International Workshop on Advances in Lattice QCD, Institute of Modern Physics, Huizhou, China, Oct. 14, 2025

Lee-Yang zeros near critical points

Masakiyo Kitazawa
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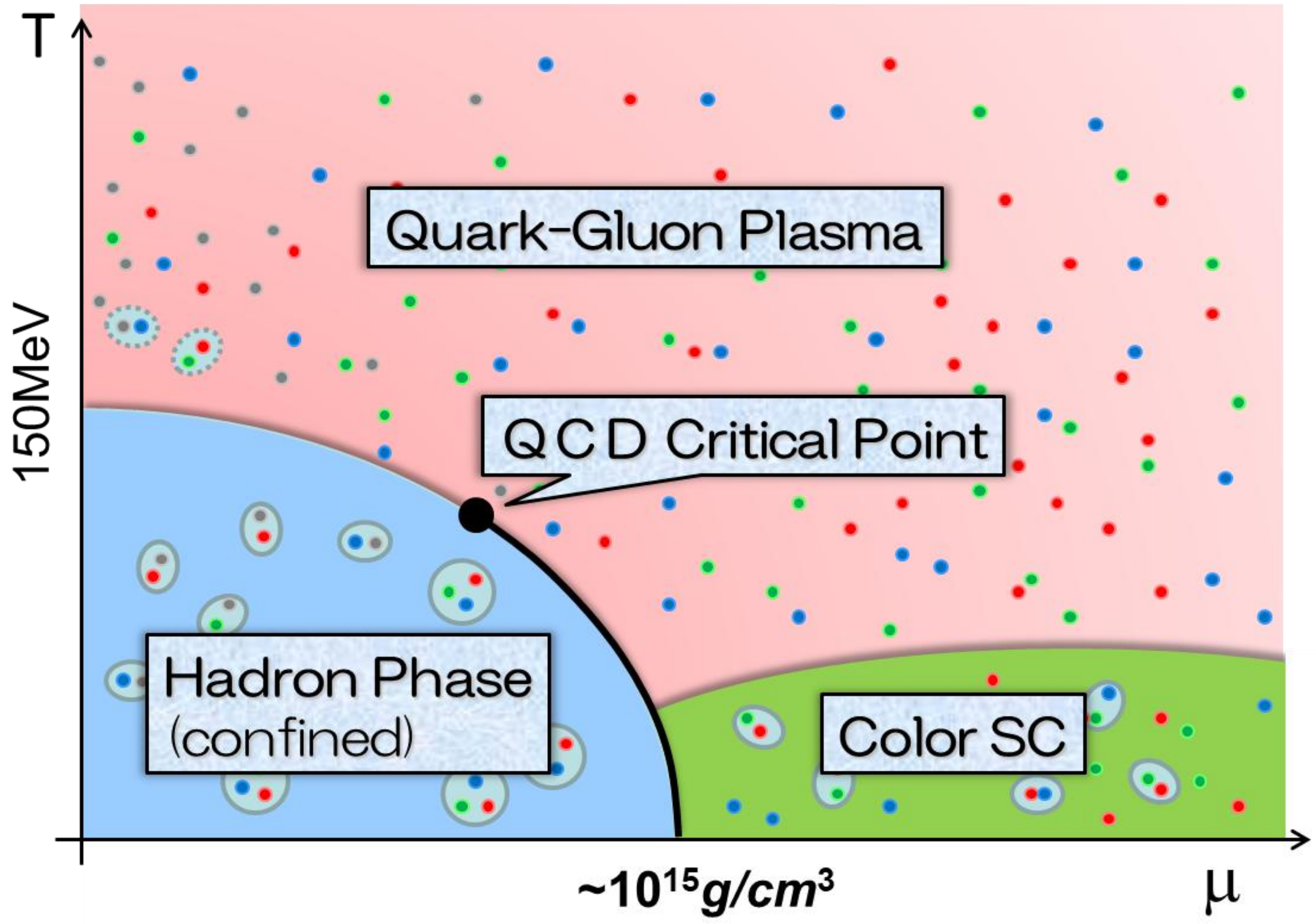
In collab. with Tatsuya Wada, Kazuyuki Kanaya

Special thanks to S. Ejiri

Wada, MK, Kanaya, PRL**134**, 162302 ('25), arXiv:2508. 20422

QCD Phase Diagram



Rich phase structure in QCD

— QCD critical point(s)
— color superconductivity

Sign problem

— difficulty in lattice QCD Monte-Carlo simulations at $\mu \neq 0$

Various approaches

— Taylor expansion method
— Imaginary chem. pot.
— Complex Langevin
— Lifschetz thimble
— Lee-Yang edge singularity
— …

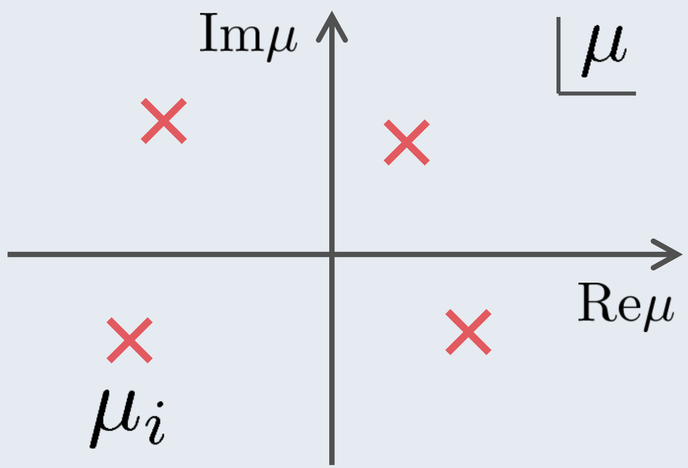
Lee-Yang Zeros

Yang, Lee; Lee, Yang ('52)

Partition Function $Z(T,\mu)$

Finite V ➤ Polynomial of μ (or T)

$$Z(T,\mu)=\prod_i(\mu-\mu_i)$$



➤ **zeros** on the complex plane

=Lee-Yang Zeros

Lee-Yang Zeros

Yang, Lee; Lee, Yang ('52)

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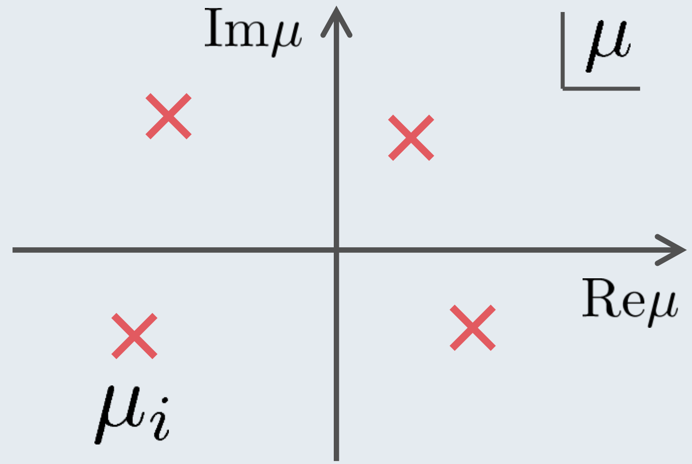
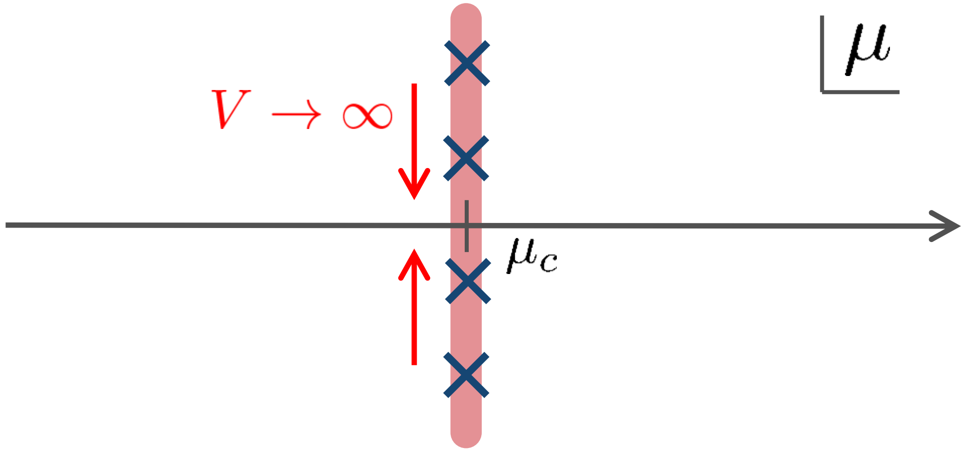


➤ **zeros** on the complex plane

=**Lee-Yang Zeros**

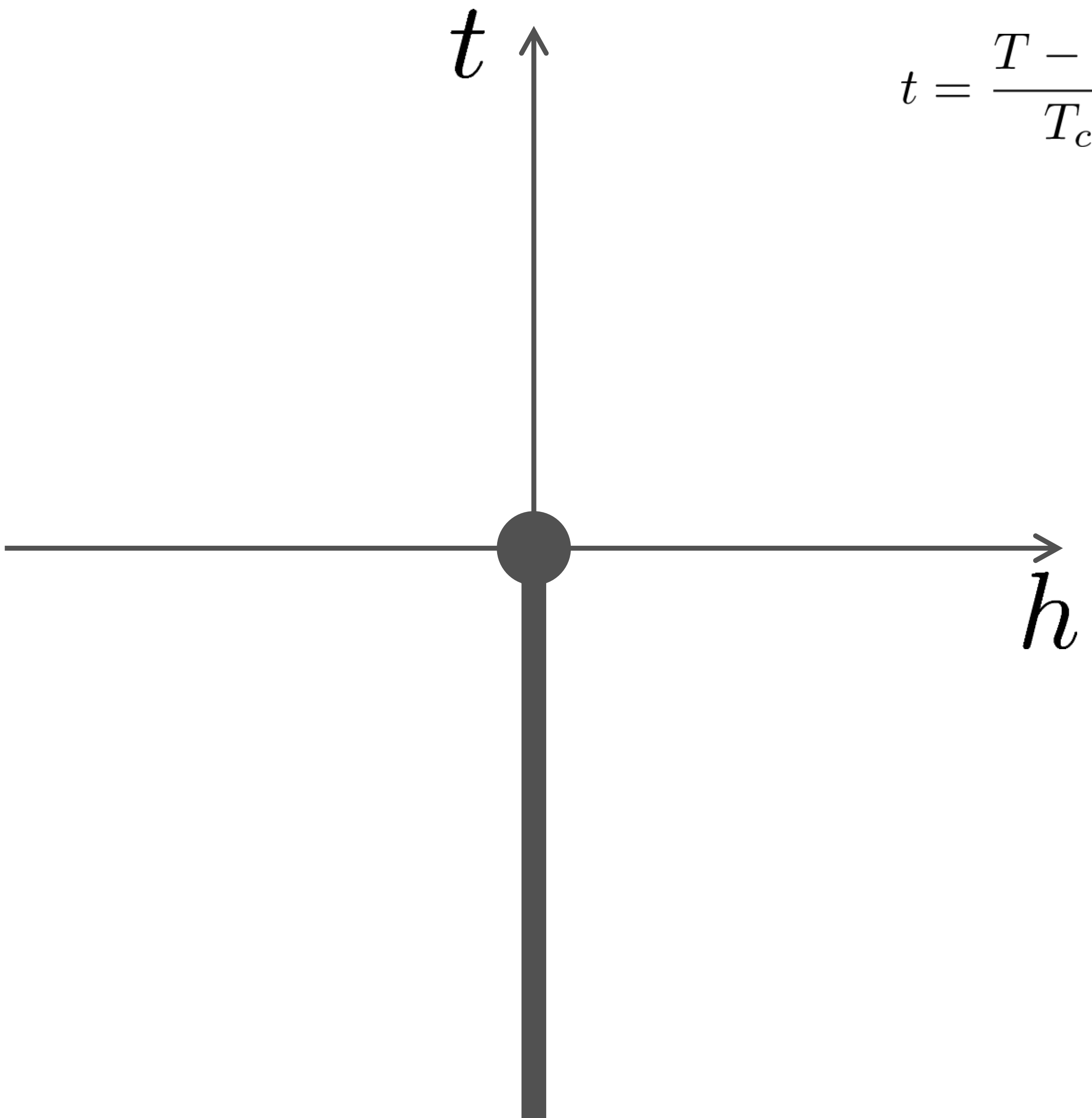
Phase Transition & LYZ

First-order transition
at $\mu=\mu_c$



— For $V\to\infty$, LYZs are accumulated on the line crossing the real axis at $\mu=\mu_c$.

LYZ around a Critical Point in Ising Model



$$t = \frac{T - T_c}{T_c}$$

1st-transition

singularity on the real h axis

Crossover

no singularity on the real axis

Note:
LYZ in complex-h plane are purely imaginary.

