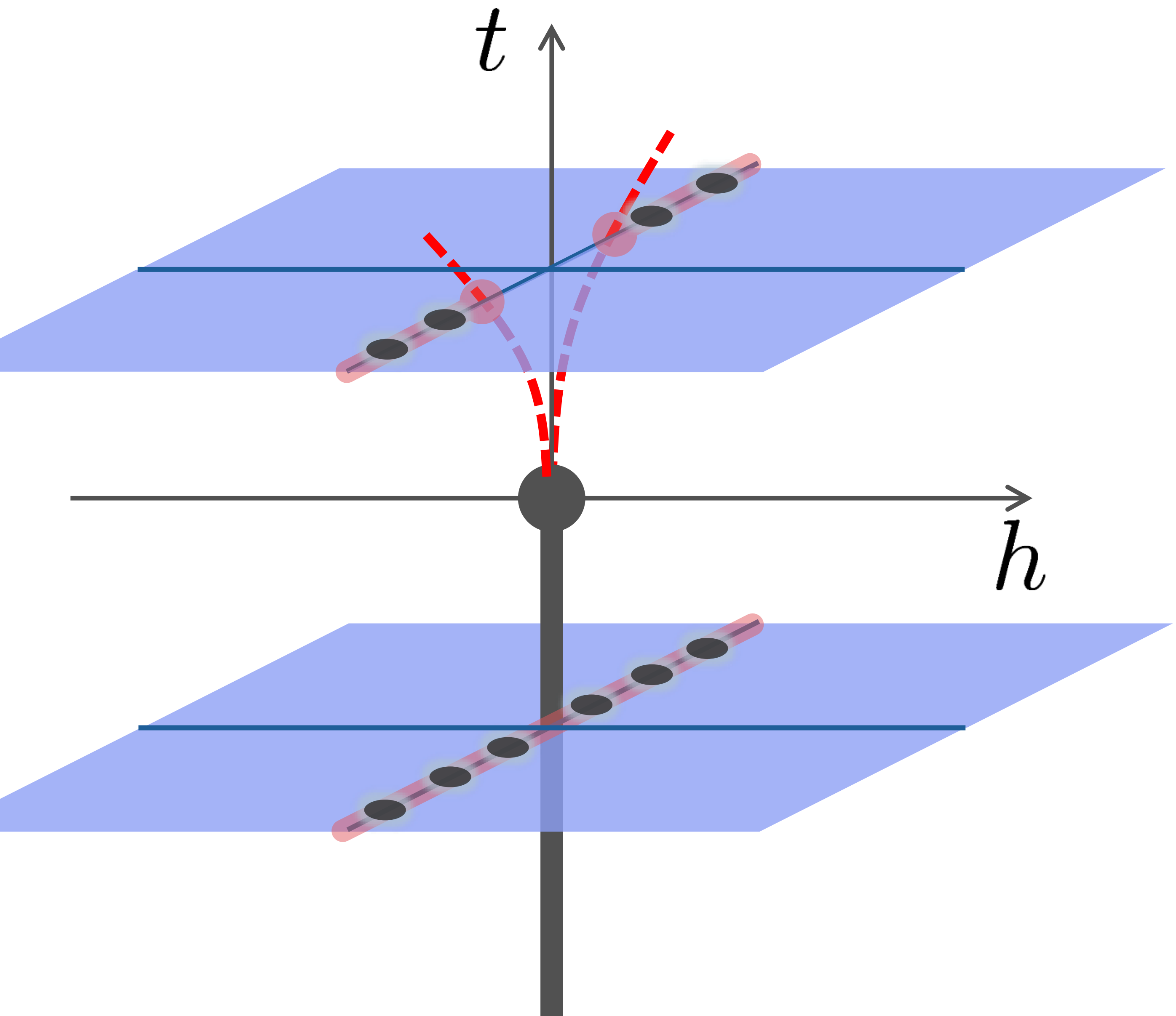


1st-transition

singularity on the real h axis

Crossover

no singularity on the real axis

LY edge singularity

starting from the CP

Edge singularity is determined by the analytic property of the scaling function.

Recent Topics in LYZ

and LY edge singularity

Analytic Structure

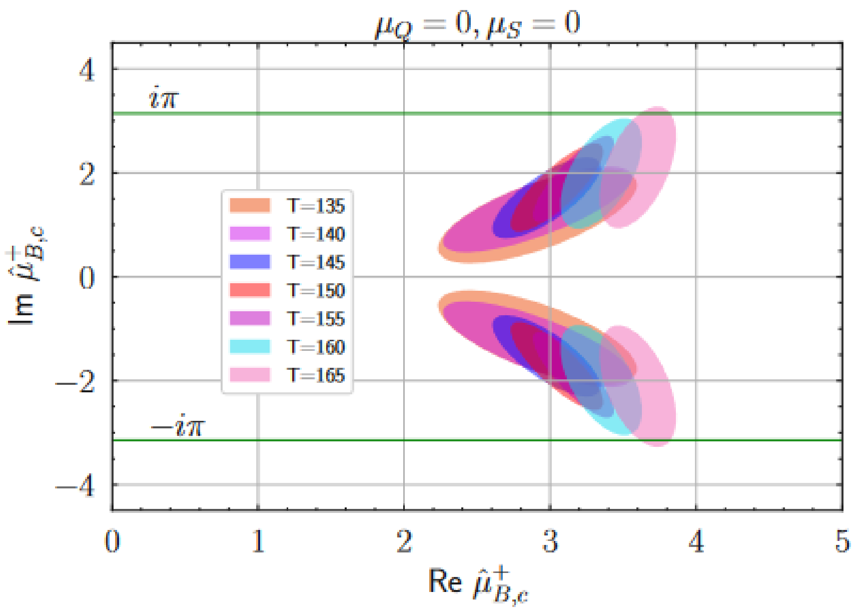
— Scaling functions, FRG, ...

An, Mesterhazy, Stephanov ('16)
Johnson, Rennecke, Skokov ('23)
Karsch, Schmidt, Singh ('23)
...

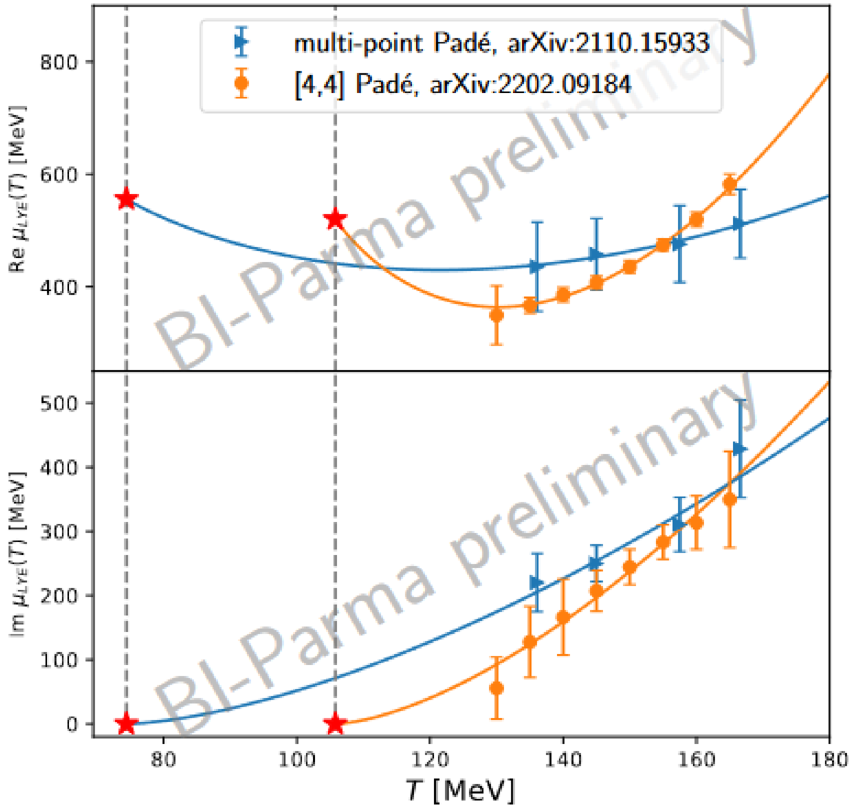
Locating QCD-CP at $\mu \neq 0$ on the lattice?

— Taylor exp. + Imaginary μ + Pade approx.

