Lee-Yang zeros and edge singularity in QCD

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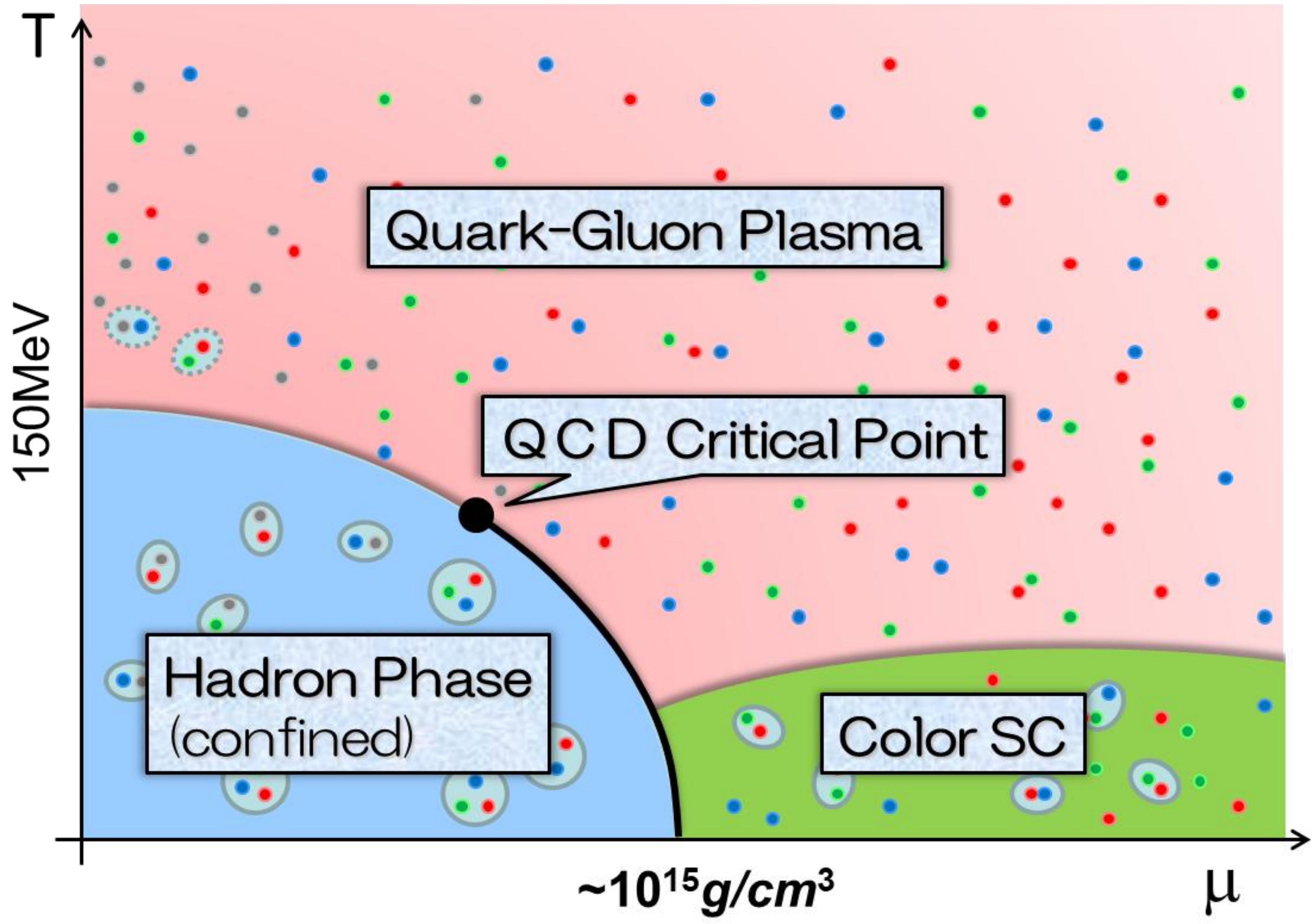
In collab. with
Tatsuya Wada, Gyozo Kovacs, Takahiro M. Doi, Kazuyuki Kanaya

Wada, Kovacs, MK, Doi, arXiv:2605.19964 [hep-ph]
Wada, MK, Kanaya, PRL **134**, 162302 ('25); ibid. JPSJ **95**, 024002 ('26)

QCD Phase Diagram



Rich phase structure in QCD

- QCD critical point(s)
- color superconductivity
- Quest by heavy-ion collisions
- Difficulty in lattice simulations due to sign problem

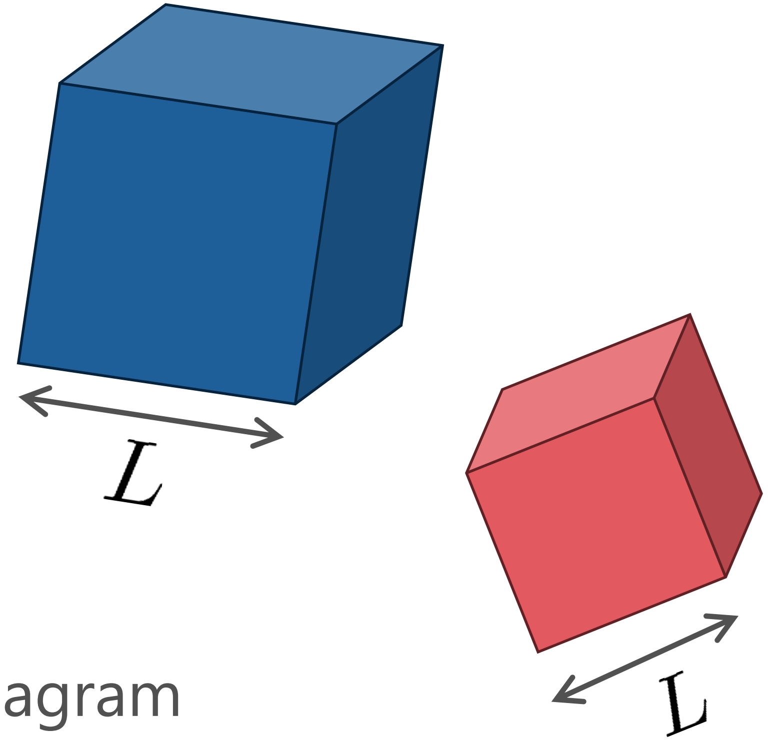
Importance of effective-model studies

- NJL(-like) model, functional RG, …
- Phase structure
- Interpreting HIC/lattice results

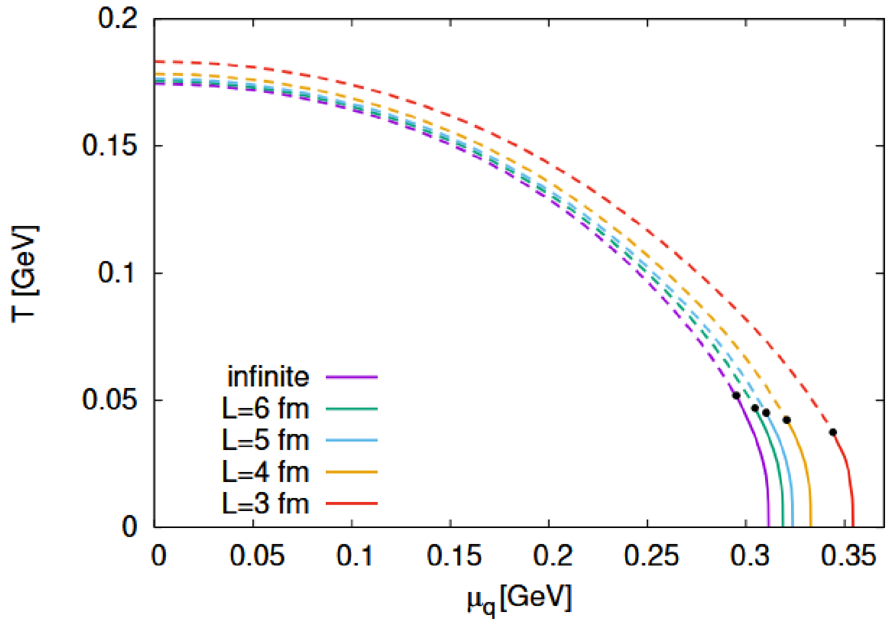
Finite-Size Effects

Why are they important?

— Heavy-ion collisions create FS systems
— Monte-Carlo simulations only for FS systems



FS effects in effective models



Kovacs+, PRD ('23)

L-dependent phase diagram

Caution

— No phase transitions in FS systems.
— Inconsistency with the FS scaling near the critical point.
— Lee-Yang zeros

Lee-Yang Zeros and QCD Phase Diagram

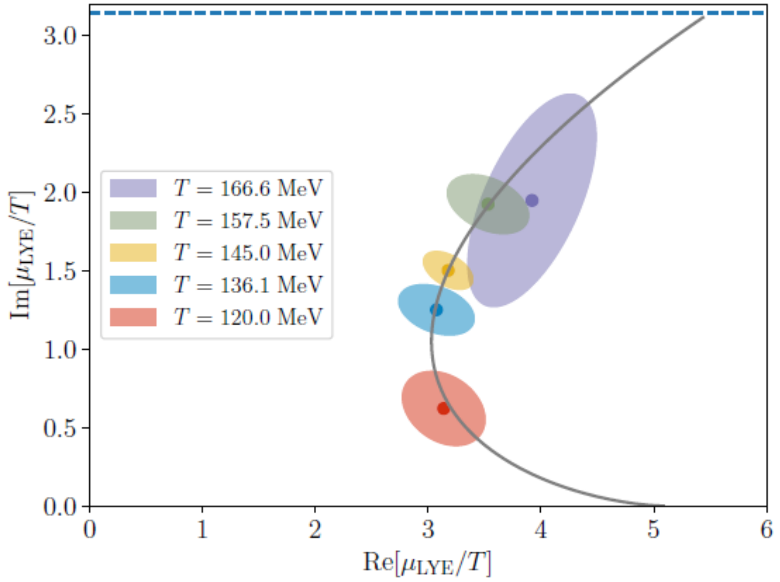
Locating QCD-CP using Lattice data

Clarke+, 2405.10196; Adam+, 2507.13254; ...