Lee-Yang, 1952

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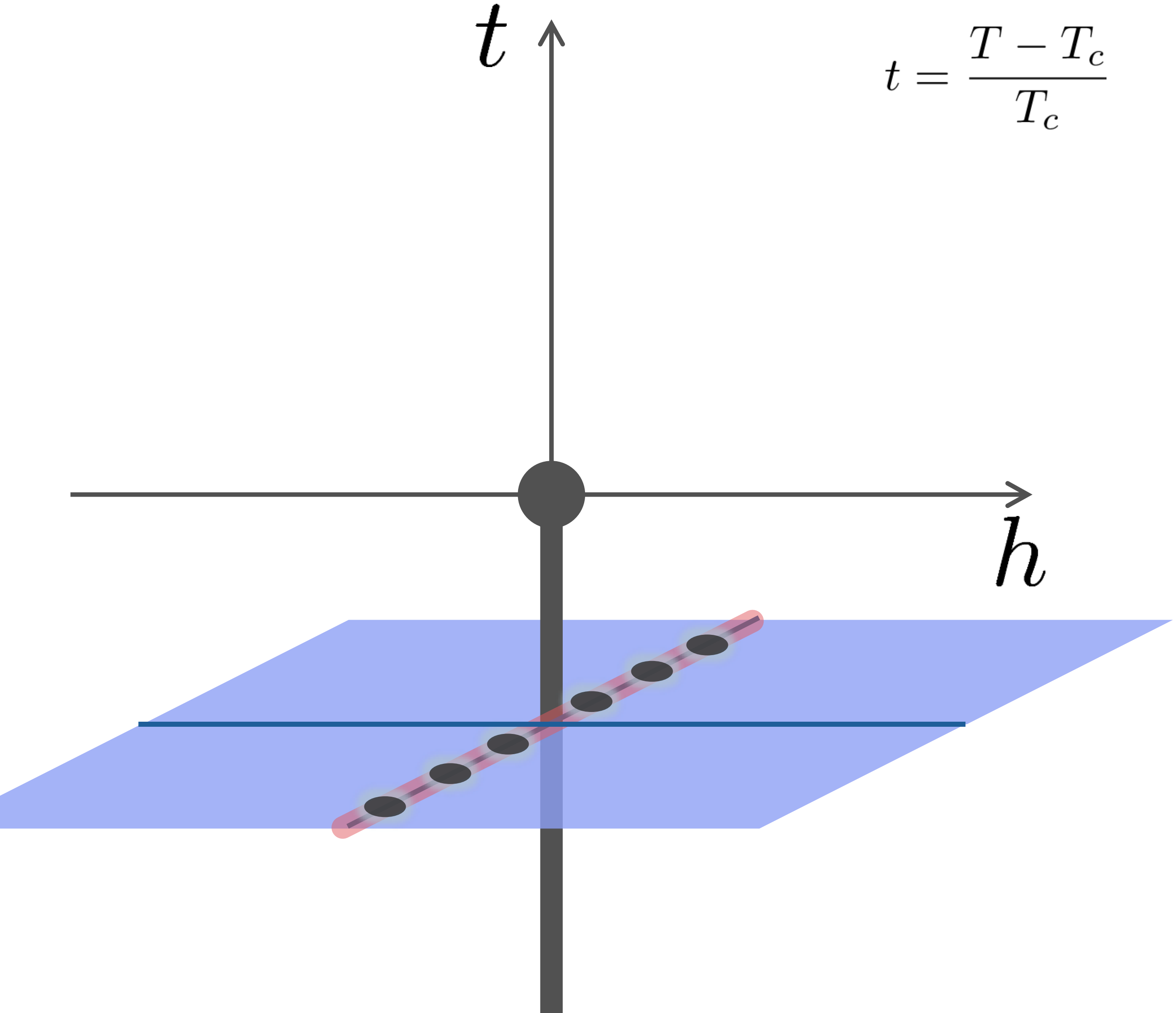

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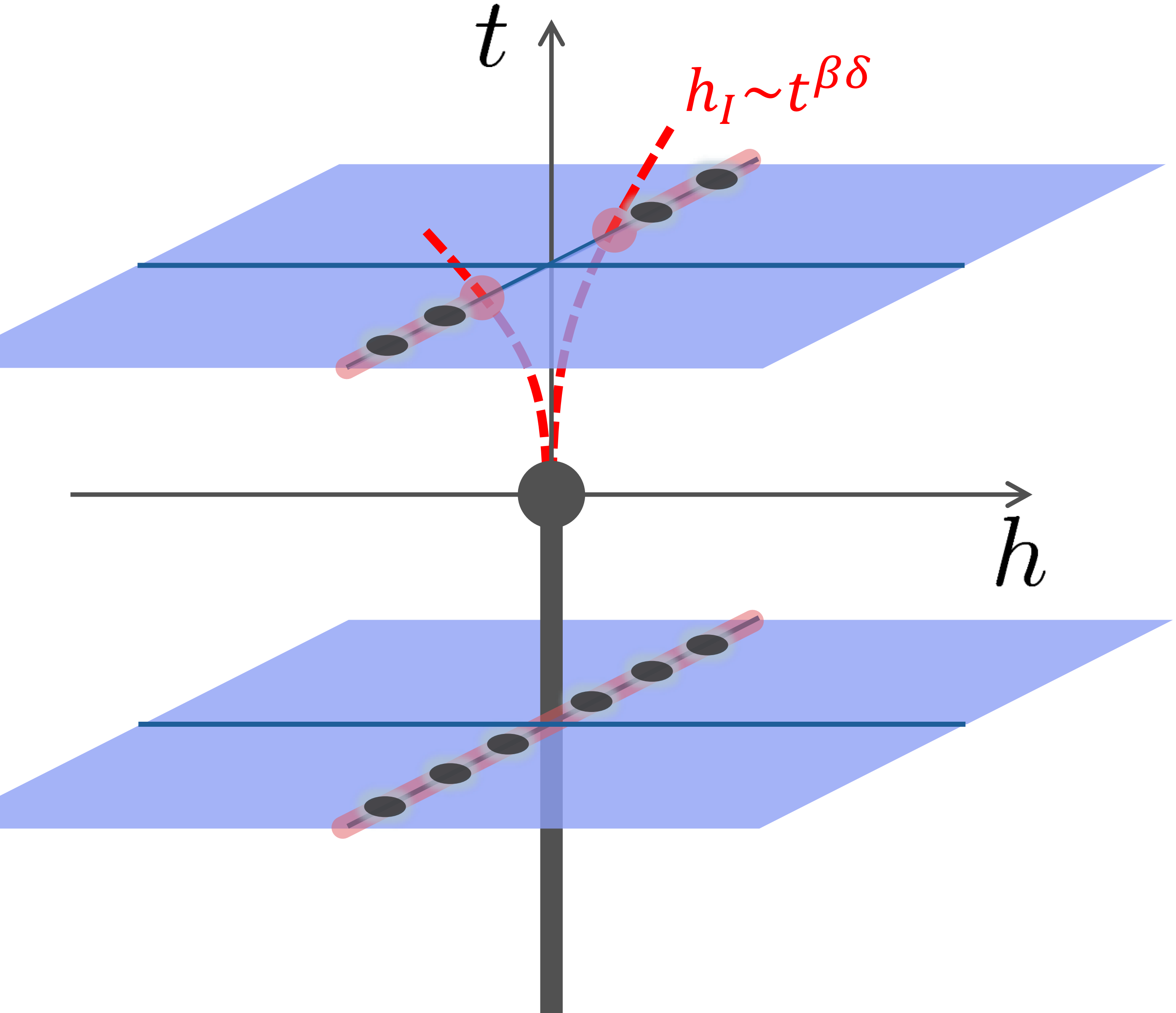

1st-transition

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LY edge singularity

Starting from the CP

Its behavior is governed by the the scaling function.

$$h_I \sim t^{\beta\delta}$$

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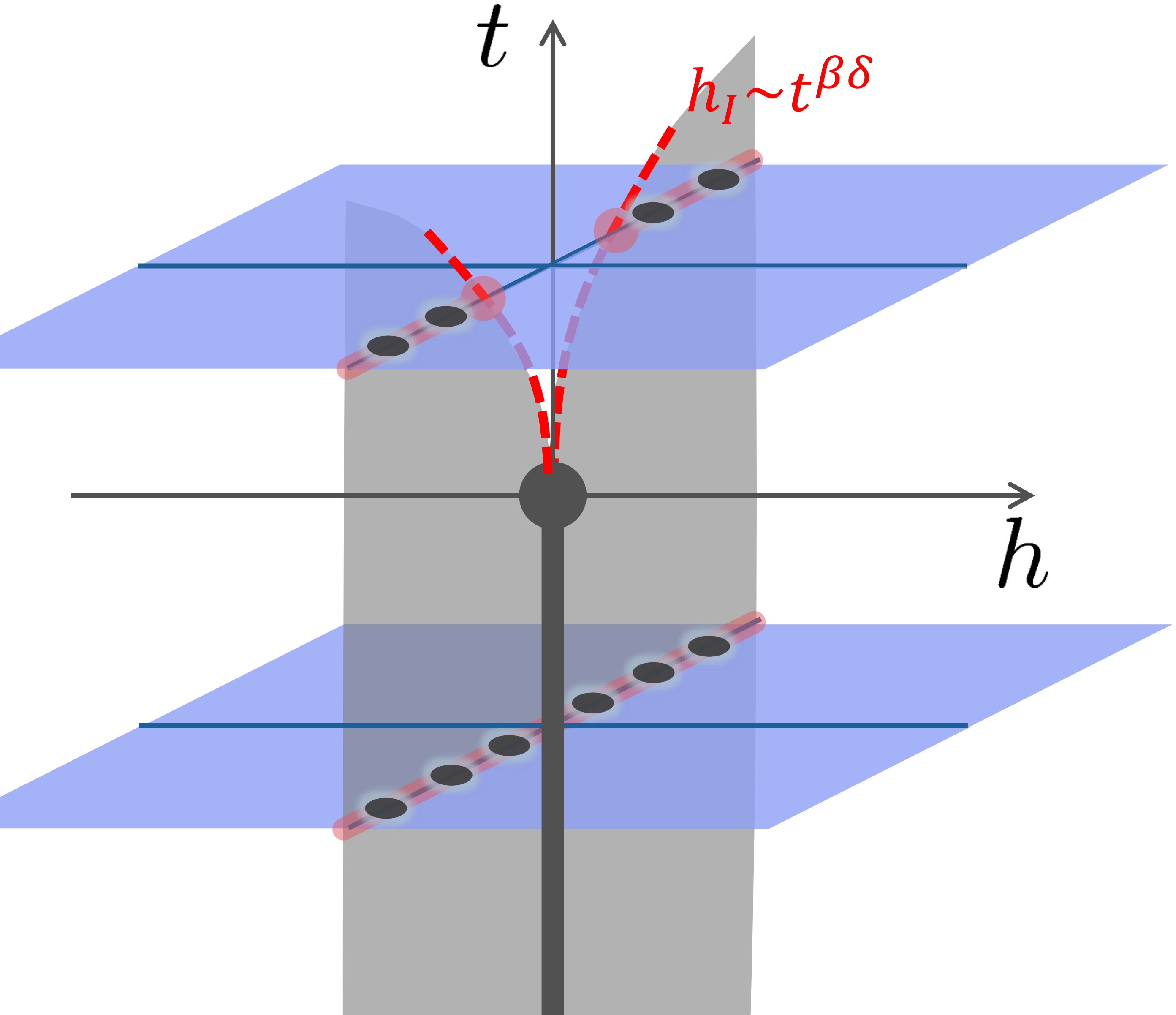

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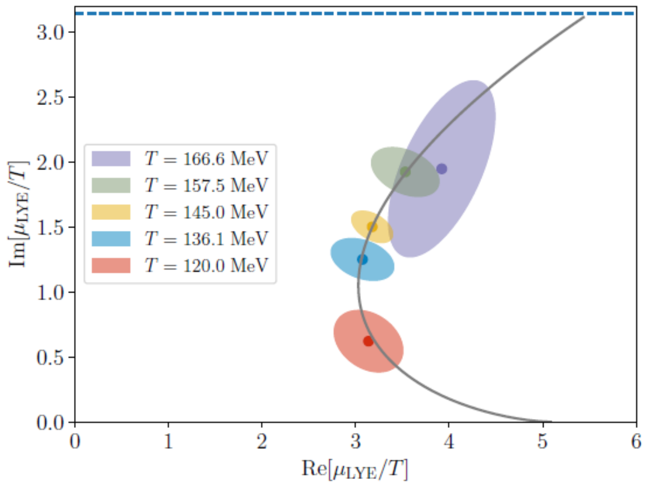
Recent Progress in LYZ/LYES and Lattice

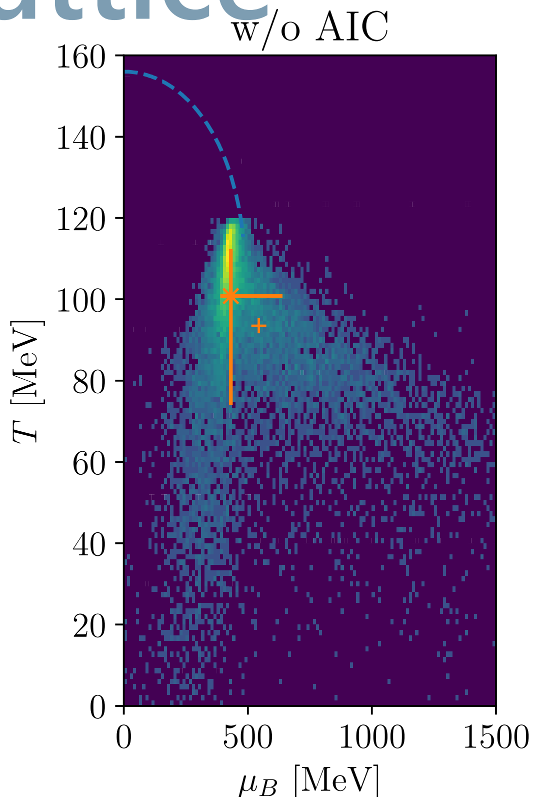
Analytic Structure

— Scaling functions, FRG, …

An, Mesterhazy, Stephanov ('16)
Johnson, Rennecke, Skokov ('23)
Karsch, Schmidt, Singh ('23)
…

