YITP Lunch Seminar, Kyoto, 2025/11/19

Lee-Yang Zeros and Critical Points

Masakiyo Kitazawa

In collab. with Tatsuya Wada, Kazuyuki Kanaya

Wada, MK, Kanaya, PRL ('25), arXiv:2508.20422

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Lee-Yang Zeros and

PHYSICAL REVIEW LETTERS **134,** 162302 (2025)

Editors' Suggestion

Locating Critical Points Using Ratios of Lee-Yang

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[3] *Tomonaga Center for the History of the Universe, University of Tsukuba, Tsukuba,*

京都大学 KYOTO UNIVERSITY

ホーム > 最新の研究成果を知る > 多様な系に現れる臨界点の性質を系統的に決定する新手法を開発―リーヤンゼロを用いた一般的手法―

多様な系に現れる臨界点の性質を系統的に決定する新手法を開発―リーヤンゼロを用いた一般的手法―

企業・研究者の方

公開日： 2025年05月02日

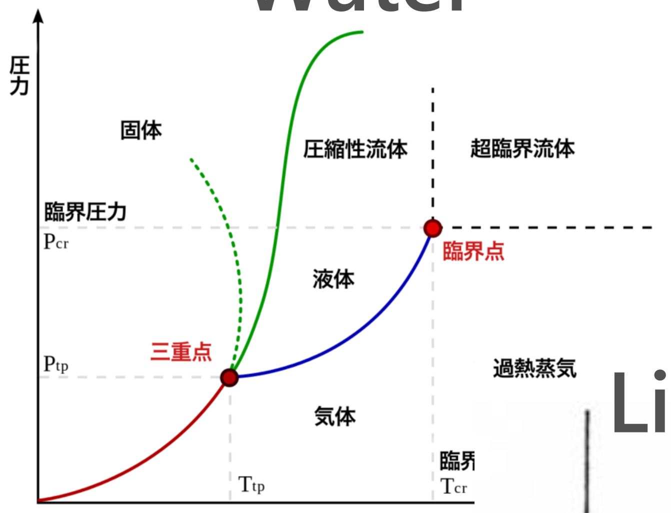
臨界点は、宇宙から素粒子まで様々なスケールの相転移現象に現れる普遍的な概念です。臨界点近傍で

In collab. with Tatsuya

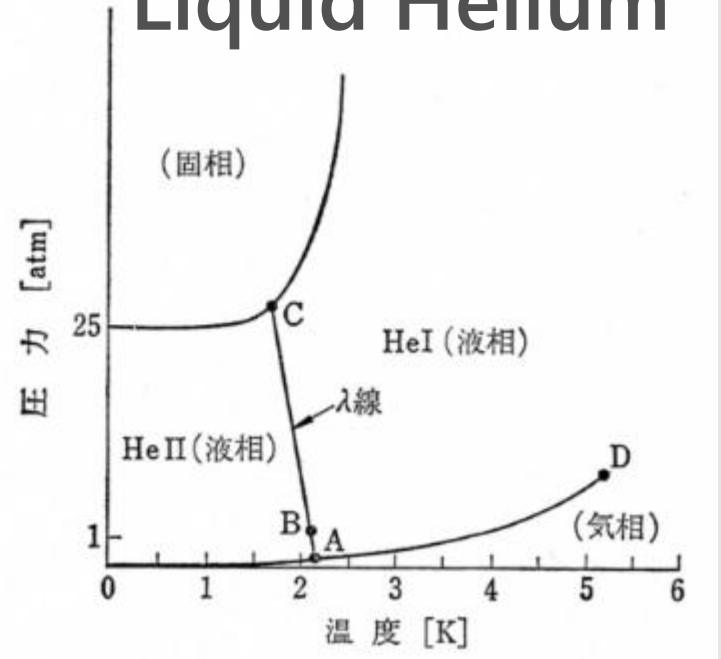
Wada, MK, Kanaya, PRL

Critical Points Everywhere!

Water

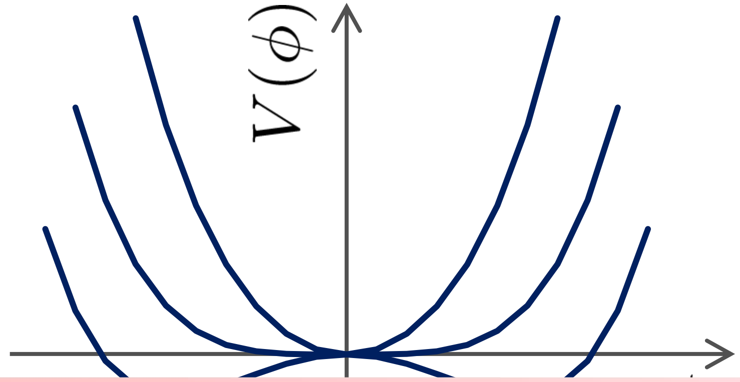


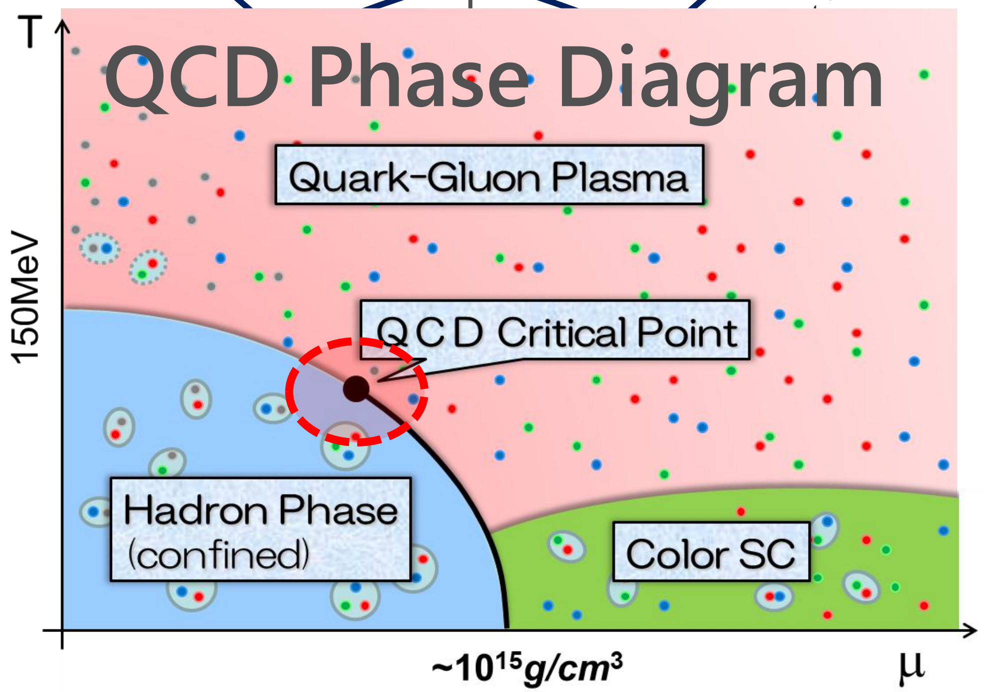
Liquid Helium



Ising

$$V(\phi) = -a\phi^2 + b\phi^4$$





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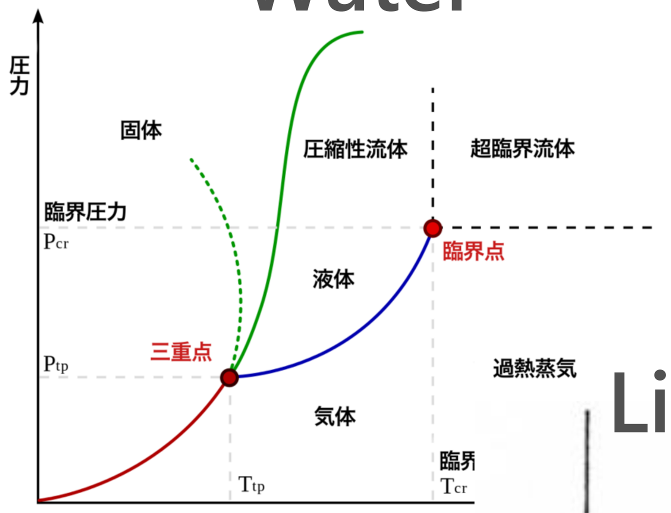

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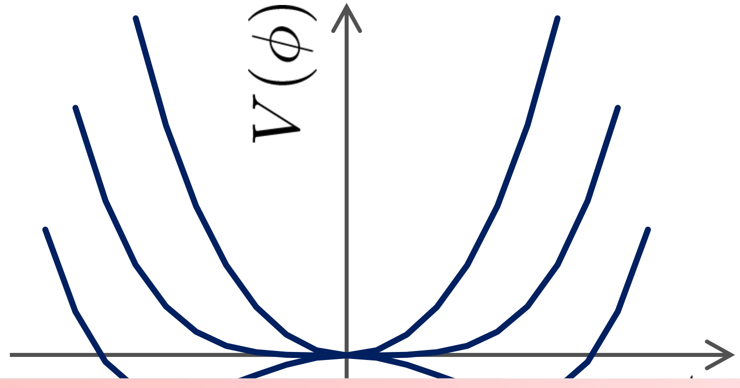

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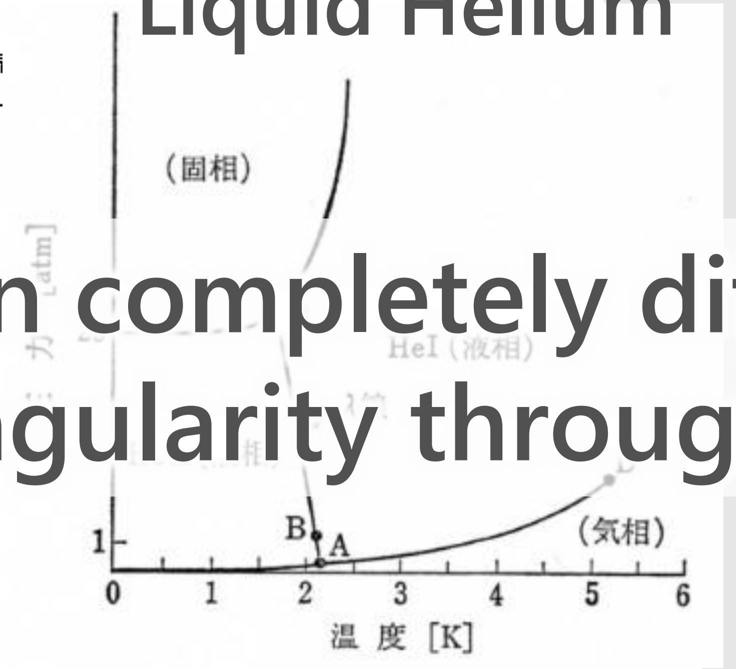
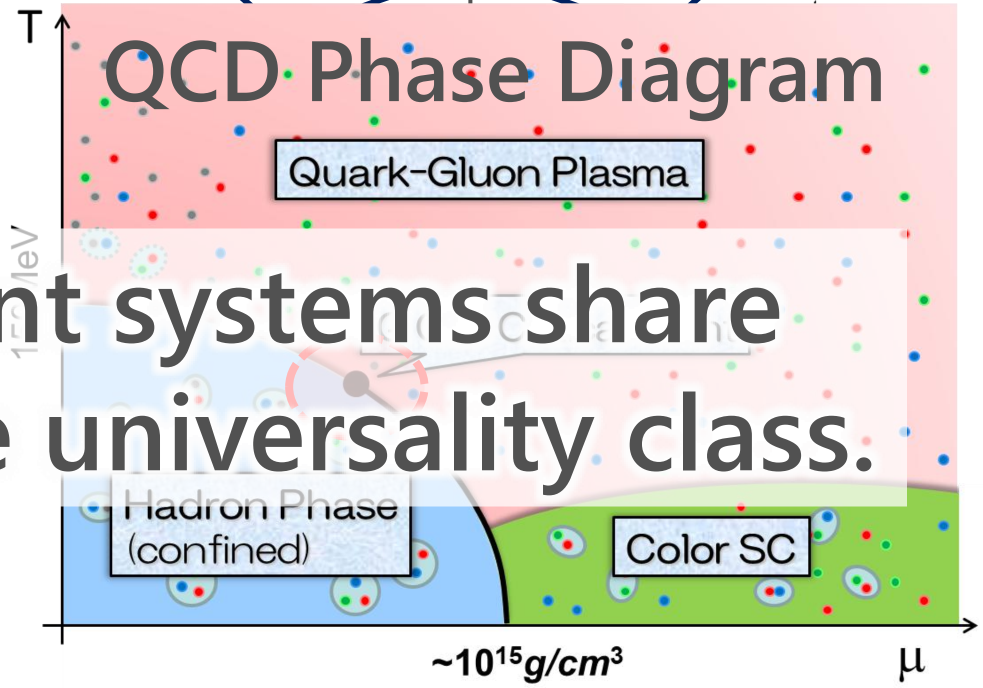


QCD Phase Diagram

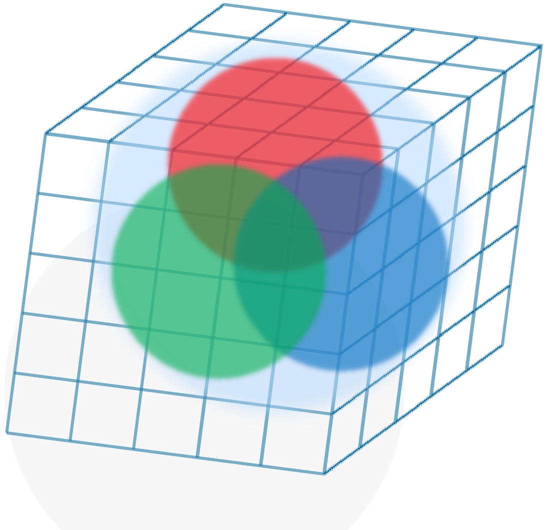


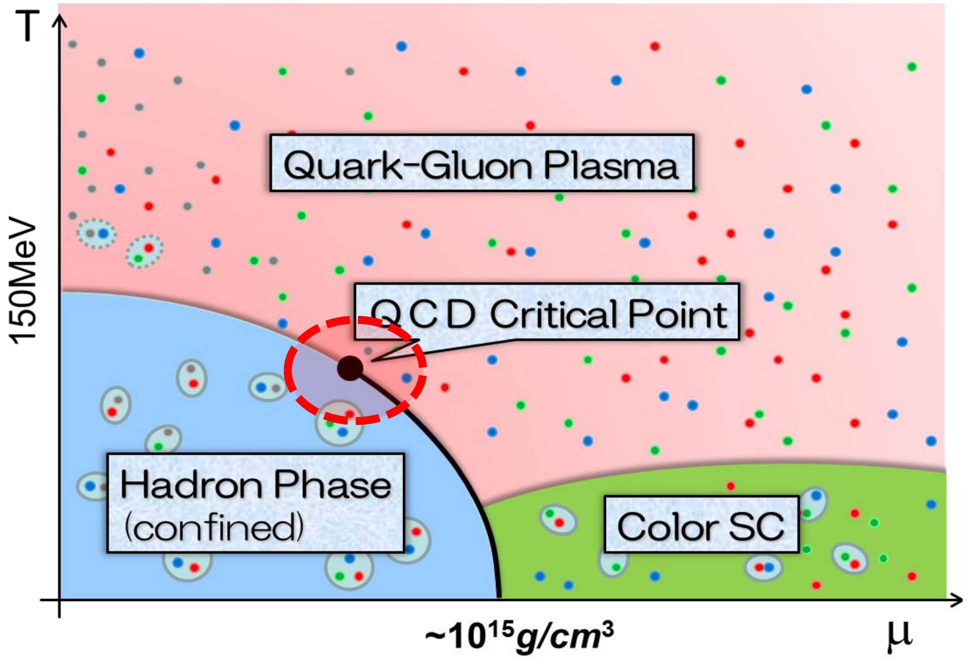
CPs in completely different systems share the singularity through the universality class.

