臨界点近傍でのスケーリング関数の埋め込みにおけるLee-Yangゼロの活用

北沢正清
（京大基研）

和田辰也、金谷和至

Wada, MK, Kanaya, in preparation

このトークで伝えたいこと

Lee-Yangゼロを利用して臨界点の位置を決める方法の提案

ビンダーキュムラント法に似た方法です

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自己紹介

- ― 専門：原子核理論
- ― 有限温度・有限密度QCD
- ― 格子QCD数値解析など

非ガウスゆらぎで探る宇宙最高密度の相転

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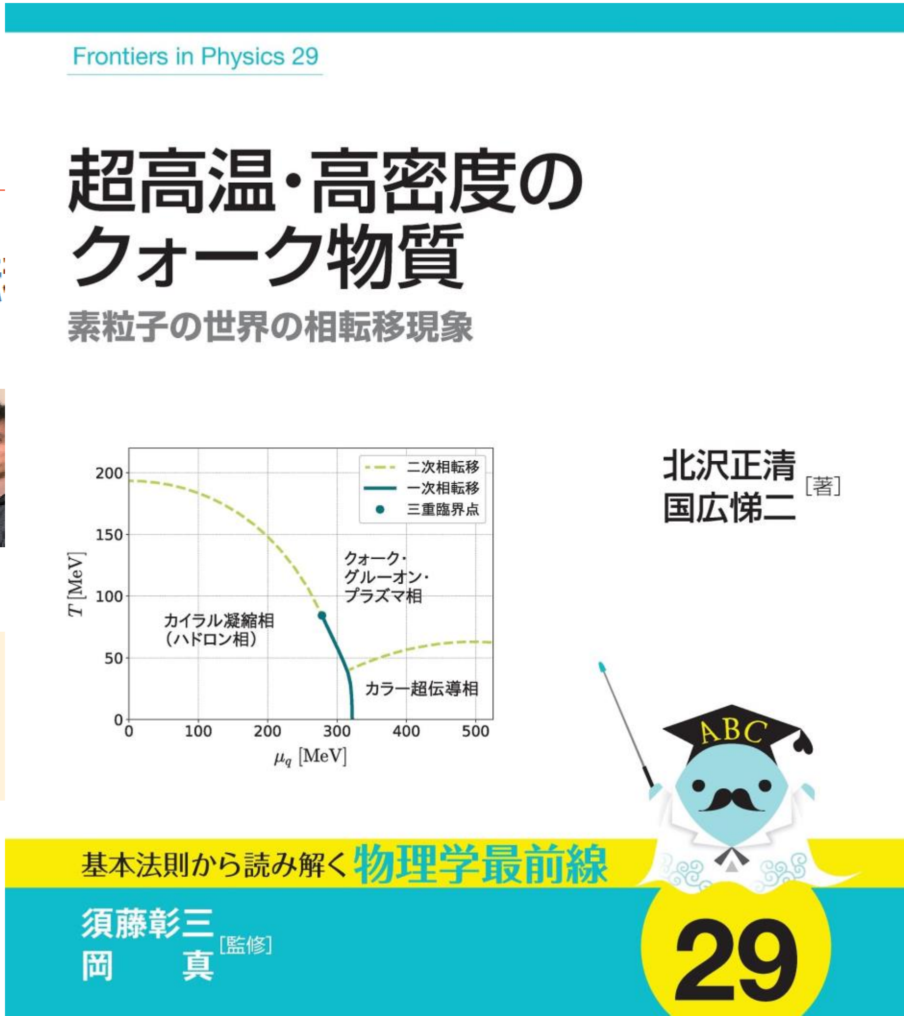
現在，およそ10^{15} g/cm^{3}という超高密度で実現するとされる相転移の実験的探索が世界各地の実験施設で行われているのをご

これら一連の実験が目指す最重要課題が，ビームエネルギー走査による高密度領域の相構造探索である．

物理学会誌2021年8月号



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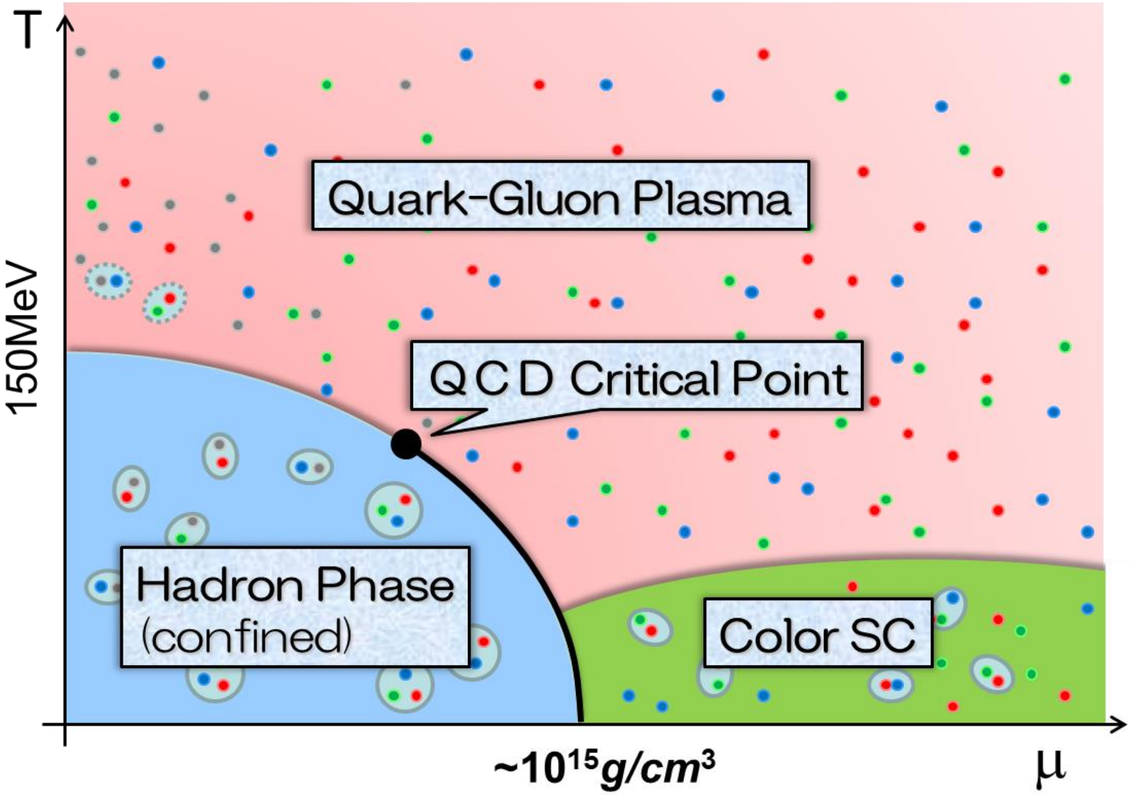