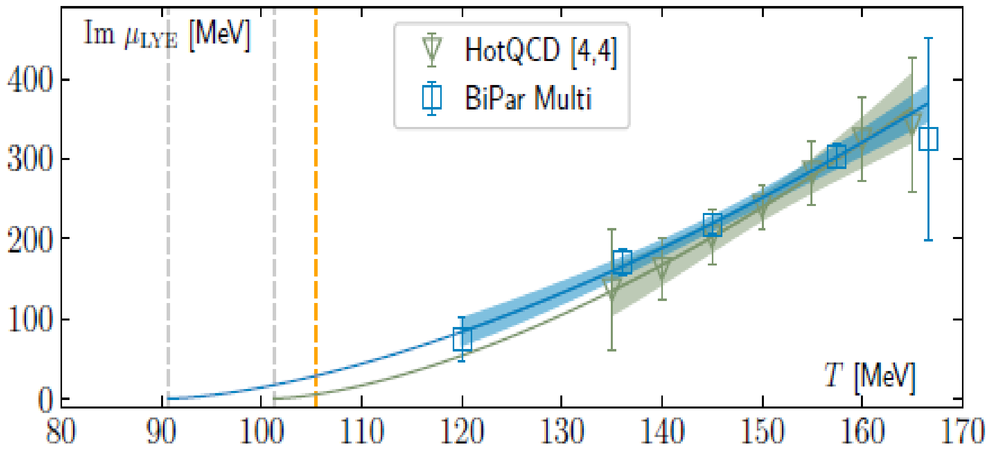
Procedure
1. Determine 1st LYZ numerically
2. Identify it as the LY edge singularity
3. Extrapolate to the CP

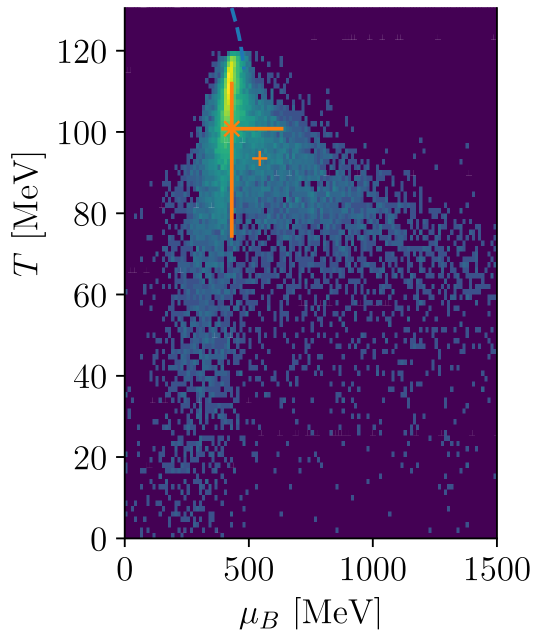
— Imaginary μ + Pade approx.

A similar analysis in an effective model should be useful for a better understanding of this procedure.



Clarke+, 2405.10196





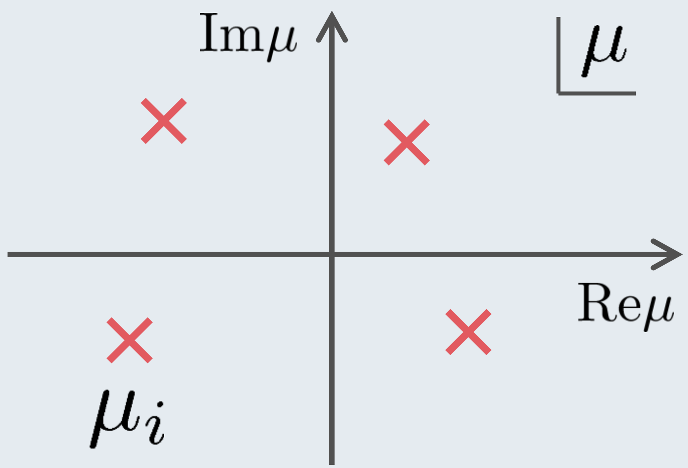
Lee-Yang Zeros

Yang, Lee; Lee, Yang ('52)

Partition Function $Z(T,\mu)$

Finite V ➤ Polynomial of μ (or T)

$$Z(T,\mu) = \prod_i (\mu - \mu_i)$$



➤ **zeros** on the complex plane

=**Lee-Yang Zeros**

Lee-Yang Zeros

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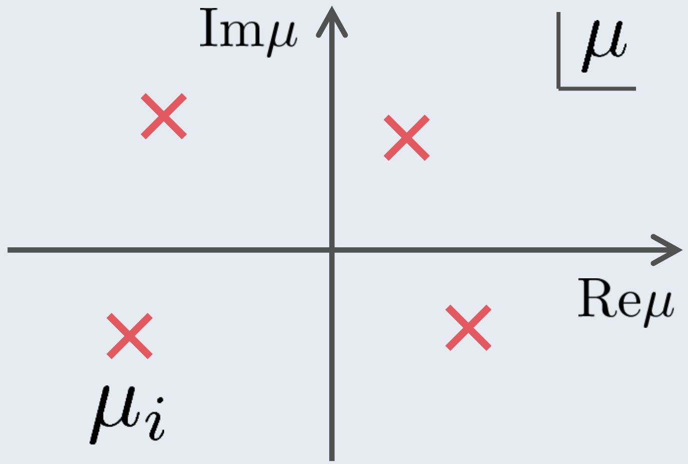

➤ **zeros** in the complex plane

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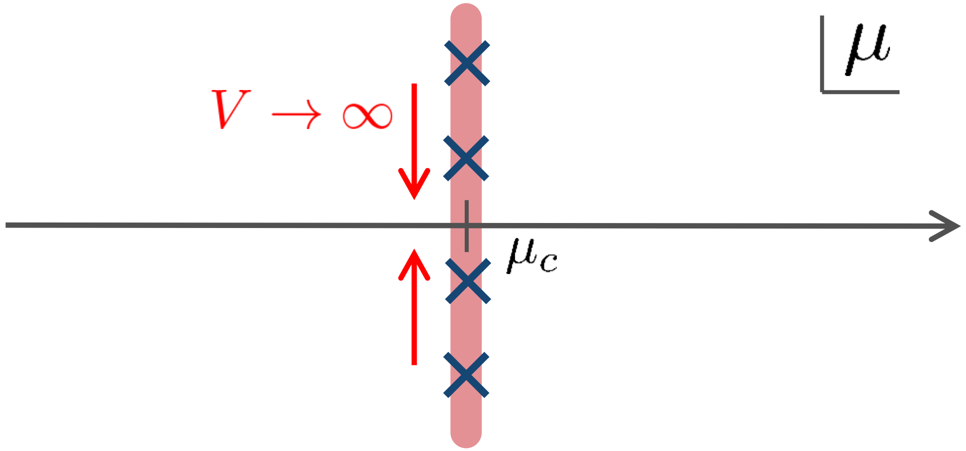
Phase Transition & LYZ

First-order transition
at $\mu=\mu_c$

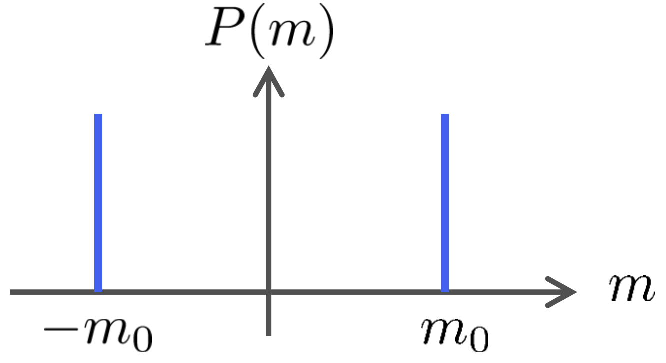


— For $V\to\infty$, LYZs are accumulated on the line crossing the real axis at $\mu=\mu_c$.