Search for QCD-CP with Lee-Yang Zeros

in Lattice QCD Simulations

…;
Dimopoulos+, PRD ('22);
Clarke+, 2405.10196;
Adam+, 2507.13254; …





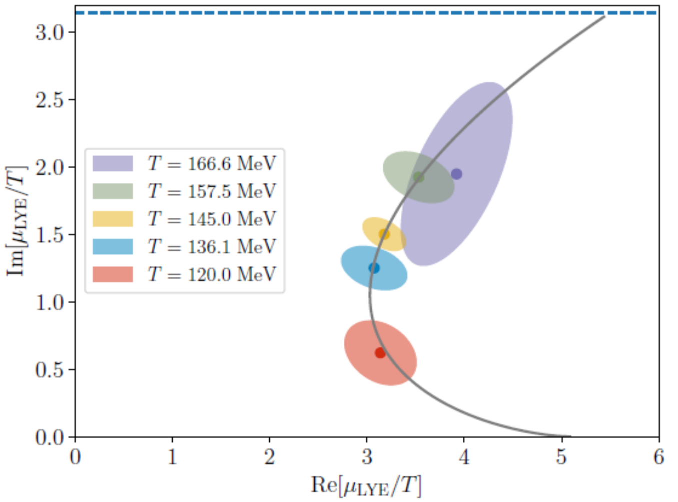
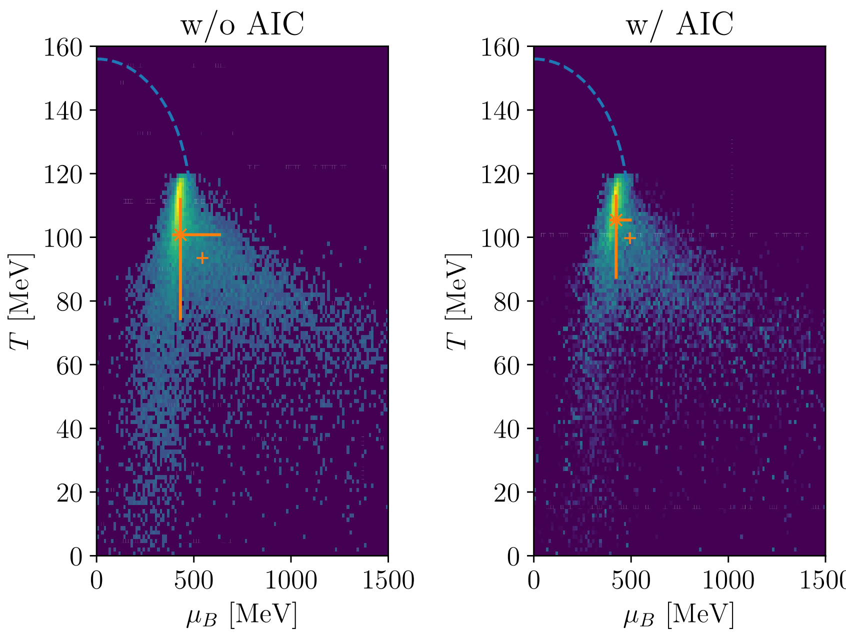


FIG. 3. Singularities at T = 166.6, 157.5, 145.0, 136.1 and 120.0 MeV. The dashed line lies at $\hat{\mu}_B = i\pi$.

arXiv:2405.10196v1 [hep-lat] 16 May 2024



$$\begin{cases} \mu^{\rm CEP} = 422^{+80}_{-35}\,\mathrm{MeV} \\ T^{\rm CEP} = 105^{+8}_{-18}\,\mathrm{MeV} \end{cases}$$

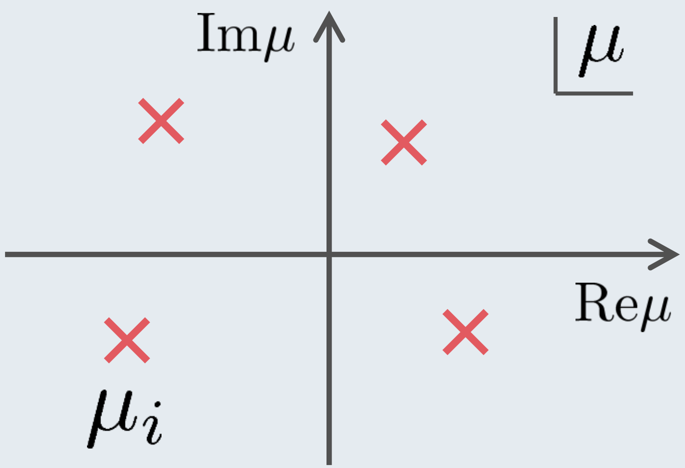
Lee-Yang Zeros

Yang, Lee; Lee, Yang ('52)

Partition Function $Z(T,\mu)$

Finite V ➤ Polynomial of μ (or T)

$$Z(T,\mu)=\prod_i(\mu-\mu_i)$$



➤ **zeros** on the complex plane

=Lee-Yang Zeros

Lee-Yang Zeros

Yang, Lee; Lee, Yang ('52)

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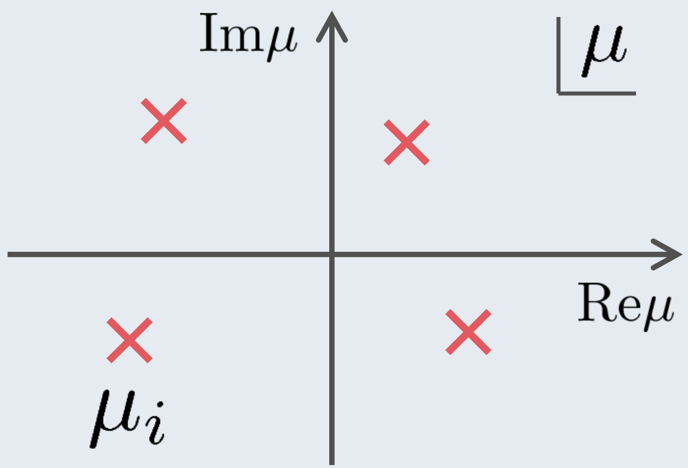


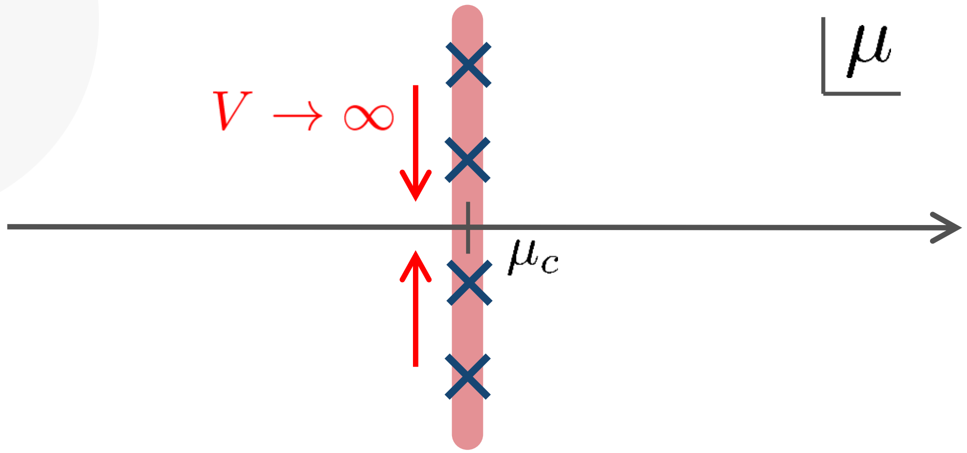
➤ **zeros** on the complex plane

=Lee-Yang Zeros

Phase Transition & LYZ

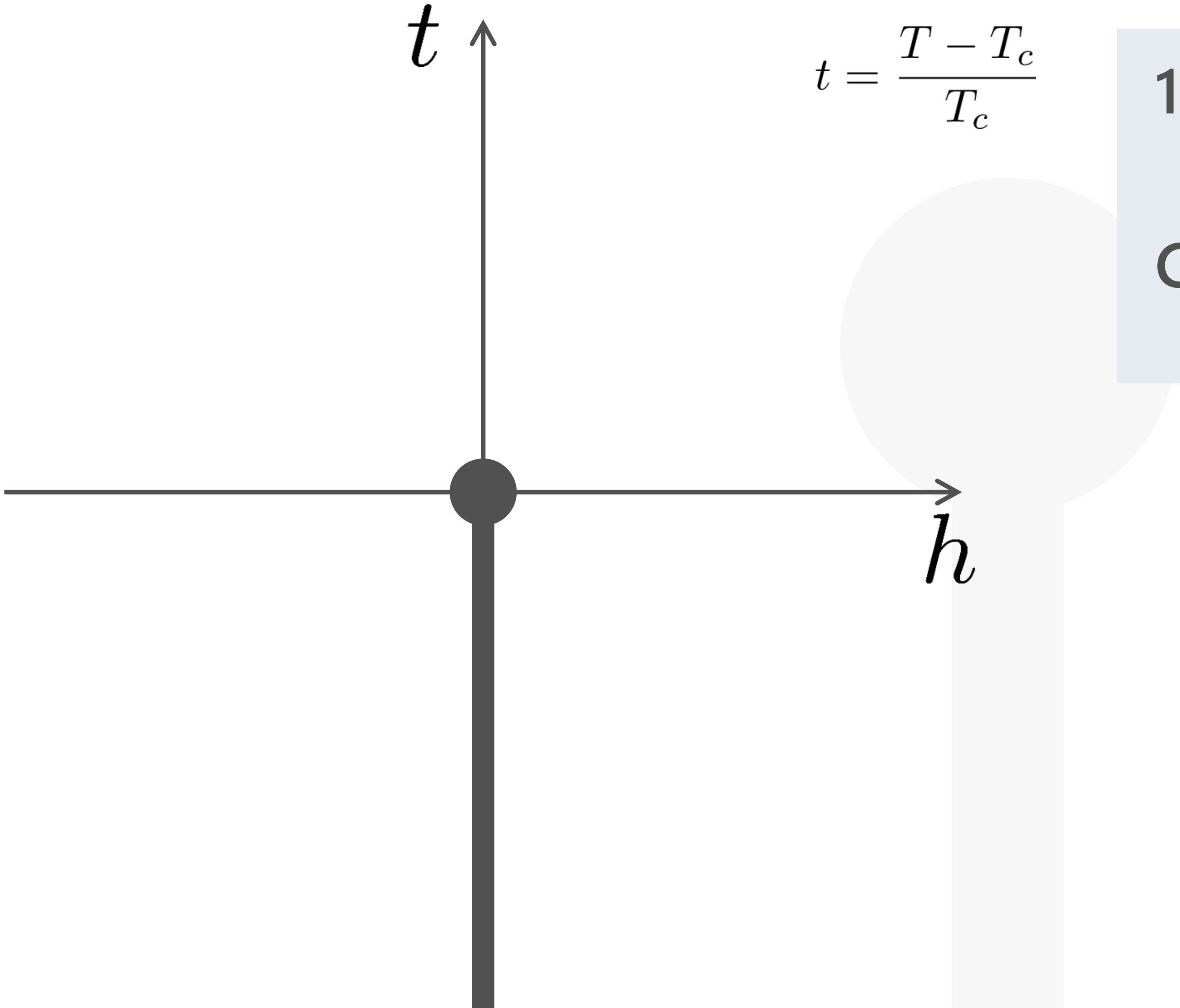
First-order transition

at $\mu = \mu_c$



— For $V \to \infty$, LYZs gather on the line crossing the real axis at $\mu = \mu_c$.