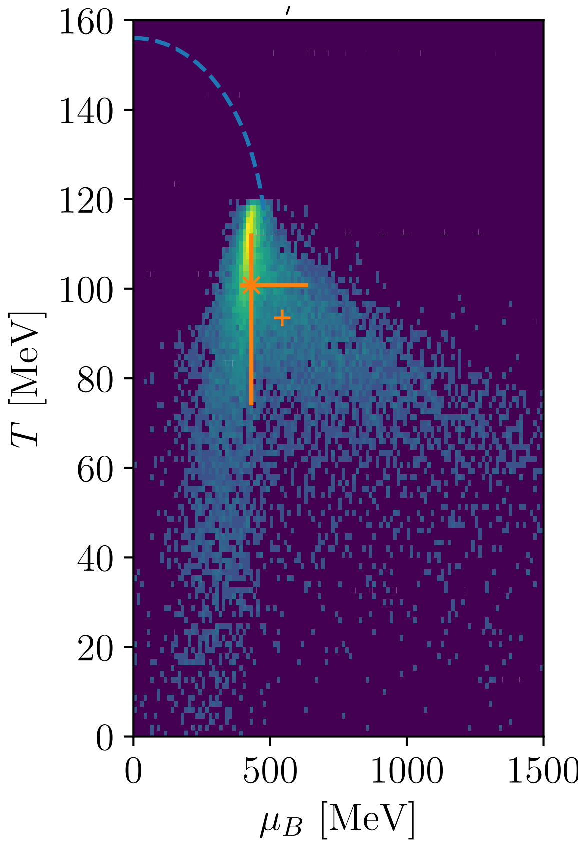
格子QCD数値計算の**有限密度系**への適用は**複素位相問題**により困難

最近の進展

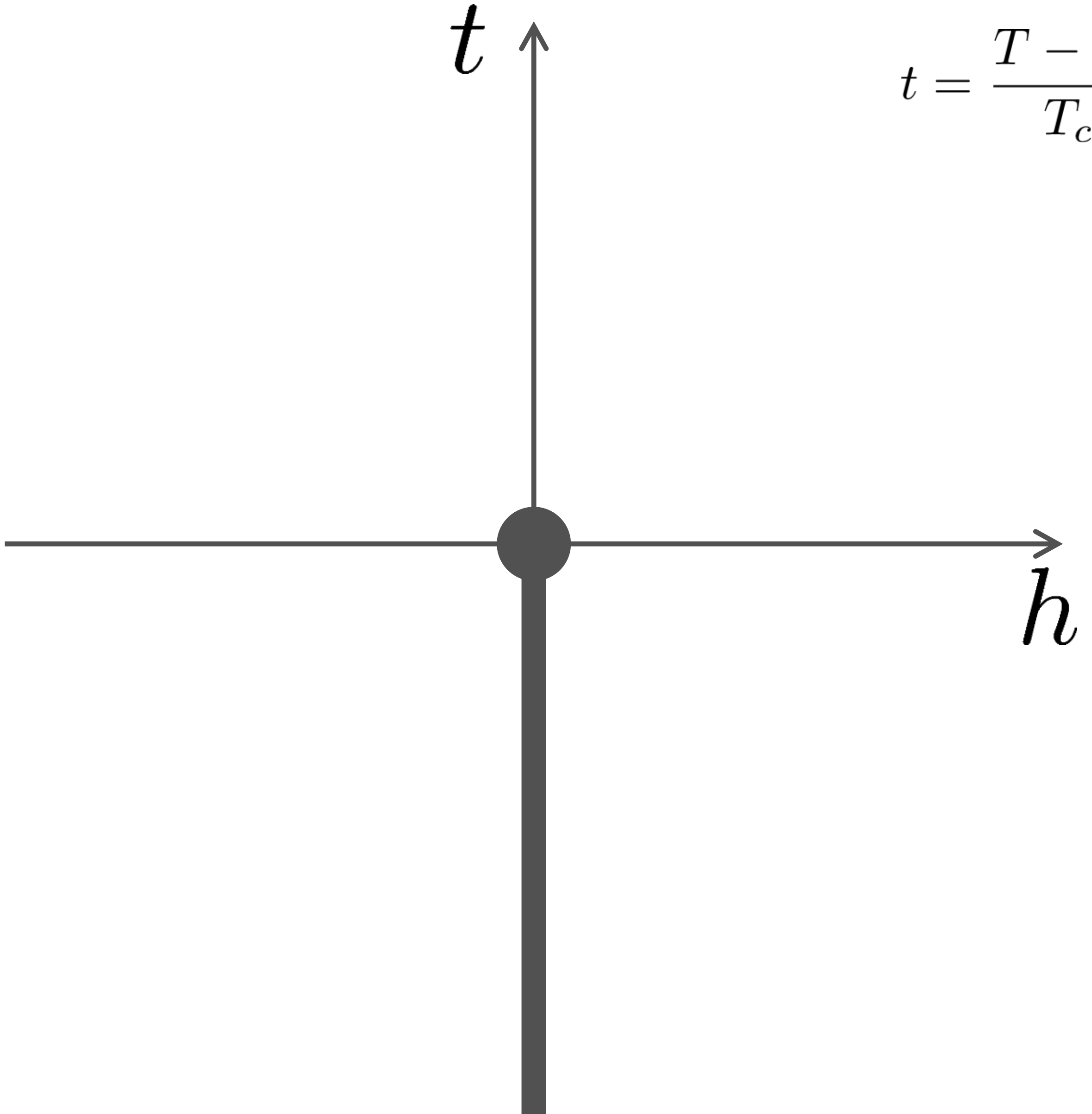
リーヤンゼロを利用して格子QCD数値計算で臨界点の位置決定に成功？

$$\begin{cases} \mu^{\rm CEP} = 422^{+80}_{-35}{\rm MeV} \\ T^{\rm CEP} = 105^{+8}_{-18}{\rm MeV} \end{cases}$$

arXiv:2405.10196 [hep-lat]



LYZ around Critical Point in Ising Model



$$t = \frac{T - T_c}{T_c}$$

1st-transition

singularity on the real h axis

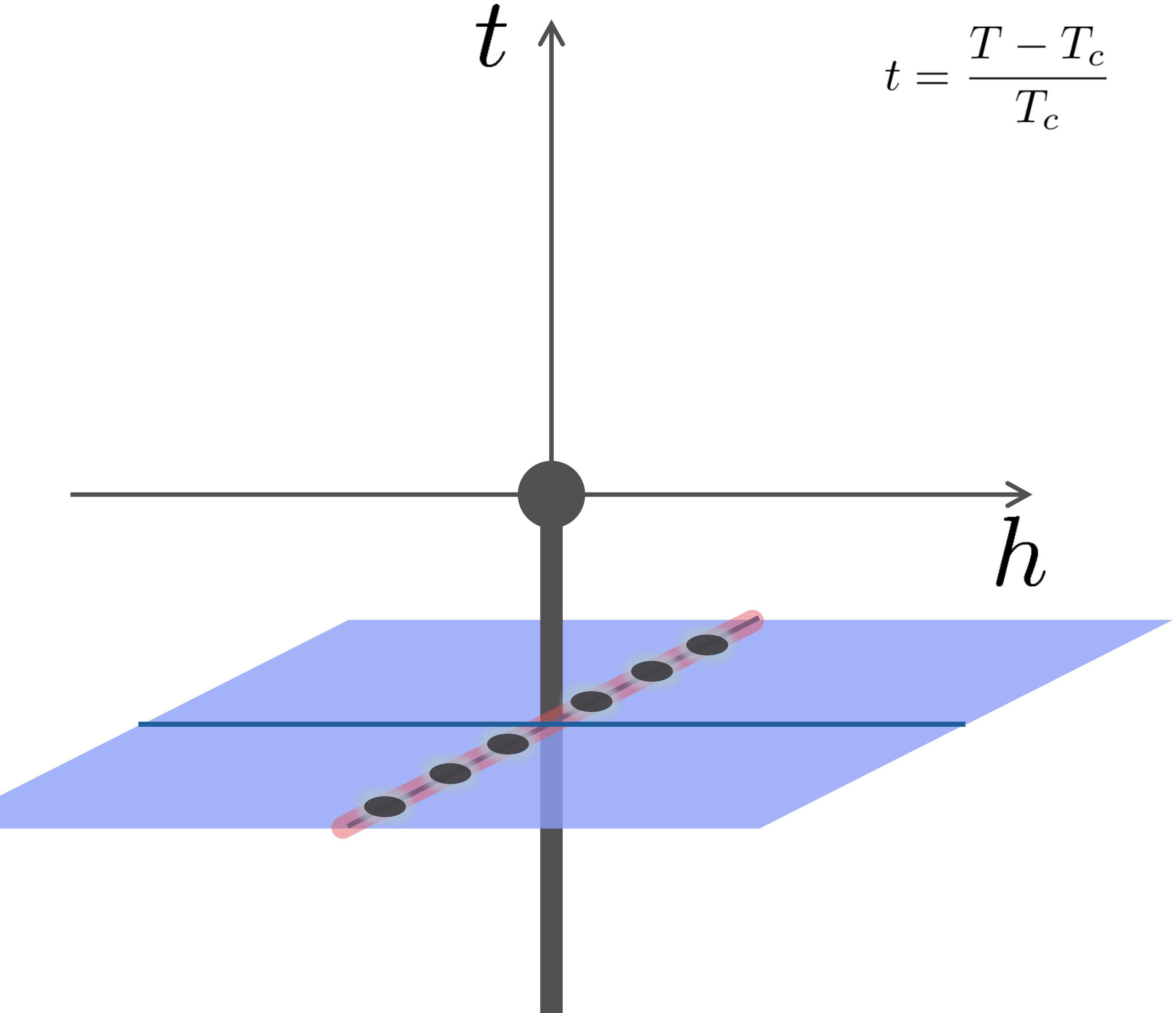
Crossover

no singularity on the real axis

Note:
LYZ in complex-h plane are purely imaginary.

Lee-Yang, 1952

LYZ around Critical Point in Ising Model



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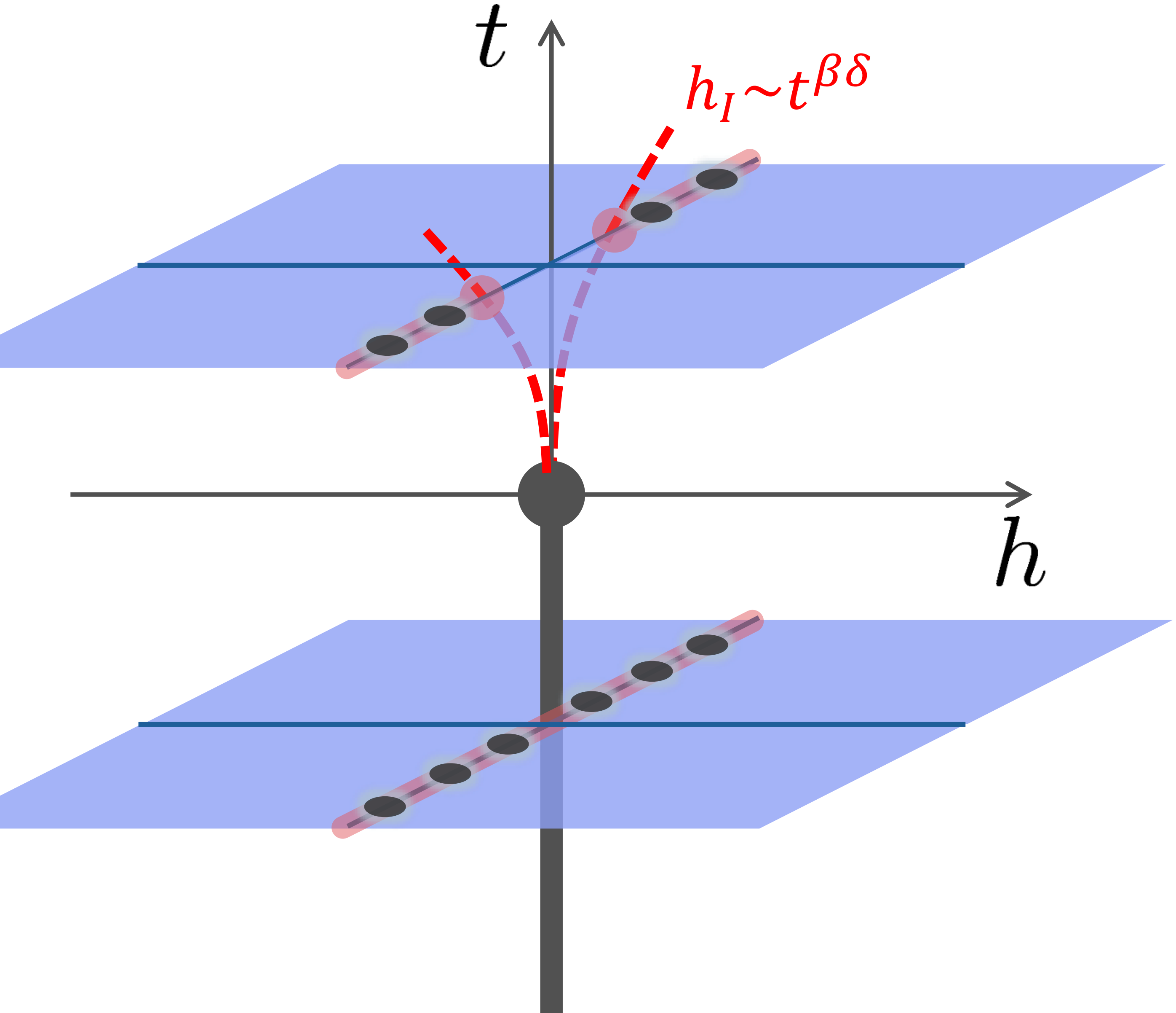