Clarke+, 2405.10196

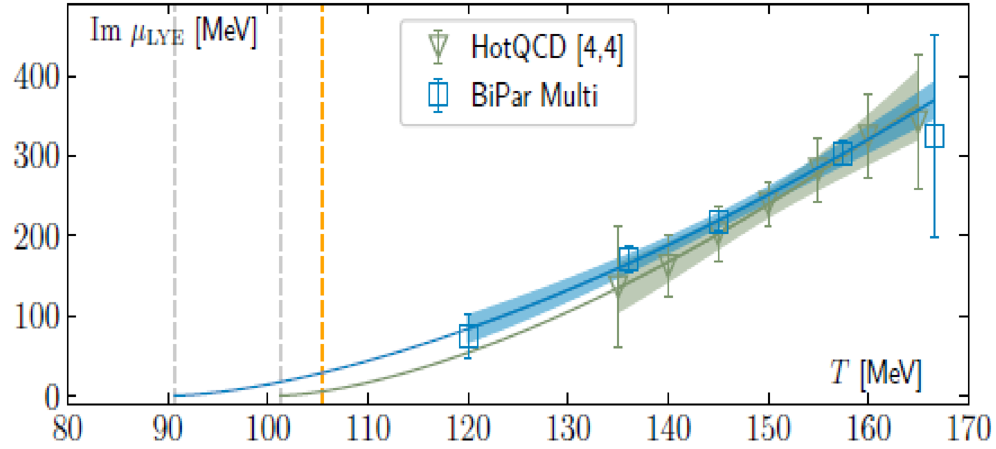




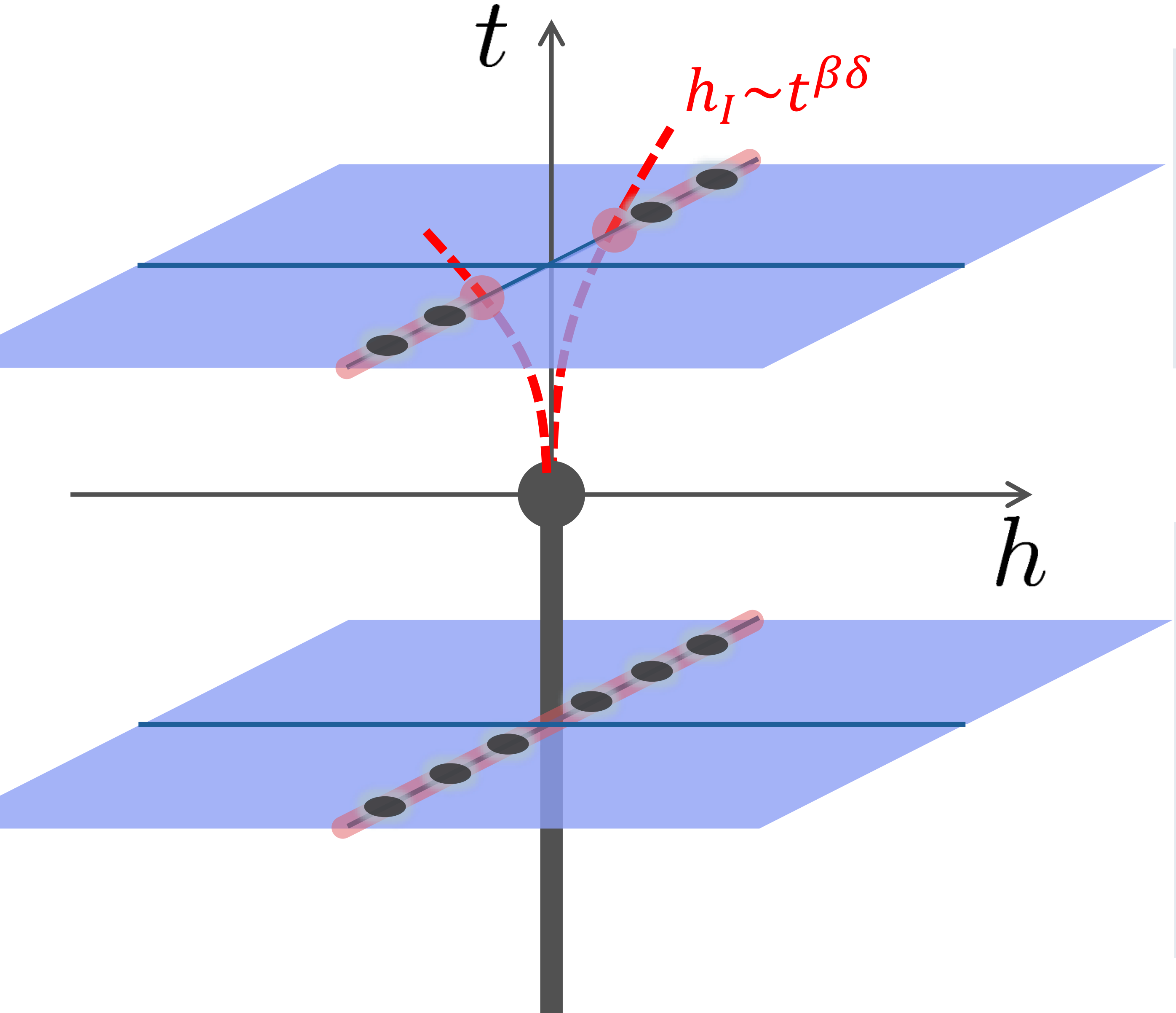
Locating QCD-CP at $\mu \neq 0$ on the lattice?

Clarke+, 2405.10196; Adam+, 2507.13254

— Taylor exp. + Imaginary μ + Pade approx.
— Identify the 1st LYZ to be LYES



Purpose of This Study



On finite volume,

1st LYZ ≠ LYES

Our Purpose

— Understand finite-volume effects of LYZ

— Exploit them for the CP searches

Finite-Size Scaling

Scaling Hypothesis

$$F_{\rm sing}(t, h, L^{-1}) = \tilde{F}_{\rm sing}(L^{y_t} t, L^{y_h} h)$$

$$Z_{\rm sing}(t, h, L^{-1}) = \tilde{Z}_{\rm sing}(L^{y_t} t, L^{y_h} h)$$

$$F = F_{\rm sing} + F_{\rm reg}$$

$$Z = Z_{\rm sing} \times Z_{\rm reg}$$

LYZ in the scaling region on finite volume

$$Z(t, h, L^{-1}) \sim \tilde{Z}_{\rm sing}(L^{y_t} t, L^{y_h} h) = 0$$

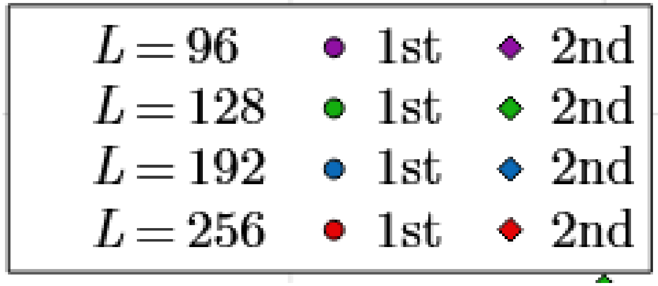
$$L^{y_h} h_{\rm LY}^{(n)}(t) = \tilde{h}_{\rm LY}^{(n)}(L^{y_t} t)$$

Itzykson, Nucl. Phys. 220 (1983)

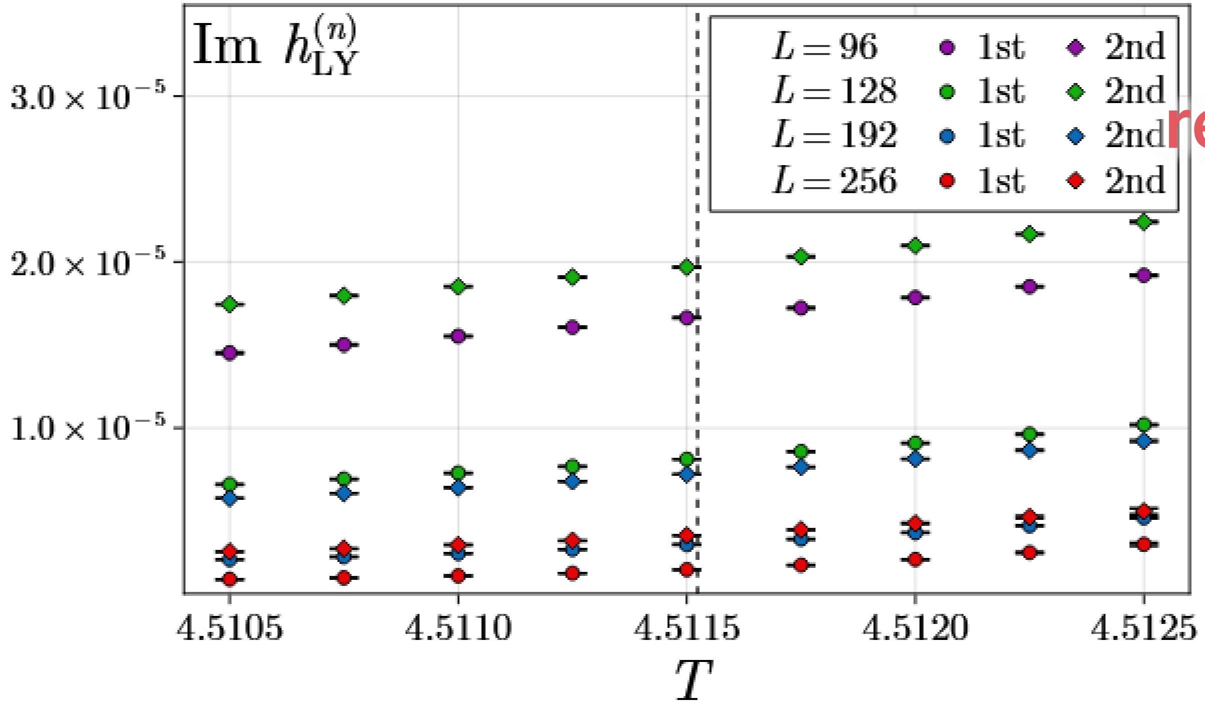
LYZ in 3d Ising Model

Wada, MK, Kanaya, arXiv:2508. 20422

$$L^{y_h} h_{\mathrm{LY}}^{(n)}(t) = \tilde{h}_{\mathrm{LY}}^{(n)}(L^{y_t} t)$$



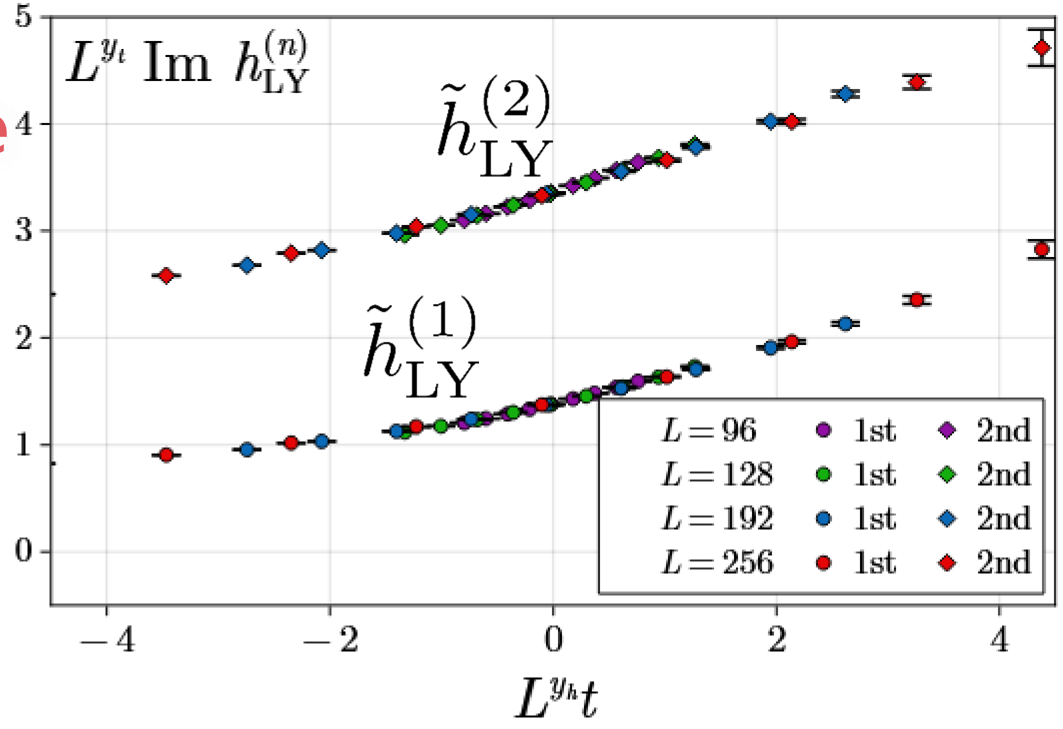
1st & 2nd LYZ



T_c: Ferrenberg ('18)

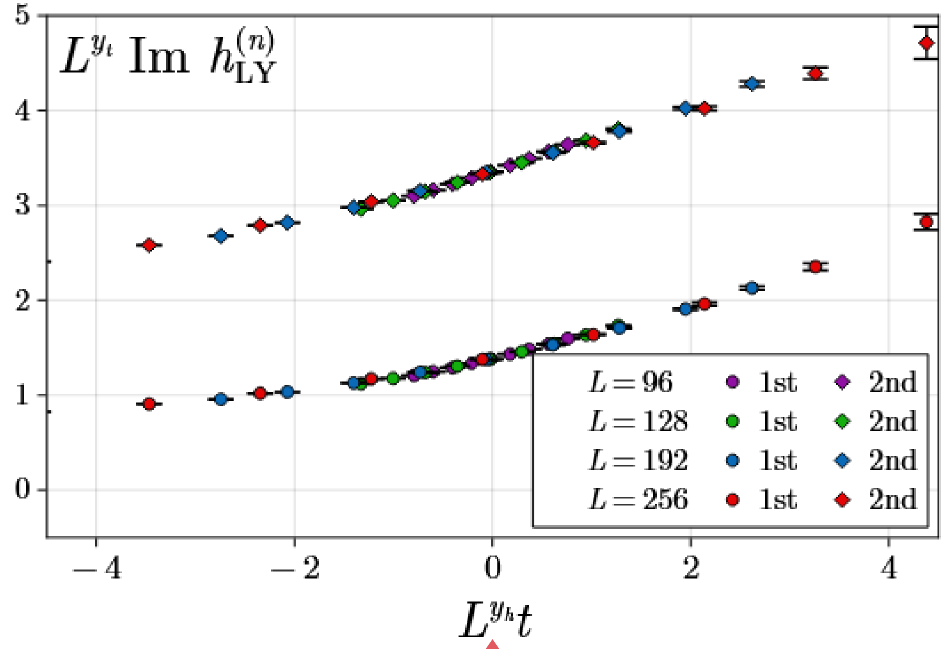
rescale

Rescaled



Where is QCD Critical Point?