HotQCD ('22)
D.A. Clarke, Lattice 2023



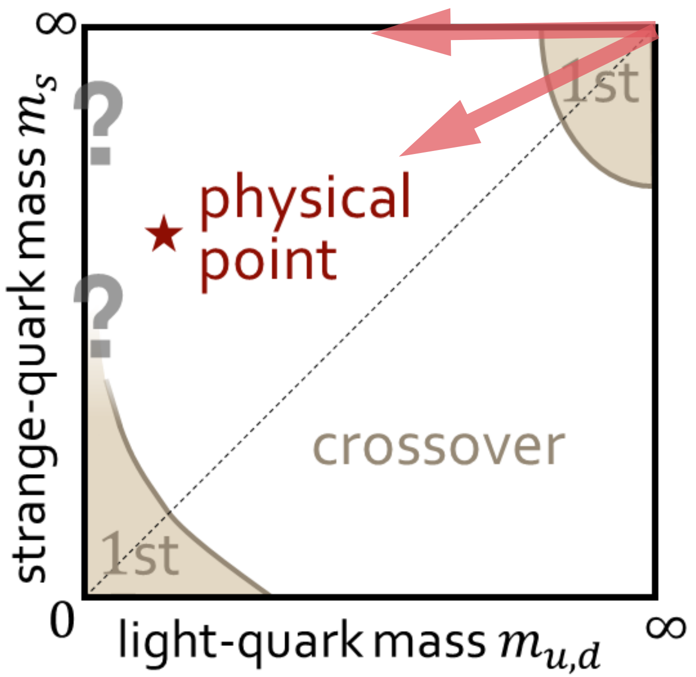
HotQCD ('22)



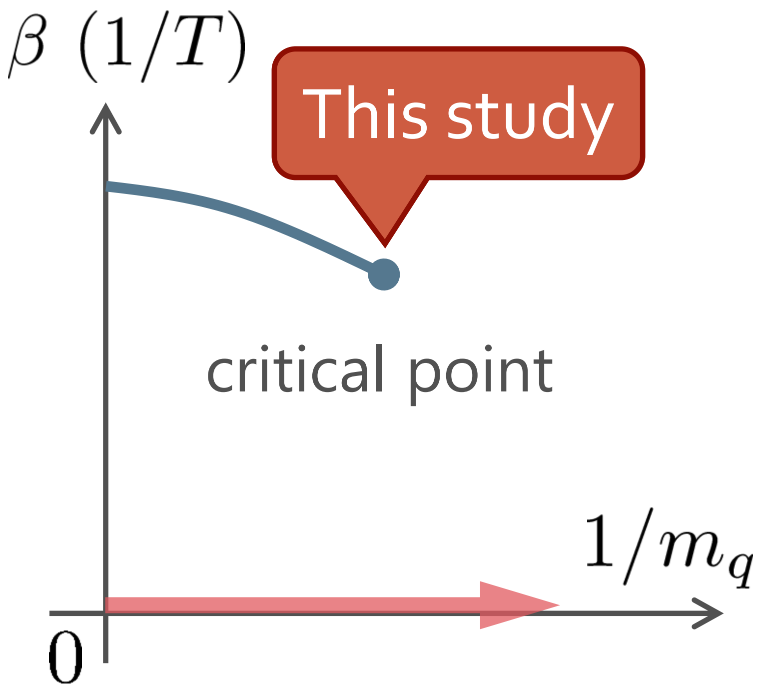
D.A. Clarke, Lattice 2023

Present Study: Heavy-Quark QCD

Columbia Plot



Phase Diagram



CP in heavy-quark QCD

– $\mu_q = 0$ & large m_q

Easy to handle in lattice simulations!

We study the LYZ around the HQ-QCD-CP.

Hopping-Parameter Expansion (HPE)

~ $1/m_q$ expansion

Wilson Fermion

$$S_q = \sum_{x,y} \bar{\psi}_x M_{xy} \psi_y \qquad \kappa \sim \frac{1}{2m_q a}$$: hopping parameter

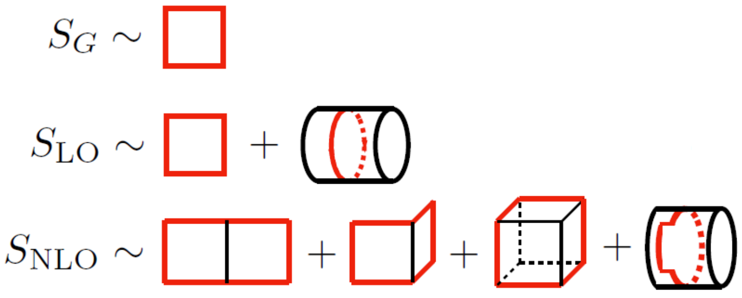
$$M_{xy} = \delta_{xy} - \kappa B_{xy} \qquad B_{xy} = \sum_{\mu=1}^{4} \left[(1-\gamma_\mu) U_{x,\mu} \delta_{y,x+\hat{\mu}} + (1+\gamma_\mu) U^\dagger_{y,\mu} \delta_{y,x-\hat{\mu}} \right]$$

nonzero only for neighboring (x, y)

Hopping-Parameeer Expansion

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int \mathcal{D}U_\mu \mathcal{O} e^{-S_g + \mathrm{tr} \ln M(\kappa)}$$

$$\mathrm{tr} \ln M(\kappa) = -\sum_{n=1}^{\infty} \frac{1}{n} \mathrm{tr}[B^n] \kappa^n$$



nth order terms in the HPE: closed trajectories of length n.

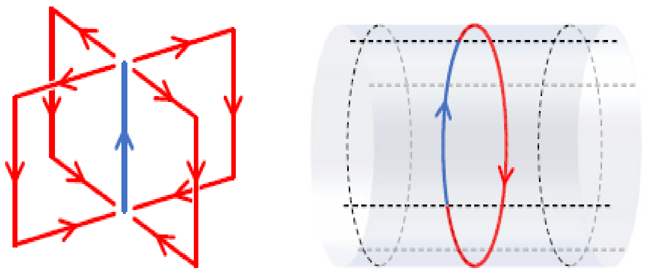
Higher-Order Terms in HPE

Monte Carlo Simulation @ LO

$$S_{\rm LO} = -6N_{\rm site}\beta^*\hat{P} - \lambda N_s^3\hat{\Omega}_{\rm R}$$

heat bath & over relaxation with modified staple

- Numerical cost is almost the same as the pure YM!



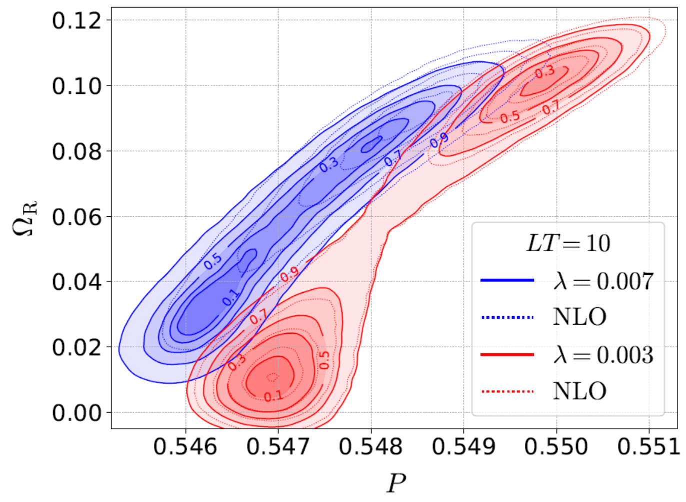
$\hat{P}$: plaquette
$\hat{\Omega}$: Polyakov loop
$\lambda = 2^{N_t+2}N_c\kappa^{N_t}$

NLO by Reweighting

$$\langle\mathcal{O}\rangle_{\rm NLO} = \frac{\langle\hat{O}e^{-S_{\rm NLO}}\rangle_{\rm LO}}{\langle e^{-S_{\rm NLO}}\rangle_{\rm LO}}$$

Overlapping problem is well suppressed due to the LO confs.