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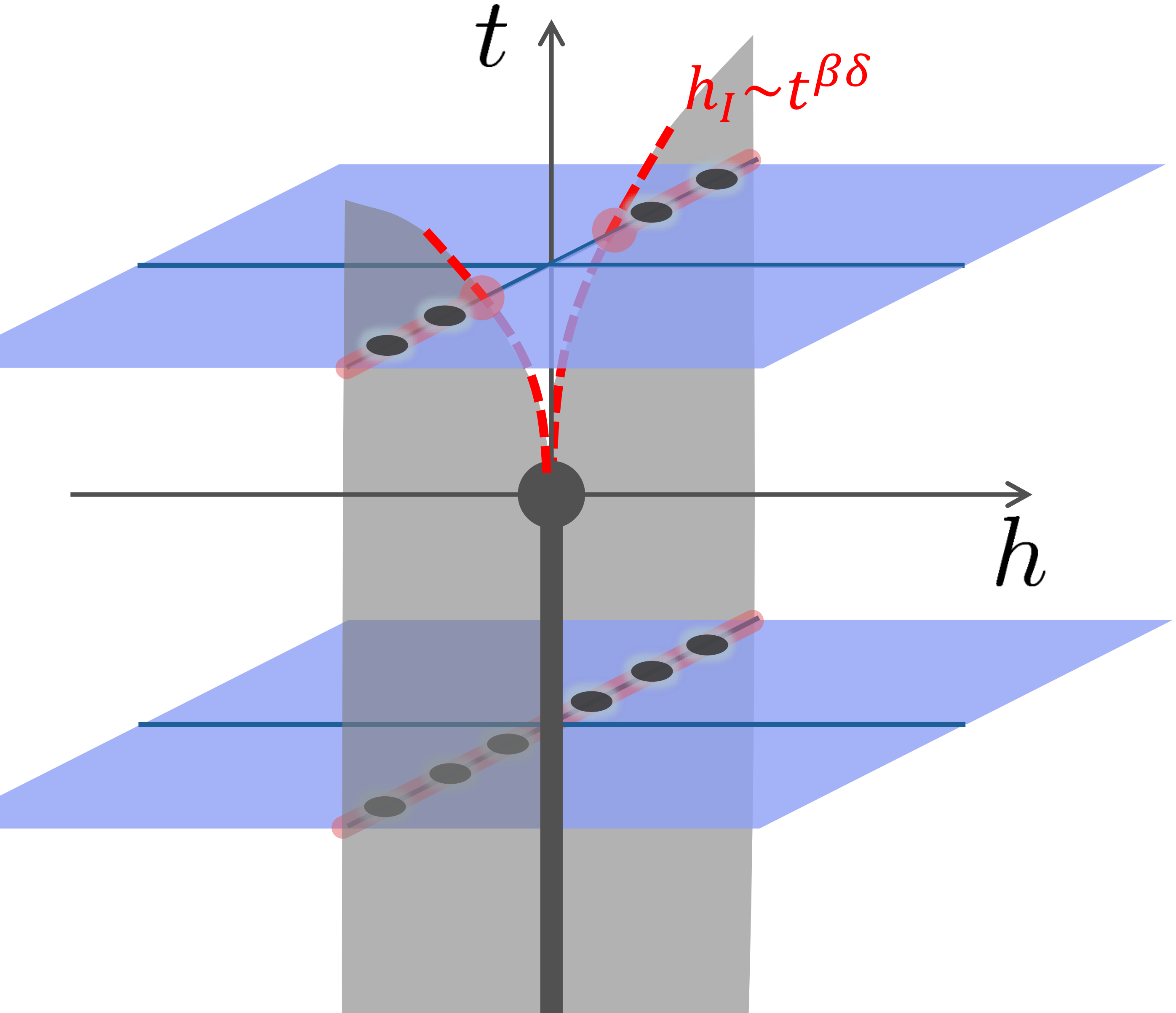
LY edge singularity

Starting from the CP

Its behavior is governed by the the scaling function.

$$h_I \sim t^{\beta\delta}$$

LYZ around Critical Point in Ising Model



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Starting from the CP

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Recent Progress in LYZ/LYES and Lattice

Analytic Structure

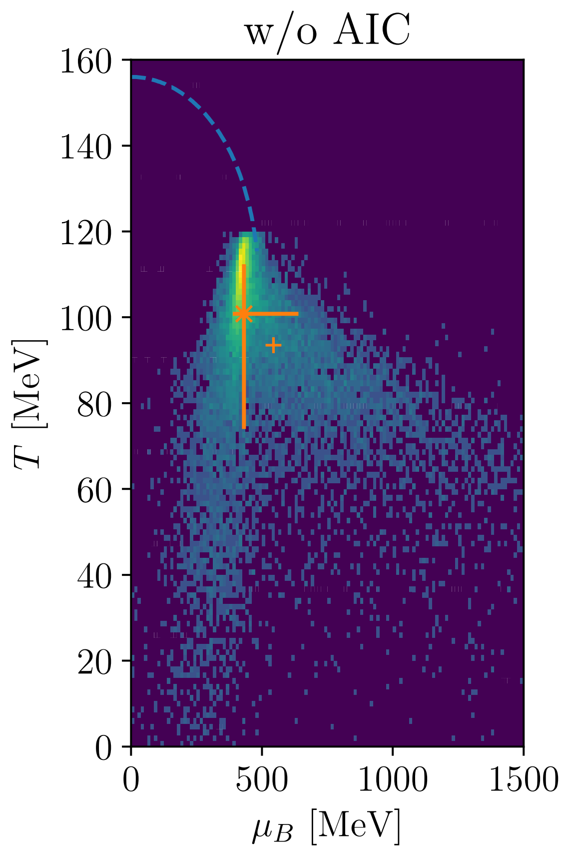
— Scaling functions, FRG, …

An, Mesterhazy, Stephanov ('16)
Johnson, Rennecke, Skokov ('23)
Karsch, Schmidt, Singh ('23)
…

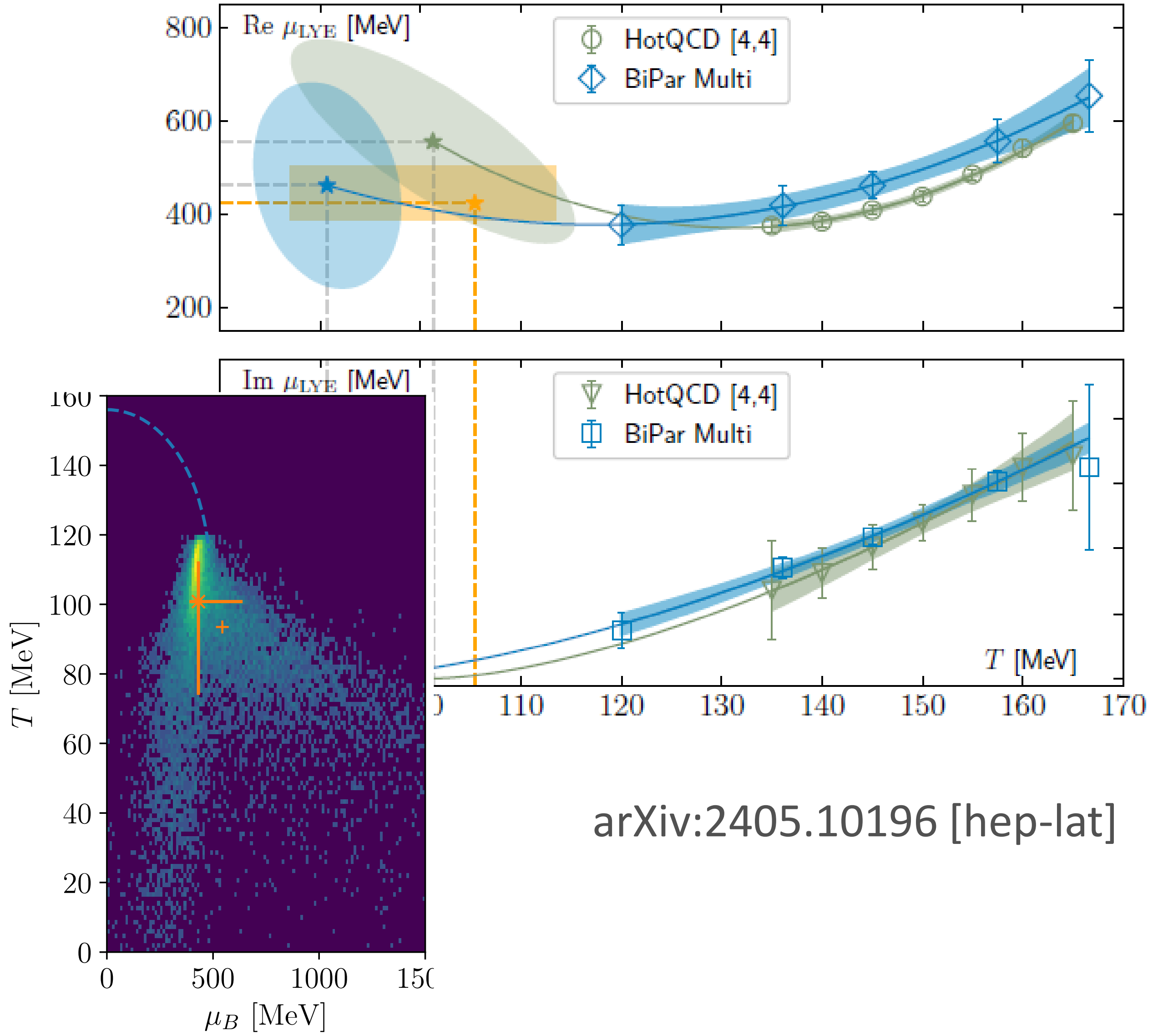
Locating QCD-CP at $\mu \neq 0$ on the lattice?