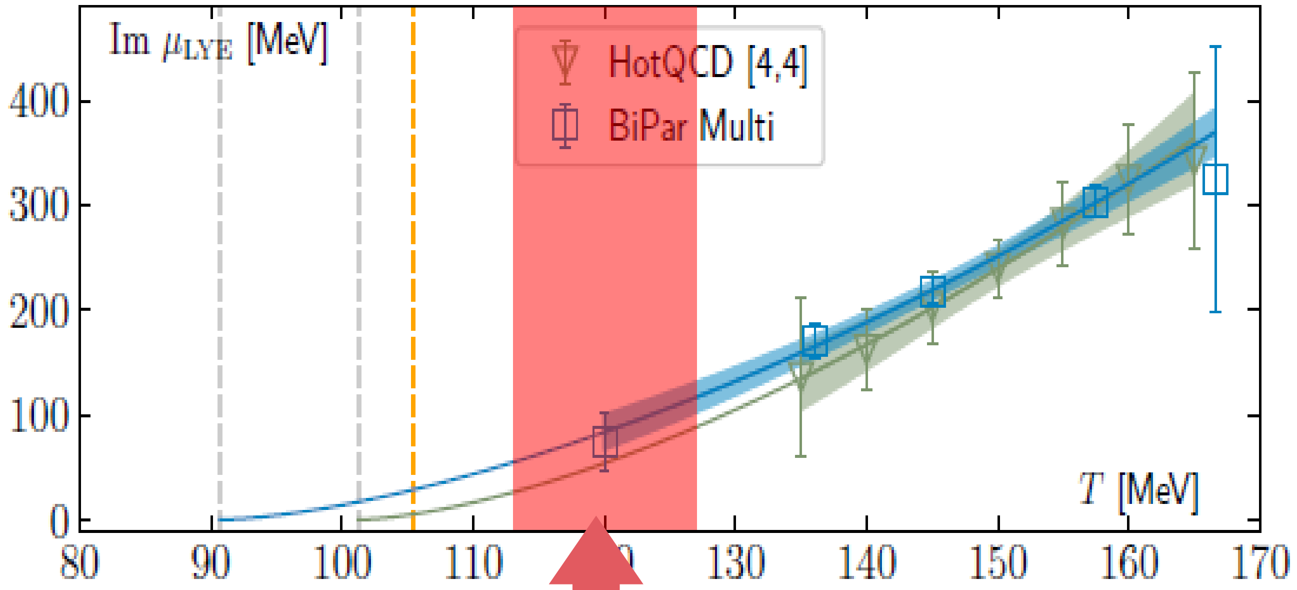
Ising model



$T = T_c$

LYZ in QCD

Clarke+, arXiv:2405.10196



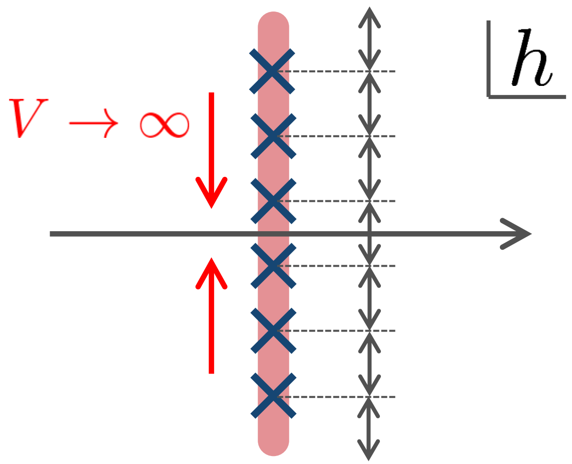
CP around here?

Wada, MK, Kanaya, PRL ('25)

Lee-Yang Zero Ratios

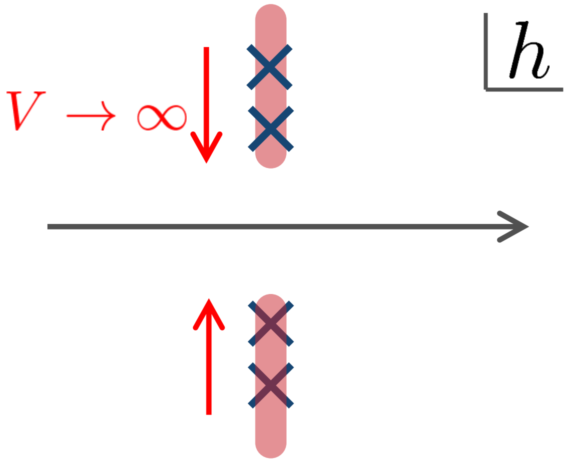
$$R_{nm}(t) = \frac{h^{(n)}(t)}{h^{(m)}(t)}$$

First-Order Side $(t < 0)$



$$R_{nm}(t) \xrightarrow[V\to\infty]{} \frac{2n-1}{2m-1}$$

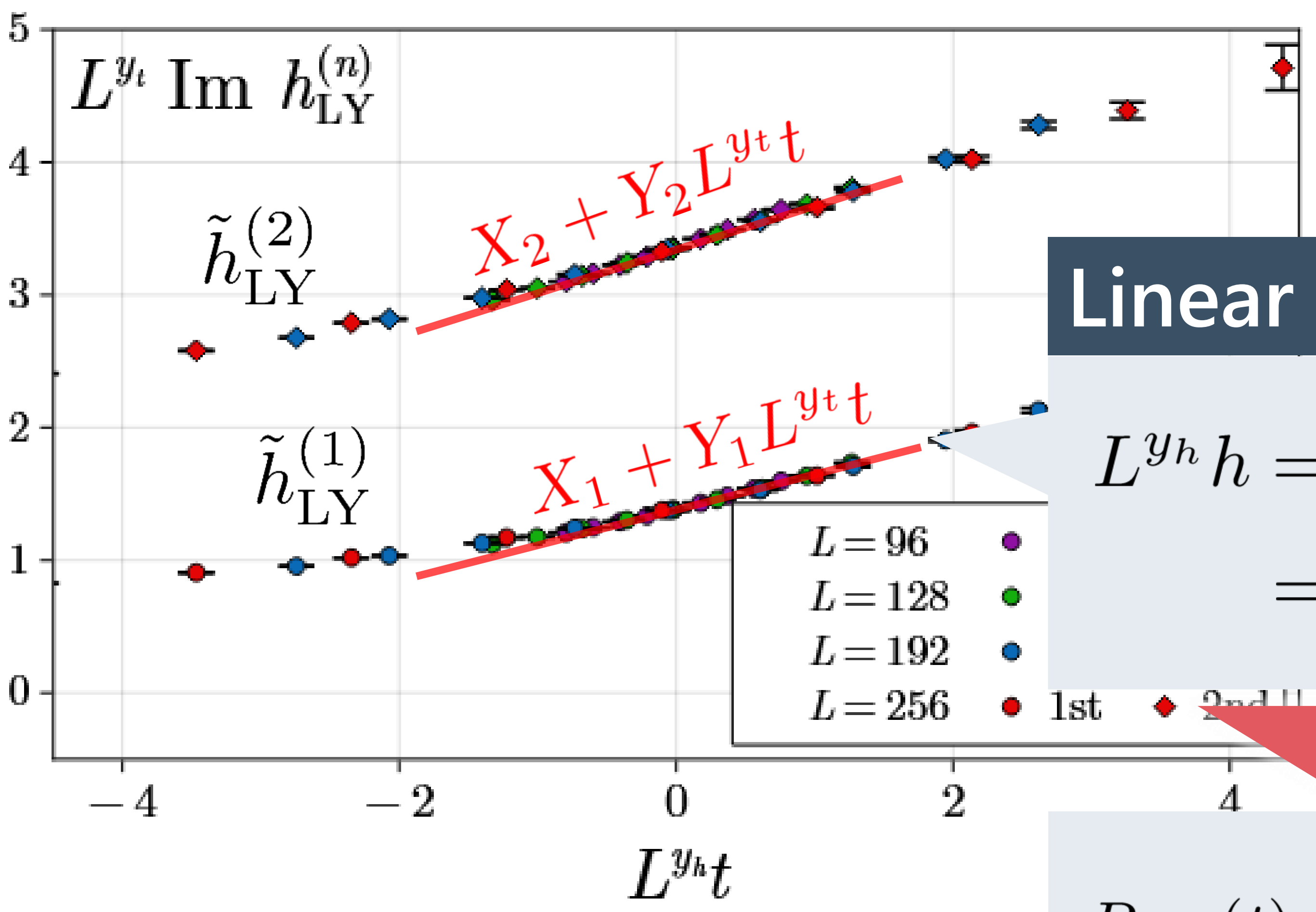
Crossover Side $(t > 0)$



$$R_{nm}(t) \xrightarrow[V\to\infty]{} 1$$

Wada, MK, Kanaya, PRL ('25)

LYZ near $t = 0$



Linear Approx. at $t = 0$

$$L^{y_h} h = \tilde{h}_{\rm LY}^{(i)}(L^{y_t} t)$$
$$= X_i + Y_i L^{y_t} t + \mathcal{O}(t^2)$$

$$R_{nm}(t) = \frac{X_n}{X_m}\left(1 + C_{nm} t L^{y_t} + \mathcal{O}(t^2)\right)$$

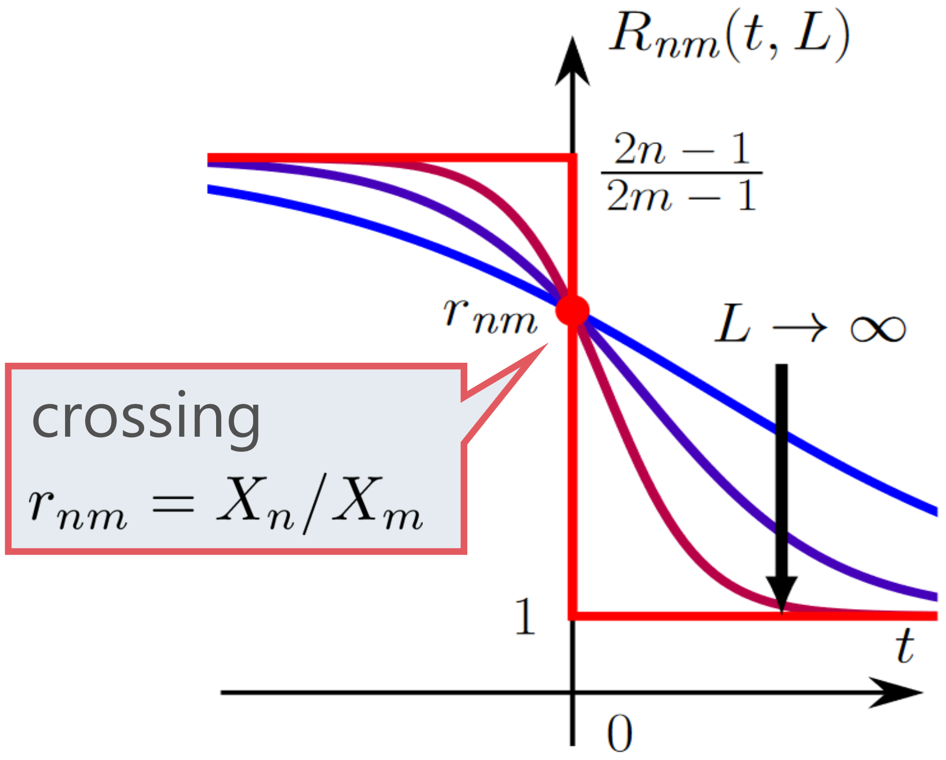
LYZ Ratios $R_{nm}(t) = \dfrac{h^{(n)}(t)}{h^{(m)}(t)}$

Wada, MK, Kanaya, PRL, '25

$$R_{n1}(t) \xrightarrow[V\to\infty]{} \begin{cases} 2n-1 & t<0 \ \text{(1st order)} \\ 1 & t>0 \ \text{(crossover)} \end{cases}$$

$$R_{nm}(t) = r_{nm}\left(1 + C_{nm} t L^{y_t} + \mathcal{O}(t^2)\right)$$

near $t = 0$



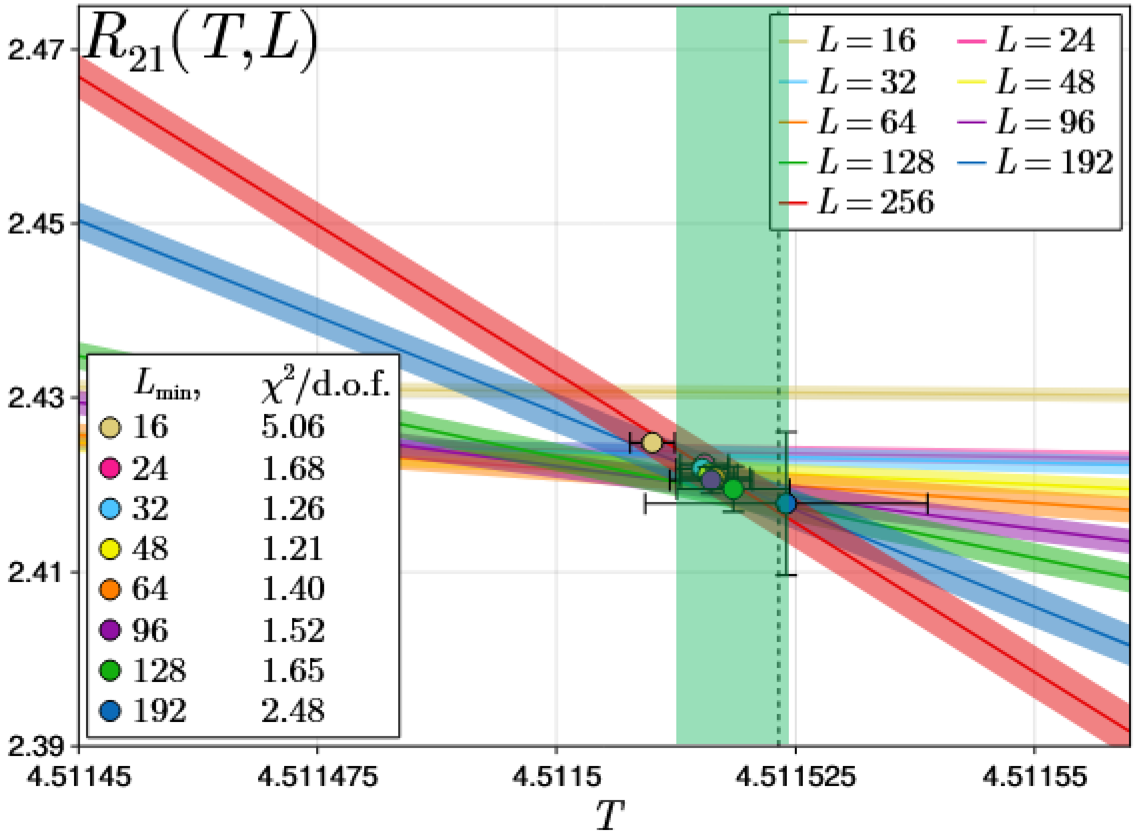
— $R(0)$ is L independent, the universal value.

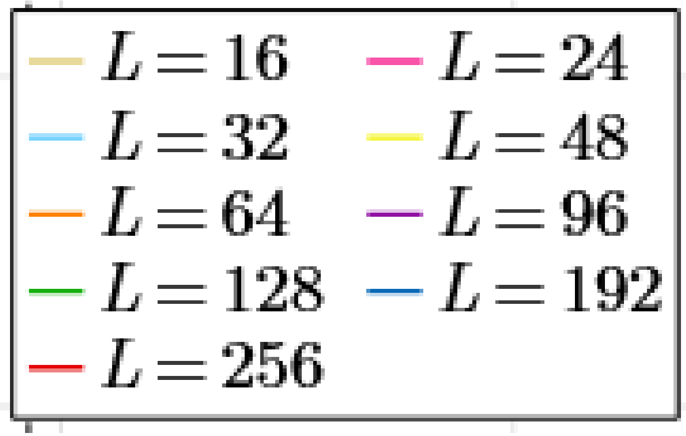
— Intersection point of various L gives the CP.