LYZ and 1st-Order Phase Transitions

- 1st PT ➔ coexistence of two phases

  Magnetization $M = \langle m \rangle V = \pm m_0 V$

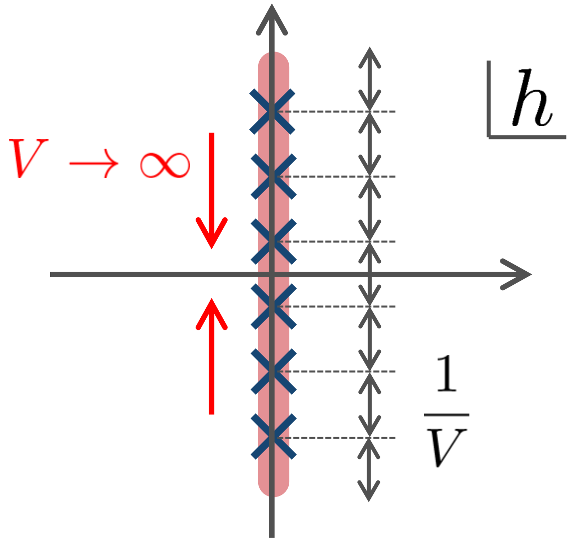


- Partition function

$$Z = \mathrm{Tr}[e^{-\beta(H_0 - Mh)}]$$
$$\propto e^{-\beta V m_0 h} + e^{\beta V m_0 h}$$
$$= e^{-\beta V m_0 h}(1 + e^{2\beta V m_0 h})$$

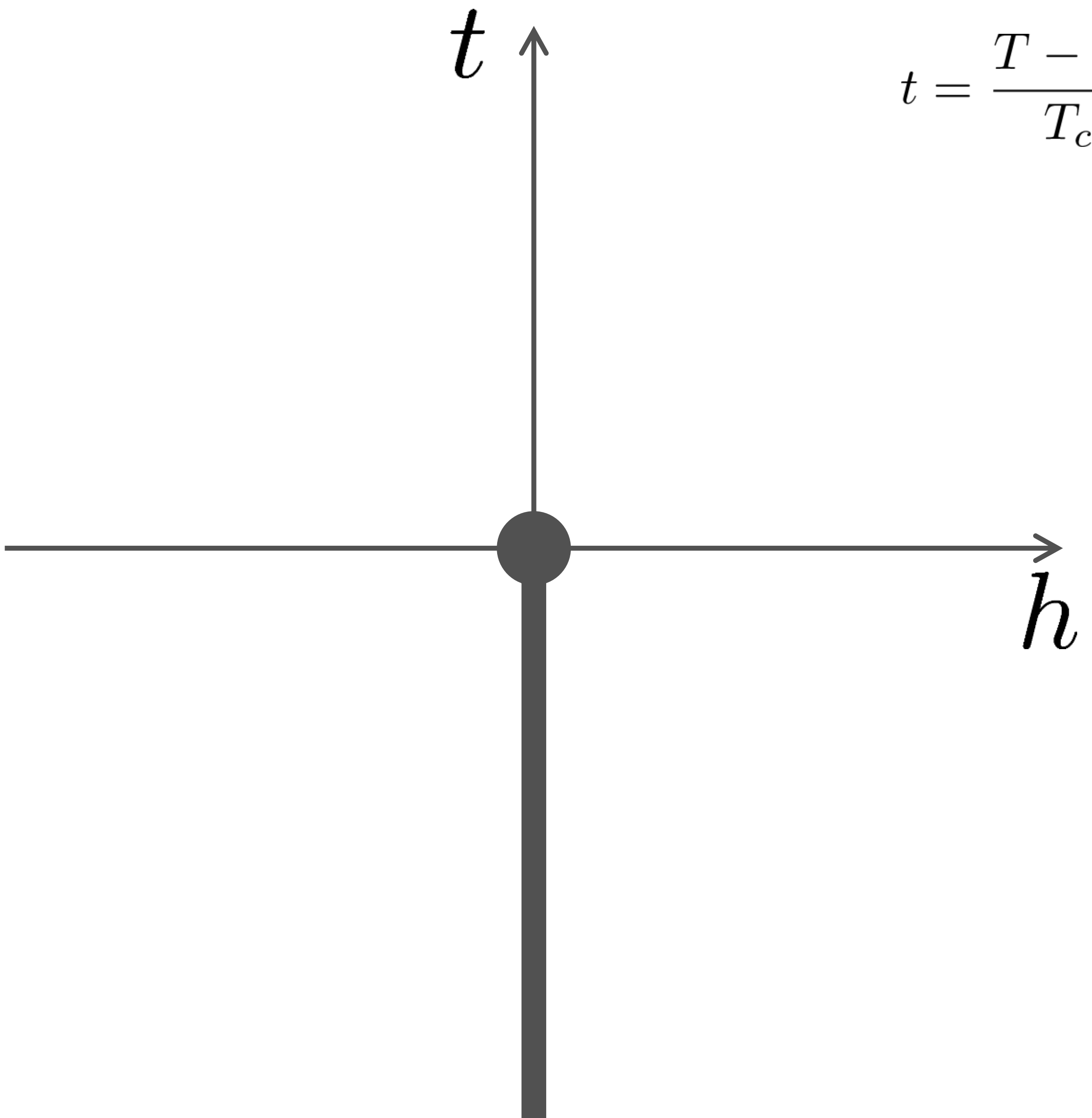
➔LYZs ($Z = 0$) at

$$h = i\left(n + \frac{1}{2}\right)\frac{1}{\beta V m_0}$$



Note: LYZs become continuous for $V \to \infty$.

LYZ around a Critical Point in Ising Model



$$t = \frac{T - T_c}{T_c}$$

1st-transition

singularity on the real h axis

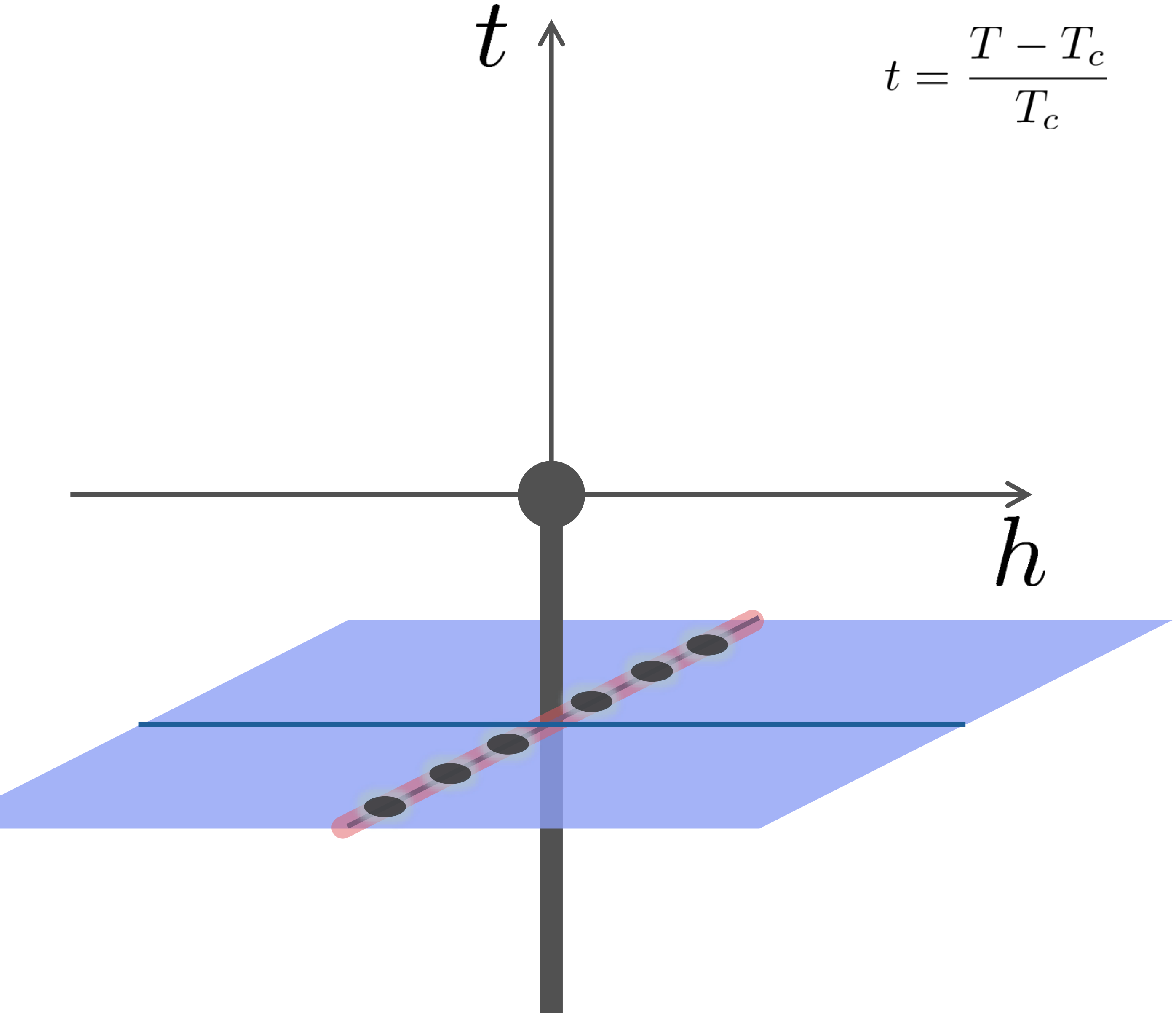
Crossover

no singularity on the real axis

Note:
LYZ in complex-h plane are purely imaginary.