Clarke+, arXiv:2405.10196

— Taylor exp. + Imaginary μ + Pade approx.
— Identify the 1st LYZ to be LYES



LYZによるQCD臨界点探索



arXiv:2405.10196 [hep-lat]

1st LYZ = LY edge singuarity

を仮定して$\mathrm{Im}\mu_{\rm LYZ} \to 0$へ外挿

$$\begin{cases} \mathrm{Re}\mu_{\rm LYZ} = \mu_{\rm CP} + c_1 \Delta t + c_2 \Delta t^2 \\ \mathrm{Im}\mu_{\rm LYZ} = c_3 \Delta t^{\beta\delta} \end{cases}$$

有限サイズ効果は？

Finite-Size Scaling

Scaling Hypothesis

$$F_{\rm sing}(t,h,L^{-1}) = \tilde{F}_{\rm sing}(L^{y_t}t, L^{y_h}h)$$

$$Z_{\rm sing}(t,h,L^{-1}) = \tilde{Z}_{\rm sing}(L^{y_t}t, L^{y_h}h)$$

$$F = F_{\rm sing} + F_{\rm reg}$$

$$Z = Z_{\rm sing} \times Z_{\rm reg}$$

LYZ in the scaling region on finite volume

$$Z(t,h,L^{-1}) \sim \tilde{Z}_{\rm sing}(L^{y_t}t, L^{y_h}h) = 0$$

$$L^{y_h}h^{(i)} = \tilde{h}^{(i)}_{\rm LY}(L^{y_t}t)$$

LYZ in 3d-Ising Model

$$H = -\sum_{\langle i,j\rangle} s_i s_j - h\sum_i s_i$$

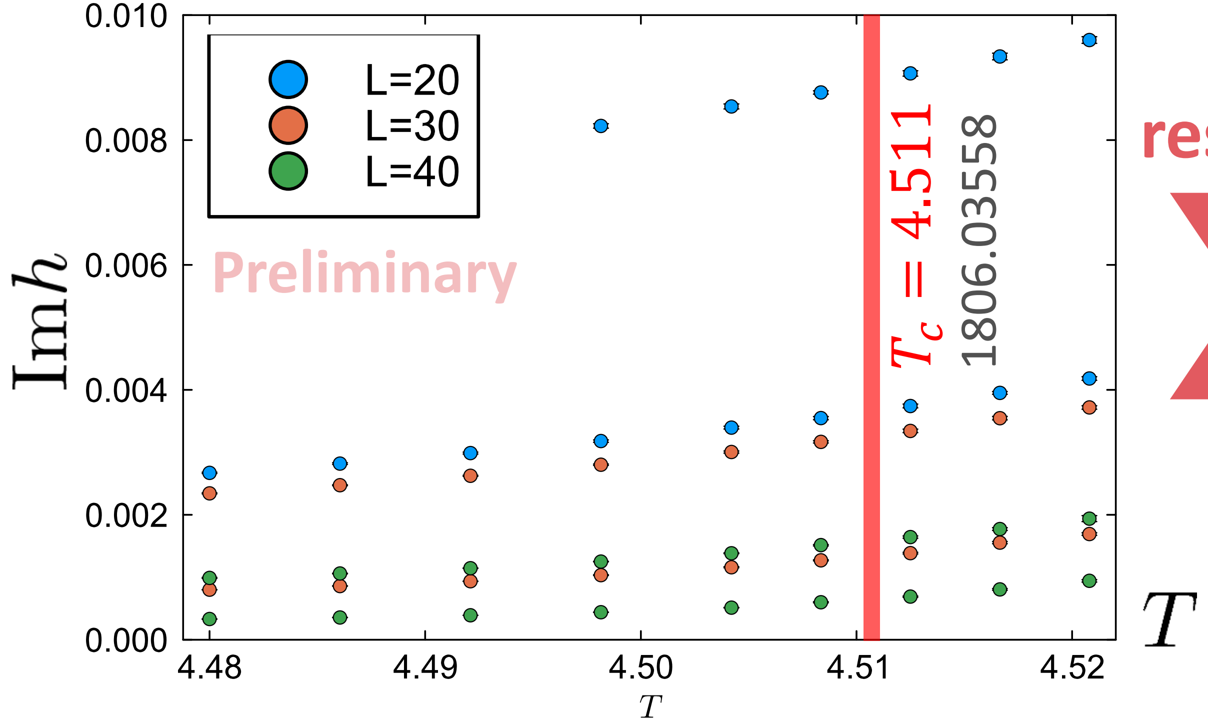
Monte-Carlo + reweighting

LYZ

$$\frac{Z(t,\tilde{h})}{Z(t,h)} = 0$$

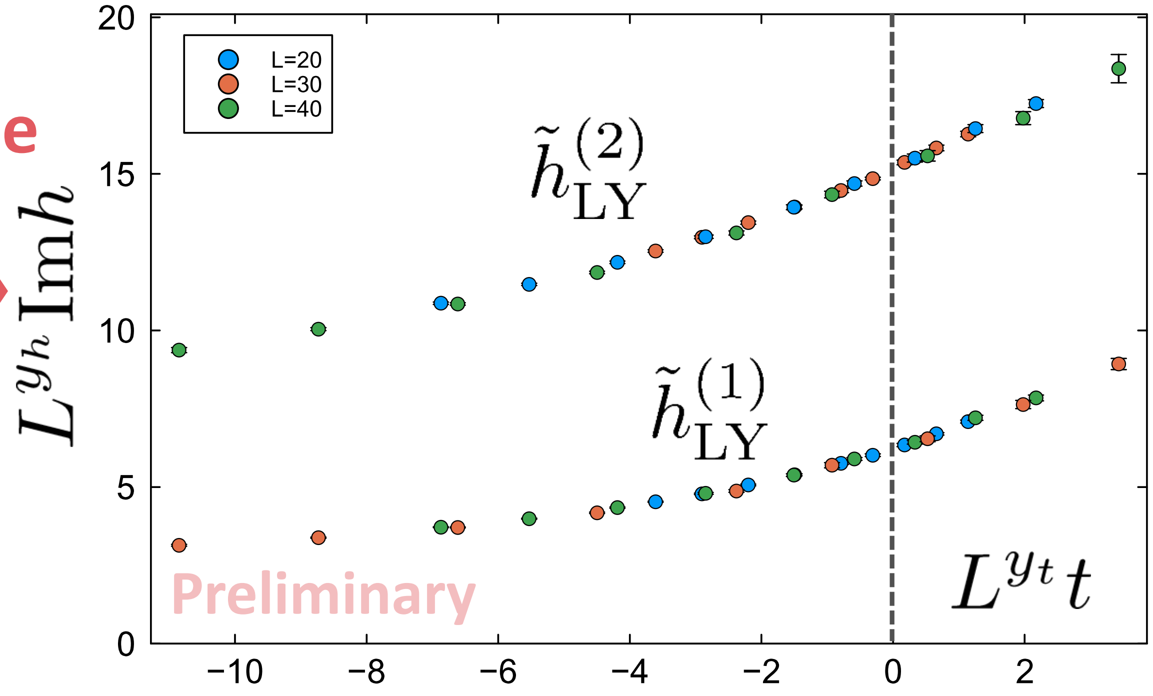
$$L^{y_h} h^{(i)} = \tilde{h}^{(i)}_{\rm LY}(L^{y_t} t)$$