LYZ around a Critical Point in Ising Model



$$t = \frac{T - T_c}{T_c}$$

1st-transition

singularity on the real h axis

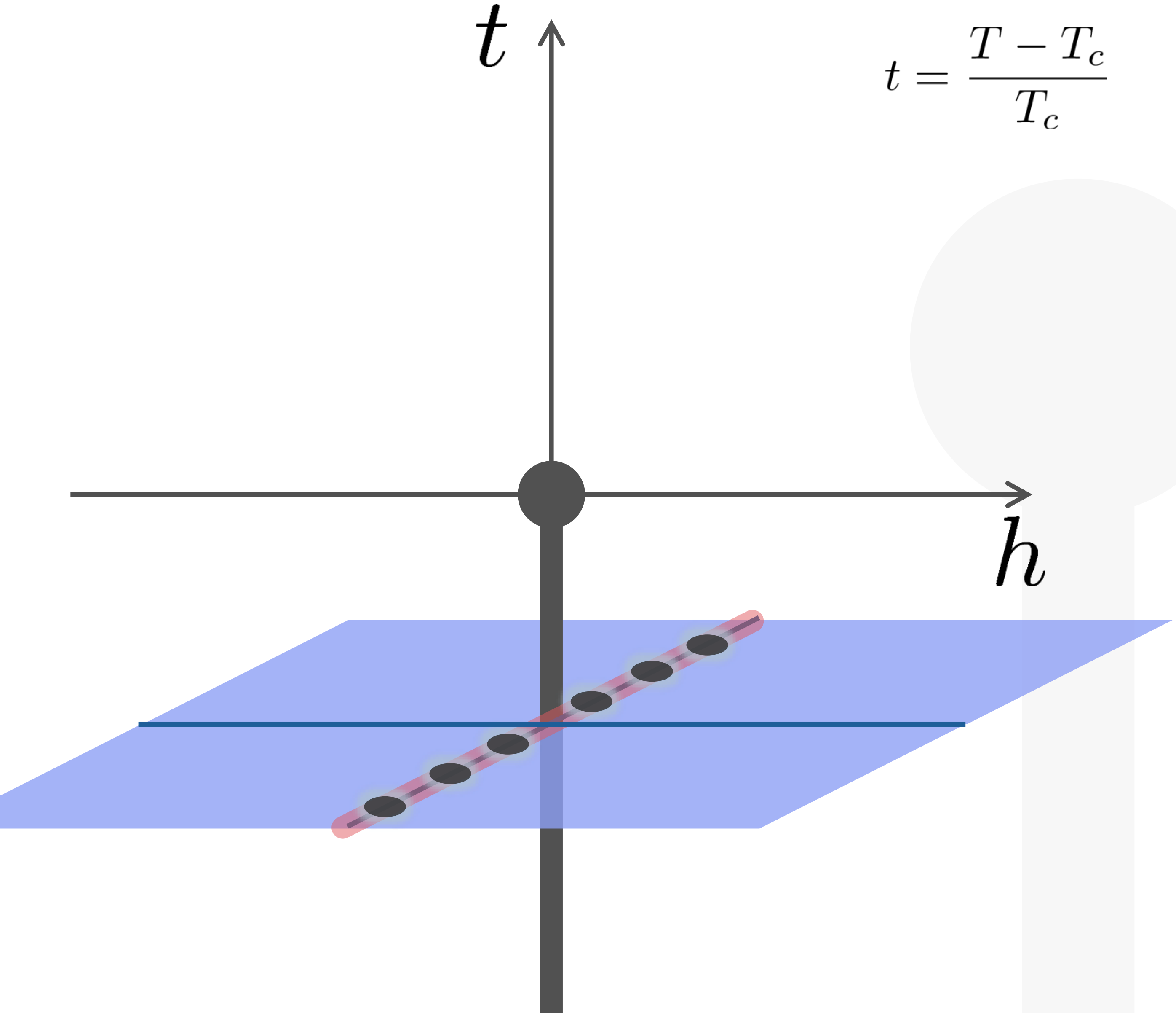
Crossover

no singularity on the real axis

Note:
LYZ in complex-h plane are purely imaginary.

Lee-Yang, 1952

LYZ around a Critical Point in Ising Model



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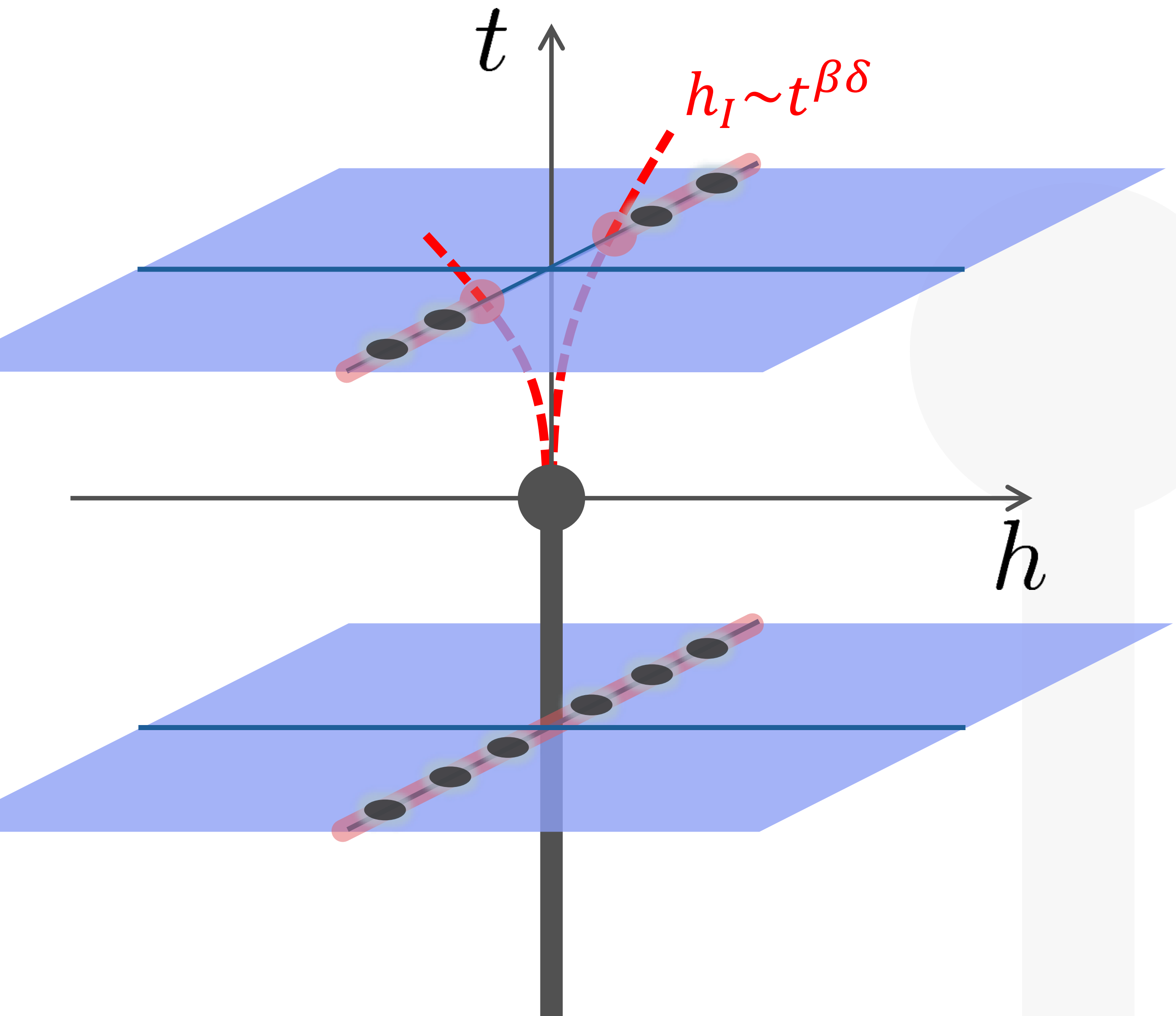

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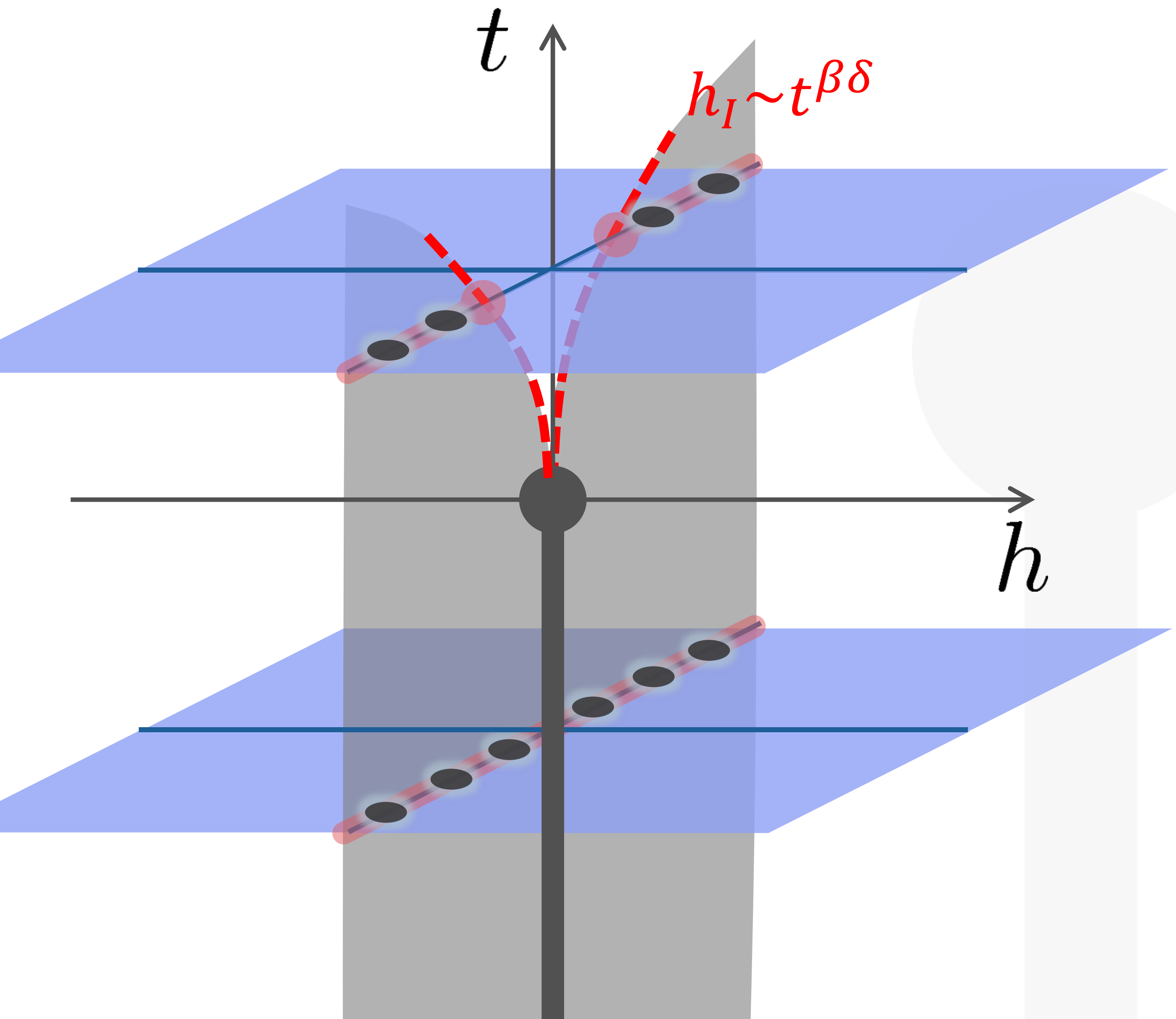
LY edge singularity

Starting from the CP

Its behavior is governed by the the scaling function.

$h_I \sim t^{\beta\delta}$

LYZ around a Critical Point in Ising Model



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Starting from the CP

Its behavior is governed by the the scaling function.

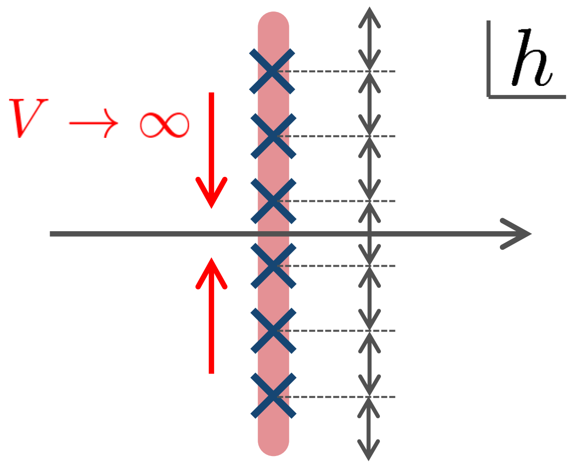
$h_I \sim t^{\beta\delta}$

Lee-Yang Zero Ratios

Wada, MK, Kanaya, PRL ('25)

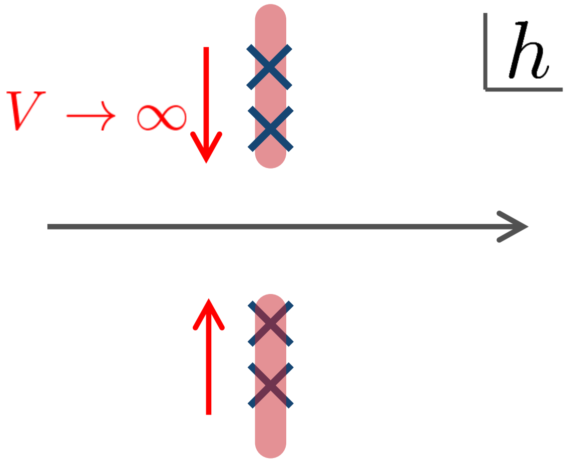
$$R_{nm}(t) = \frac{h^{(n)}(t)}{h^{(m)}(t)}$$