Lee-Yang, 1952

LYZ around a Critical Point in Ising Model



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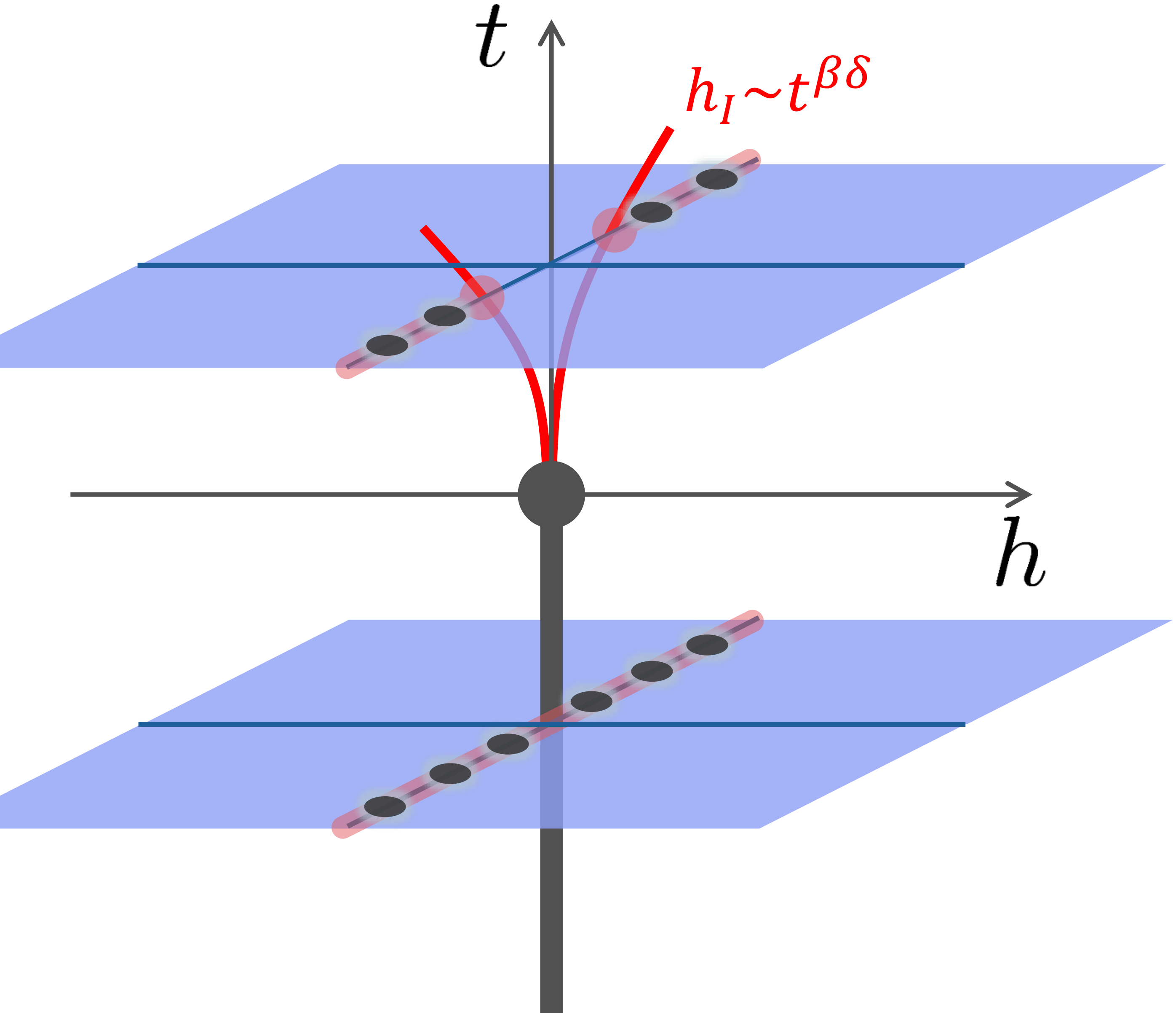

1st-transition

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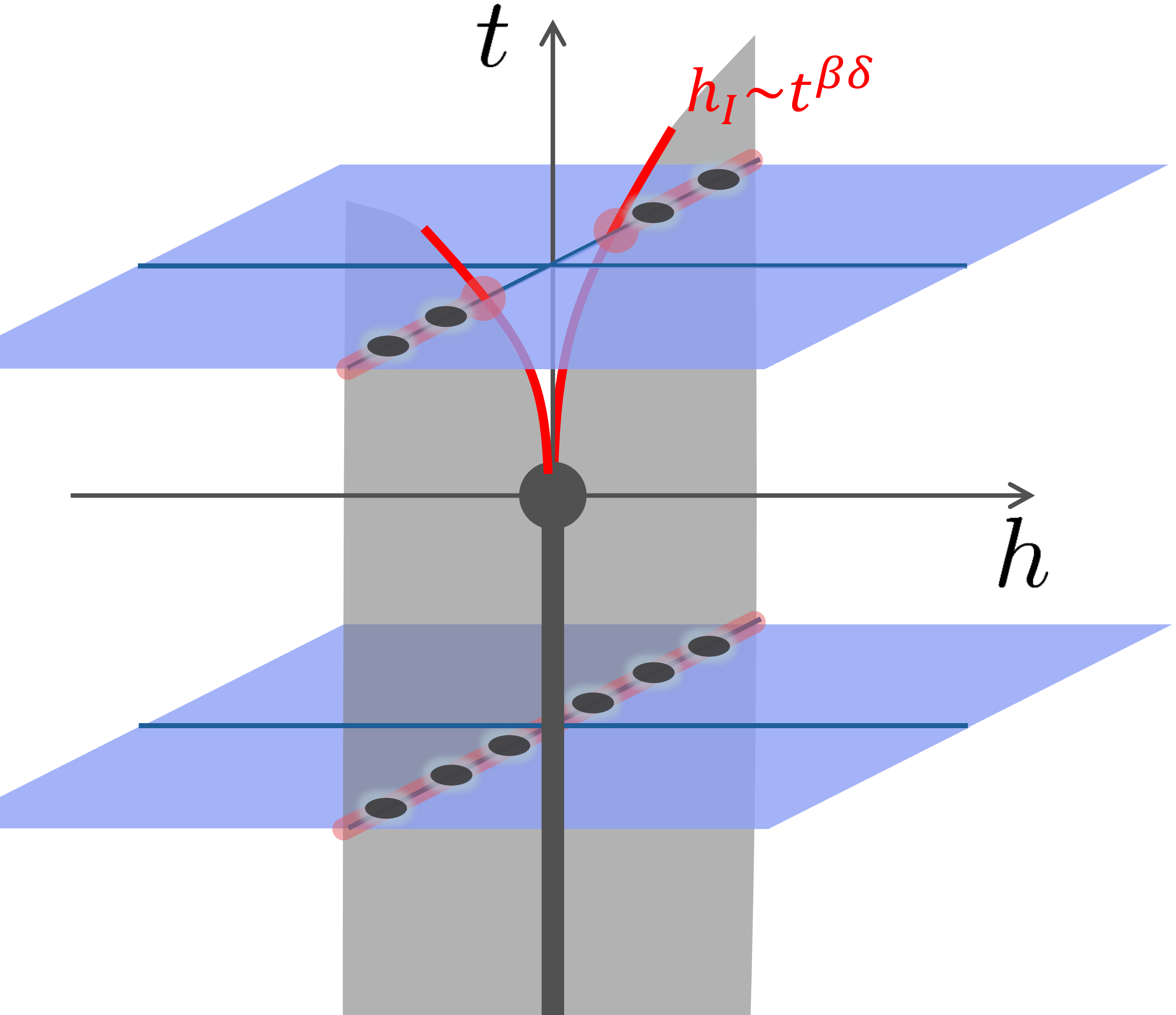
LY edge singularity

Starting from the CP

Its behavior is governed by the the scaling function.

$$h_I \sim t^{\beta\delta}$$

LYZ around a Critical Point in Ising Model



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singularity on the real h axis

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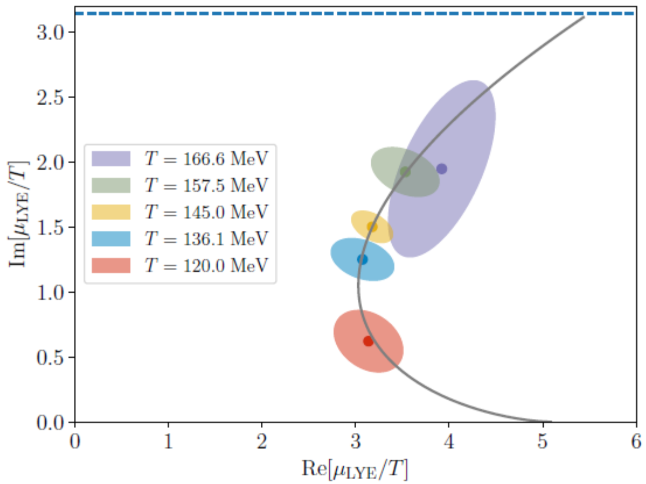
Recent Progress in LYZ/LYES and Lattice

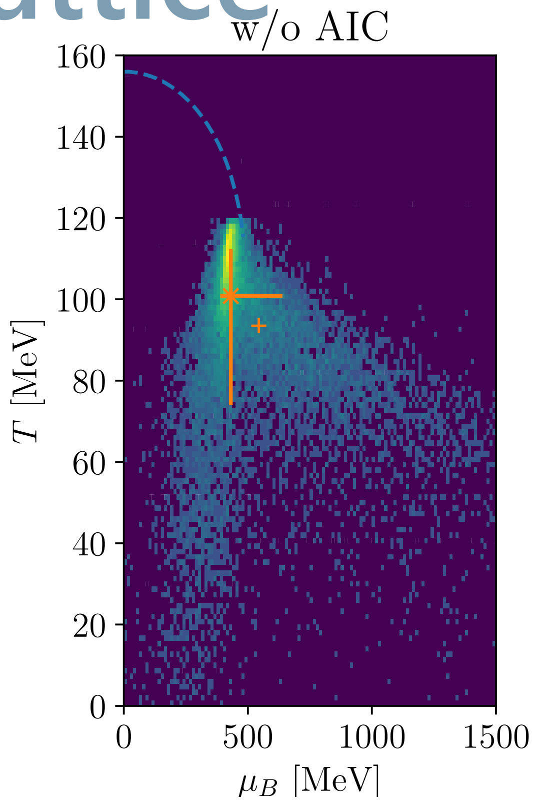
Analytic Structure

— Scaling functions, FRG, …

An, Mesterhazy, Stephanov ('16)
Johnson, Rennecke, Skokov ('23)
Karsch, Schmidt, Singh ('23)
…