- Realize high statistical analysis

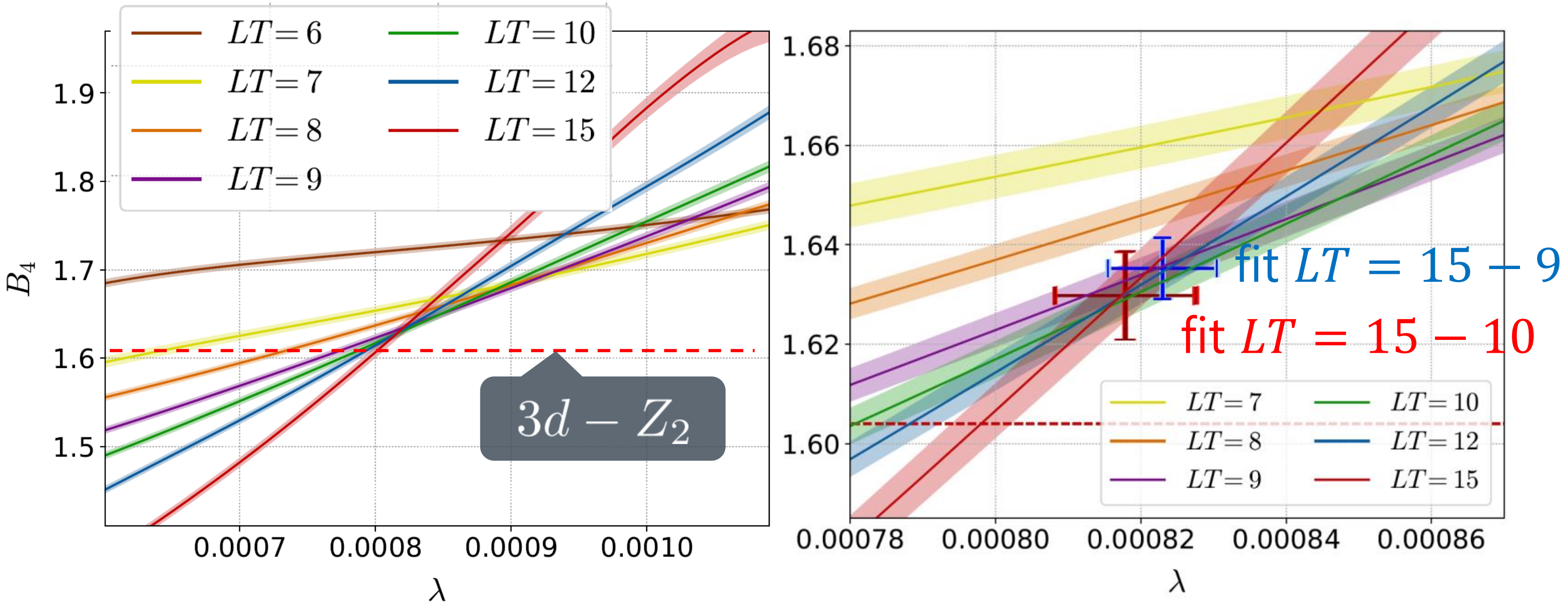


Binder Cumulant Analysis

for the HQ-QCD-CP at $N_t = 6$

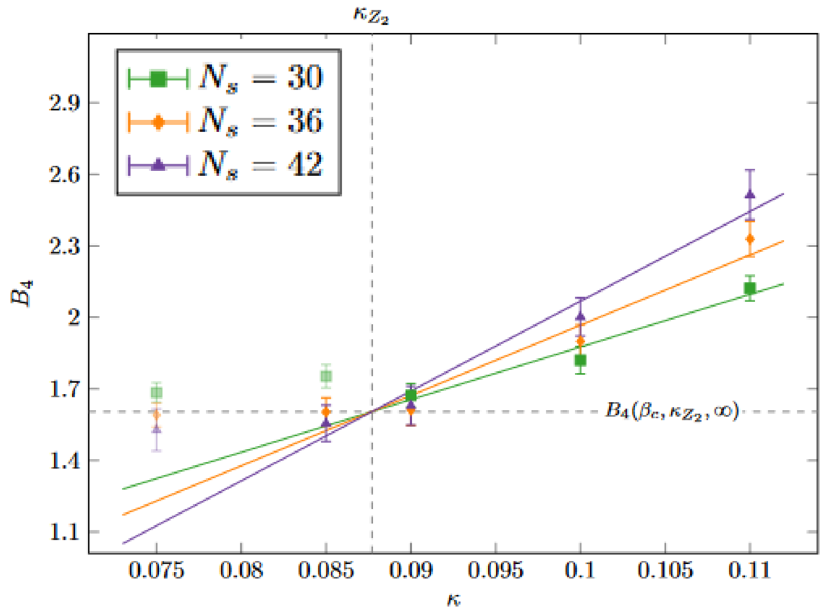
$$LT = N_x / N_t$$
$$\lambda = 2^{N_t+2} N_c \kappa^{N_t}$$

Our result (HPE-NLO)



MK, Ashikawa, Ejiri, Kanaya, Lattice 2023

w/ Dynamical Fermions



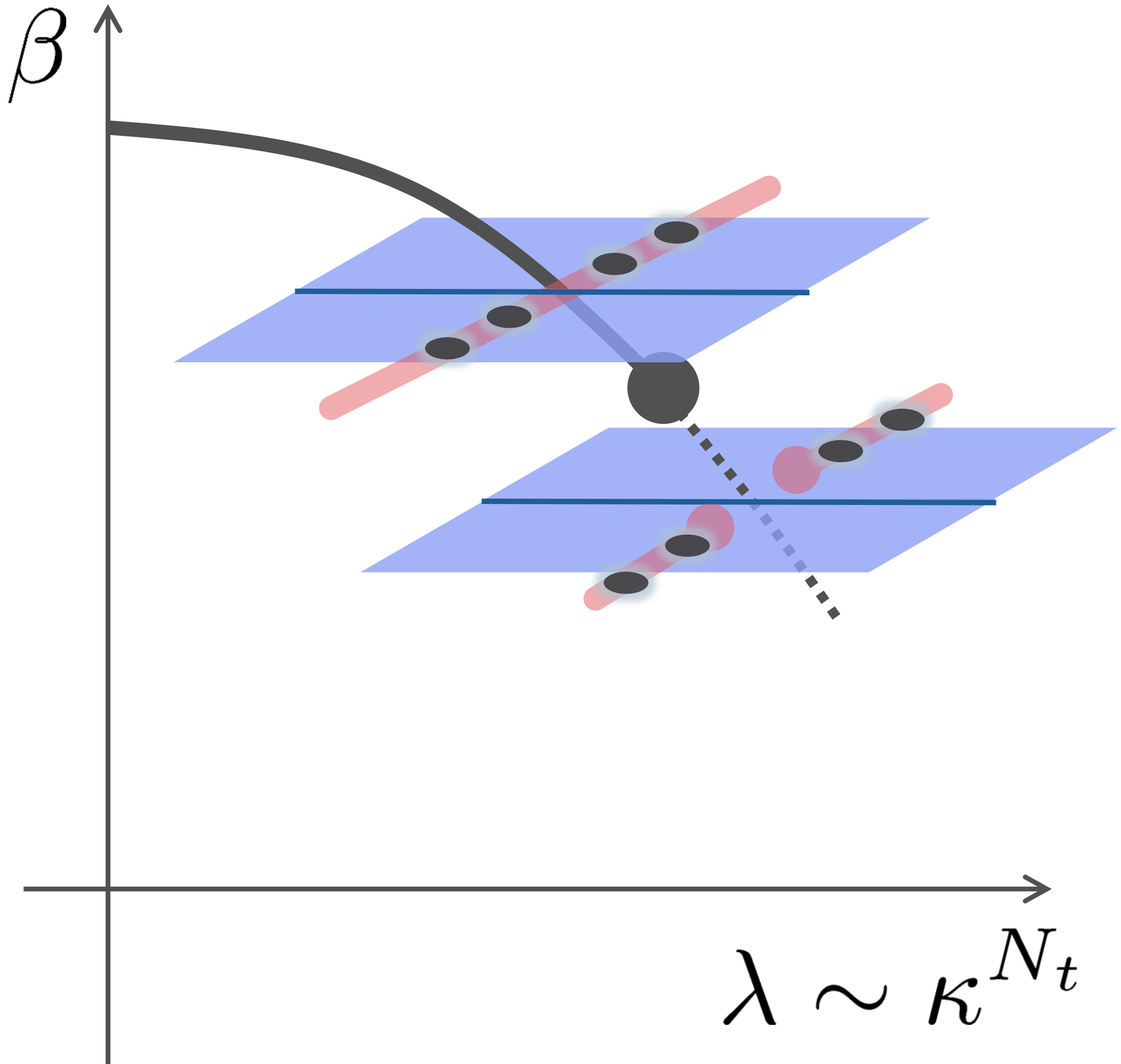
Cuteri , Philipsen , Schön, Sciarra, '21

One order smaller statistical errors on more than twice larger LT!