— Reminiscent of Binder-cumulant analysis

LYZR vs Binder Cumulant in 3d-Ising

LYZR

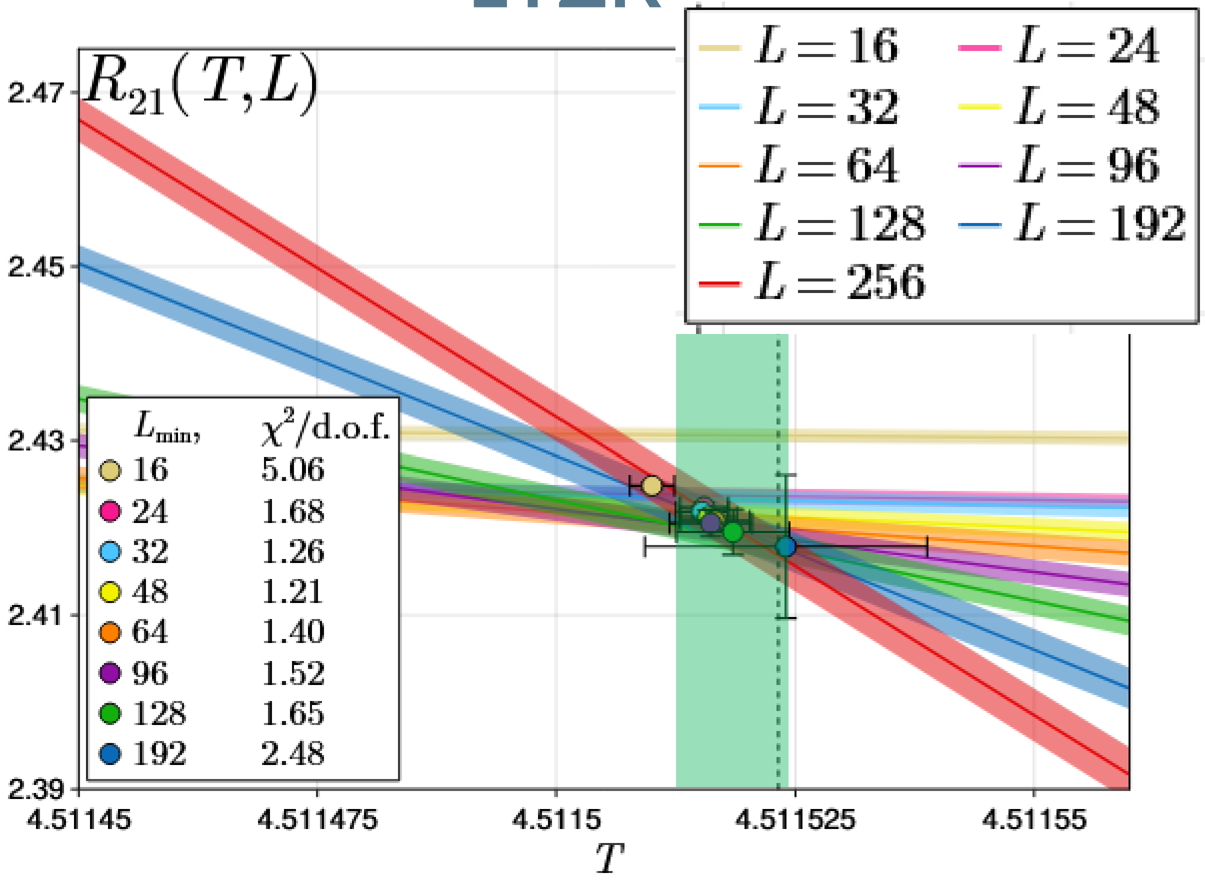




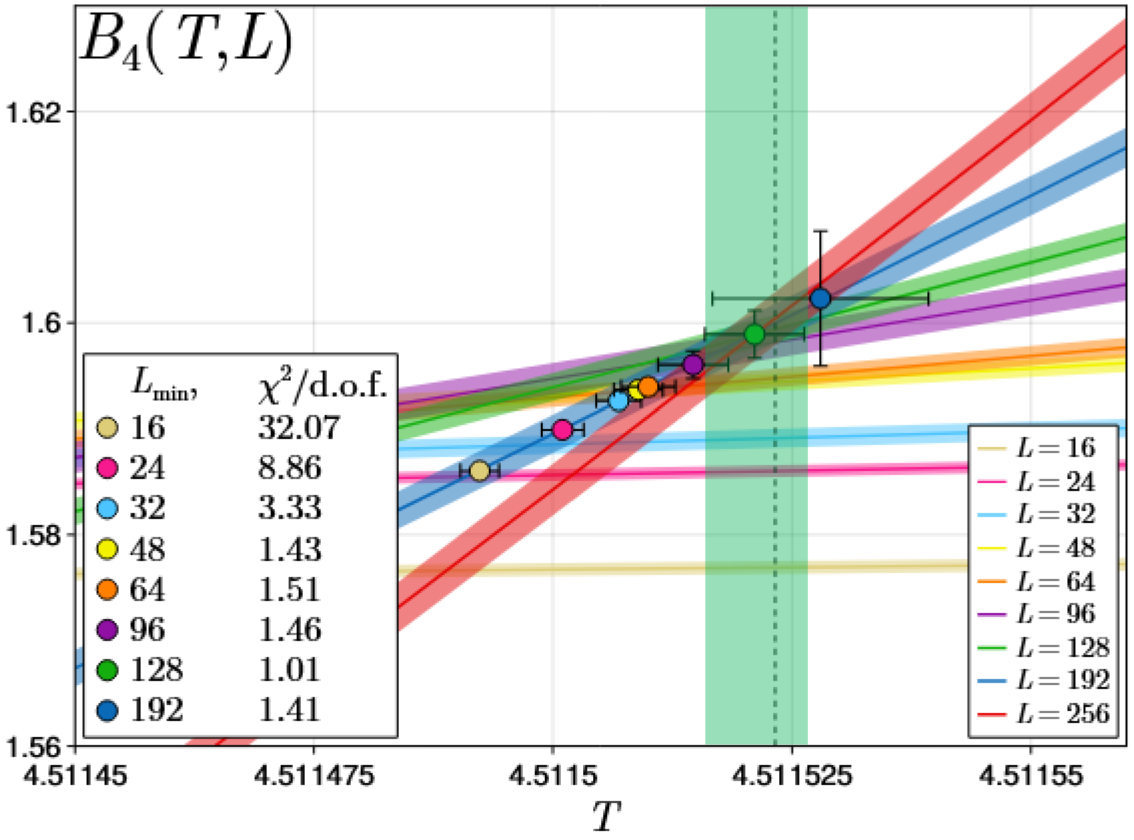
Wada, MK, Kanaya, arXiv:2508.20422

LYZR vs Binder Cumulant in 3d-Ising

LYZR



Binder cumulant

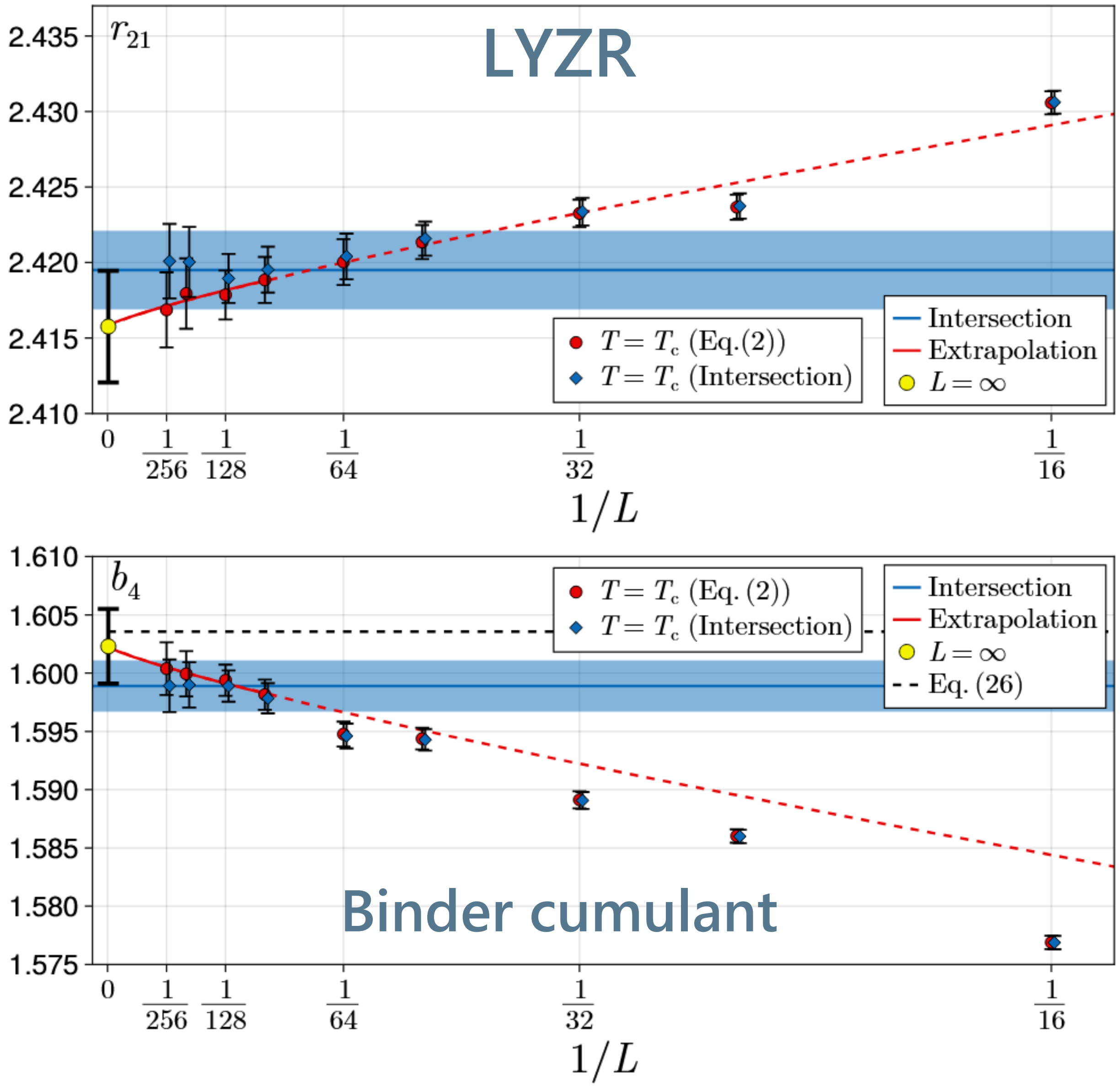


Faster convergence of the violation of FSS in LYZ?

Wada, MK, Kanaya, arXiv:2508.20422

Convergence at $T = T_c$

Wada, MK, Kanaya, arXiv:2508. 20422



Red: T_c of Ferrenberg ('18)
Blue: T_c of intersection point

$$R_{21}(0, L) = r_{21}(1 + cL^{-\omega})$$

In 3d-Ising,
violation of FSS is more quickly suppressed in the LYZR than the Binder cumulant for $L \to \infty$.

$$Z = Z_{\text{sing}} \times Z_{\text{reg}}$$

Curvature

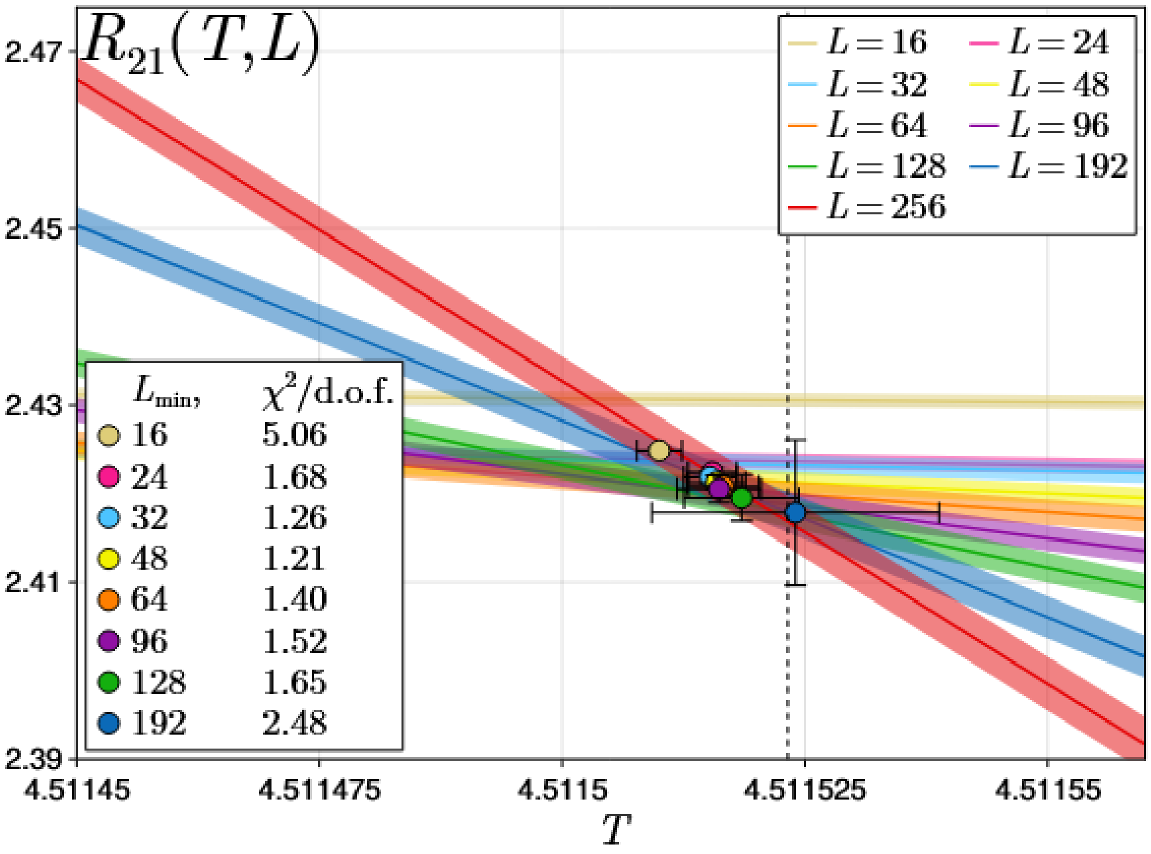
Wada, MK, Kanaya, arXiv:2508.20422

Our fits to $R_{nm}(t,L), B_4(t,L)$ need a non-linear term.

$$R_{n1}(T,L) = r_{n1} + c_{n1} L^{y_t}\left(\frac{T-T_c}{T_c}\right) + d_{n1} L^{2y_t}\left(\frac{T-T_c}{T_c}\right)^2$$

Measure of non-linearity

$$C_f = L^{-y_t}\frac{\partial^2 f/\partial T^2}{\partial f/\partial T}\bigg|_{T=T_c}$$