First-Order Side $(t < 0)$



$$R_{nm}(t) \xrightarrow[V\to\infty]{} \frac{2n-1}{2m-1}$$

Crossover Side $(t > 0)$



$$R_{nm}(t) \xrightarrow[V\to\infty]{} 1$$

Finite-Size Scaling

Scaling Hypothesis

$$F_{\rm sing}(t, h, L^{-1}) = \tilde{F}_{\rm sing}(L^{y_t} t, L^{y_h} h)$$

$$Z_{\rm sing}(t, h, L^{-1}) = \tilde{Z}_{\rm sing}(L^{y_t} t, L^{y_h} h)$$

$$F = F_{\rm sing} + F_{\rm reg}$$

$$Z = Z_{\rm sing} \times Z_{\rm reg}$$

LYZ in the scaling region on finite volume

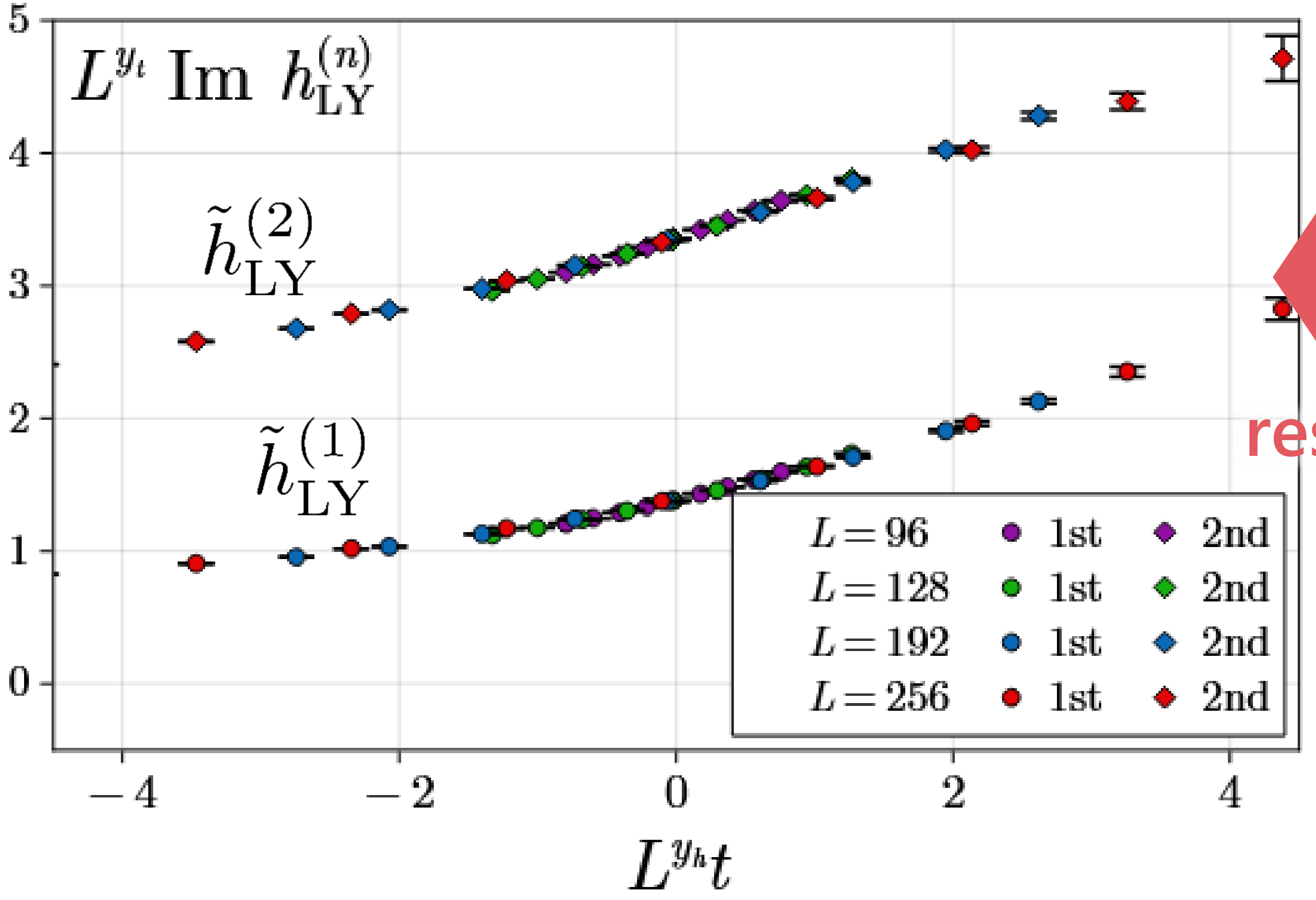
$$Z(t, h, L^{-1}) \sim \tilde{Z}_{\rm sing}(L^{y_t} t, L^{y_h} h) = 0$$

$$L^{y_h} h_{\rm LY}^{(n)}(t) = \tilde{h}_{\rm LY}^{(n)}(L^{y_t} t)$$

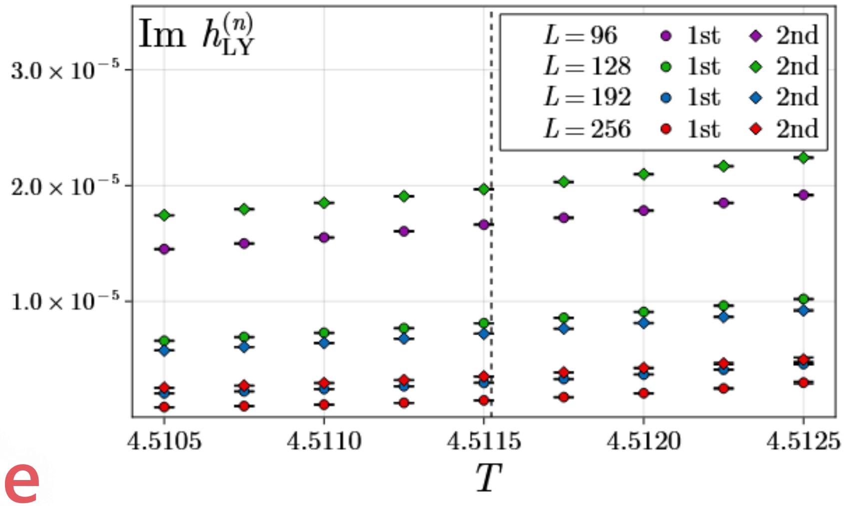
Itzykson, Nucl. Phys. 220 (1983)

LYZ near $t = 0$

Wada, MK, Kanaya, PRL ('25)

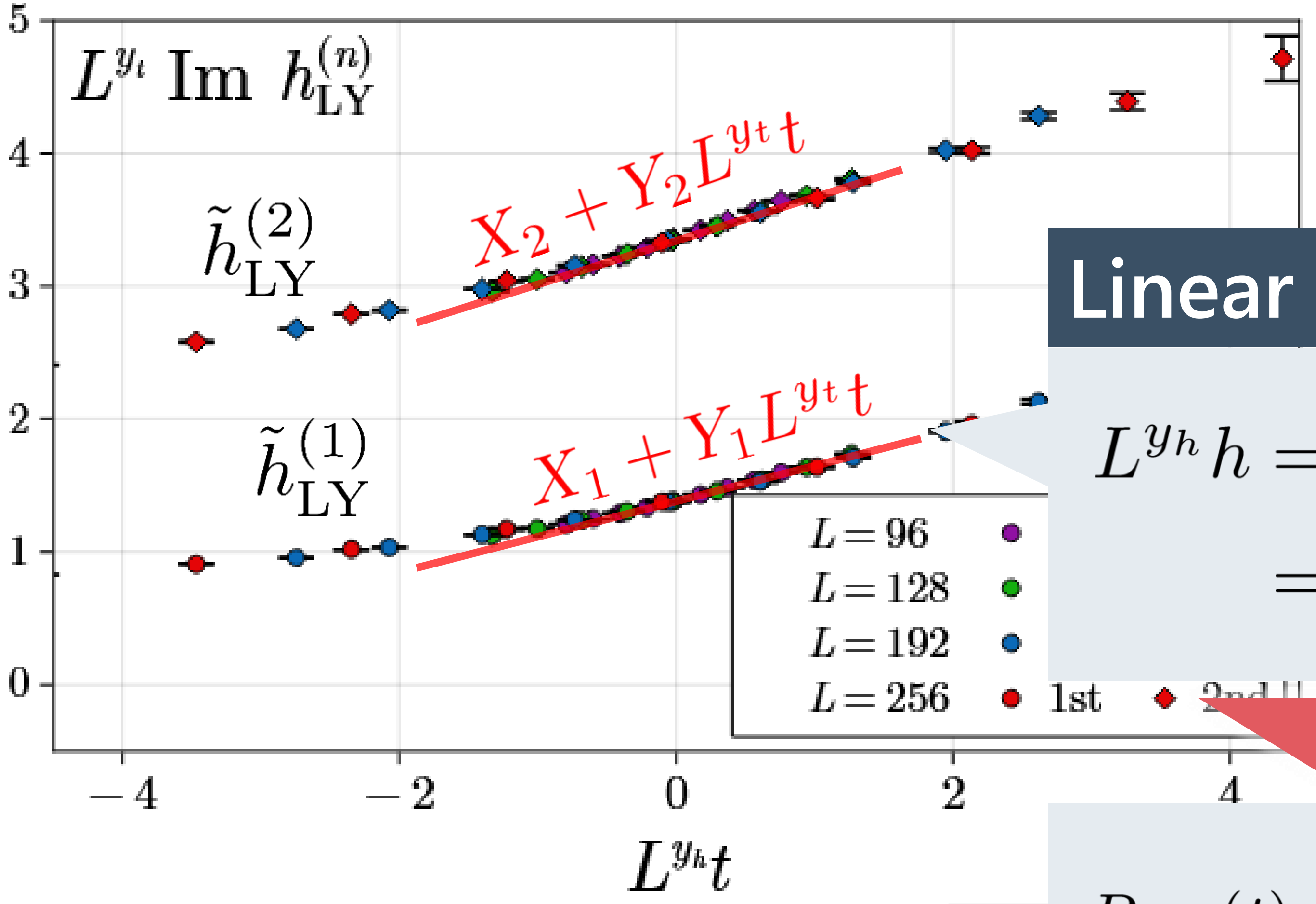


rescale



Wada, MK, Kanaya, PRL ('25)

LYZ near $t = 0$



Linear Approx. at $t = 0$

$$L^{y_h} h = \tilde{h}^{(i)}_{\mathrm{LY}}(L^{y_t} t)$$
$$= X_i + Y_i L^{y_t} t + \mathcal{O}(t^2)$$

$$R_{nm}(t) = \frac{X_n}{X_m}\left(1 + C_{nm} t L^{y_t} + \mathcal{O}(t^2)\right)$$

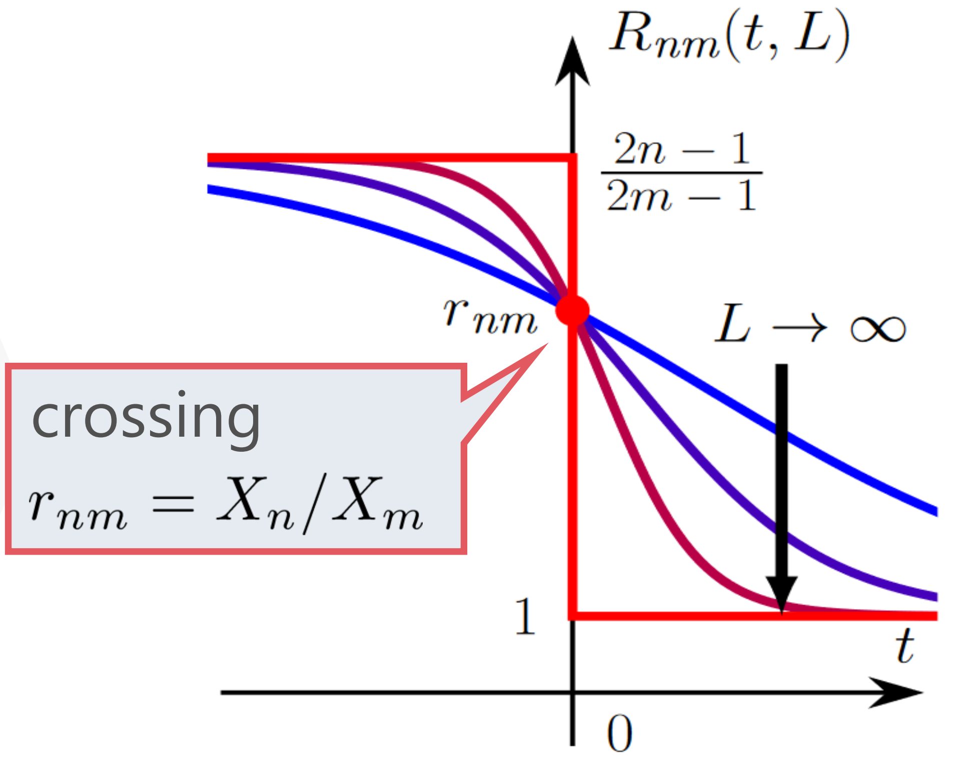
Wada, MK, Kanaya, PRL, '25

LYZ Ratios $R_{nm}(t) = \frac{h^{(n)}(t)}{h^{(m)}(t)}$

$$R_{n1}(t) \xrightarrow[V\to\infty]{} \begin{cases} 2n-1 & t<0 \text{ (1st order)} \\ 1 & t>0 \text{ (crossover)} \end{cases}$$