Precise determination of the location of the CP

LY Zeros on Complex λ

$$\lambda = 2^{N_t+2} N_c \kappa^{N_t}$$



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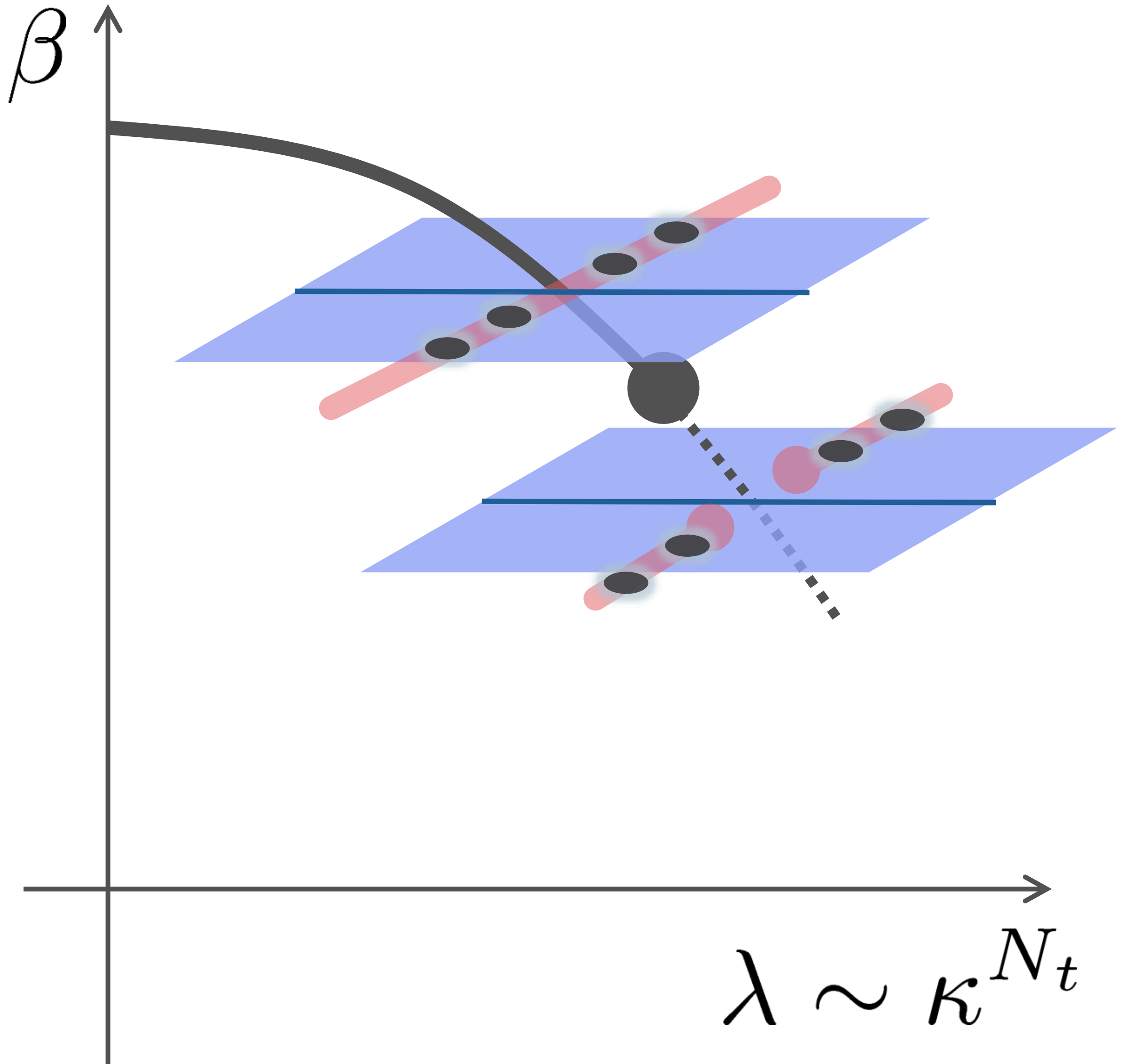
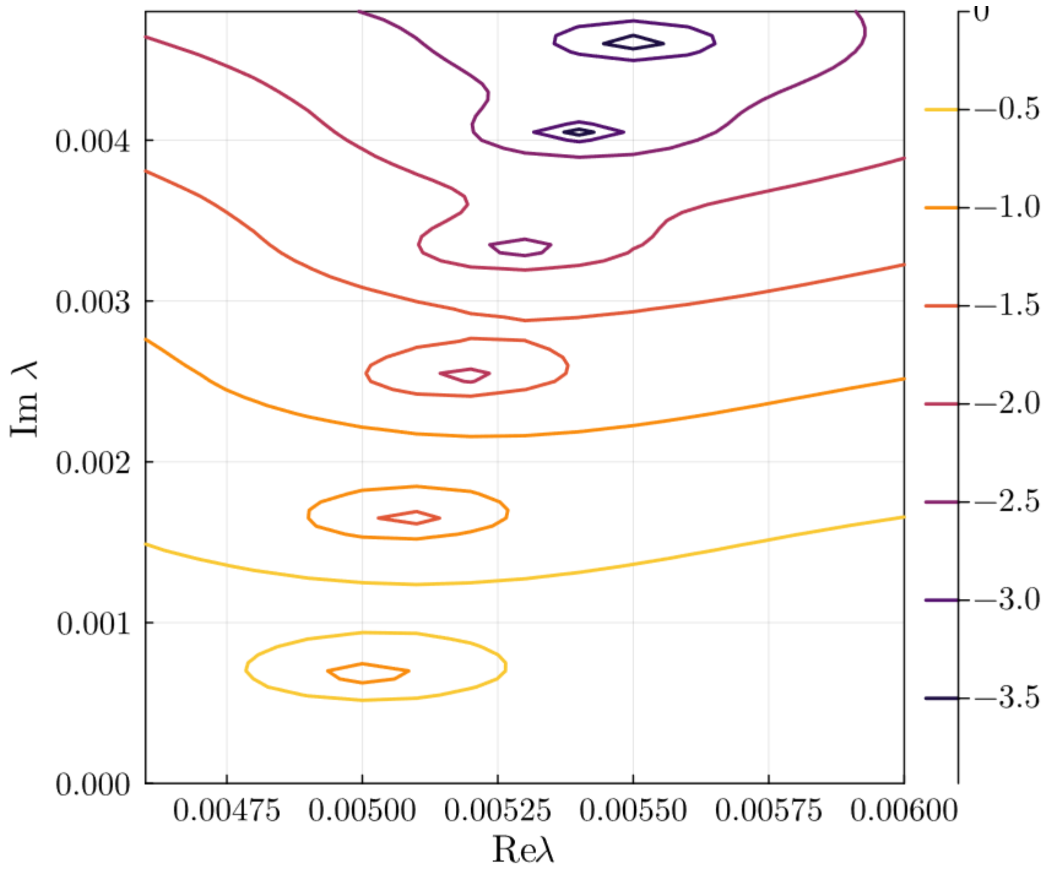


Partition func. Z on complex λ

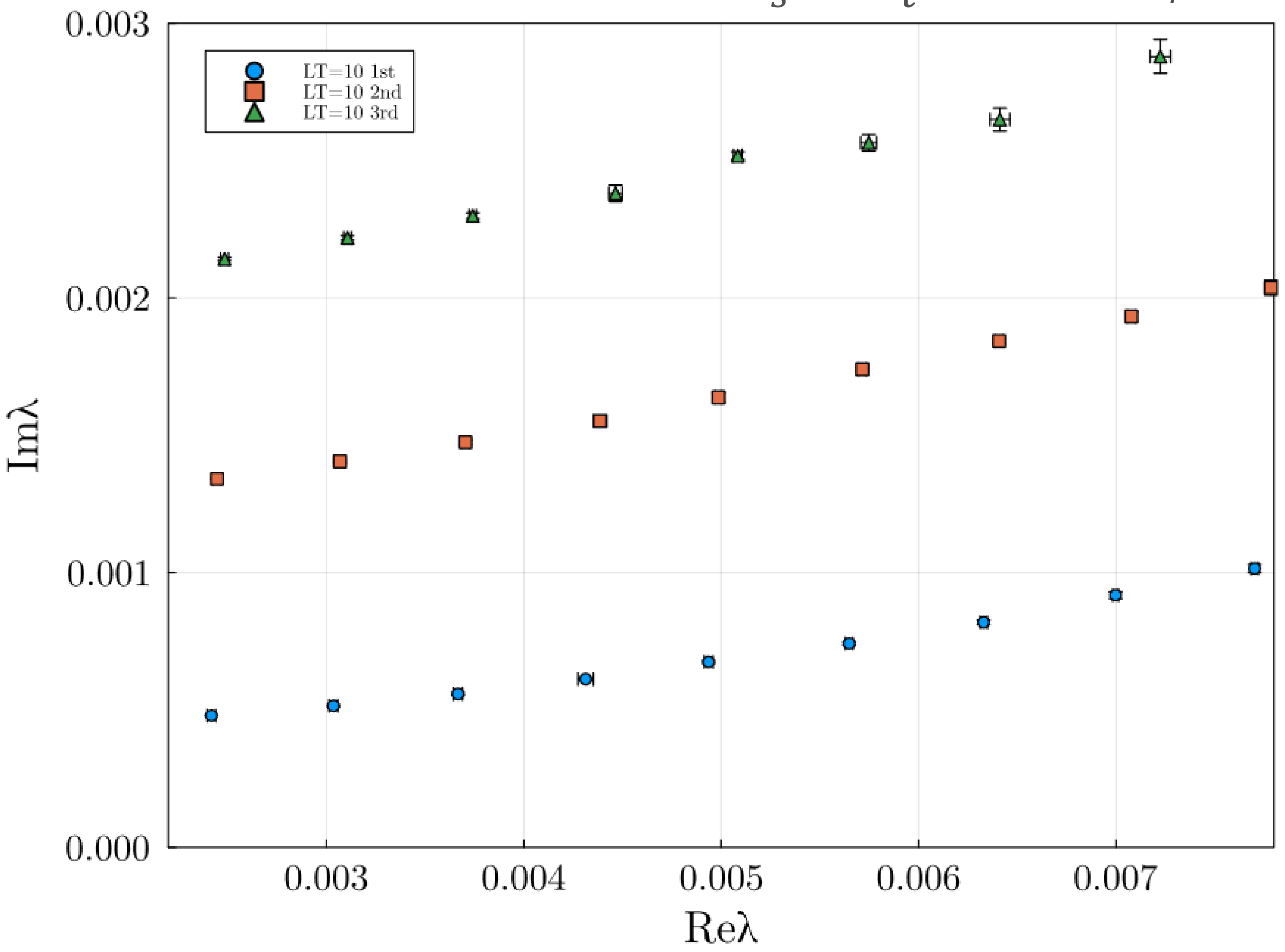
$N_s^3 \times N_t = 40^3 \times 4,\ \beta = 5.6861$ LO