Curvature

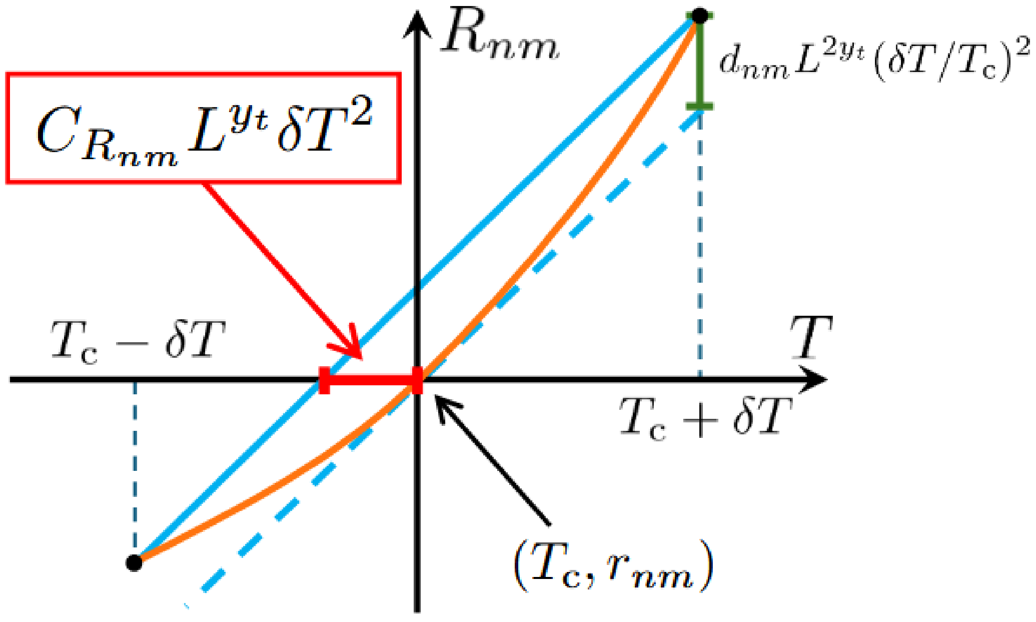
Wada, MK, Kanaya, arXiv:2508.20422

Measure of non-linearity

$$C_f = L^{-y_t} \frac{\partial^2 f/\partial T^2}{\partial f/\partial T}\bigg|_{T=T_c}$$

f	R_{21}	R_{31}	R_{41}	B_4
C_f	0.0093(17)	0.0053(17)	-0.0010(34)	0.0364(37)

f	$h_{\mathrm{LY}}^{(1)}$	$h_{\mathrm{LY}}^{(2)}$	$\langle M^2\rangle_c$
C_f	0.07149(59)	0.04230(101)	-0.07209(44)



C_f is invariant under

$$\begin{pmatrix} t' \\ h' \end{pmatrix} = \begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix} \begin{pmatrix} t \\ h \end{pmatrix}$$

Wada, MK, Kanaya, PRL ('25)

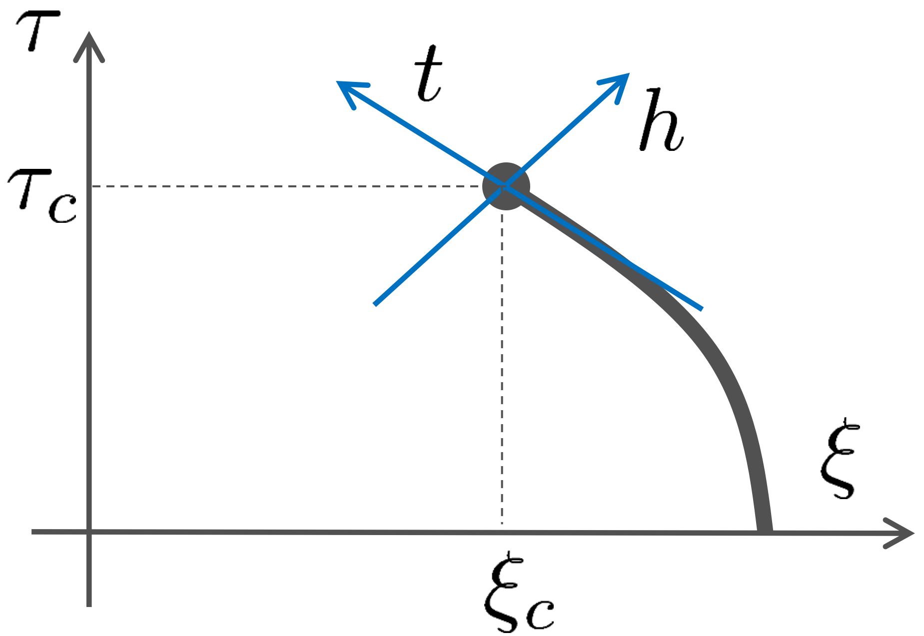
$$\bar{y} = y_t - y_h = -0.894$$

CP in a General System

- CP on a $\tau - \xi$ plane
- **LYZ on the complex ξ plane**

$$\begin{pmatrix} t \\ h \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} \tau - \tau_c \\ \xi - \xi_c \end{pmatrix} = A \begin{pmatrix} \delta\tau \\ \delta\xi \end{pmatrix}$$

$$L^{y_h} h^{(n)}(t) \simeq X_n + Y_n L^{y_t} t$$



$$Z_{\rm sing}(\tau, \xi) = \tilde{Z}_{\rm Ising}\left(t(\tau, \xi), \xi(\tau, \xi)\right)$$

CP in a General System

Wada, MK, Kanaya, PRL ('25)

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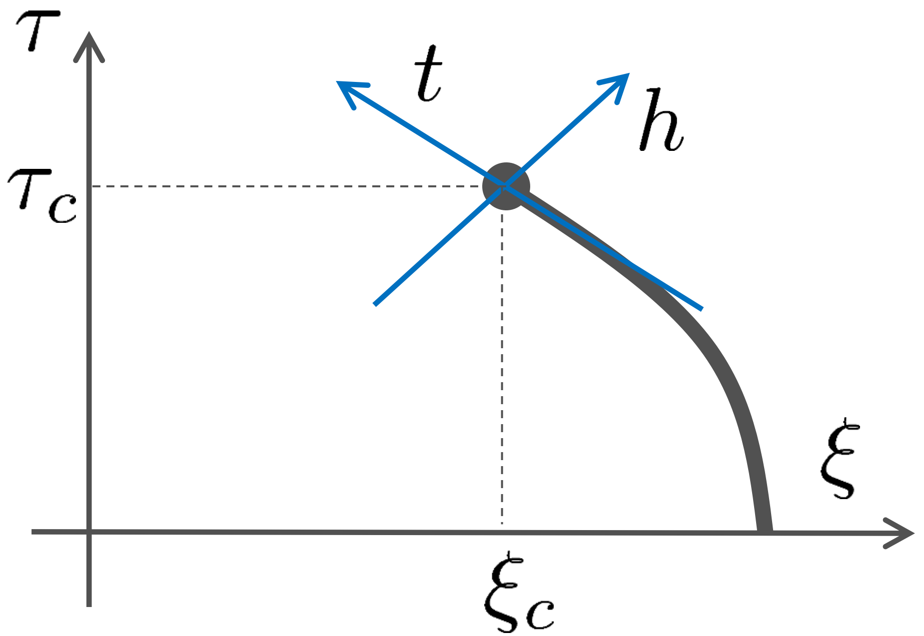


$$\begin{cases} \xi_{\mathrm{R}}^{(n)} = \xi_c - \dfrac{a_{21}}{a_{22}} \delta\tau + \mathcal{O}(L^{2\bar{y}}) \\ \xi_{\mathrm{I}}^{(n)} = \dfrac{X_n}{a_{22}} L^{-y_h} + \dfrac{\det AY_n}{a_{22}^2} \delta\tau L^{\bar{y}} + \mathcal{O}(L^{2\bar{y}}) \end{cases}$$

$L \to \infty$

generalization to finite V

LY Edge Singularity

$$\begin{cases} \mathrm{Re}\xi_{\mathrm{LYES}} \simeq c_1 \tau \\ \mathrm{Im}\xi_{\mathrm{LYES}} \simeq c_2 \tau^{\beta\delta} \end{cases}$$

Stephanov, 2006

LYZ Ratios for General CP

Wada, MK, Kanaya, PRL ('25)

$\bar{y} = y_t - y_h = -0.894$

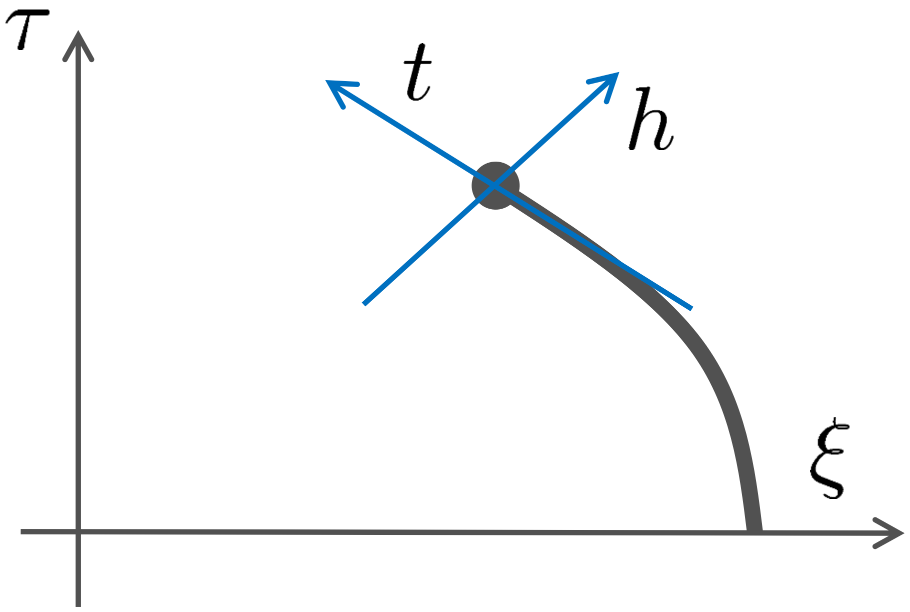
LYZ Ratio

$$R_{nm}(\tau) = \frac{\xi_{\rm I}^{(n)}(\tau)}{\xi_{\rm I}^{(m)}(\tau)} = r_{nm}\left(1 + CL^{y_t}\delta\tau + \mathcal{O}(\delta\tau^2)\right)\left(1 + DL^{2\bar{y}} + \mathcal{O}(L^{4\bar{y}})\right)$$

nonzero for $a_{12} \neq 0$

$$r_{nm} = \frac{X_n}{X_m}, \quad C = \frac{\det A}{a_{22}}\left(\frac{Y_2}{X_2} - \frac{Y_1}{X_1}\right), \quad D = \frac{a_{12}^2}{a_{22}^2}(Y_1^2 - Y_2^2)$$

$$\begin{pmatrix} t \\ h \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} \tau - \tau_c \\ \xi - \xi_c \end{pmatrix} = A \begin{pmatrix} \delta\tau \\ \delta\xi \end{pmatrix}$$



r_{nm} are universal constants specific to universality classes.

LYZ Ratios vs Binder Cumulant

Wada, MK, Kanaya, PRL ('25)

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Binder cumulant

Jin+, PRD**86**, 2017

$$B_4(t) = b_4\left(1 + c\tau L^{y_t} + \mathcal{O}(t^2)\right)\left(1 + dL^{\bar{y}} + \mathcal{O}(L^{2\bar{y}})\right)$$

nonzero for $a_{12} \neq 0$

r_{nm} are universal constants specific to universality classes.

Deviation at $t = 0$ due to $a_{12} \neq 0$ converges faster in LYZ ratio.

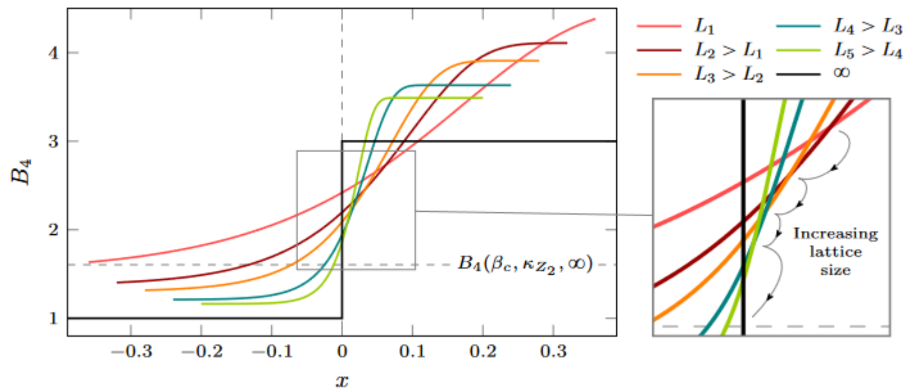
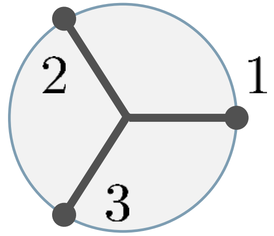


fig from Cuteri+, PRD ('21)

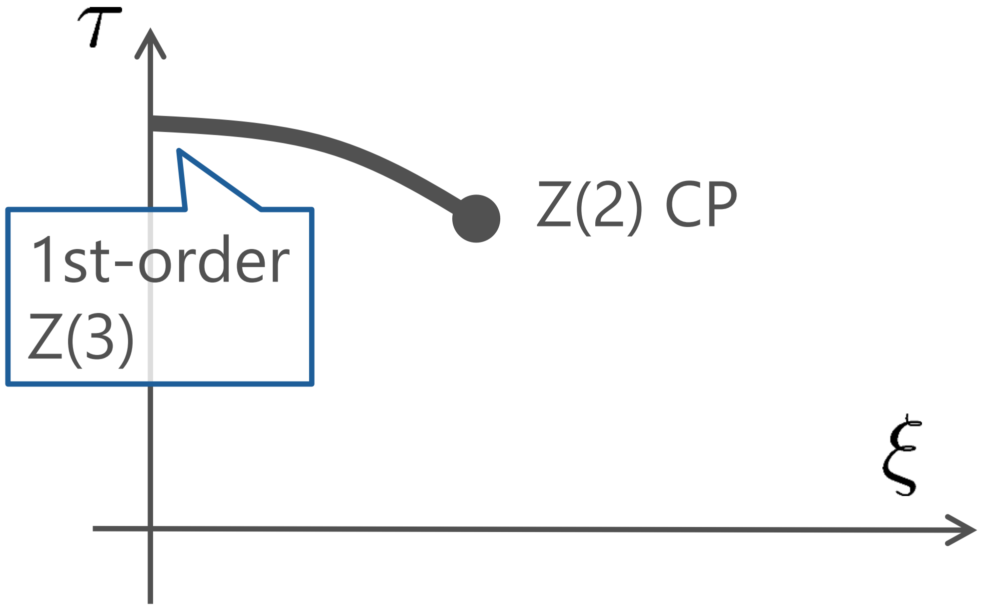
Numerical Analysis: 3d 3-State Potts Model

$$H = -\tau\sum_{\langle i,j\rangle}\delta_{\sigma_i,\sigma_j} - \xi\sum_i \delta_{\sigma_i,1} \quad \sigma_i = 1,2,3$$