Clarke+, 2405.10196

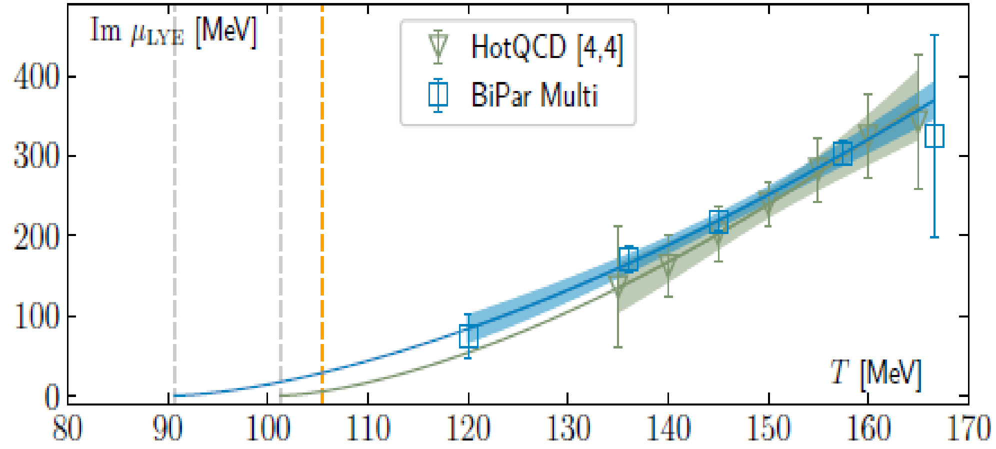




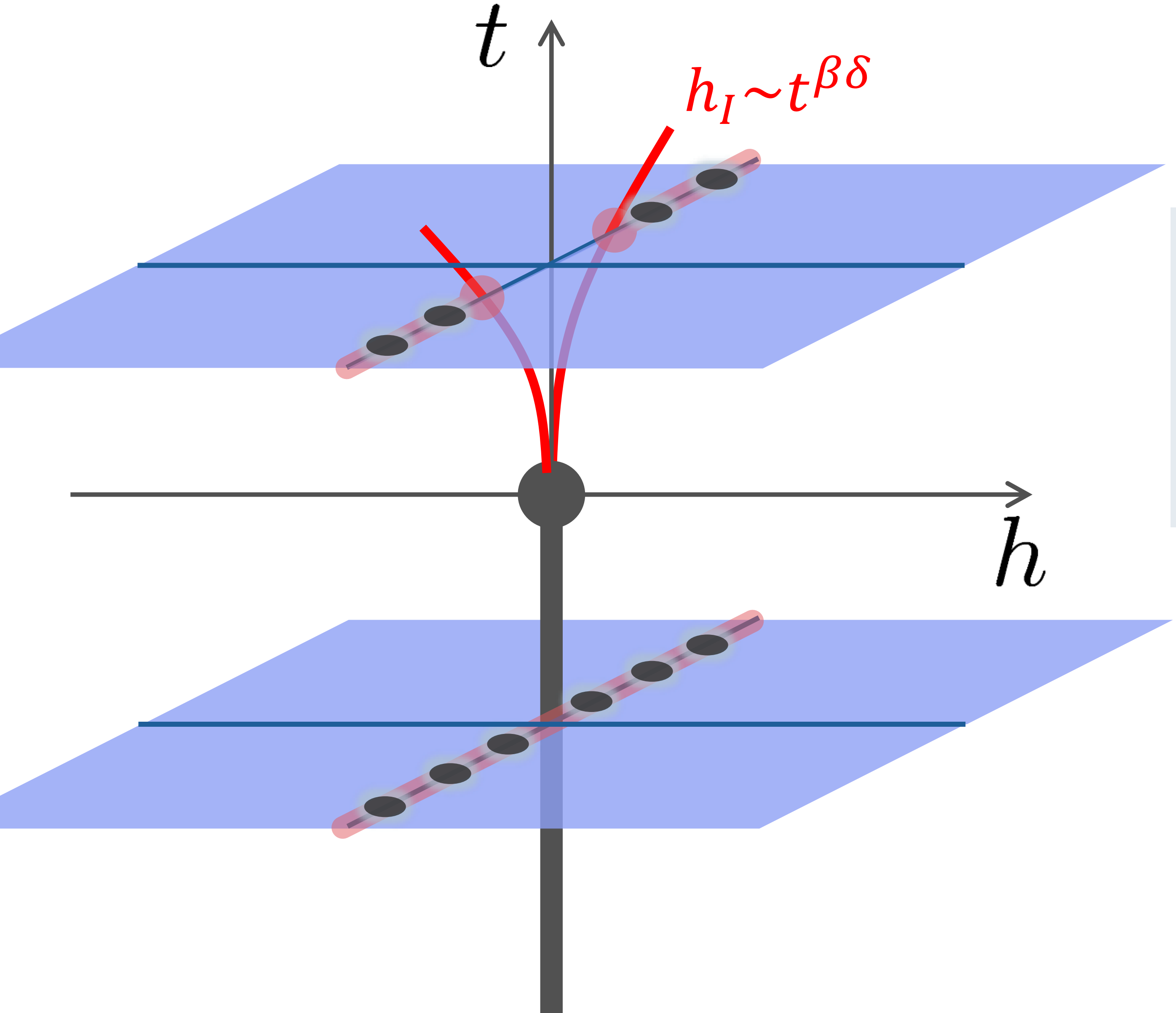
Locating QCD-CP at $\mu \neq 0$ on the lattice?

Clarke+, 2405.10196; Adam+, 2507.13254

— Taylor exp. + Imaginary μ + Pade approx.
— Identify the 1st LYZ to be LYES



Caveat



On finite volume,

1st LYZ ≠ LYES

Linear Sigma Model $(N_c = 3, N_f = 1)$

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - U(\phi) + \bar{\psi}(i\gamma^\mu\partial_\mu - \gamma_0\mu_q - g\phi)\psi$$

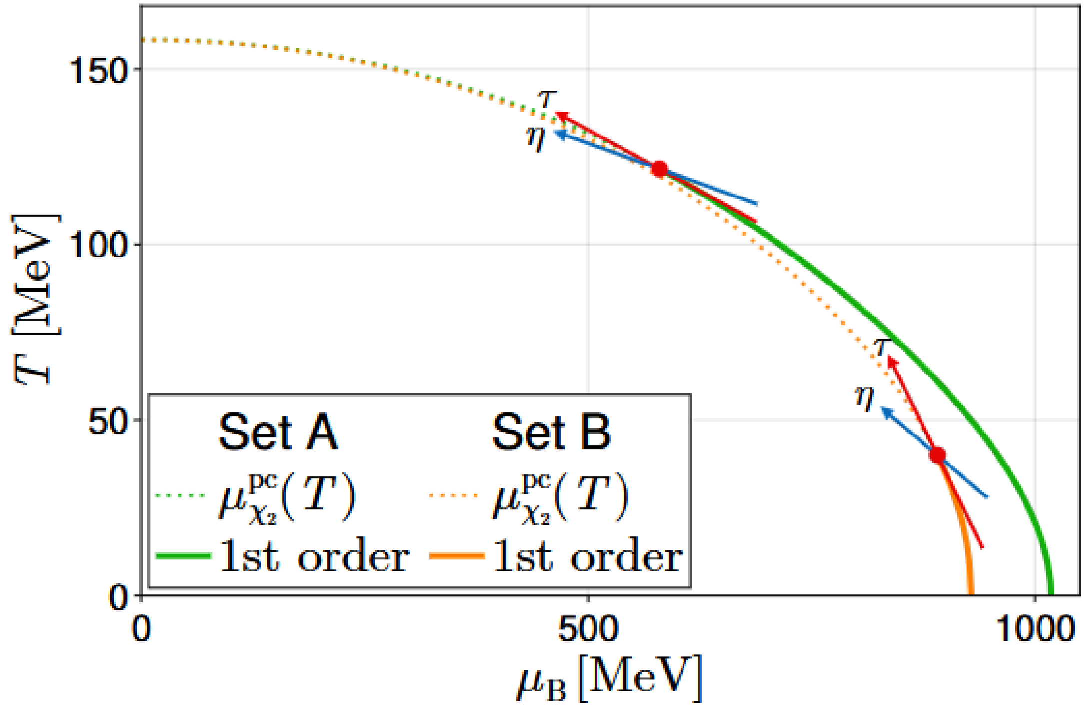
$$U(\phi) = \frac{m^2}{2}\phi^2 + \frac{\lambda}{4}\phi^4 - h\phi$$

ϕ: scalar boson
ψ: quark

	λ	g	$T_{\rm CP}$	$\mu_{\rm CP}$	$T_0^{\rm pc}$
set A	29.0	4.55	121.53	579.97	158.46
set B	46.0	4.35	40.04	890.84	158.38

Mean-field approximation

1. assume spatially uniform field $\phi(x) = \bar{\phi}$
2. integrating out ψ neglecting $\delta\phi^2$
3. effective potential $U = U(\bar{\phi})$
4. find its global minimum (solve the self-consistent equation)
5. free-energy $f = U(\bar{\phi}_{\rm min})$