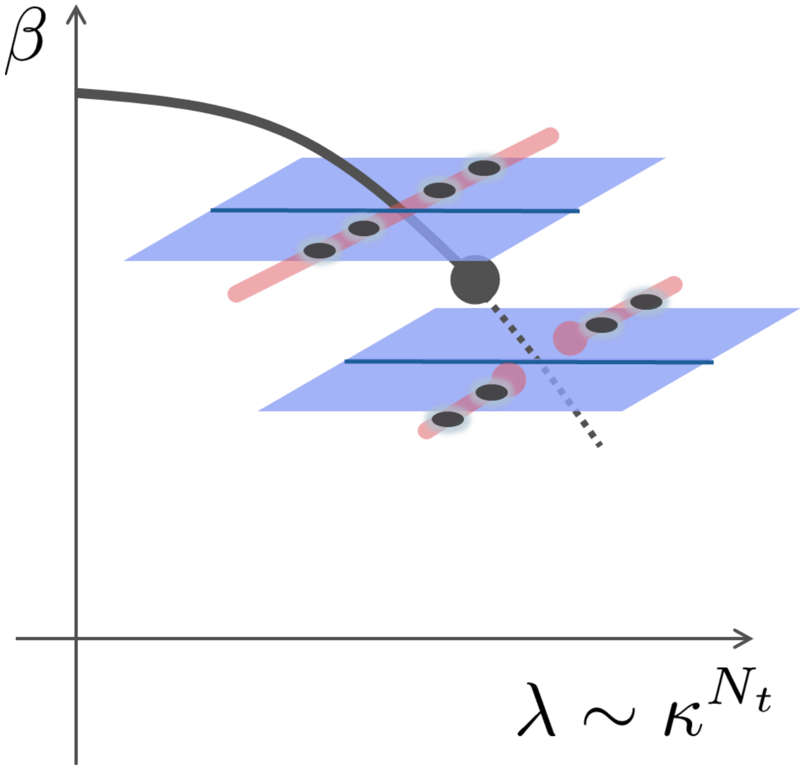


Good statistics to find many LYZ

LY Zeros

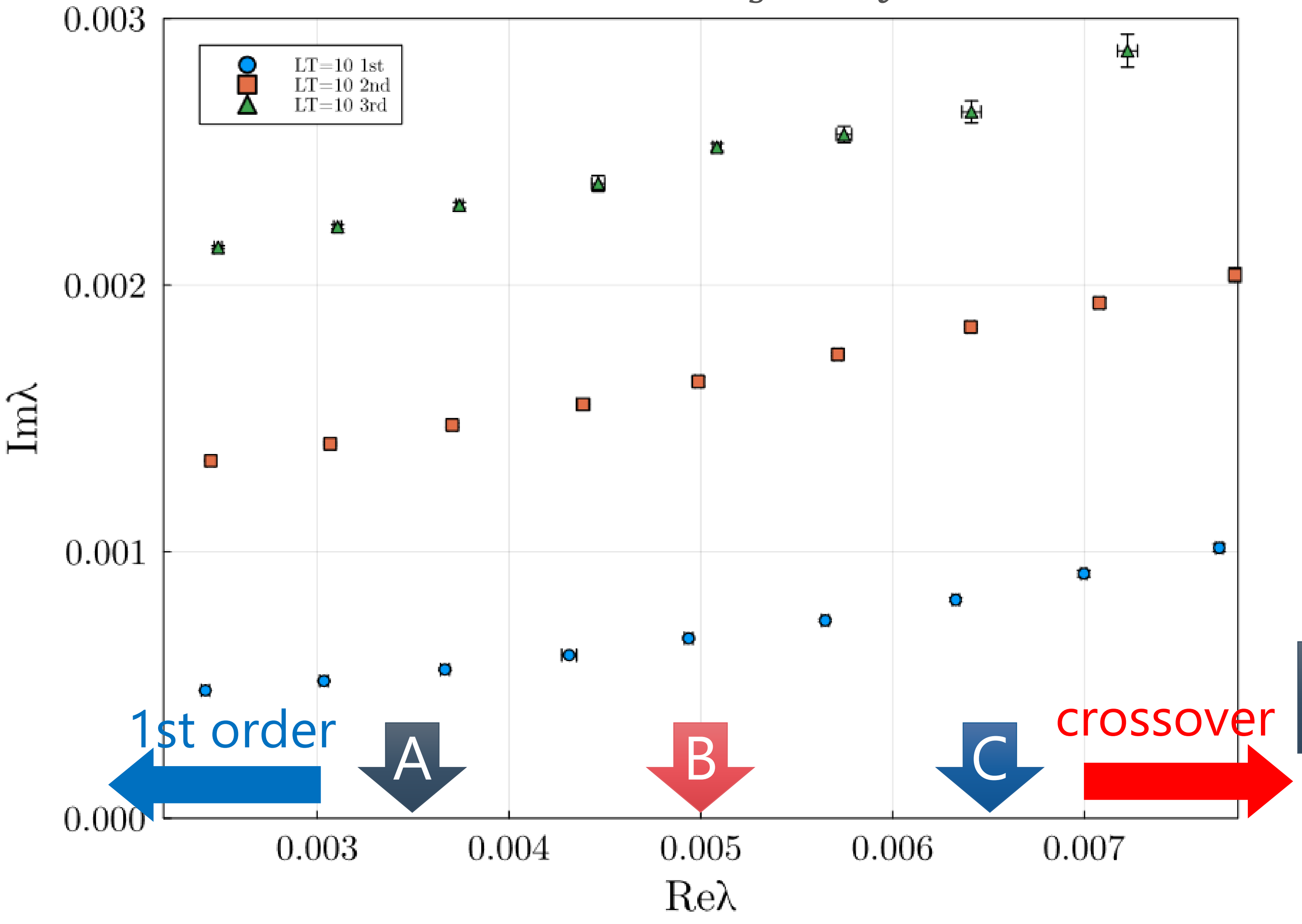
$N_s^3 \times N_t = 40^3 \times 4$, LO

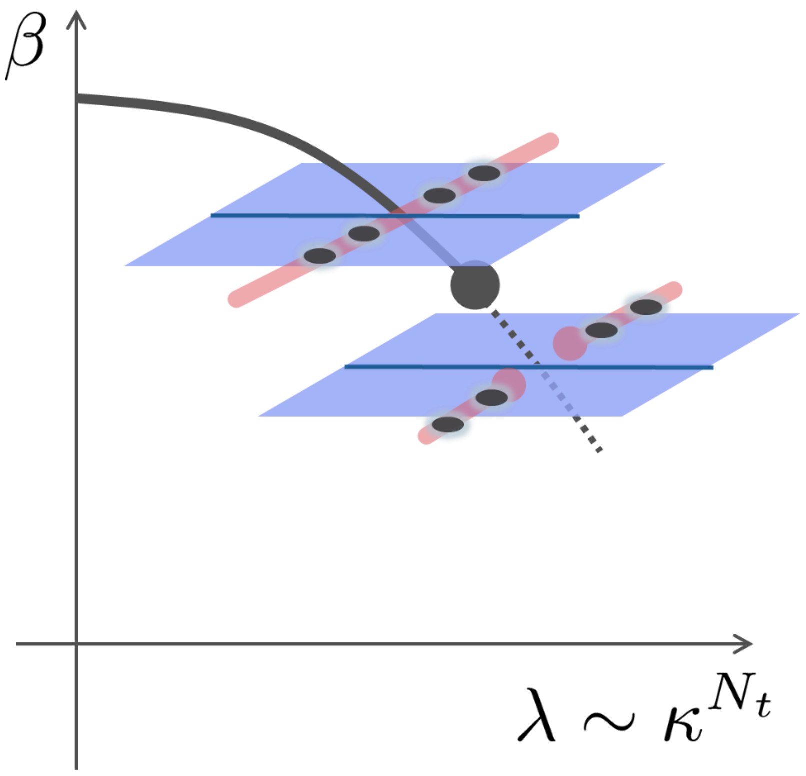




LY Zeros

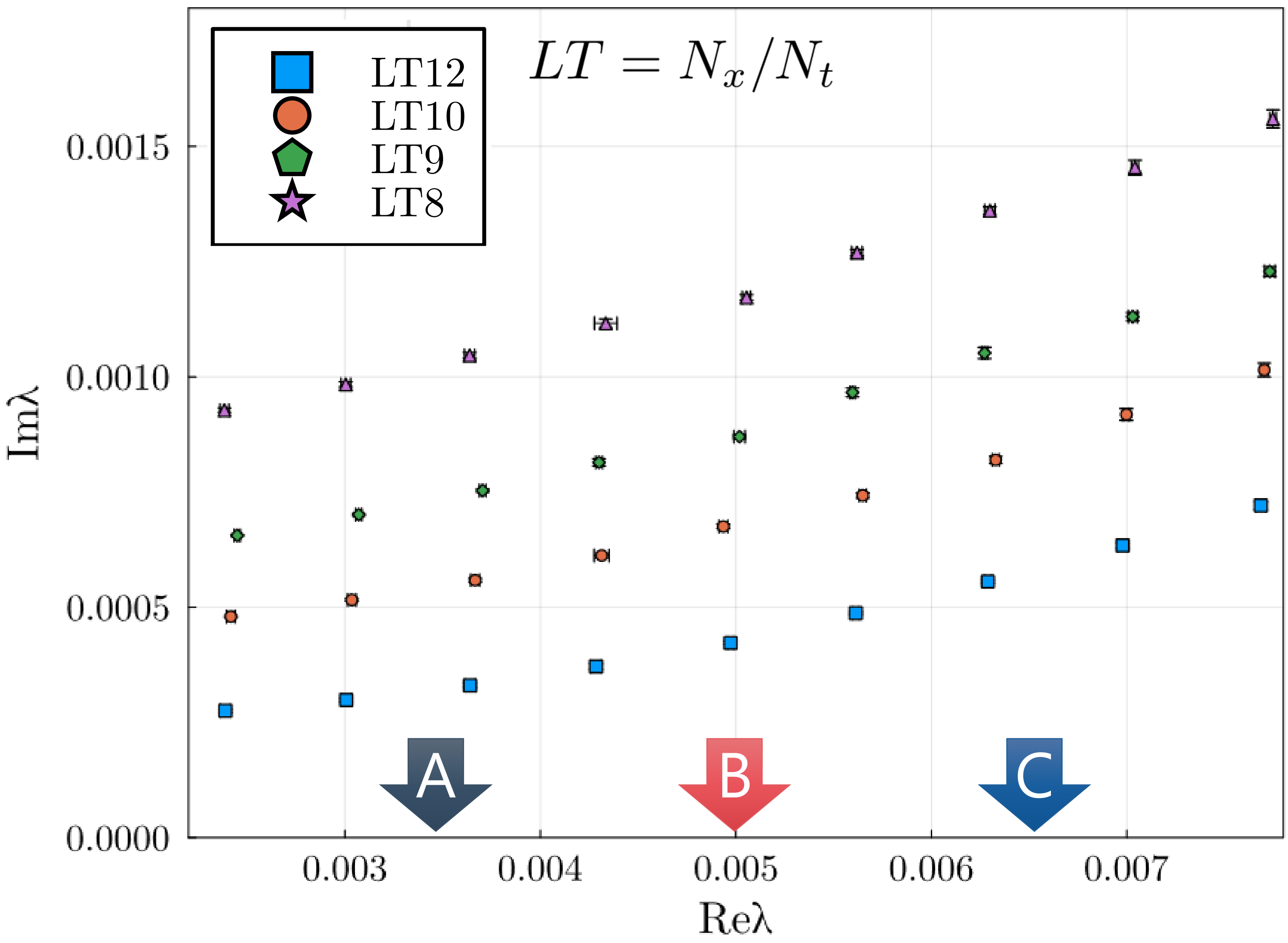