$$R_{nm}(t) = r_{nm}\left(1 + C_{nm} t L^{y_t} + \mathcal{O}(t^2)\right)$$

near $t = 0$



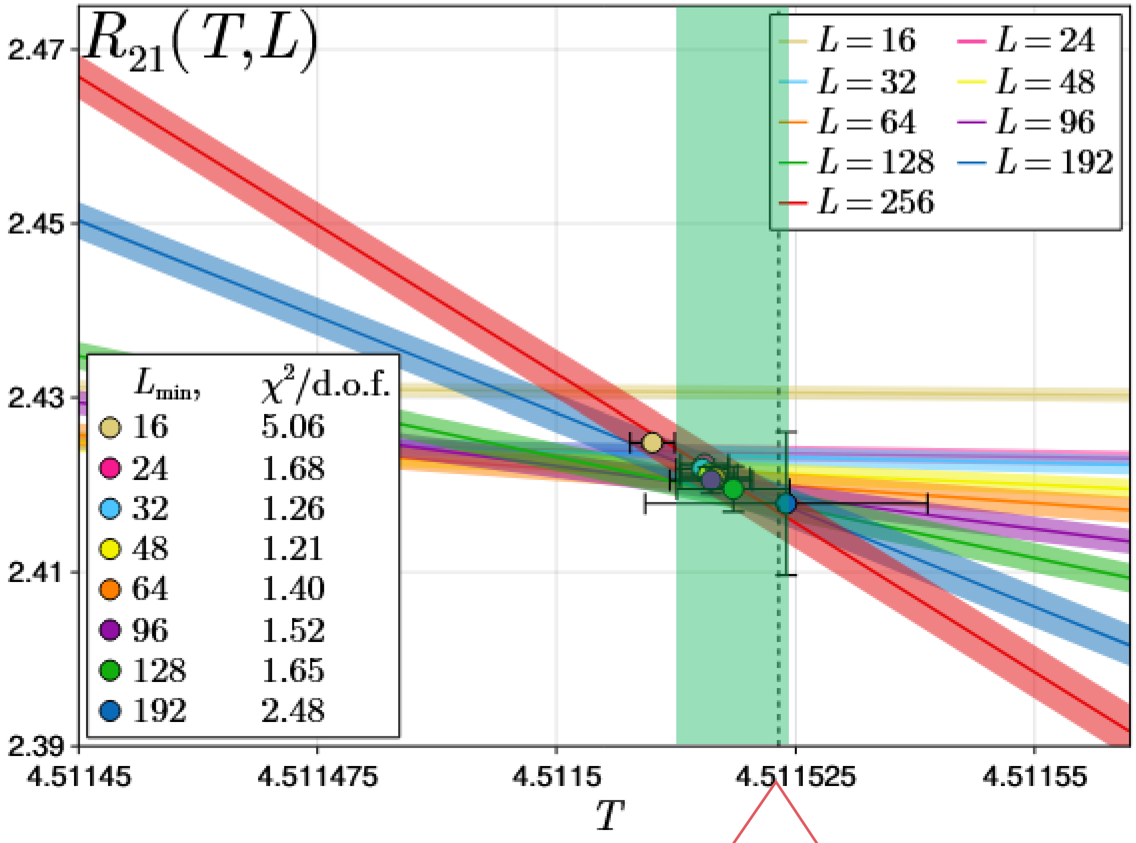
— $R(0)$ is L independent, the universal value.

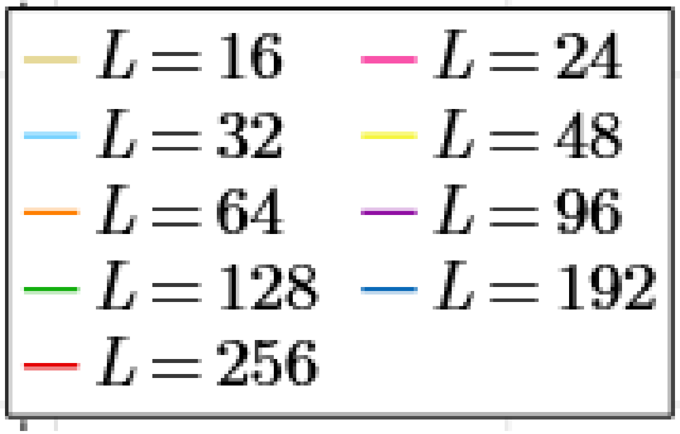
—Intersection point of various L gives the CP.

—Reminiscent of Binder-cumulant analysis

LYZR in 3d-Ising

LYZR



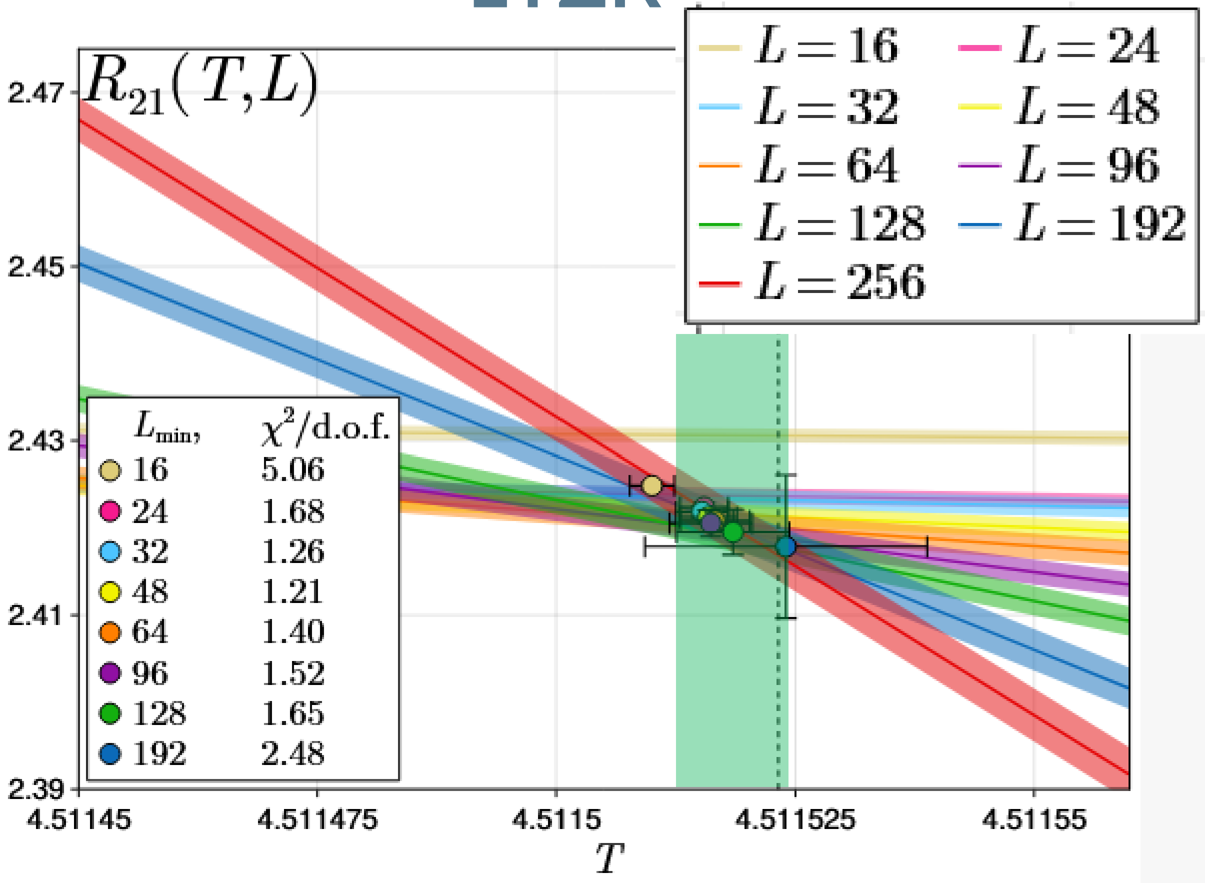


T_c Ferrenberg ('18)

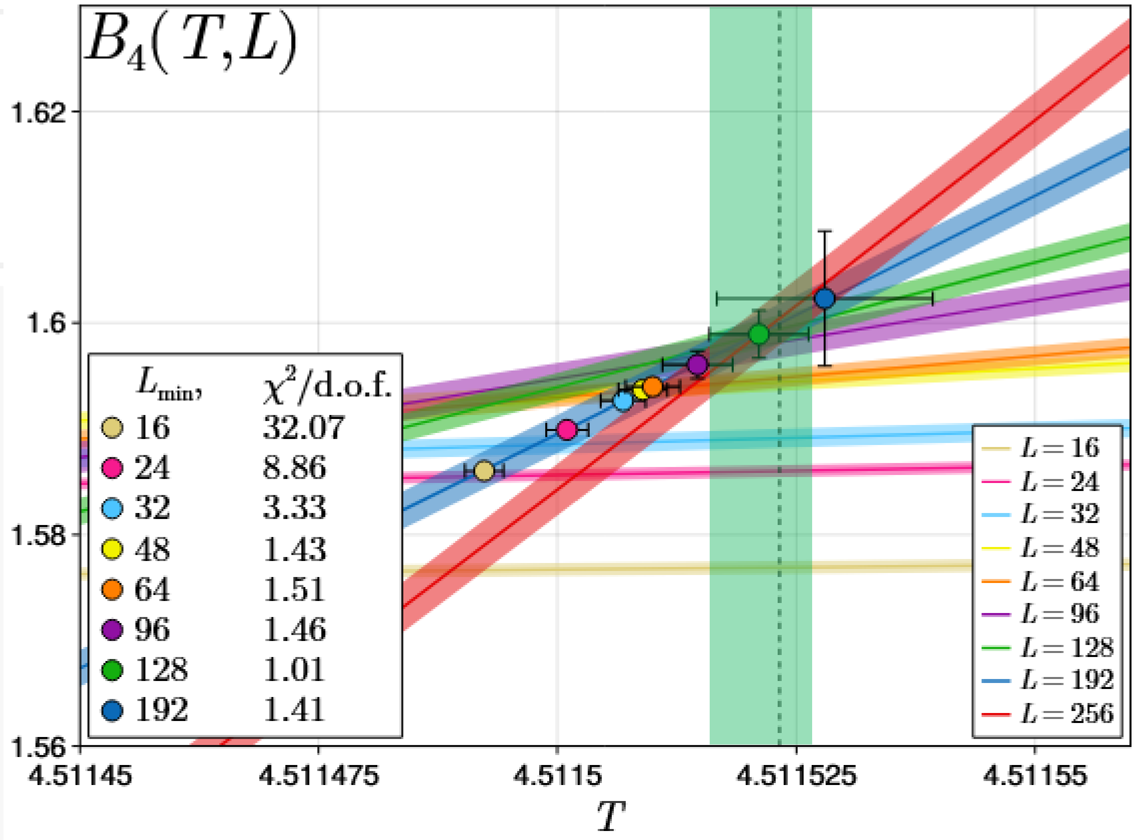
Wada, MK, Kanaya, arXiv:2508.20422

LYZR vs Binder Cumulant in 3d-Ising

LYZR



Binder cumulant

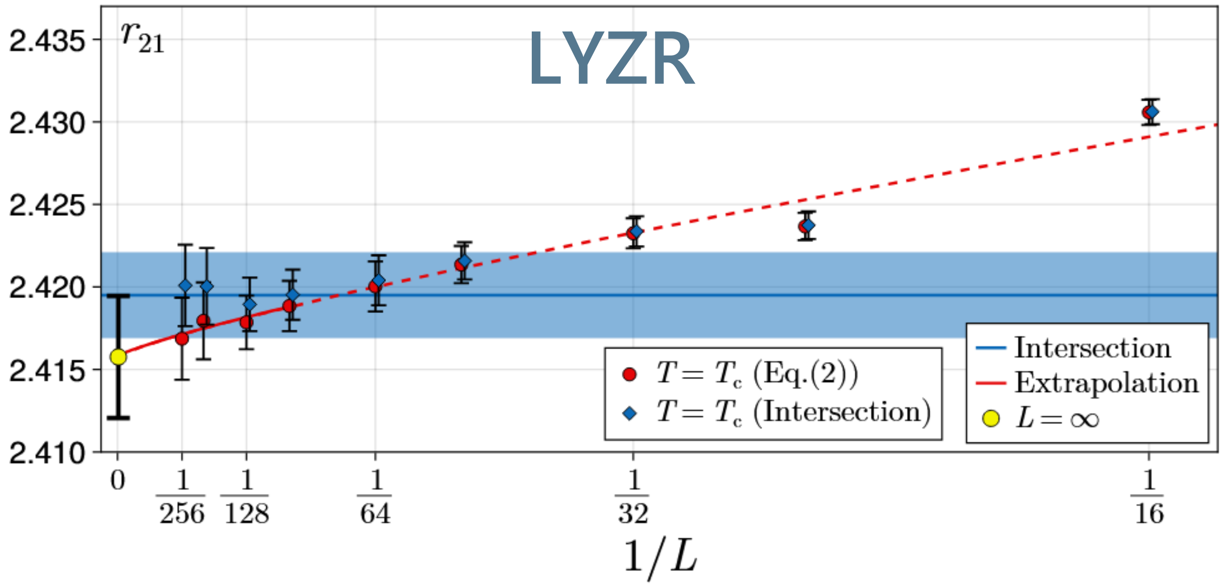


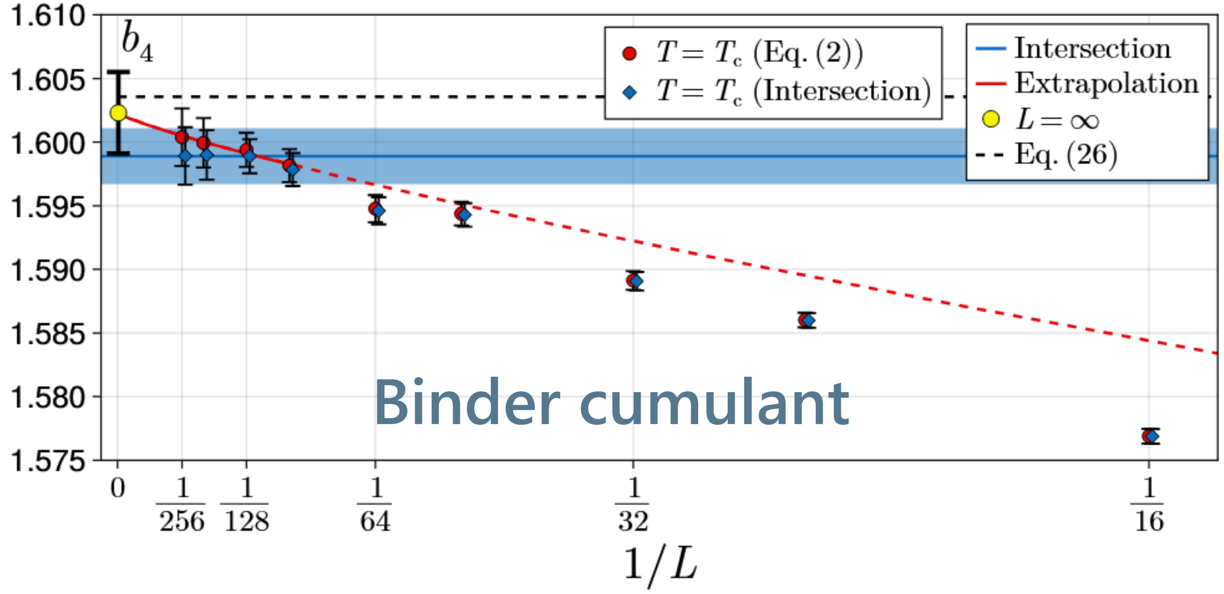
Faster convergence of the violation of FSS in LYZ?