FS Effects in Mean-Field Approach

Old Ideas: Momentum Discretization

$$U_{\bar{q}q}(\bar{\phi}; T, \mu_{\rm B}) = -2N_{\rm c}TV \int \frac{d^3p}{(2\pi)^3} f(p)$$

$$\xrightarrow[L:\text{finite}]{} -2N_{\rm c}T \sum_{\boldsymbol{p}} f(p)$$

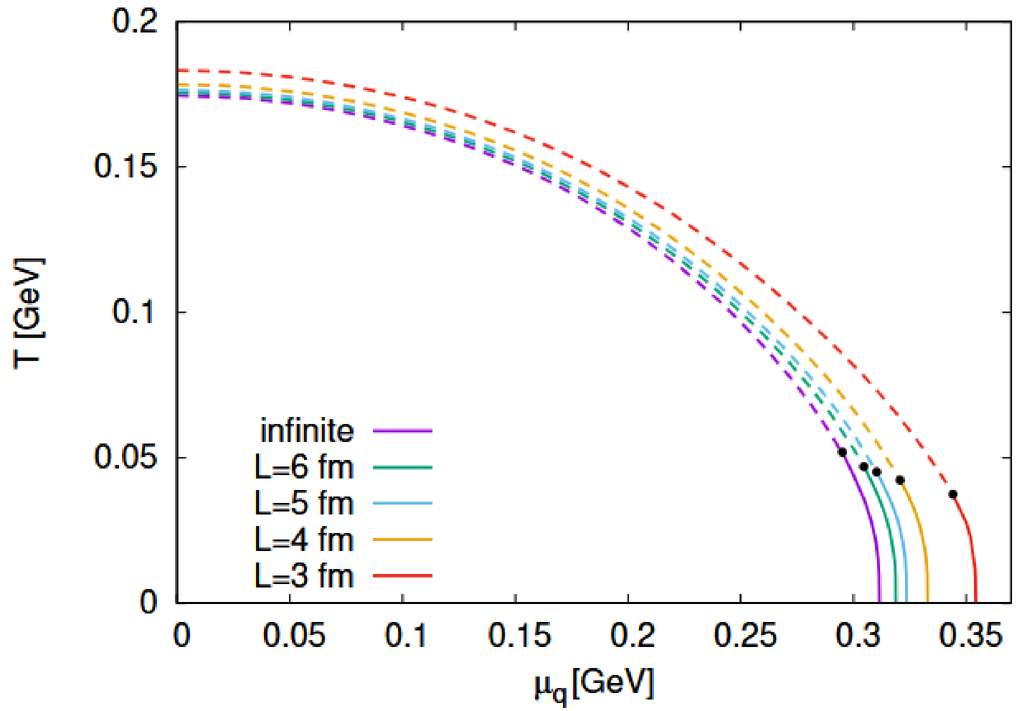
$$f(p) = \sum_{s=\pm 1} \log(1 + e^{-(E(p) - s\mu_q)/T})$$

Taking the same procedure as before, we obtain thermodynamics in finite volume.

Problems

— Sharp phase transition appears even in FS systems.

— LYZs do not appear in this approach.



Kovacs+, PRD ('23)

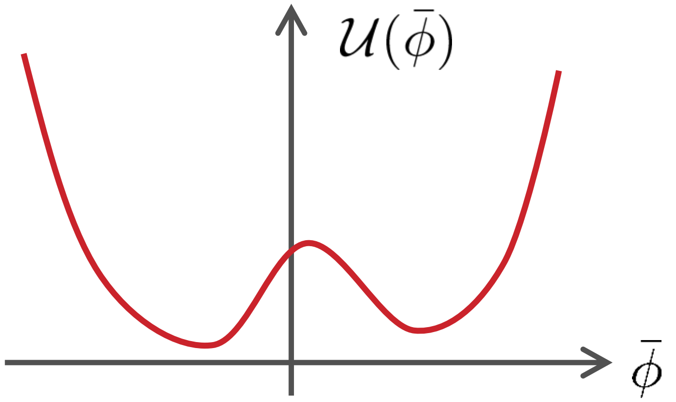
Finite-Size MF Approach

Kovacs+, 2510.24507

Incorporate fluctuations of $\bar{\phi}$

$$f = -\frac{T}{V}\ln Z \qquad Z = \int d\bar{\phi}\, e^{-V\mathcal{U}(\bar{\phi})/T}$$

— Fluctuations of $\bar{\phi}$ is incorporated into f.
– f becomes a smooth function for finite V.
— Conventional MFA is recovered for $V \to \infty$.
— LYZs naturally emerges.



Let's apply this method to the study of LYZs in this model!

Wada, Kovacs, MK, Doi ('26)

Finite-Size Scaling (FSS)

Landau potential

$$U_{\text{Landau}}(\bar\phi;\tau,\eta)=\frac{\tau}{2}\bar\phi^2+\frac{b}{4!}\bar\phi^4-\eta\bar\phi$$

$$\mathcal{Z}_{\text{Landau}}(\tau,\eta,L)\sim\int d\bar\phi\, e^{-L^d(\tau\bar\phi^2/2+b\bar\phi^4/4!-\eta\bar\phi)/T}$$

Scaling Transformation

$$\tau\to\ell^{y_\tau}\tau,\quad \eta\to\ell^{y_\eta}\eta,$$
$$\bar\phi\to\ell^{y_\phi}\bar\phi,\quad L\to\ell^{-1}L$$

$$\mathcal{Z}_{\text{Landau}}(\tau,\eta,L)\sim\mathcal{Z}_{\text{Landau}}(\ell^{y_\tau}\tau,\ell^{y_\eta}\eta,\ell^{-1}L)$$
$$\sim\tilde{\mathcal{Z}}(L^{y_\tau}\tau,L^{y_\eta}\eta)$$

:FSS

MF scaling exponents:

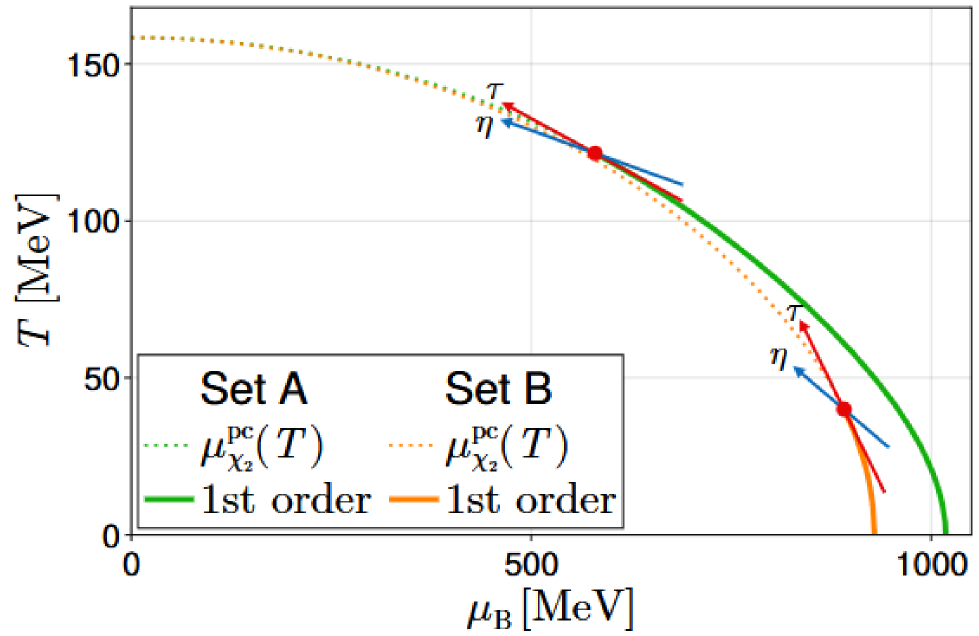
$$y_\tau=\frac{d}{2},\quad y_\eta=\frac{3d}{4},\quad y_\phi=\frac{d}{4}$$

FSS in MF near a CP is obtained in FS-MF approach.

FSS in General Systems

Expansion of $U(\phi)$ around the CP

$$\mathcal{U}(\bar\phi, T, \mu_{\rm B})$$
$$= u_0 - (a_{21}\delta T + a_{22}\delta\mu_{\rm B})\delta\bar\phi + \frac{a_{11}\delta T + a_{12}\delta\mu_{\rm B}}{2}\delta\bar\phi^2$$
$$+ \frac{b}{4!}\delta\bar\phi^4 + \sum_{n=5}^{\infty}\frac{\alpha_n^{\rm CP}}{n!}\delta\bar\phi^n$$



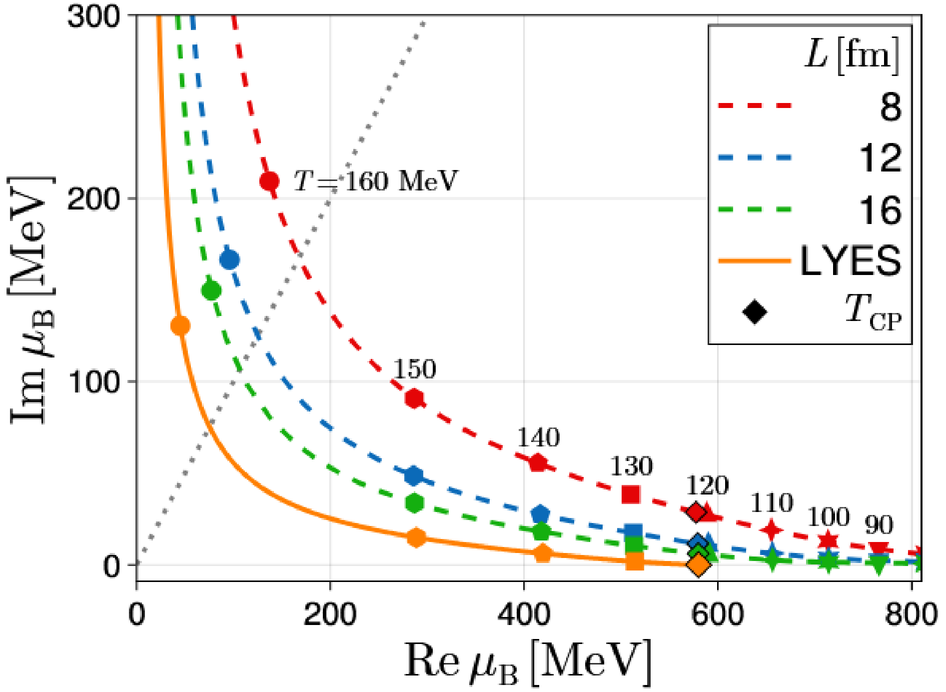
Linear-mapping relation to Landau potential

$$\begin{pmatrix}\tau\\ \eta\end{pmatrix} = \begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}\begin{pmatrix}T - T_{\rm CP}\\ \mu_{\rm B} - \mu_{\rm CP}\end{pmatrix} \equiv A\begin{pmatrix}\delta T\\ \delta\mu_{\rm B}\end{pmatrix}$$