$N_s^3 \times N_t = 40^3 \times 4$, LO

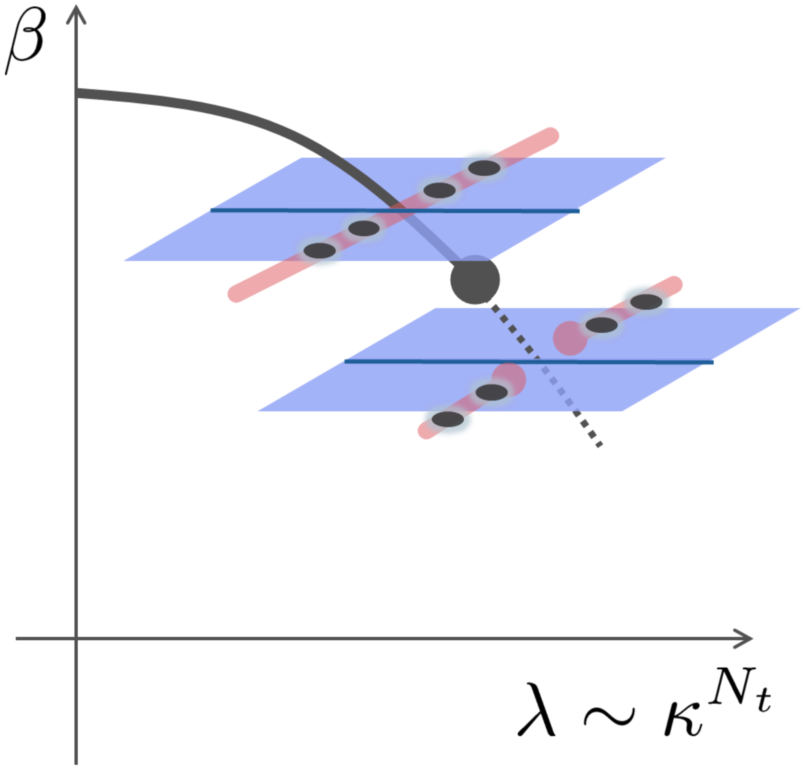




Quiz: Where is λ_c?

LT Dependence of 1st LYZ





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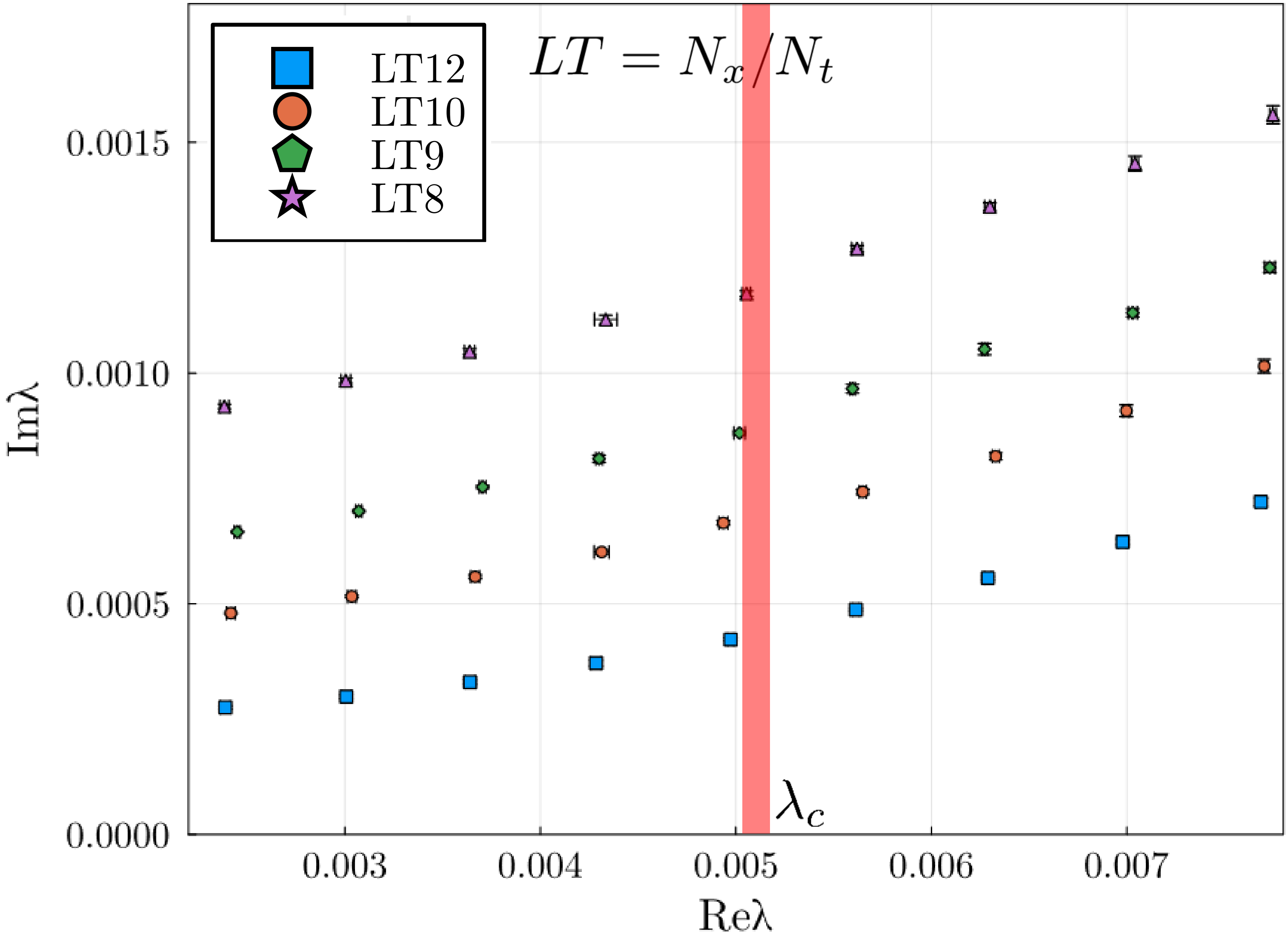
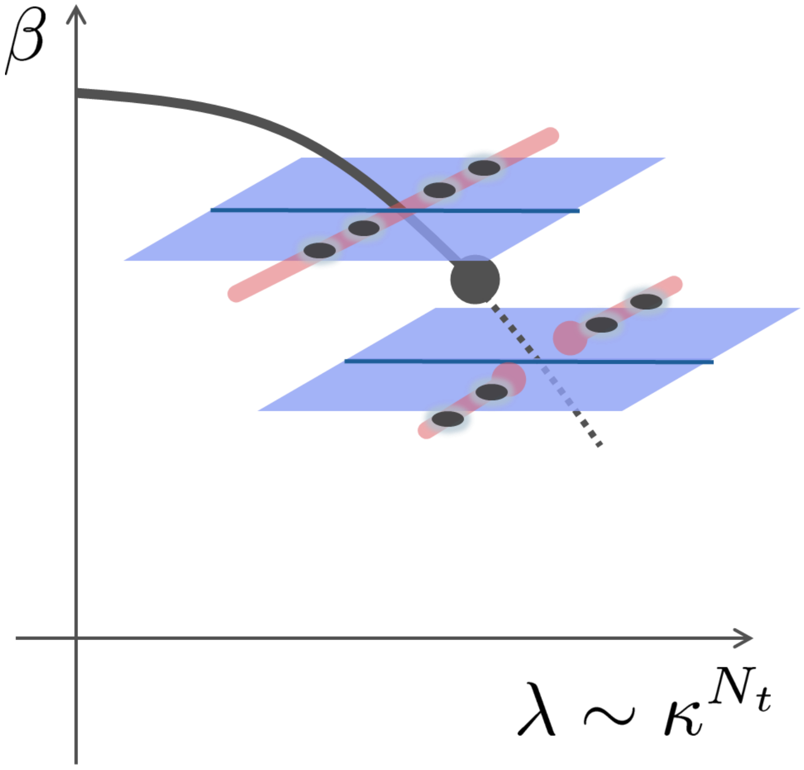




Quiz: Where is λ_c?