1st & 2nd LYZ

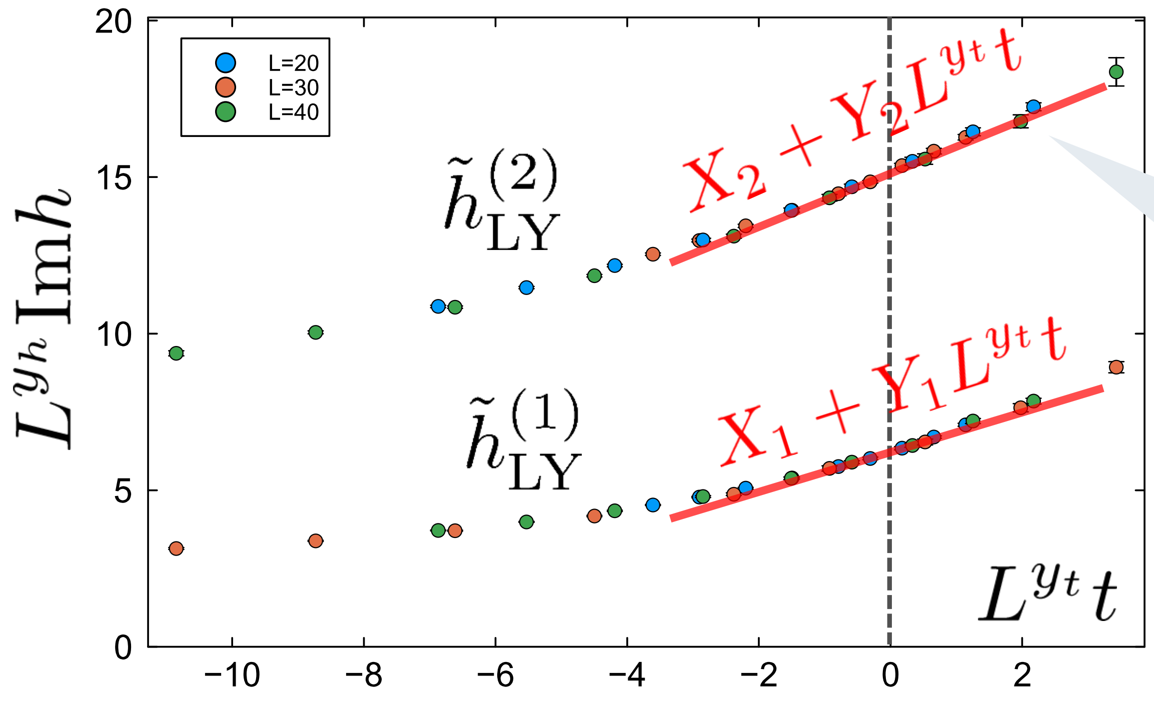


rescale

Rescaled



Linear Approximation & LYZ Ratio



Linear Approx. at $t = 0$

$$L^{y_h} h = \tilde{h}^{(i)}_{\mathrm{LY}}(L^{y_t} t)$$
$$= X_i + Y_i L^{y_t} t + \mathcal{O}(t^2)$$

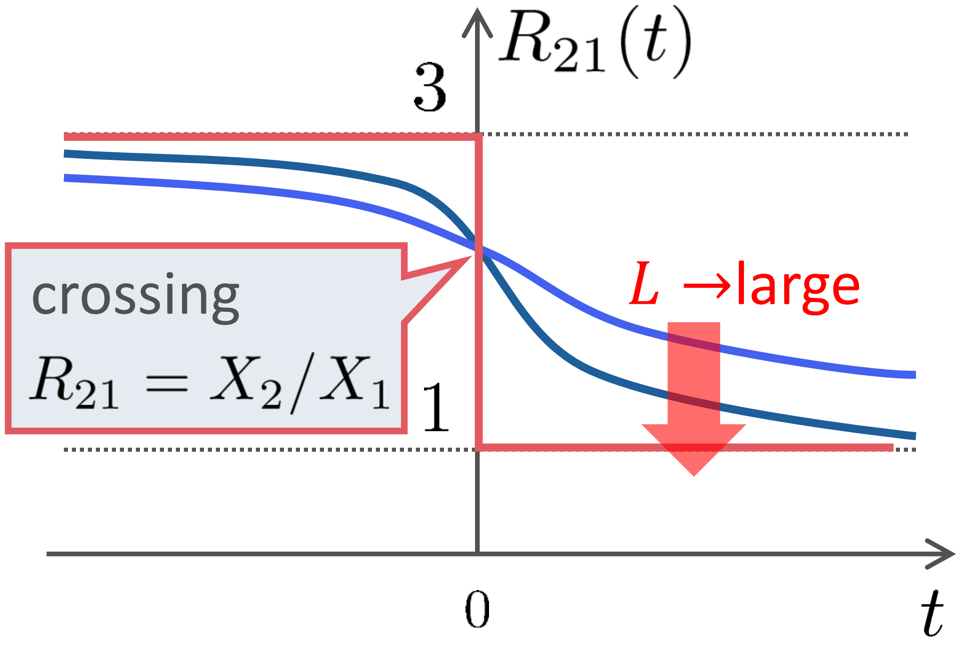
Take Ratio between nth/mth

$$R_{nm}(t) = \frac{h^{(n)}(t)}{h^{(m)}(t)} = \frac{X_n}{X_m}\left(1 + C_{nm} t L^{y_t} + \mathcal{O}(t^2)\right) \qquad C_{nm} = \frac{Y_n}{X_n} - \frac{Y_m}{X_m}$$

LYZ Ratio

$$R_{nm}(t) = \frac{h^{(n)}(t)}{h^{(m)}(t)} = \frac{X_n}{X_m}\left(1 + C_{nm} t L^{y_t} + \mathcal{O}(t^2)\right)$$

$$R_{n1}(t) = \begin{cases} 2n-1 & t \to -1 \quad \text{(1st order)} \\ X_n/X_1 & t = 0 \\ 1 & t \to \infty \quad \text{(crossover)} \end{cases}$$



— $R(0)$ is L independent, the universal value.

— Crossing point of various L gives the CP.

— Reminiscent of Binder-cumulant analysis

LYZ Ratio

$$R_{nm}(t) = \frac{h^{(n)}(t)}{h^{(m)}(t)} = \frac{X_n}{X_m}\left(1 + C_{nm} t L^{y_t} + \mathcal{O}(t^2)\right)$$

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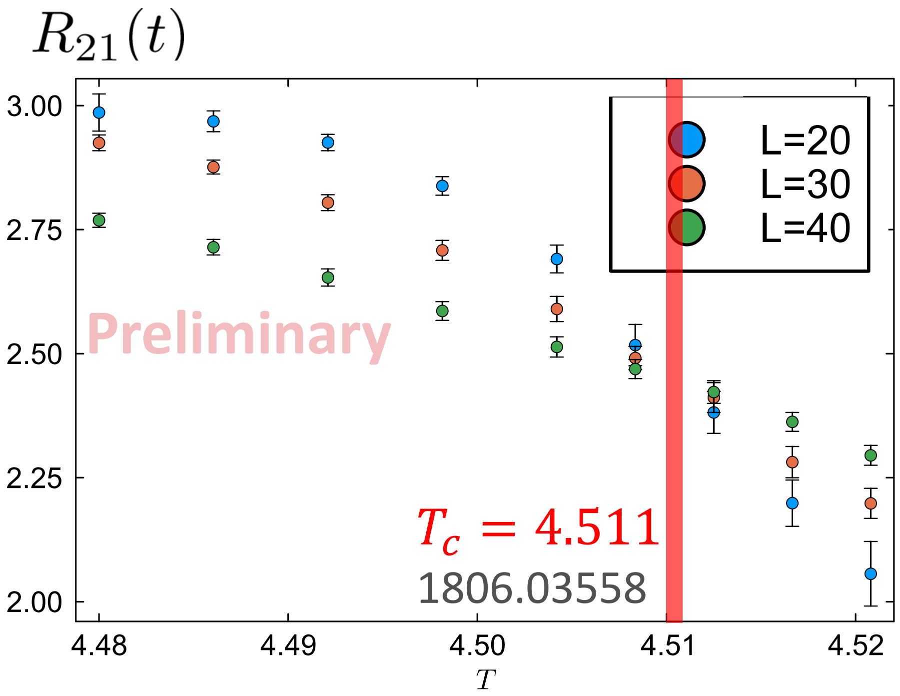
Numerical Result in 3d-Ising



$R_{21}(0) \simeq 2.42$

— **$R(0)$ is L independent, the universal value.**

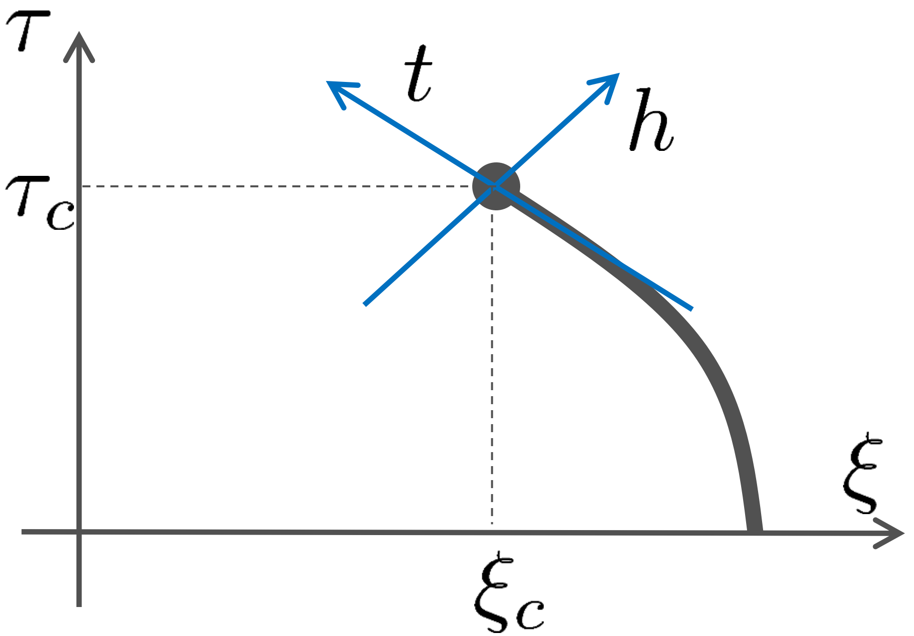
— **Crossing point of various L gives the CP.**