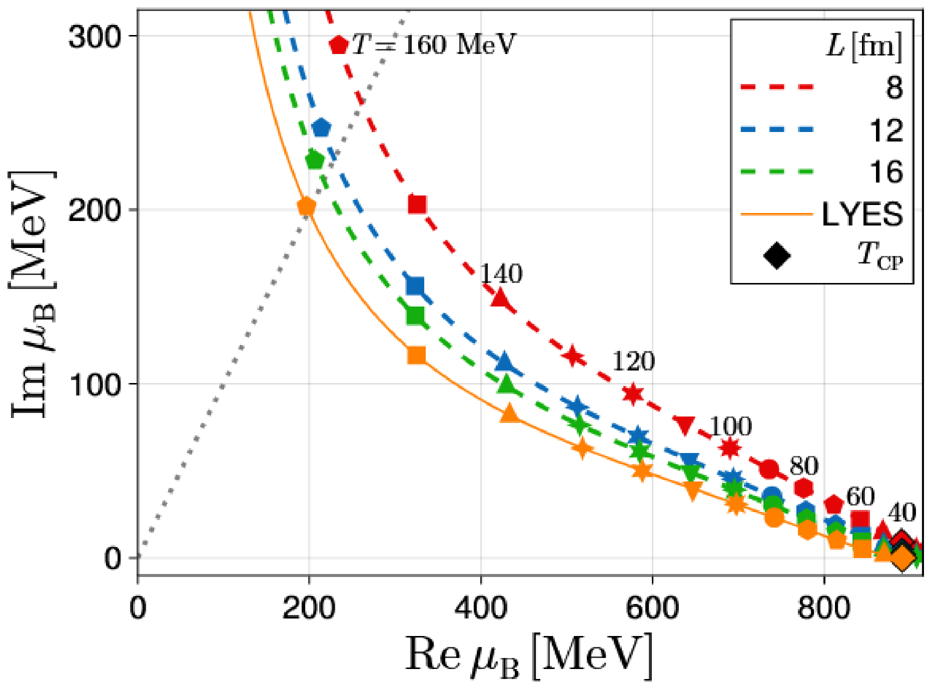
$$a_{11} = \partial_T\partial_{\bar\phi}^2\mathcal{U}\,, \quad a_{12} = \partial_{\mu_{\rm B}}\partial_{\bar\phi}^2\mathcal{U}\,,$$
$$a_{21} = \partial_T\partial_{\bar\phi}\mathcal{U}\,, \quad a_{22} = \partial_{\mu_{\rm B}}\partial_{\bar\phi}\mathcal{U}$$

1st LYZ in the complex-μ_B plane

Set A



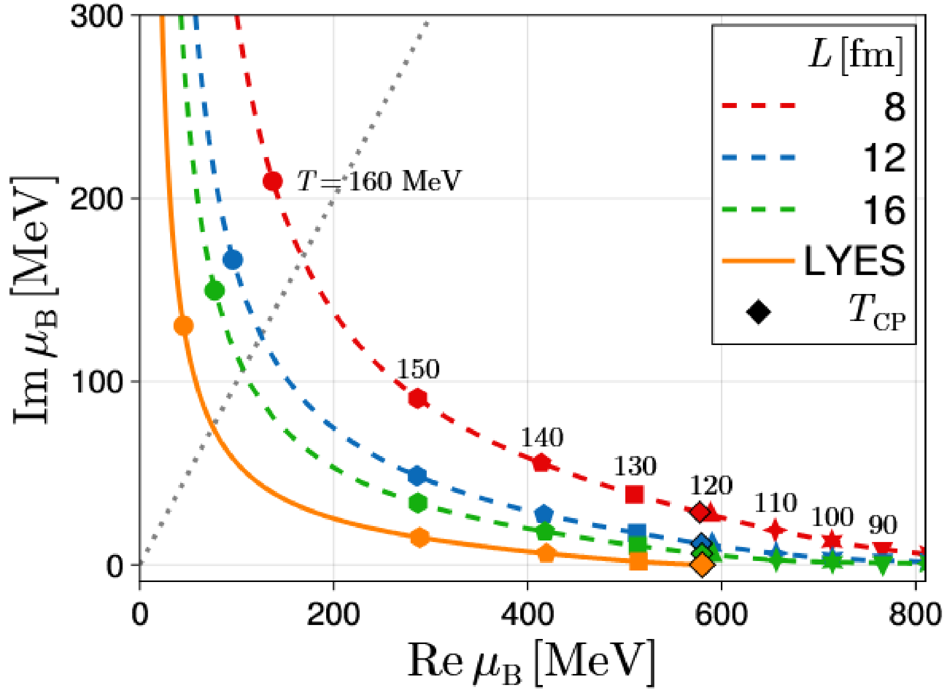
Set B



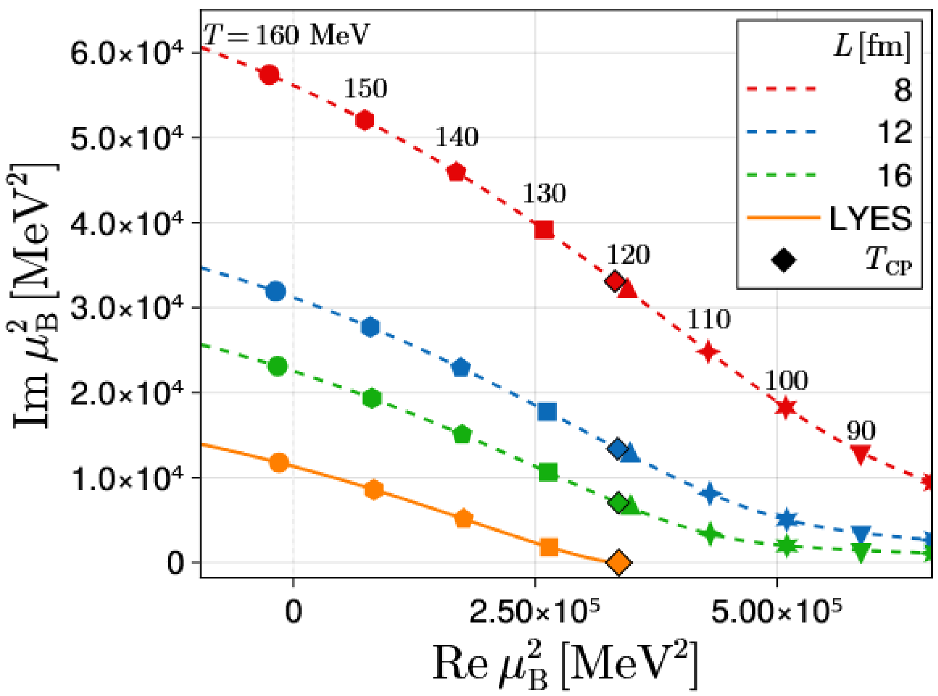
— The 1st LYZ on finite L is away from the LYES. It never touches the real axis.
— The trajectories of LYES/LYZ move upward for small $\mathrm{Re}\mu_B$.

1st LYZ: complex-μ_B/μ_B^2 planes: Set A

Complex-μ_B



Complex-μ_B^2

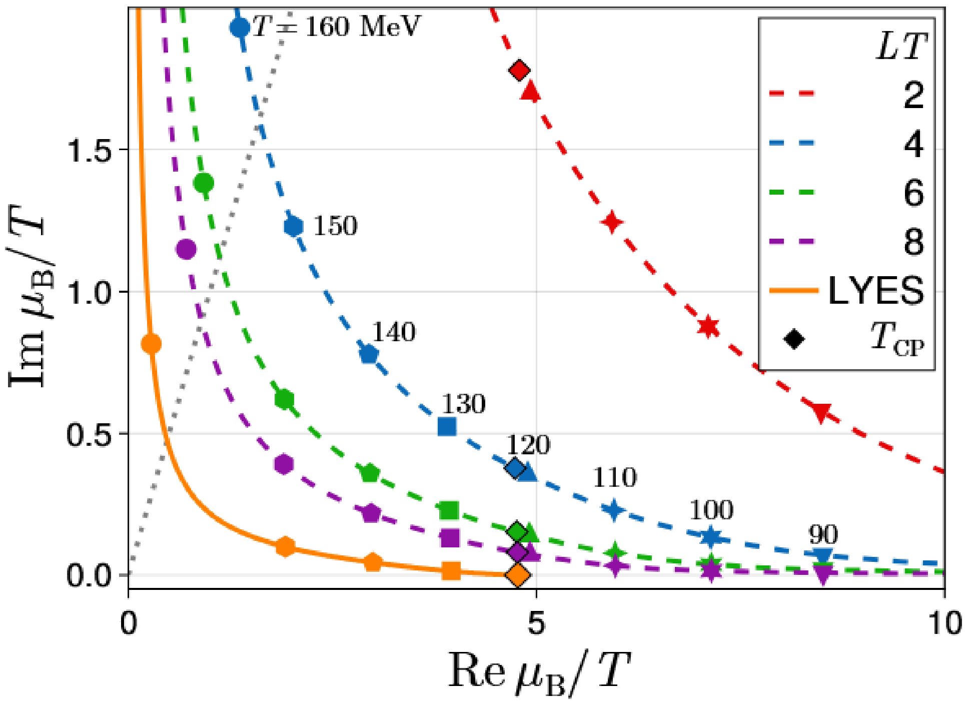


— Charge conjugation symmetry leads to $Z(\mu) = Z(-\mu)$ ➔ Z is an even function.