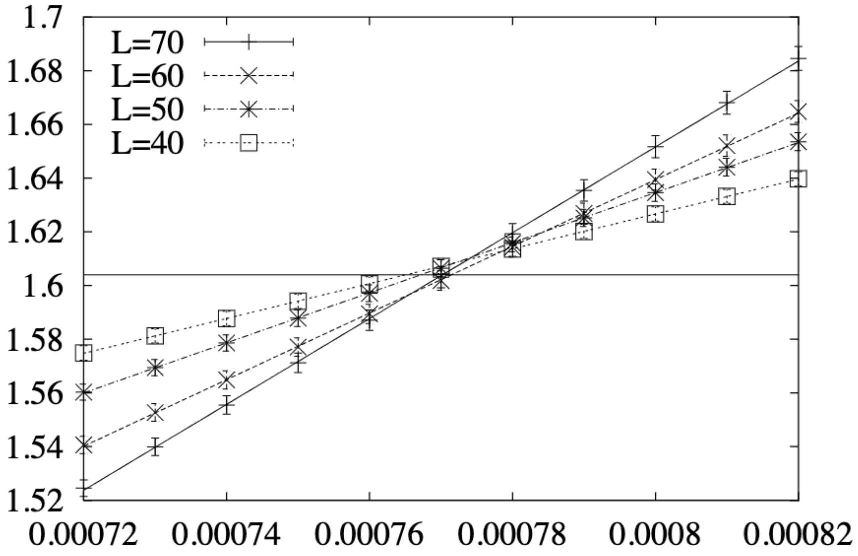
Monte-Carlo + reweighting

$\sigma_i =$ 

Phase Diagram



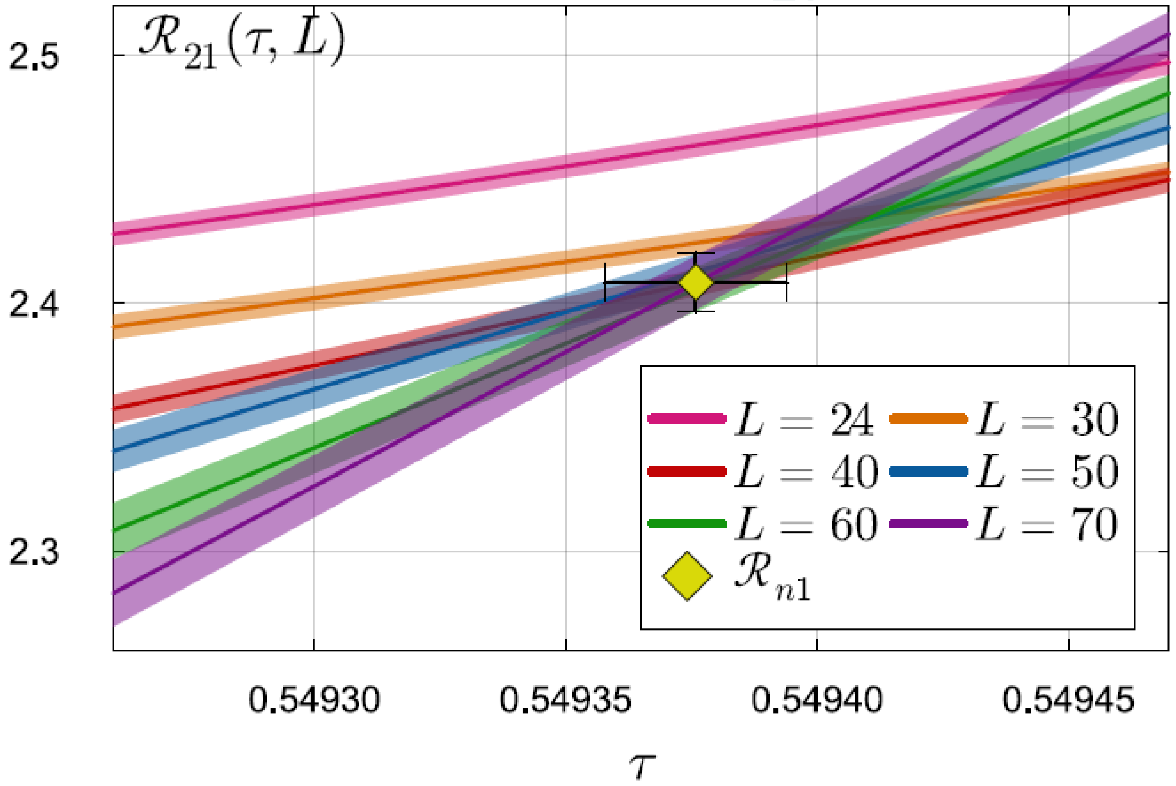
Binder-Cumulant Analysis



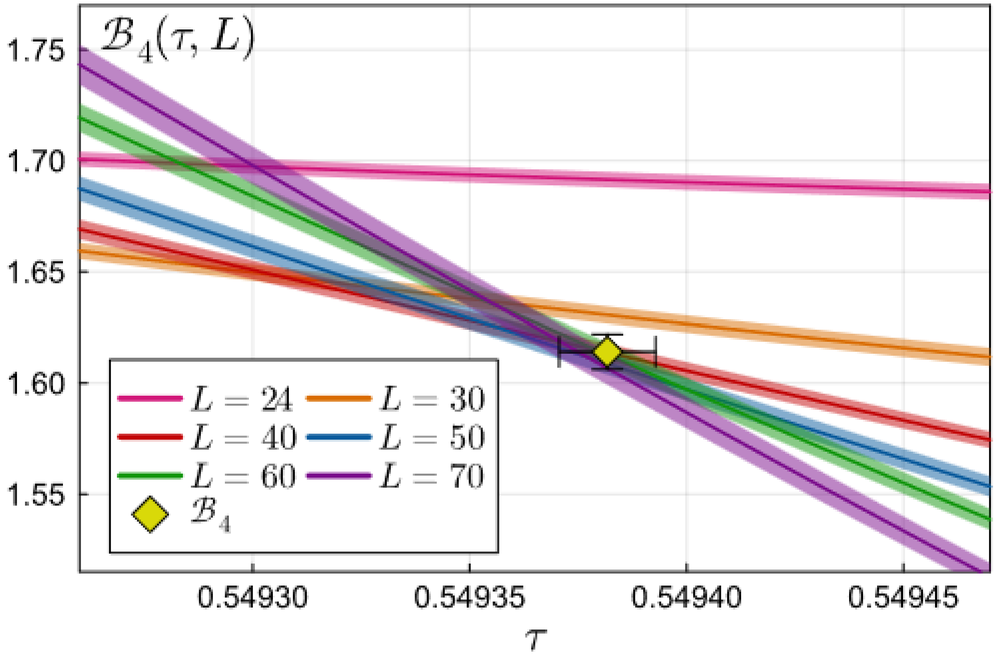
Karsch, Stickan, 2000

3d 3-State Potts Model: LYZ Ratio

LYZ Ratio (R_{21})



Binder Cumulant



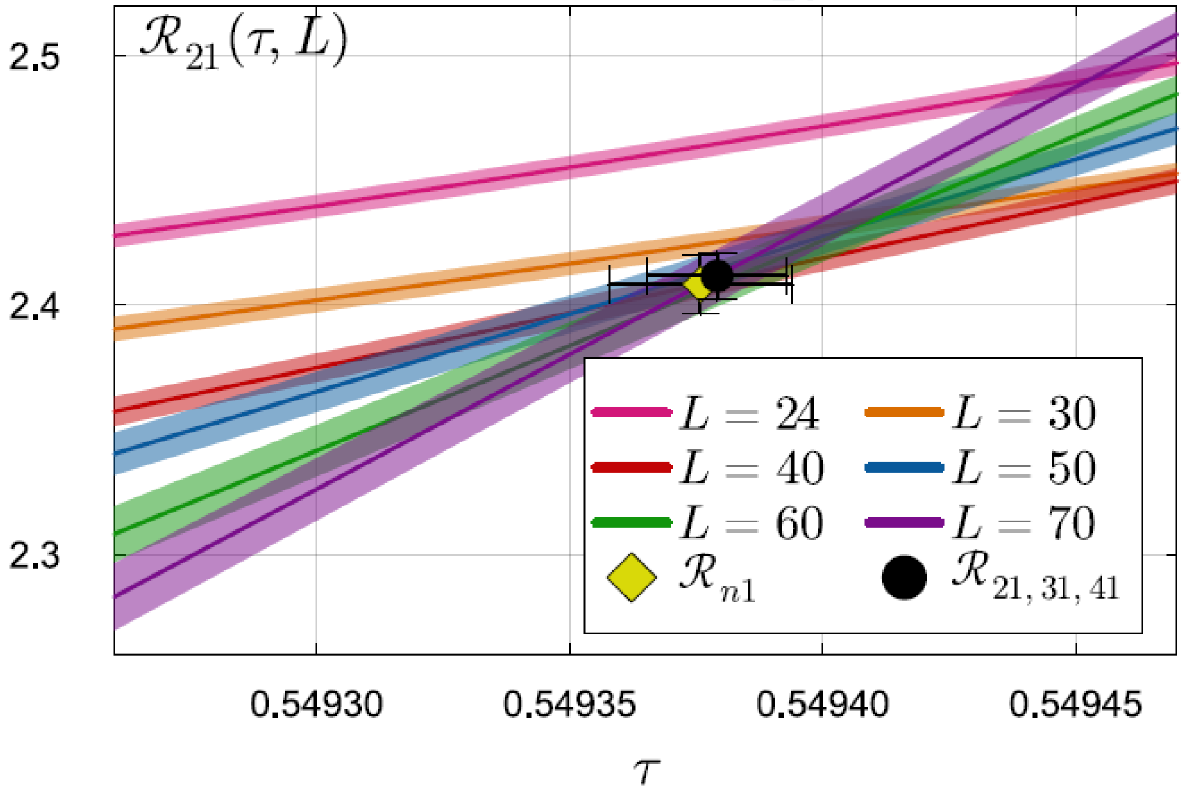
Wada, MK, Kanaya, PRL, '25

fit data	τ_c	y_t	r_{n1} or b_4	χ^2/dof
$\mathcal{R}_{21}$	0.549375(18)	1.53(19)	2.408(12)	0.38
$\mathcal{B}_4$	0.549382(11)	1.63(13)	1.614(8)	0.69

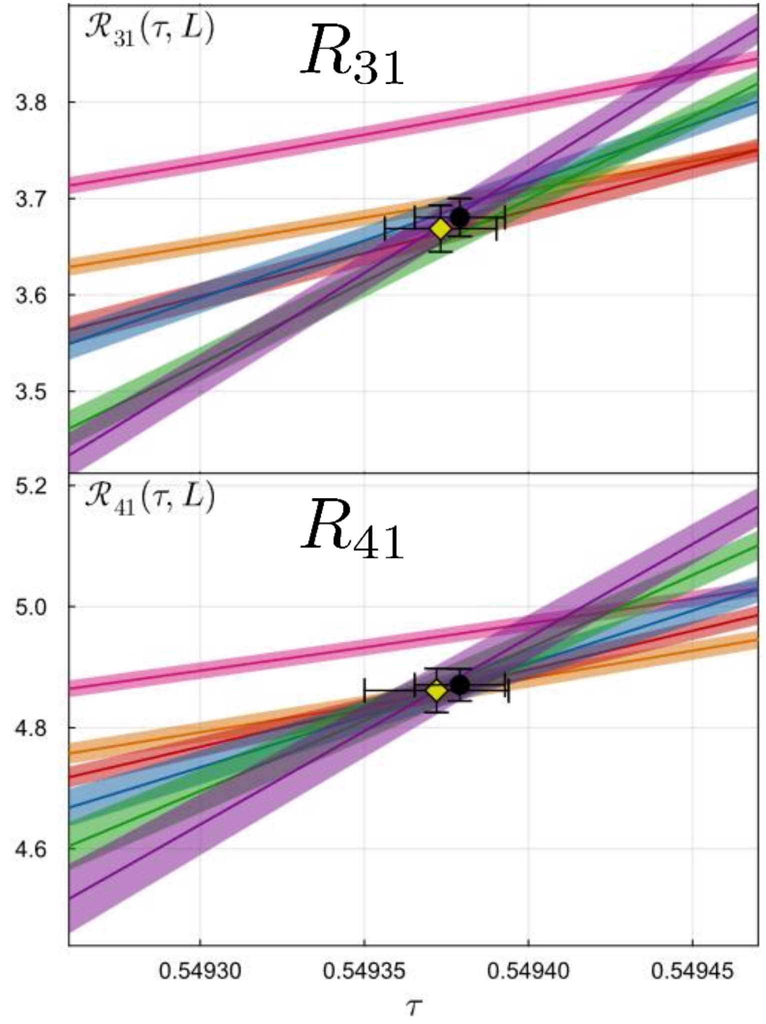
R_{n1} and B_4 give the same value of τ_c within statistics.