Wada, MK, Kanaya, arXiv:2508.20422

Convergence at $T = T_c$

Wada, MK, Kanaya, arXiv:2508. 20422





Red: T_c of Ferrenberg ('18)
Blue: T_c of intersection point

$$R_{21}(0, L) = r_{21}(1 + cL^{-\omega})$$

In 3d-Ising,
violation of FSS is more quickly suppressed in the LYZR than the Binder cumulant for $L \to \infty$.

$$Z = Z_{\mathrm{sing}} \times Z_{\mathrm{reg}}$$

Wada, MK, Kanaya, PRL ('25)

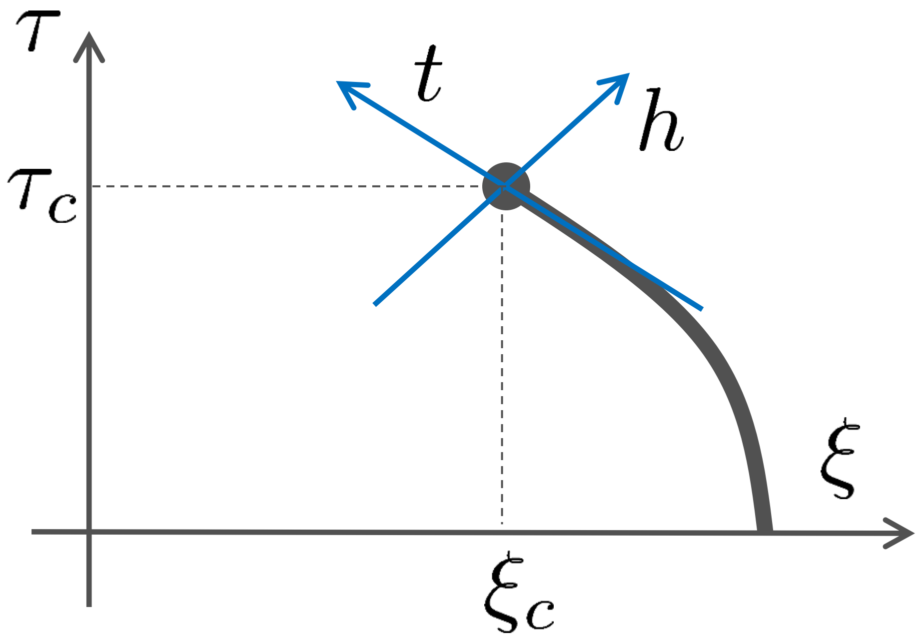
$\bar{y} = y_t - y_h = -0.894$

CP in a General System

- CP on a $\tau - \xi$ plane
- LYZ on the complex ξ plane

$$\begin{pmatrix} t \\ h \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} \tau - \tau_c \\ \xi - \xi_c \end{pmatrix} = A \begin{pmatrix} \delta\tau \\ \delta\xi \end{pmatrix}$$

$$L^{y_h} h^{(n)}(t) \simeq X_n + Y_n L^{y_t} t$$



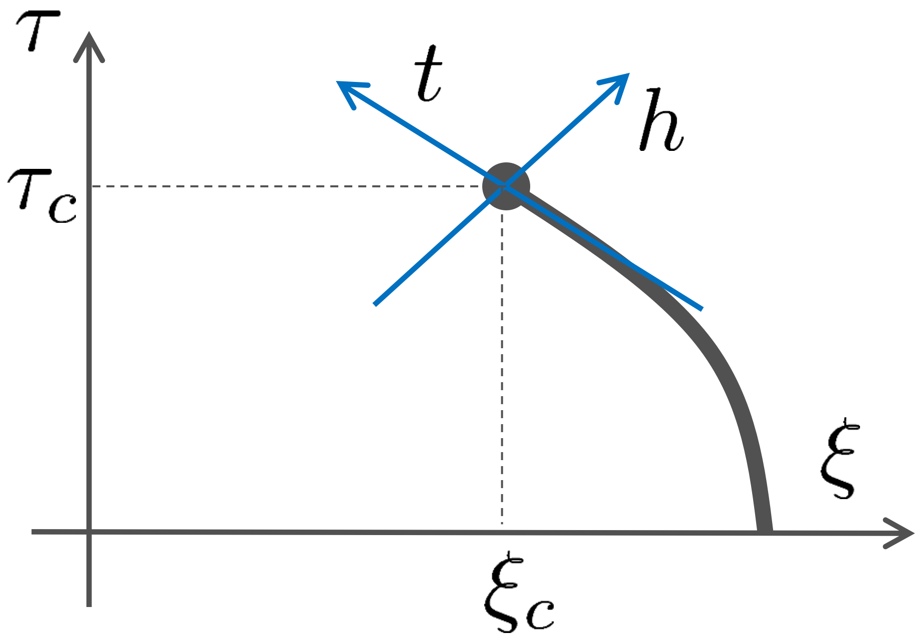
$$Z_{\mathrm{sing}}(\tau, \xi) = \tilde{Z}_{\mathrm{Ising}}\big(t(\tau,\xi), \xi(\tau,\xi)\big)$$

Wada, MK, Kanaya, PRL ('25)

CP in a General System

$$\bar{y} = y_t - y_h = -0.894$$

- CP on a $\tau - \xi$ plane
- **LYZ on the complex ξ plane**



$$\begin{pmatrix} t \\ h \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} \tau - \tau_c \\ \xi - \xi_c \end{pmatrix} = A \begin{pmatrix} \delta\tau \\ \delta\xi \end{pmatrix}$$

$$L^{y_h} h^{(n)}(t) \simeq X_n + Y_n L^{y_t} t$$

$$\begin{cases} \xi_{\mathrm{R}}^{(n)} = \xi_c - \dfrac{a_{21}}{a_{22}} \delta\tau + \mathcal{O}(L^{2\bar{y}}) \\ \xi_{\mathrm{I}}^{(n)} = \dfrac{X_n}{a_{22}} L^{-y_h} + \dfrac{\det A Y_n}{a_{22}^2} \delta\tau L^{\bar{y}} + \mathcal{O}(L^{2\bar{y}}) \end{cases}$$

$L \to \infty$

generalization to finite V

LY Edge Singularity

$$\begin{cases} \mathrm{Re}\xi_{\mathrm{LYES}} \simeq c_1 \tau \\ \mathrm{Im}\xi_{\mathrm{LYES}} \simeq c_2 \tau^{\beta\delta} \end{cases}$$

Stephanov, 2006

LYZ Ratios for General CP

Wada, MK, Kanaya, PRL ('25)

$$\bar{y} = y_t - y_h = -0.894$$

LYZ Ratio

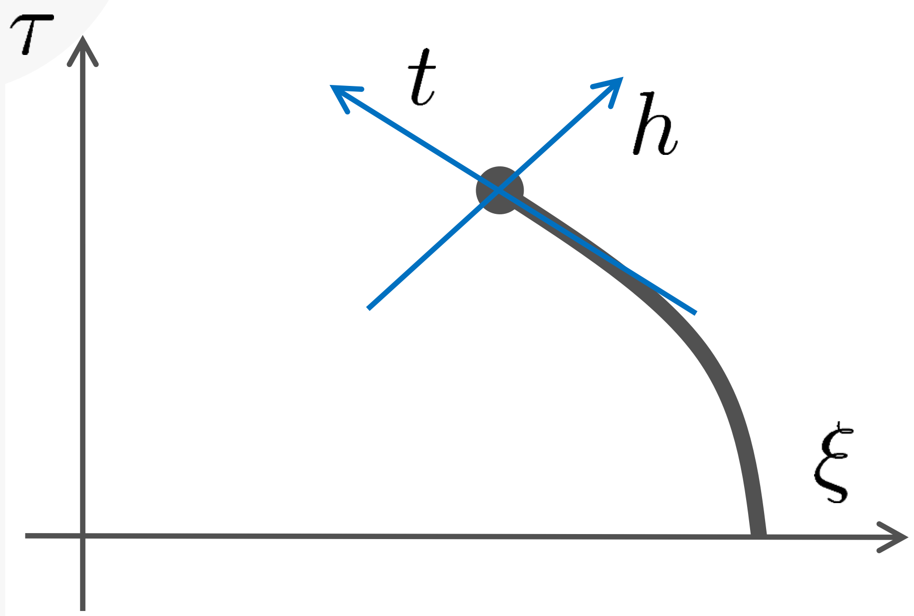
$$R_{nm}(\tau) = \frac{\xi_{\rm I}^{(n)}(\tau)}{\xi_{\rm I}^{(m)}(\tau)} = r_{nm}\left(1 + CL^{y_t}\delta\tau + \mathcal{O}(\delta\tau^2)\right)\left(1 + DL^{2\bar{y}} + \mathcal{O}(L^{4\bar{y}})\right)$$

nonzero for $a_{12} \neq 0$

$$r_{nm} = \frac{X_n}{X_m}, \quad C = \frac{\det A}{a_{22}}\left(\frac{Y_2}{X_2} - \frac{Y_1}{X_1}\right), \quad D = \frac{a_{12}^2}{a_{22}^2}(Y_1^2 - Y_2^2)$$

$$\begin{pmatrix} t \\ h \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} \tau - \tau_c \\ \xi - \xi_c \end{pmatrix} = A \begin{pmatrix} \delta\tau \\ \delta\xi \end{pmatrix}$$

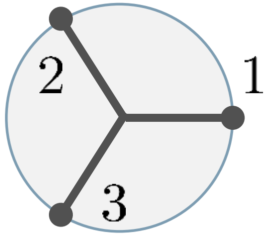
r_{nm} are universal constants specific to universality classes.



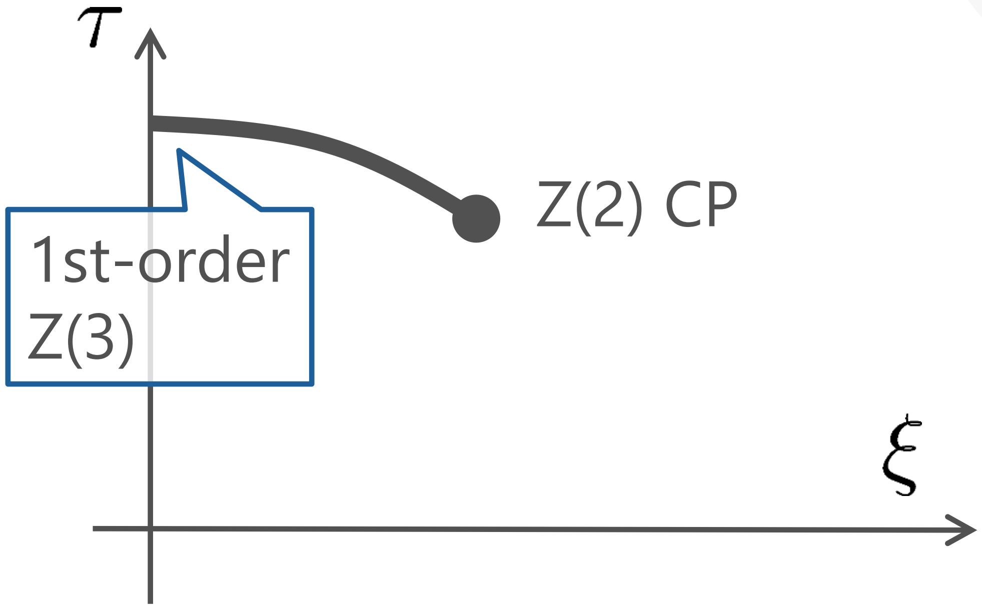
Numerical Analysis: 3d 3-State Potts Model

$$H = -\tau \sum_{\langle i,j \rangle} \delta_{\sigma_i,\sigma_j} - \xi \sum_i \delta_{\sigma_i,1} \quad \sigma_i = 1,2,3$$