A: $\lambda_c = 0.00516(15)$

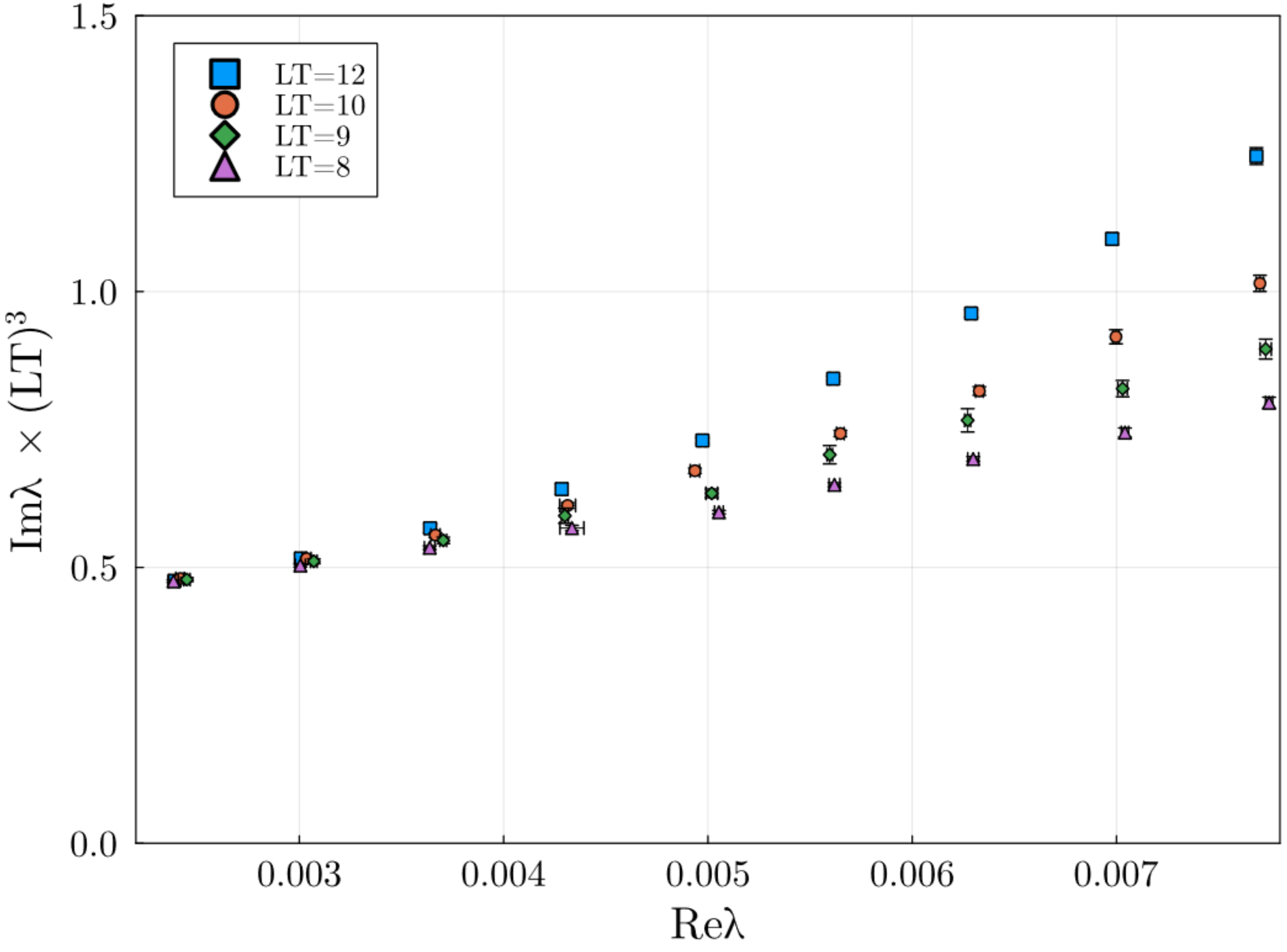
Scaling at the First-Order Side

1st Transition

coexistence of $\rho = \rho_1, \rho_2$ at $\mu = \mu_c$

$$\mu_{\mathrm{LYZ}} = \mu_c + i\frac{(2n+1)\pi T}{\Delta\rho V}$$

LYZ appears at equal distance of length proportional to $1/V$.



Derivation:

$$Z = \mathrm{Tr}e^{-\beta(H-\mu N)}$$

$$\sim e^{\beta\rho_1 V\mu_c} + e^{\beta\rho_2 V\mu_c}$$

$$e^{\beta\rho_1 V\mu}(1 + e^{\beta(\rho_2-\rho_1)V\mu}) = 0$$

Finite-Size Scaling

$$Z(t,h,L^{-1}) = \tilde{Z}(tL^{y_t}, hL^{y_h})$$

LYZs

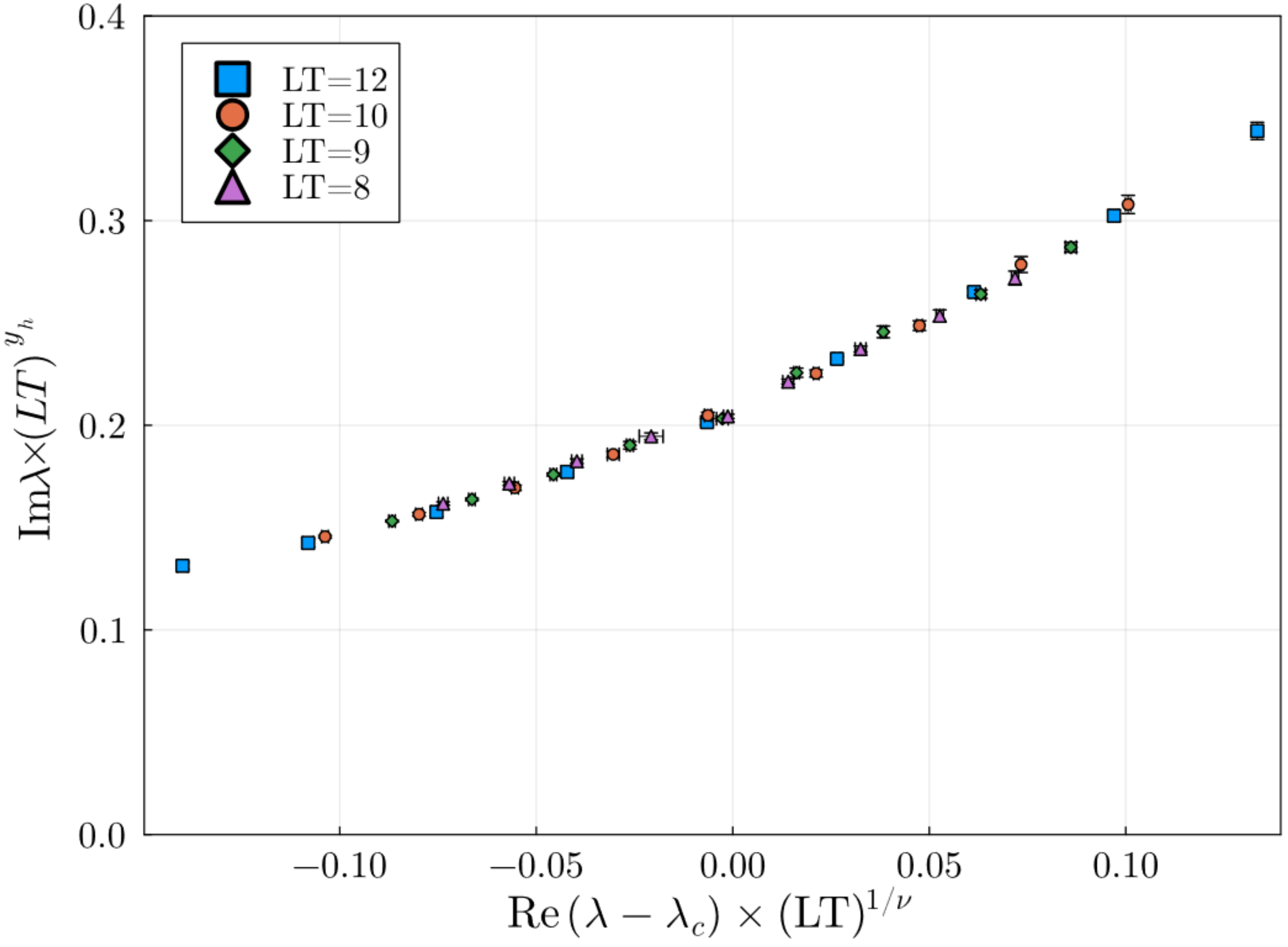
- $(t,h) = (t_1^*, h_1^*)$ at $L = L_1$
- $(t,h) = (t_2^*, h_2^*)$ at $L = L_2$

$$t_2^* = t_1^* \left(\frac{L_1}{L_2}\right)^{y_t} \qquad h_2^* = h_1^* \left(\frac{L_1}{L_2}\right)^{y_h}$$

$$\delta\lambda_{R2}^* = \delta\lambda_{R1}^* (L_1/L_2)^{y_t}$$

$$\lambda_{I2}^* = \lambda_{I1}^* (L_1/L_2)^{y_h}$$

Stephanov, '06



Summary & Outlook

- Successful numerical analysis of the LY zeros near HQ-QCD-CP.
  — owing to the use of the hopping-parameter expansion
  — direct analysis of $Z(\beta, \lambda)$ on the complex plane by reweighting