— Reminiscent of Binder-cumulant analysis

General CP

$$Y = y_t - y_h = -0.894$$

- CP on a $\tau - \xi$ plane
- Search for LYZ on the complex ξ plane



$$\begin{pmatrix} t \\ h \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} \tau - \tau_c \\ \xi - \xi_c \end{pmatrix} = A \begin{pmatrix} \delta\tau \\ \delta\xi \end{pmatrix}$$

$$L^{y_h} h^{(n)} \simeq X_i + Y_i L^{y_t} t$$

$$\begin{cases} \xi_{\mathrm{R}}^{(n)} L^{y_h} = -\dfrac{a_{21}}{a_{22}} \tau L^{y_t} + \mathcal{O}(L^{2Y}) \\ \xi_{\mathrm{I}}^{(n)} L^{y_h} = \dfrac{X_n}{a_{22}} + \dfrac{\det AY_n}{a_{22}^2} \tau L^{y_t} + \mathcal{O}(L^{2Y}) \end{cases}$$

$L \to \infty$

generalization

LY Edge Singularity

$$\begin{cases} \mathrm{Re}\xi_{\mathrm{LYES}} \simeq c_1 \tau \\ \mathrm{Im}\xi_{\mathrm{LYES}} \simeq c_2 \tau^{\beta\delta} \end{cases}$$

Stephanov, 2006

LYZ Ratio for General CP

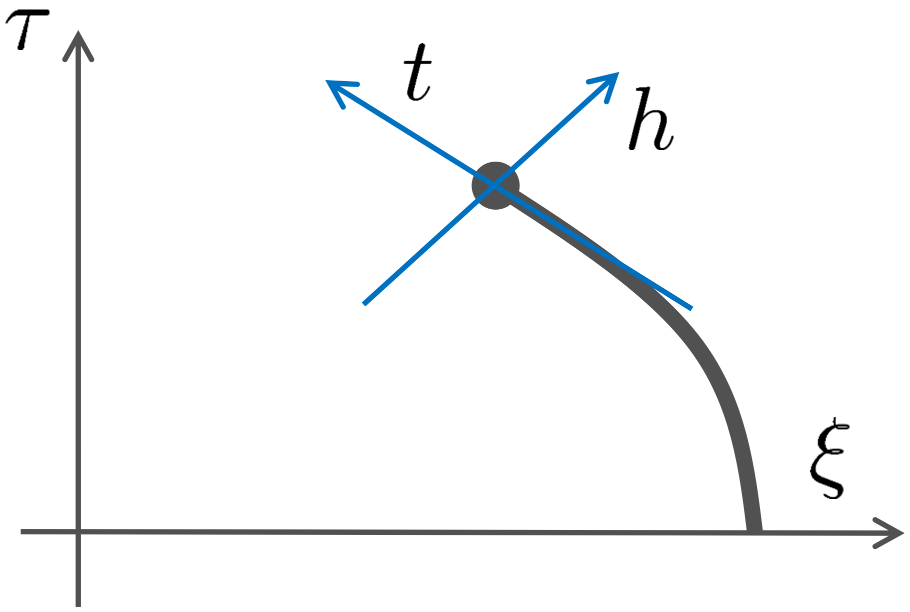
LYZ Ratio

$$R_{nm}(t) = \frac{\xi_{\mathrm{I}}^{(n)}(\tau)}{\xi_{\mathrm{I}}^{(m)}(\tau)} = \frac{X_n}{X_m}\left(1 + C\tau L^{y_t} + \mathcal{O}(t^2)\right)\left(1 + DL^{2(y_t - y_h)} + \mathcal{O}(L^{4(y_t-y_h)})\right)$$

nonzero for $a_{12} \neq 0$

$$C = \frac{\det A}{a_{22}}\left(\frac{Y_2}{X_2} - \frac{Y_1}{X_1}\right), \quad D = \frac{a_{12}^2}{a_{22}^2}(Y_1^2 - Y_2^2)$$

$$\begin{pmatrix} t \\ h \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} \tau - \tau_c \\ \xi - \xi_c \end{pmatrix} = A \begin{pmatrix} \delta\tau \\ \delta\xi \end{pmatrix}$$



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Binder cumulant

Jin+, PRD**86**, 2017

$$B_4(t) = b_4\left(1 + c\tau L^{y_t} + \mathcal{O}(t^2)\right)\left(1 + dL^{y_t - y_h} + \mathcal{O}(L^{2(y_t-y_h)})\right)$$

nonzero for $a_{12} \neq 0$

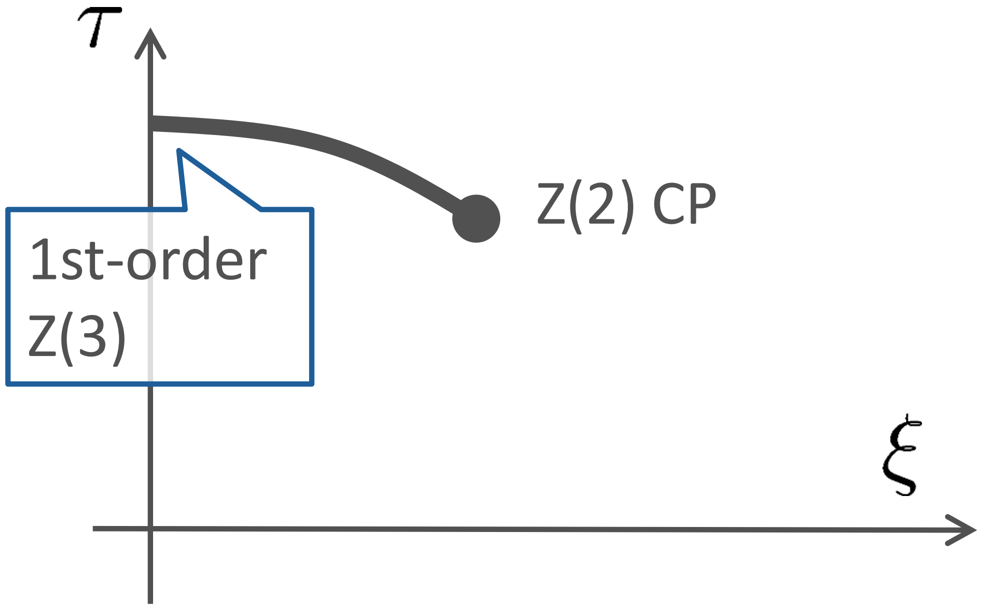
Deviation at $t = 0$ due to $a_{12} \neq 0$
converges faster in LYZ ratio.

3d 3-State Potts Model

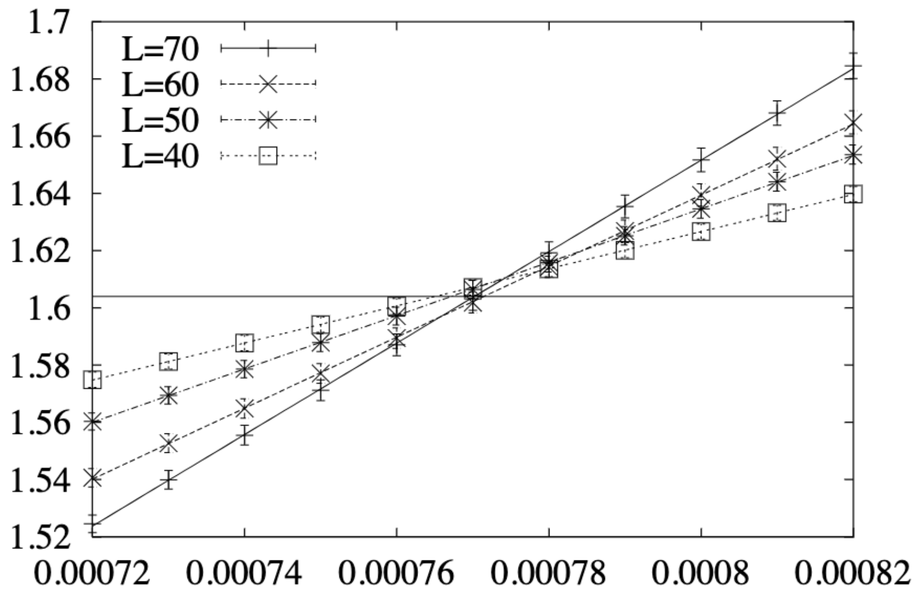
$$H = -\tau \sum_{\langle i,j \rangle} \delta_{\sigma_i,\sigma_j} - \xi \sum_i \delta_{\sigma_i,1} \quad \sigma_i = 1, 2, 3$$