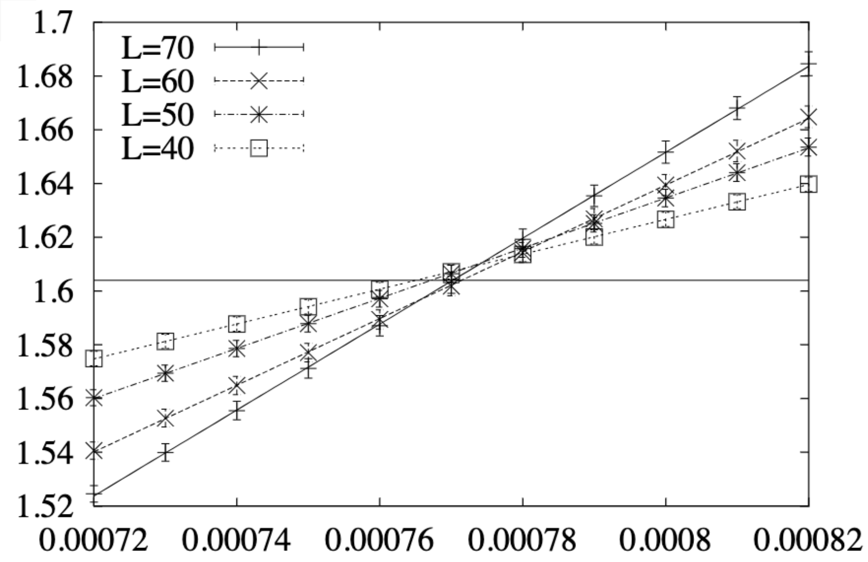
Monte-Carlo + reweighting

$\sigma_i =$ 

Phase Diagram



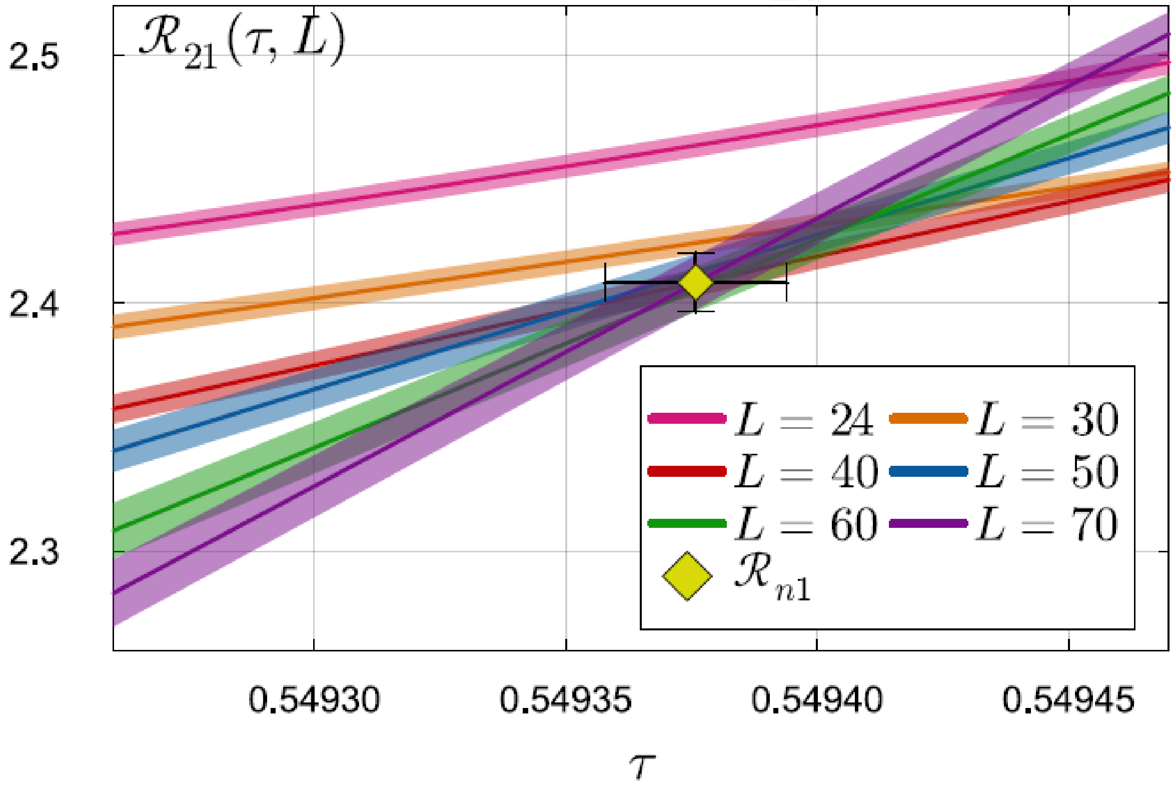
Binder-Cumulant Analysis



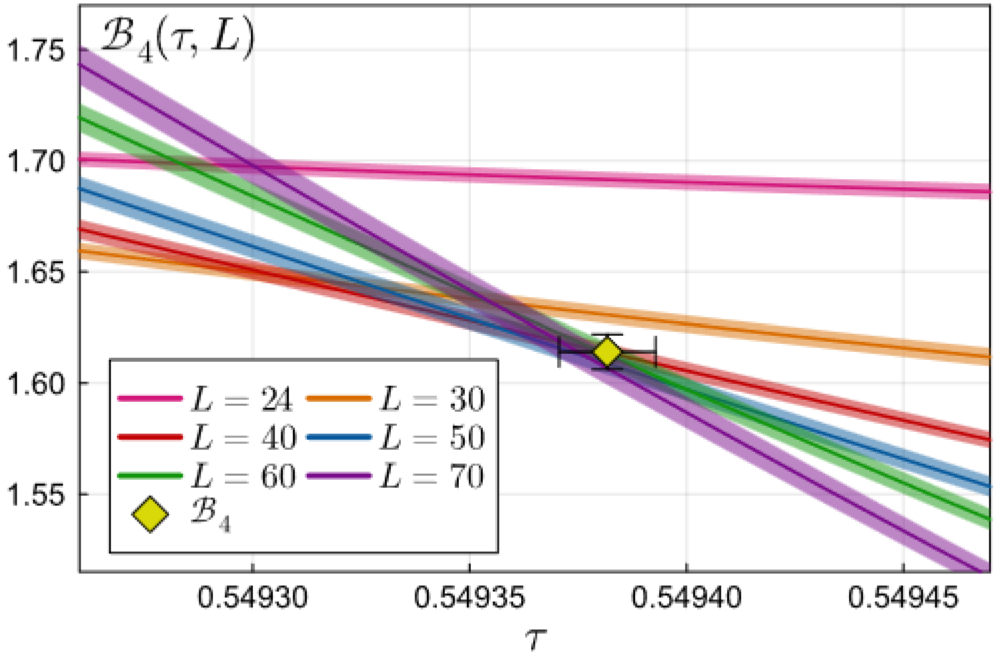
Karsch, Stickan, 2000

3d 3-State Potts Model: LYZ Ratio

LYZ Ratio (R_{21})



Binder Cumulant



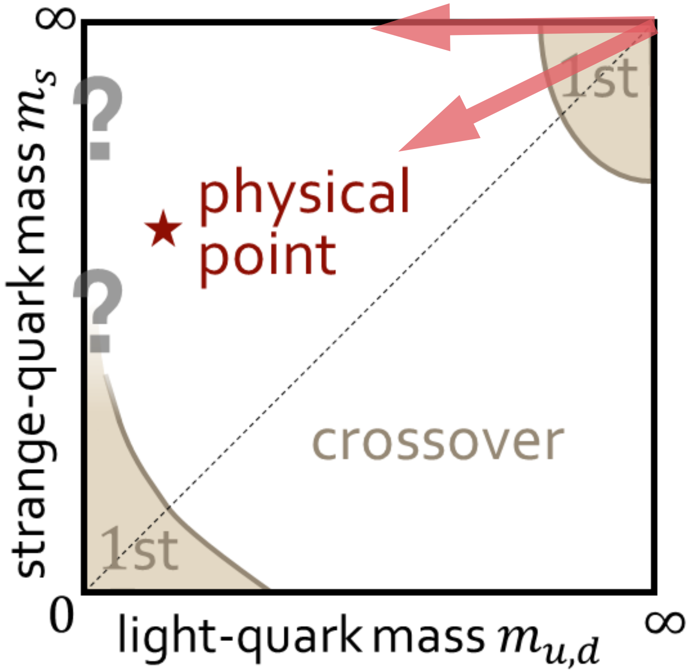
Wada, MK, Kanaya, PRL, '25

fit data	τ_c	y_t	r_{n1} or b_4	χ^2/dof
$\mathcal{R}_{21}$	0.549375(18)	1.53(19)	2.408(12)	0.38
$\mathcal{B}_4$	0.549382(11)	1.63(13)	1.614(8)	0.69

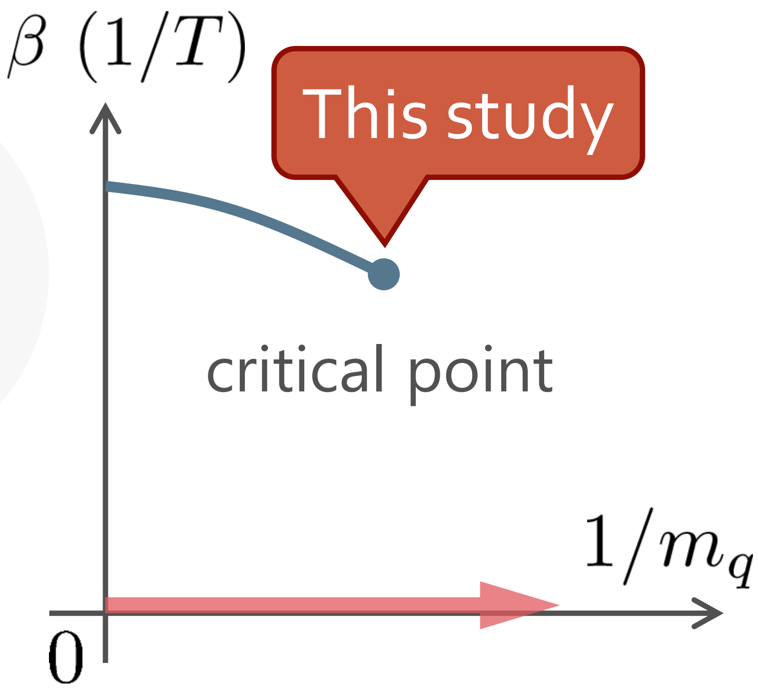
R_{n1} and B_4 give the same value of τ_c within statistics.

CP in Heavy-Quark QCD

Columbia Plot



Phase Diagram



CP in heavy-quark QCD

– $\mu_q = 0$ & large m_q

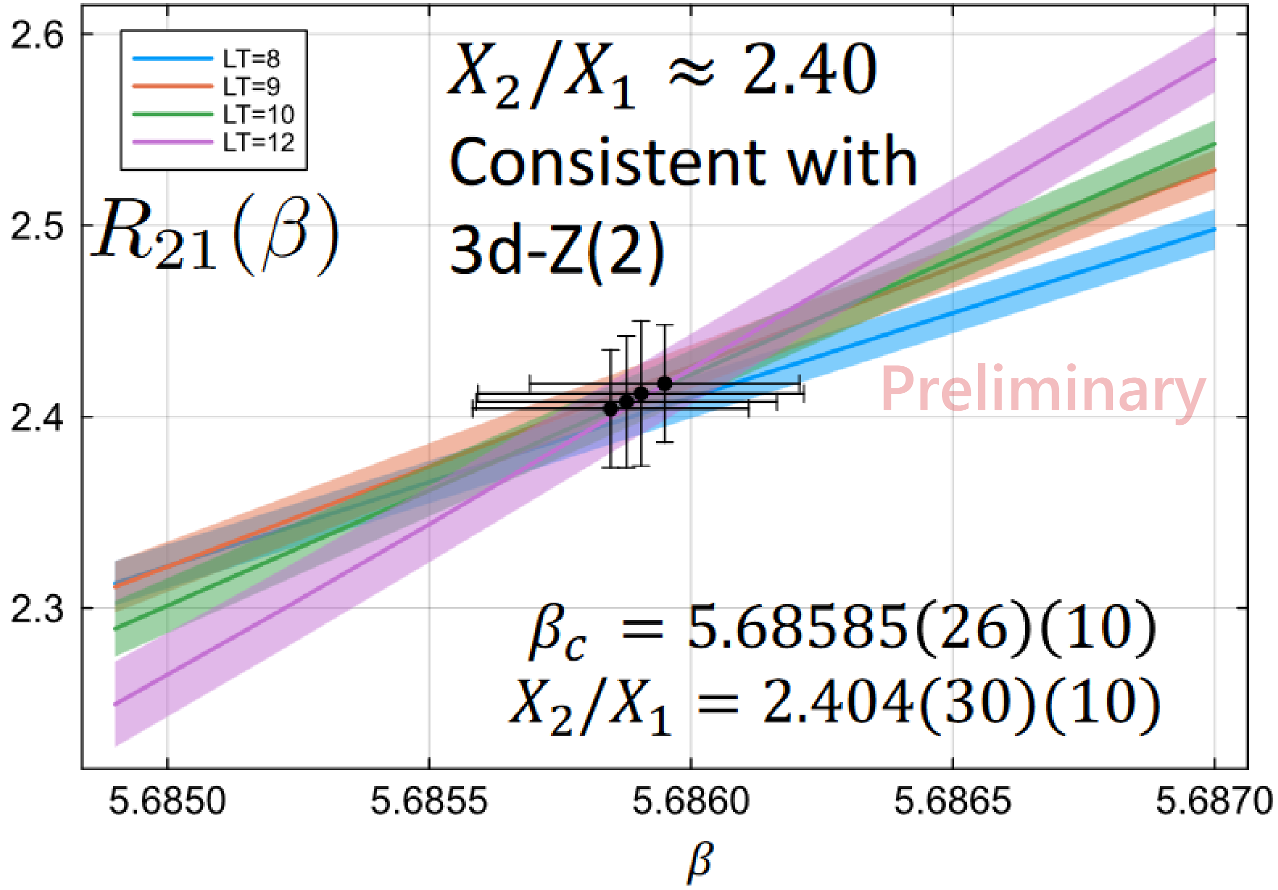
Easy to handle in lattice simulations!

We study the LYZ around the HQ-QCD-CP.