3d 3-State Potts Model: LYZ Ratio

LYZ Ratio (R_{21})



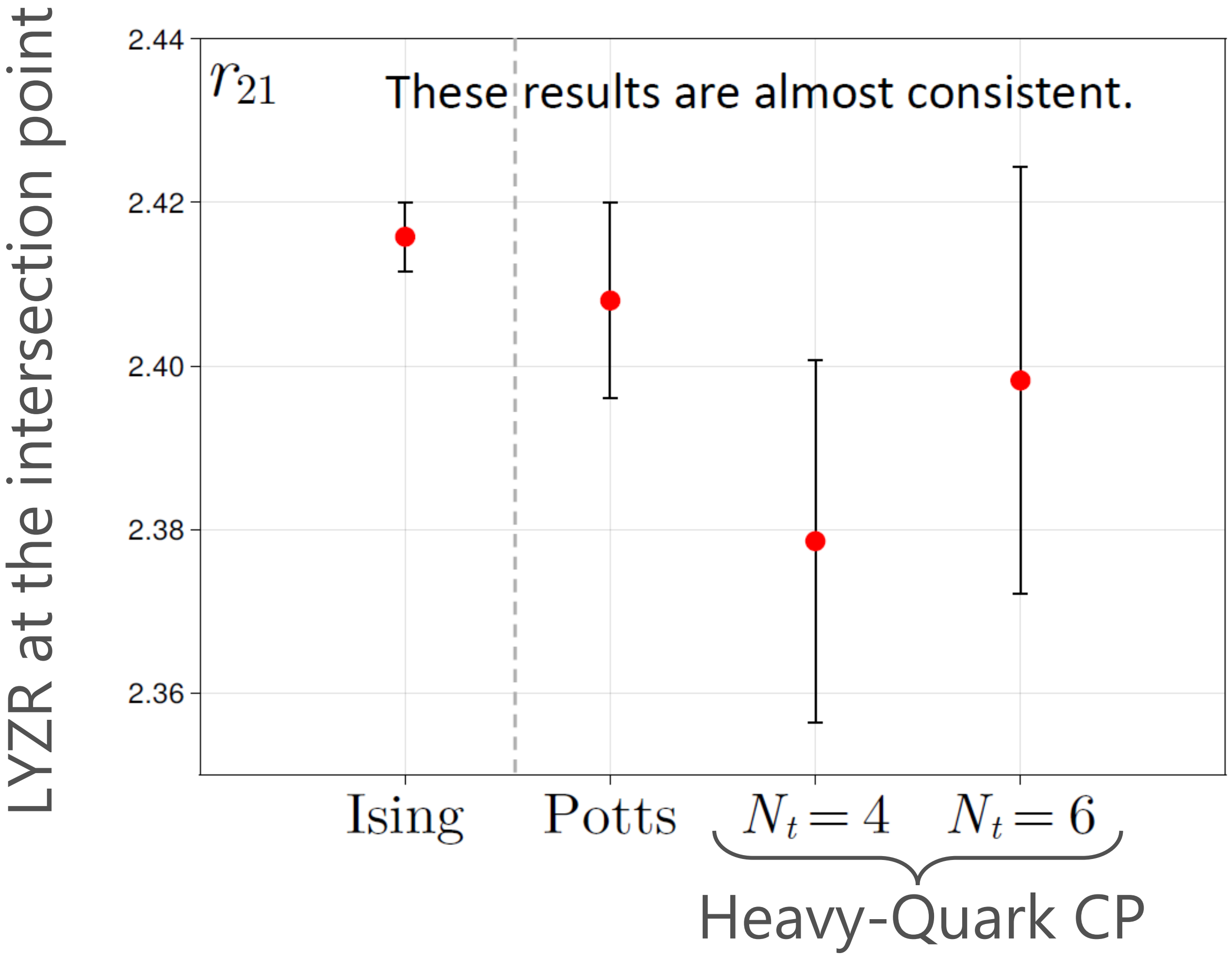
fit data	τ_c	y_t	r_{n1} or b_4	χ^2/dof
$\mathcal{R}_{21}$	0.549375(18)	1.53(19)	2.408(12)	0.38
$\mathcal{R}_{21,31,41}$	0.549379(14)	1.70(16)	–	0.56
$\mathcal{B}_4$	0.549382(11)	1.63(13)	1.614(8)	0.69



Combined use of LYZ can improve statistics!

Wada, MK, Kanaya, PRL, '25

LYZR in Various Models



CPs in 3d-Z(2) universality class

—**Ising**: 2508.20422
—**Potts**: PRL, ’25
—**HQ-CP**: 2501.18904

Consistent within statistics ➤ Confirmation of universality

Single-LYZ Method

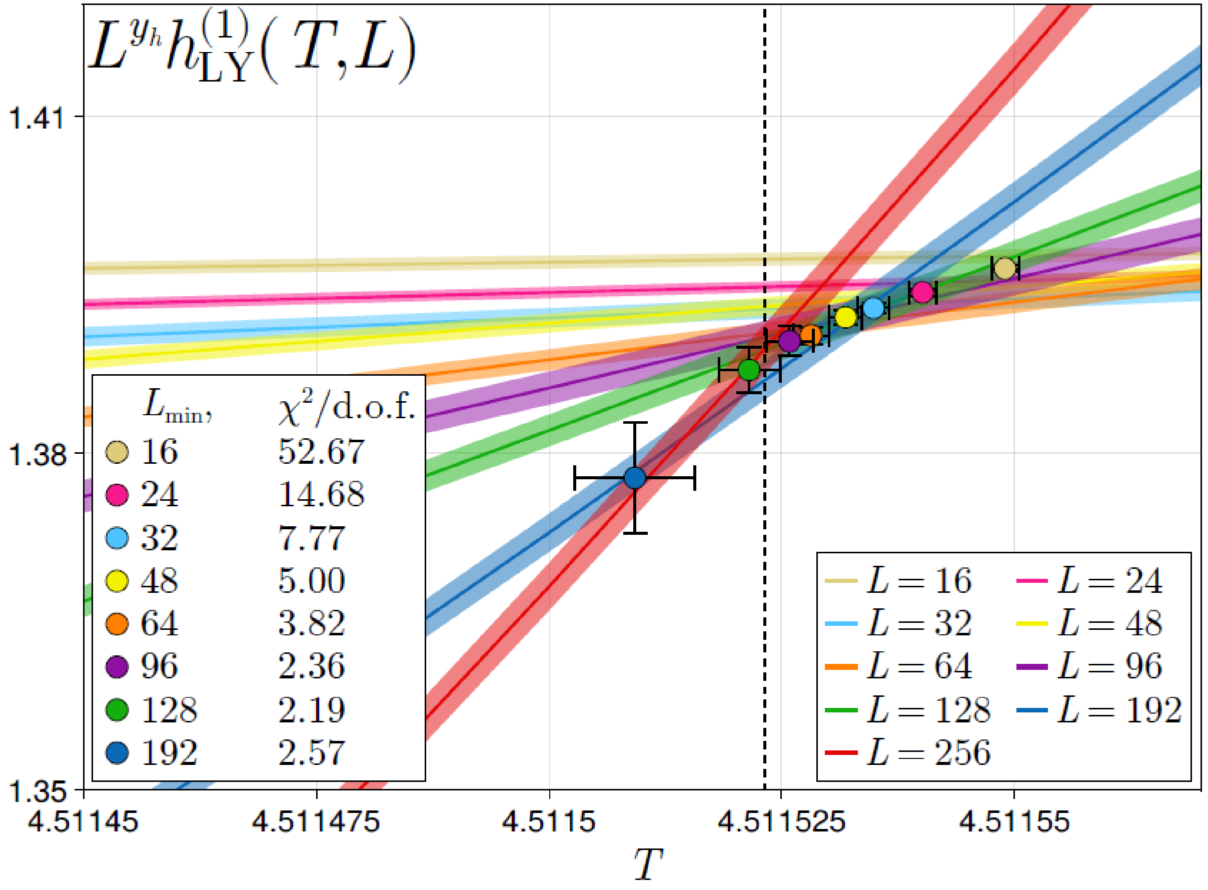
Wada, MK, Kanaya, arXiv:2508. 20422

$$L^{y_h} h^{(n)} = \tilde{h}^{(n)}_{\rm LY}(L^{y_t} t)$$

$$L^{y_h} h^{(n)} = X_n + Y_n L^{y_t} t + \mathcal{O}(t^2)$$

$L^{y_h} h^n(t)$ for various L intersect at the CP!

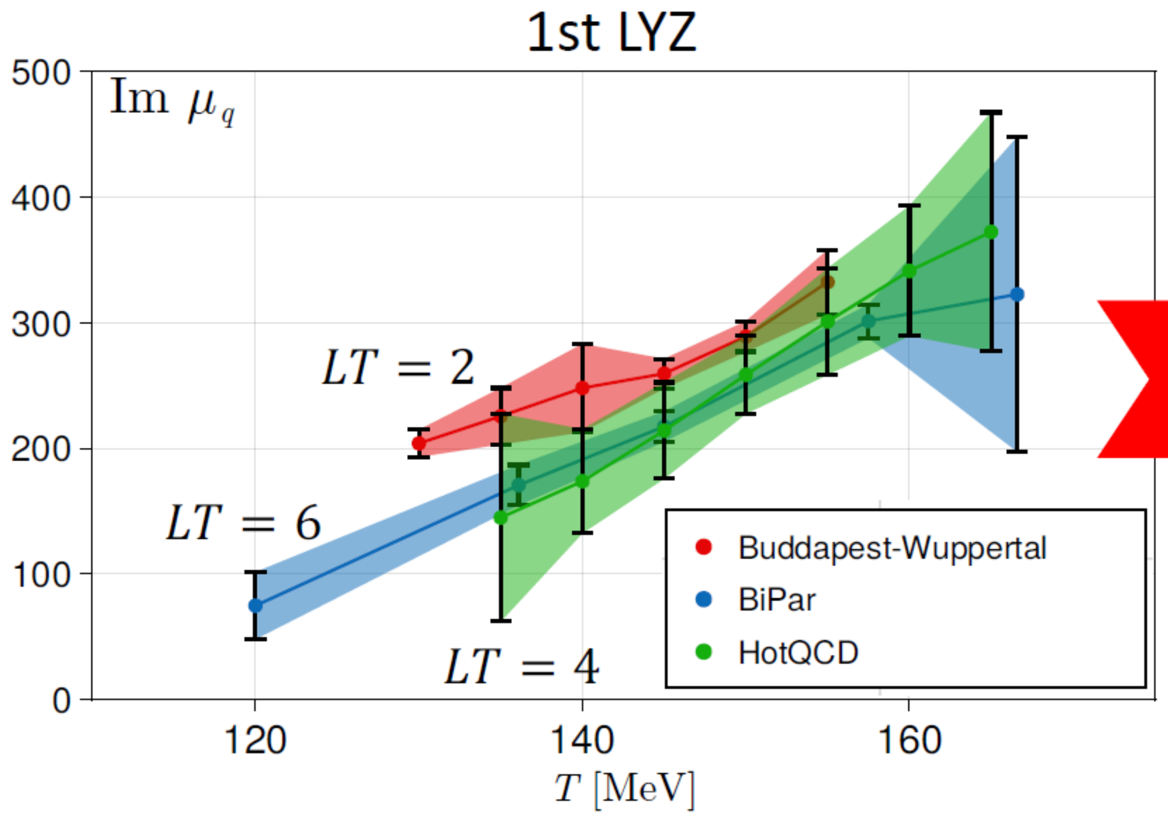
LYZ in 3d-Ising

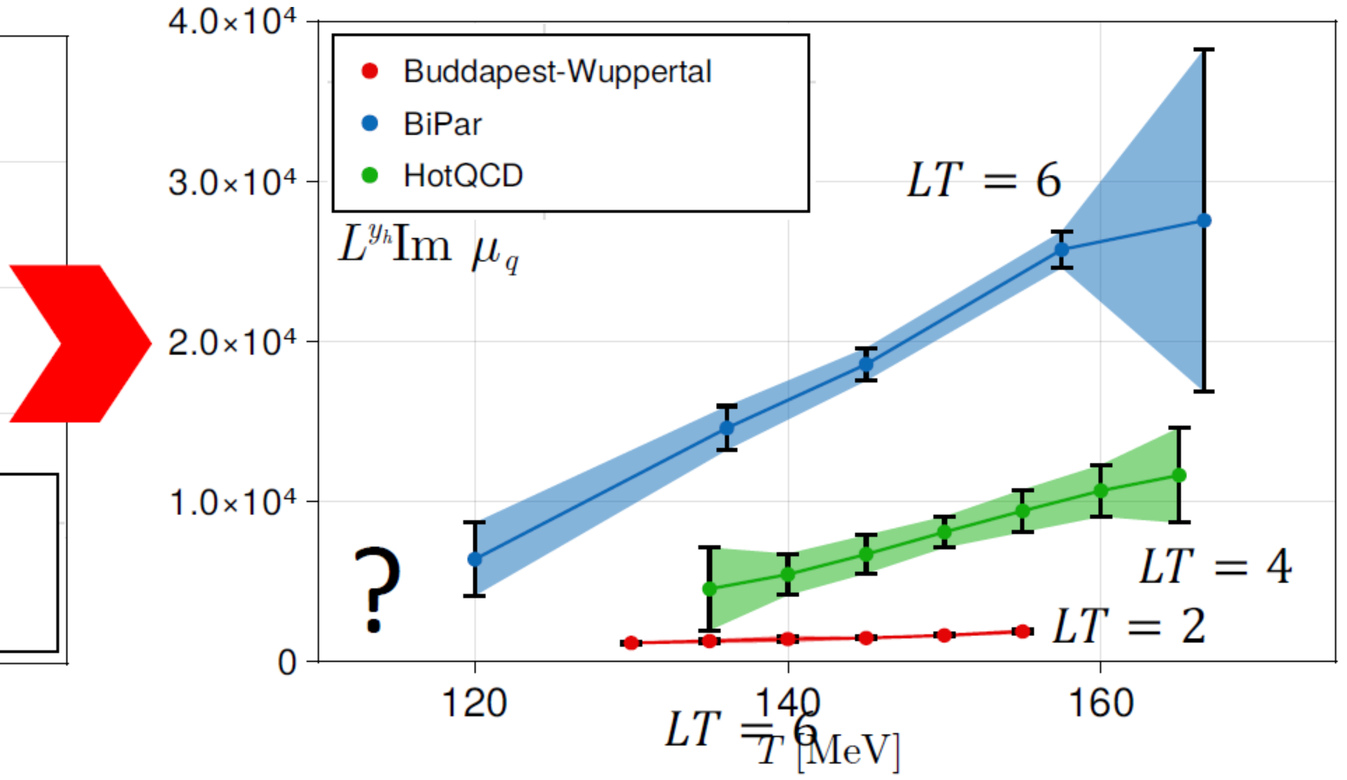


Assuming the universality class and the values of y_t, y_h, the intersection analysis can be performed only with a single LYZ.

Where is the QCD Critical Point?

Single-LYZ analysis for 2+1 flavor QCD





Intersection around $T \simeq 100$ MeV? Yet lower T data are mandatory.

Summary

Lee-Yang-zero ratio method

- A new method to utilize the **finite-size effects of Lee-Yang zeros for the CP searches**.
- Works successfully in 3d Ising and Potts models, as well as heavy-quark QCD-CP.

Outlook

- Determination of r_{nm} in each universality class
- Other quantities: ξ_c, mixing matrix A, etc.
- Application to the CP in heavy-quark QCD
- QCD-CP