Monte-Carlo + reweighting

Phase Diagram



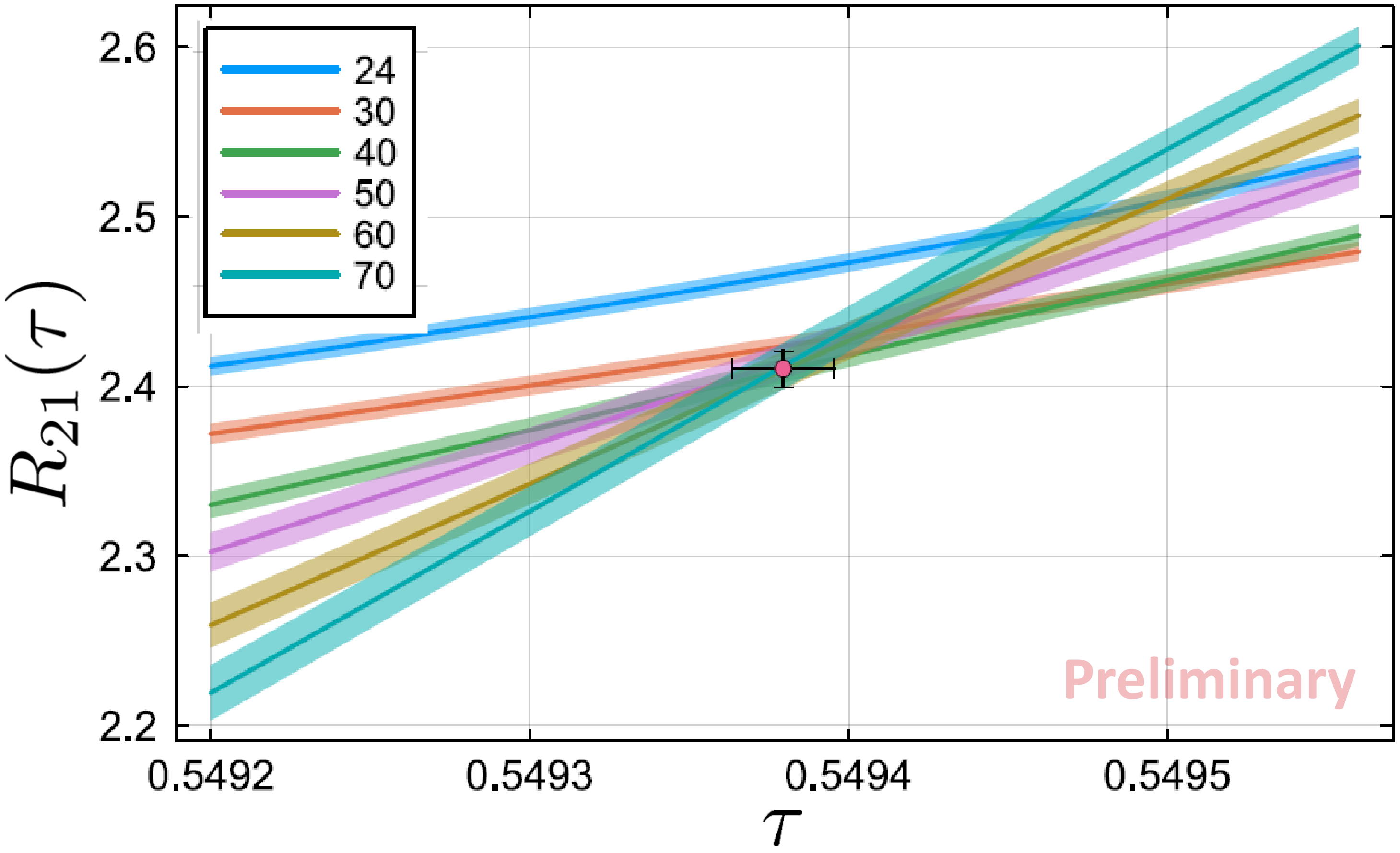
Binder-Cumulant Analysis



Karsch, Stickan, 2000

3d 3-State Potts Model: LYZ Ratio

$L = 24, 30, 40, 50, 60, 70$



Fit Results (to $L \geq 40$):

$$R_{21}(0) = 2.410(11)$$

$$y_t = 1.56(14)$$

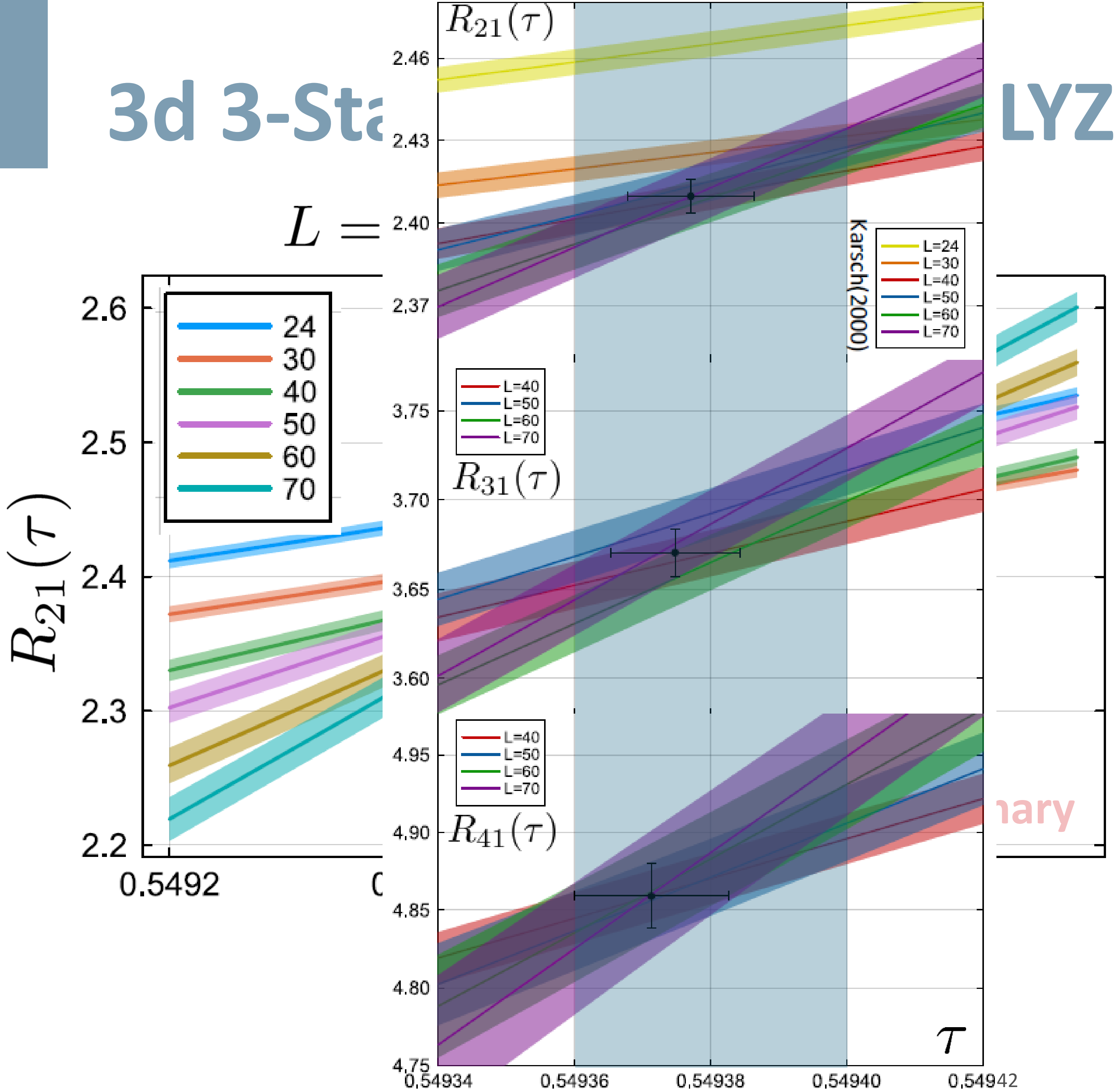
$$\tau_c = 0.549379(16)$$

Consistent with

$y_t = 1.588$ — 3d Ising

$\tau_c = 0.549380(20)$ — Karsch+, '00

3d 3-Sta… LYZ Ratio



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まとめ

リーヤンゼロの虚部の比 $$R_{nm}(\tau) = \frac{\xi_{\mathrm{I}}^{(n)}(\tau)}{\xi_{\mathrm{I}}^{(m)}(\tau)}$$

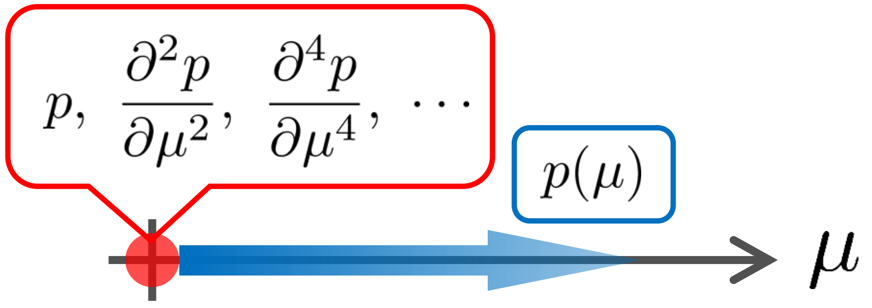
- 異なる体積で測定したR_{nm}は臨界点で交差
- **臨界点の位置決定に使える**
  – 一般のCPではビンダーキュムラントより有限体積効果を抑制

展望
– QCD臨界点探索への適用
– $R_{nm}(0)$の測定 in Ising、
– 一般の臨界点へのイジングパラメータの埋め込み

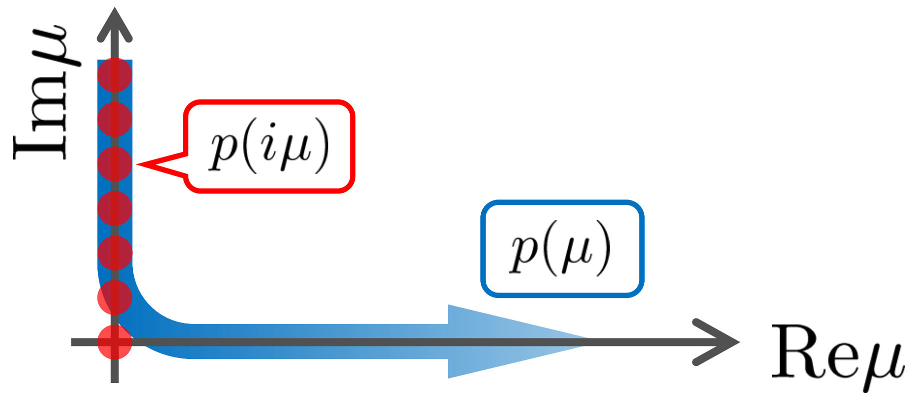
backup

Using LYZ for the QCD-CP Search

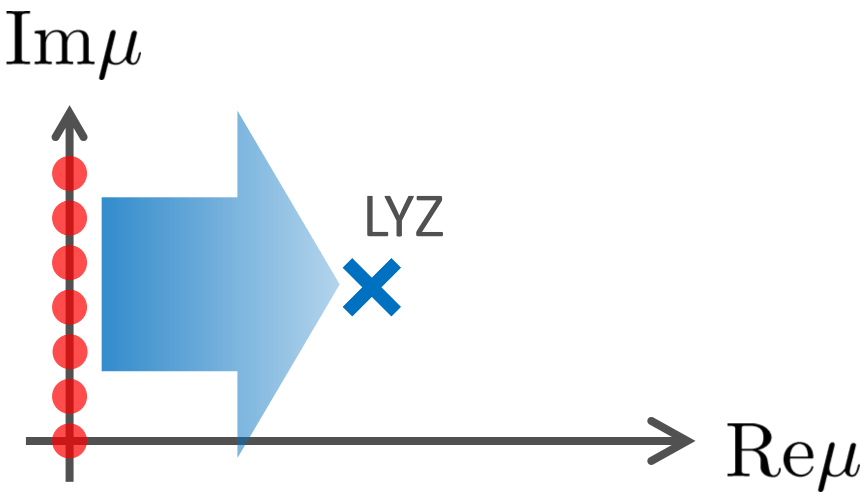
Taylor expansion



Imaginary chem. pot.