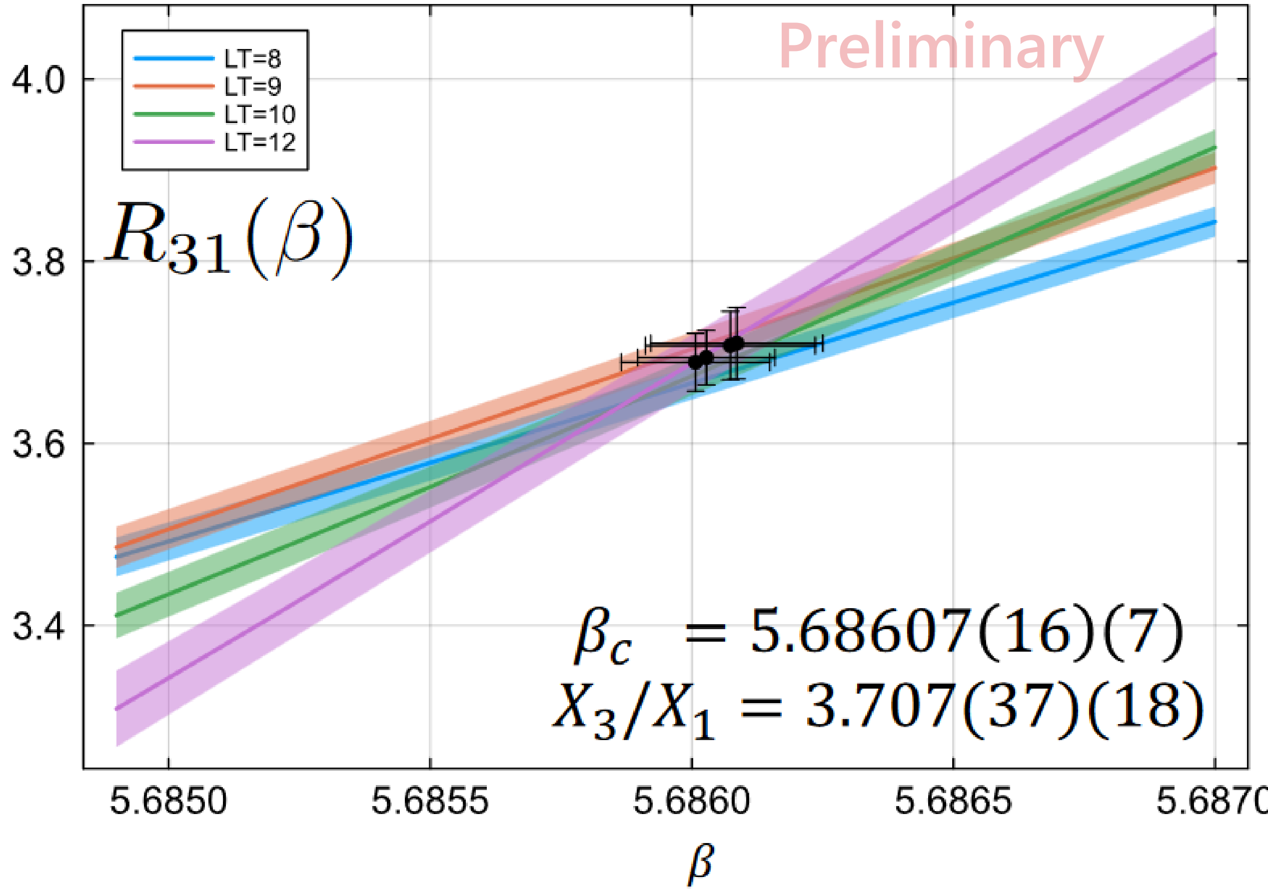
LYZ Ratio

$N_t = 4$, HPE-NLO

2nd/1st LYZ Ratio

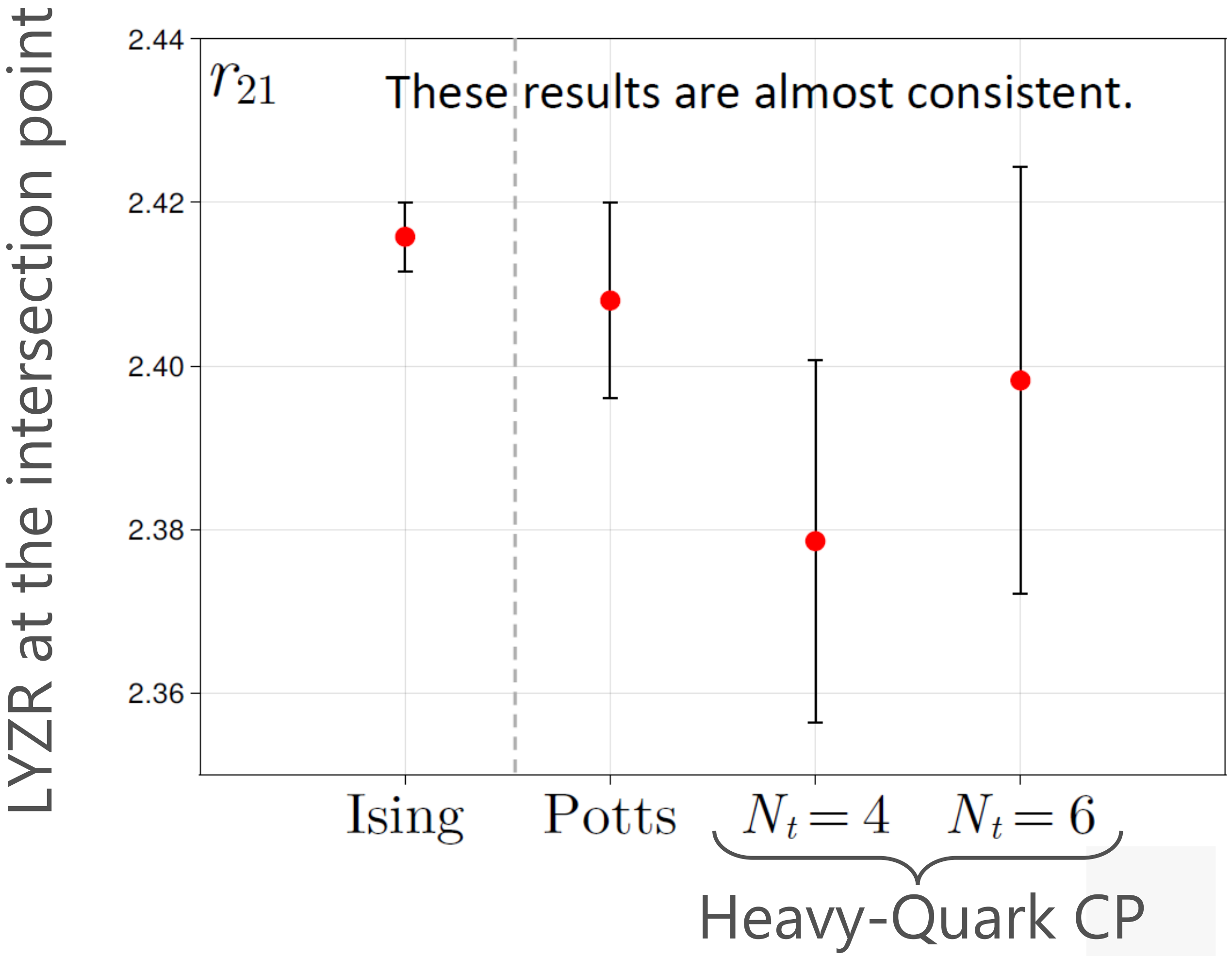


3rd /1st LYZ Ratio



Conistent with the Binder-cumulant analysis $\boldsymbol{\beta_c = 5.68578(22)}$.

LYZR in Various Models



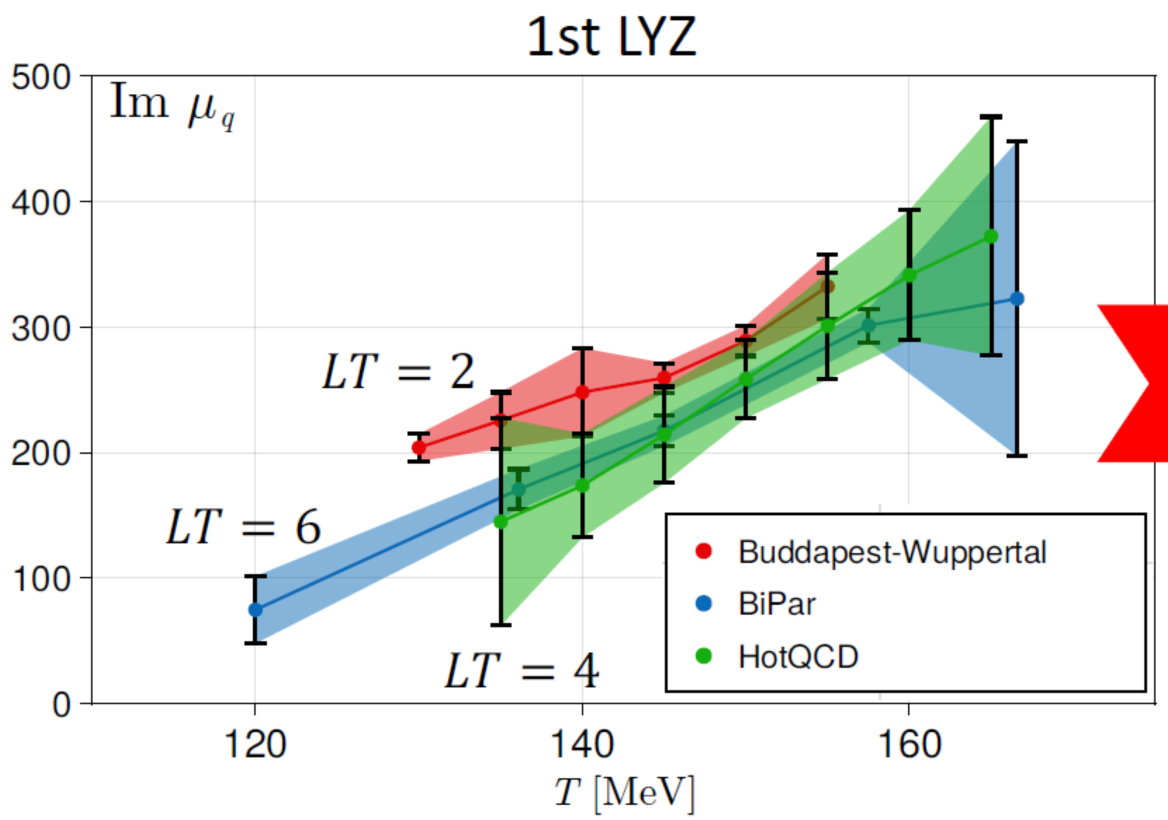
CPs in 3d-Z(2) universality class

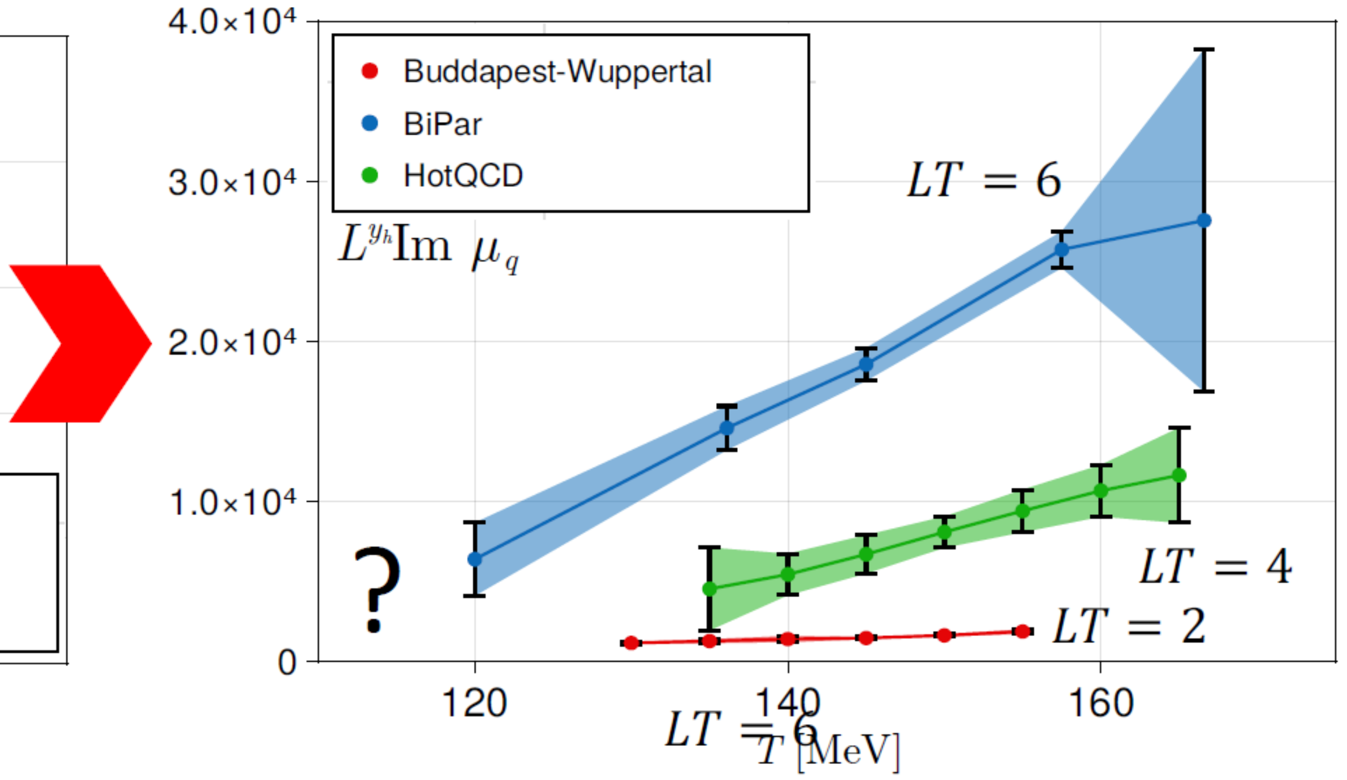
—**Ising**: 2508.20422
—**Potts**: PRL, '25
—**HQ-CP**: 2501.18904

Consistent within statistics ➤ Confirmation of universality

Application to QCD Critical Point?

Single-LYZ analysis for 2+1 flavor QCD





Intersection around $T \simeq 100$ MeV? Yet lower T data are mandatory.

Summary

Lee-Yang-zero ratios

A new tool for **the CP searches** utilizing the **finite-size scaling** of **Lee-Yang zeros**.

$$R_{nm}(t) = \frac{h^{(n)}(t)}{h^{(m)}(t)}$$

Outlook

— Determination of r_{nm} in each universality class
— More sophisticated utilization of $R_{nm}(t, L)$
— Other quantities: ξ_c, mixing matrix A, etc.
— Application to **QCD-CP** and other CPs

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