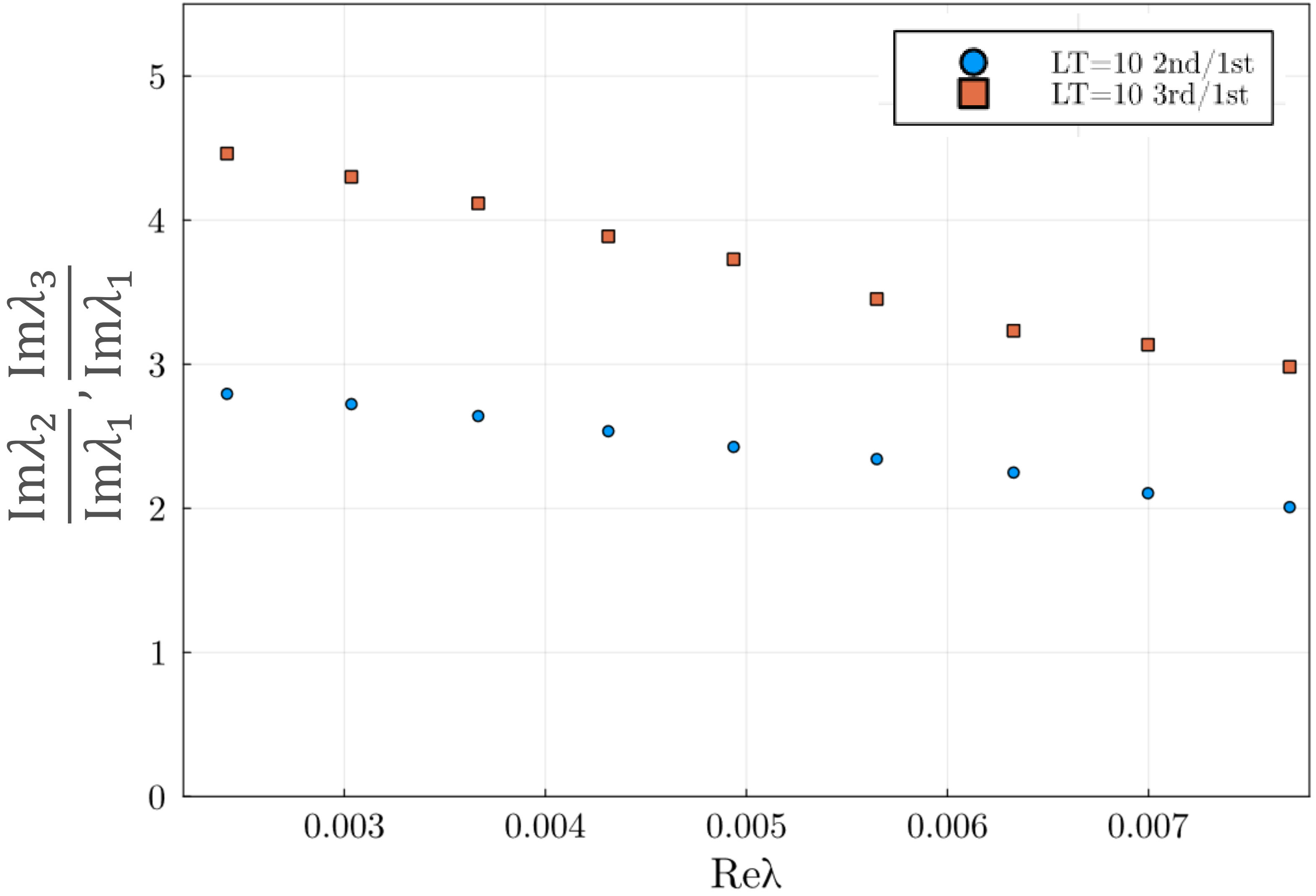
- Large finite volume effects
- Verification of the finite-size scaling

Future

- Check of Z_2 scaling / non-universal params.
- Roberge-Weiss singularity
- **How to exploit LY zeros for exploring real QCD at $\mu \neq 0$?**

Backup

Ratio of 2nd/3rd LYZs to 1st



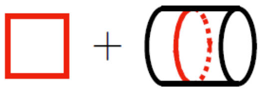
$$\mu_{\mathrm{LYZ}} = \mu_c + i\frac{(2n+1)\pi T}{\Delta\rho V}$$

Numerical Setup

	LO	NLO	NNLO⋯
Wilson type	κ^4	κ^6	$\kappa^8 \dots$
Polyakov type	κ^{N_t}	κ^{N_t+2}	$\kappa^{N_t+4} \dots$

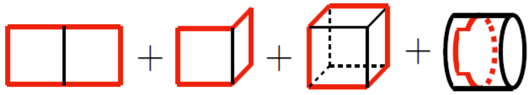
LO

Included in action
of Monte-Carlo

$S_{\mathrm{LO}} \sim$ 

NLO

reweighting

$S_{\mathrm{NLO}} \sim$ 

Kiyohara+ ('21)

NNLO+

effective
incorporation

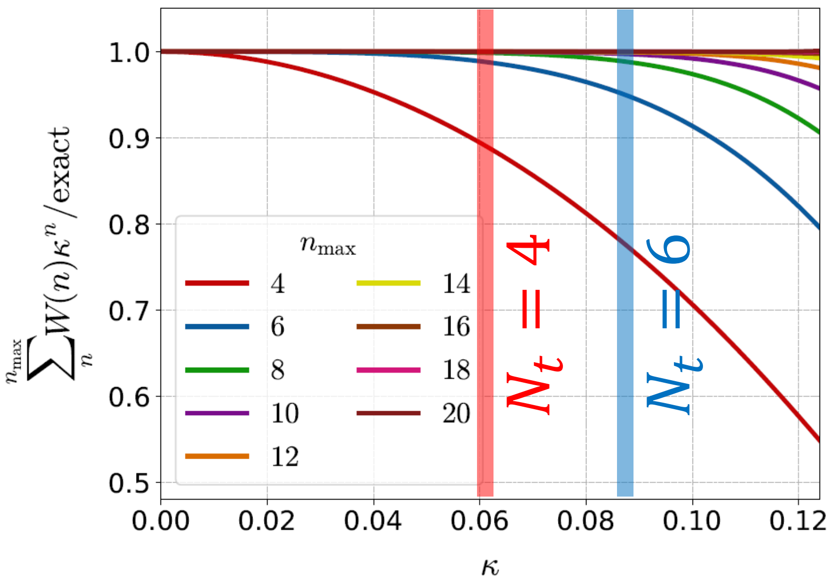
Wakabayashi+ ('22)

Convergence of HPE

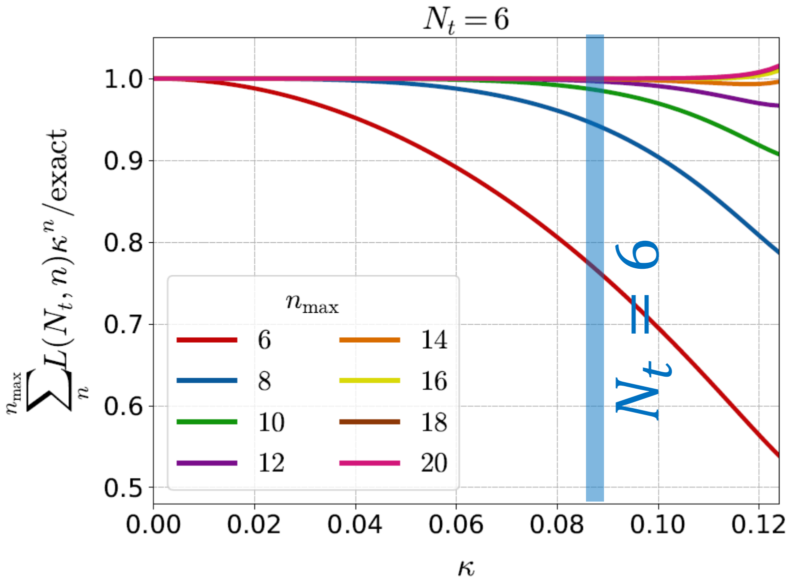
Wakabayashi+ ('22)

- HPE of free lattice field (U=1)

Wilson-loop-type



Polyakov-loop-type



$N_t = 4\ \ \kappa_c = 0.0602(4)$ Kiyohara+,'21

$N_t = 6\ \ \kappa_c = 0.0877(9)$ Cuteri+, '21

NNLO and higher

Wakabayashi+ ('22)

Numerical Simulations

$N_t = 4$
Kiyohara+, PRD104 ('21)
$N_t = 6, 8$
Ashikawa+, in prep.

- Coarse lattice: $N_t = 4, 6, 8$
- But **large spatial volume**:

$$LT = N_s \,/\, N_t \leq \mathbf{15}$$

- High statistics (~**10^6 measurements**)

- Hopping-param. ($\sim 1/m_q$) expansion
- Monte-Calro with LO action
- 4~6 simulation points for reweighting
- Lattice size:

$$N_t = 4 \quad LT = N_x/N_t = 6,8,9,10,12$$

$$N_t = 6 \quad LT = N_x/N_t = 6,7,8,9,10,12,15$$

$$N_t = 8 \quad LT = N_x/N_t = 6,8,10,12 \text{ (in prog.)}$$