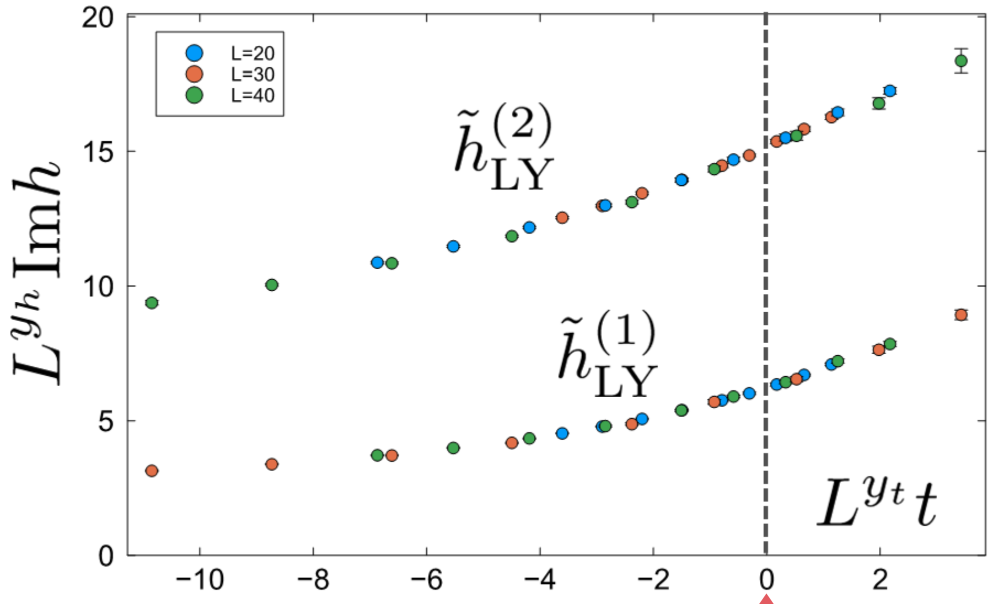
LYZ



Use of Pade approximation

Where is QCD Critical Point?

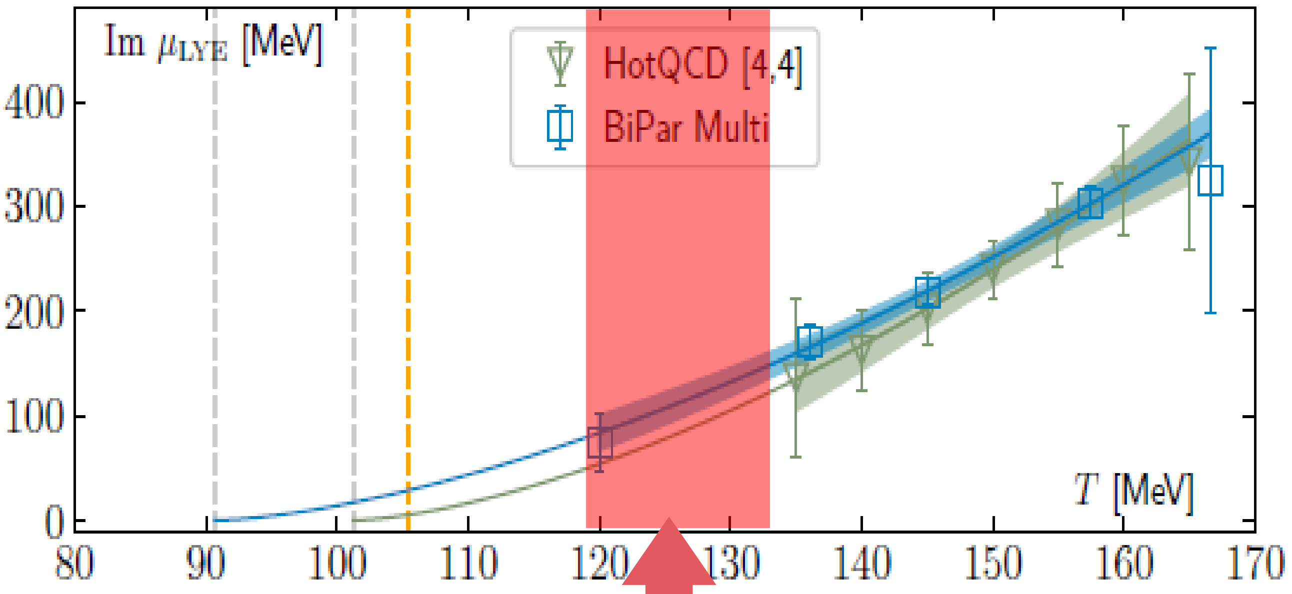
Ising model

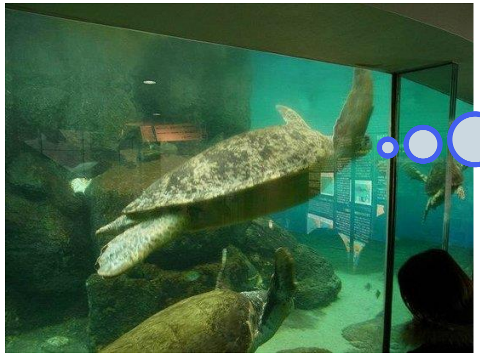


$T = T_c$

LYZ in QCD

Clarke+, arXiv:2405.10196





T_c around here??

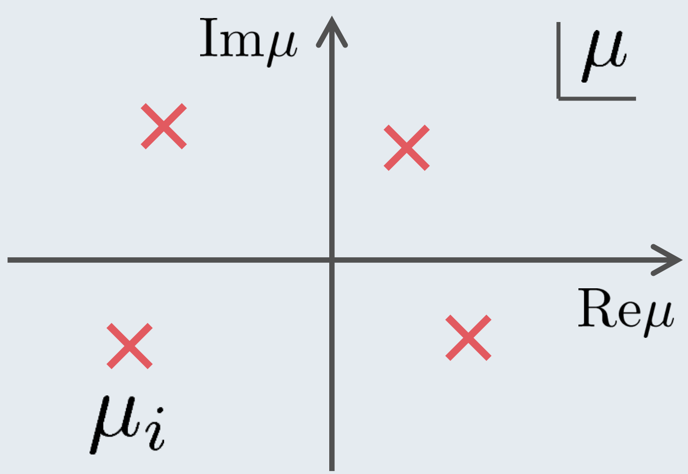
Lee-Yang Zero

Yang, Lee; Lee, Yang ('52)

Partition Function $Z(T, \mu)$

Finite V ➤ Polynomial of μ (or T)

$$Z(T,\mu) = \prod_i (\mu - \mu_i)$$



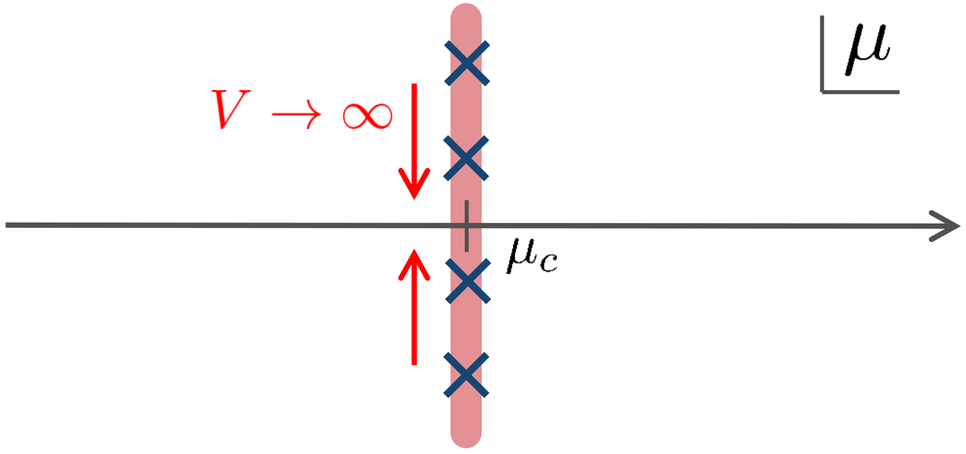
➤ **zeros** on the complex plane

=Lee-Yang Zeros

Phase Transition & LYZ

First-order transition

at $\mu = \mu_c$



— For $V \to \infty$, LYZs are accumulated on the line crossing the real axis at $\mu = \mu_c$.