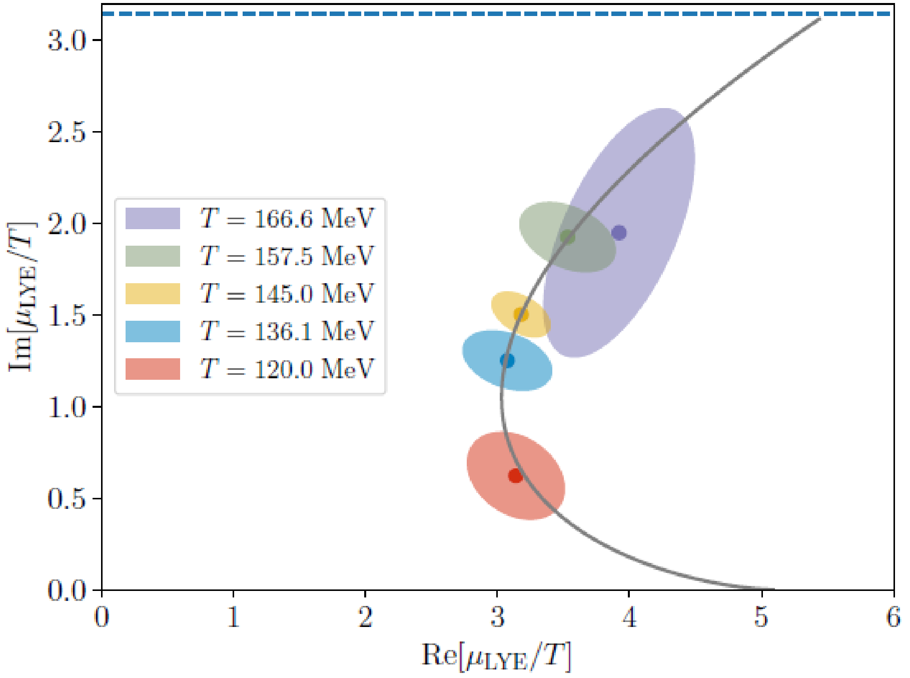
— Better scaling behavior in the complex μ_B^2 plane.

Comparison with Lattice Results

our result



lattice (Clarke+, $LT = 6$)



— Deviation of the 1st LYZ from LYES is large at $LT = 6$.

— In lattice result, $\mathrm{Re}\mu/T$ increases for high T.

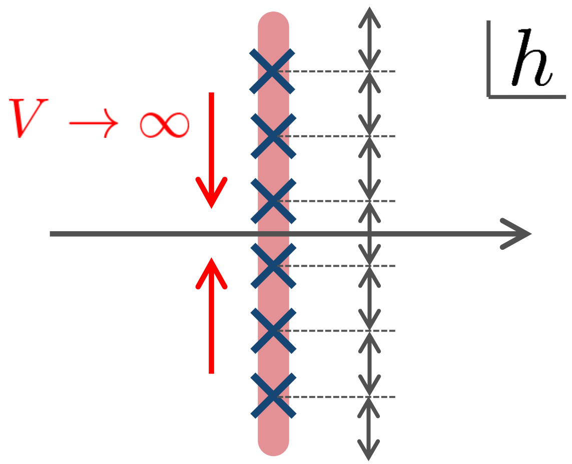
  — Effects of Roberge-Weiss symmetry?

Wada, MK, Kanaya, PRL ('25)

Lee-Yang Zero Ratios

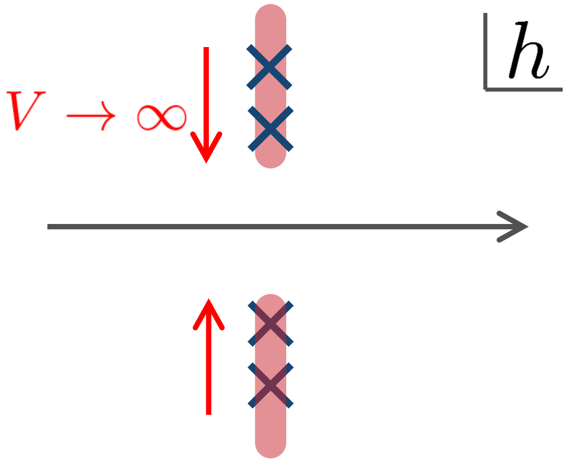
$$R_{nm}(t) = \frac{h^{(n)}(t)}{h^{(m)}(t)}$$

First-Order Side $(t < 0)$



$$R_{nm}(t) \xrightarrow[V\to\infty]{} \frac{2n-1}{2m-1}$$

Crossover Side $(t > 0)$



$$R_{nm}(t) \xrightarrow[V\to\infty]{} 1$$

Finite-Size Scaling

Scaling Hypothesis

$$F_{\mathrm{sing}}(t, h, L^{-1}) = \tilde{F}_{\mathrm{sing}}(L^{y_t} t, L^{y_h} h)$$

$$Z_{\mathrm{sing}}(t, h, L^{-1}) = \tilde{Z}_{\mathrm{sing}}(L^{y_t} t, L^{y_h} h)$$

$$F = F_{\mathrm{sing}} + F_{\mathrm{reg}}$$

$$Z = Z_{\mathrm{sing}} \times Z_{\mathrm{reg}}$$

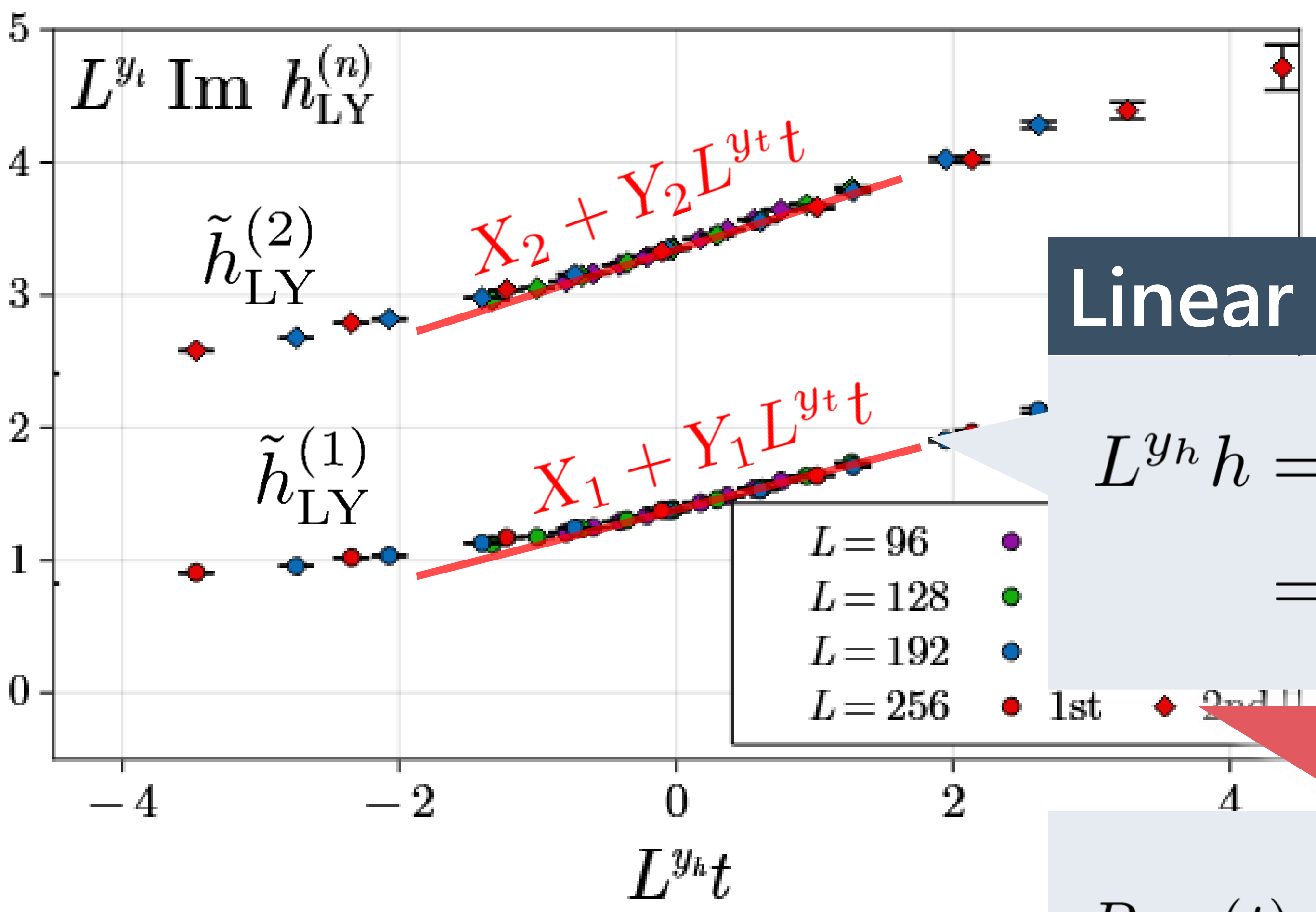
LYZ in the scaling region on finite volume

$$Z(t, h, L^{-1}) \sim \tilde{Z}_{\mathrm{sing}}(L^{y_t} t, L^{y_h} h) = 0$$

$$L^{y_h} h_{\mathrm{LY}}^{(n)}(t) = \tilde{h}_{\mathrm{LY}}^{(n)}(L^{y_t} t)$$

Itzykson, Nucl. Phys. 220 (1983)

LYZ near $t = 0$



Linear Approx. at $t = 0$

$$L^{y_h} h = \tilde{h}_{LY}^{(i)}(L^{y_t} t)$$
$$= X_i + Y_i L^{y_t} t + \mathcal{O}(t^2)$$

$$R_{nm}(t) = \frac{X_n}{X_m}\left(1 + C_{nm} t L^{y_t} + \mathcal{O}(t^2)\right)$$

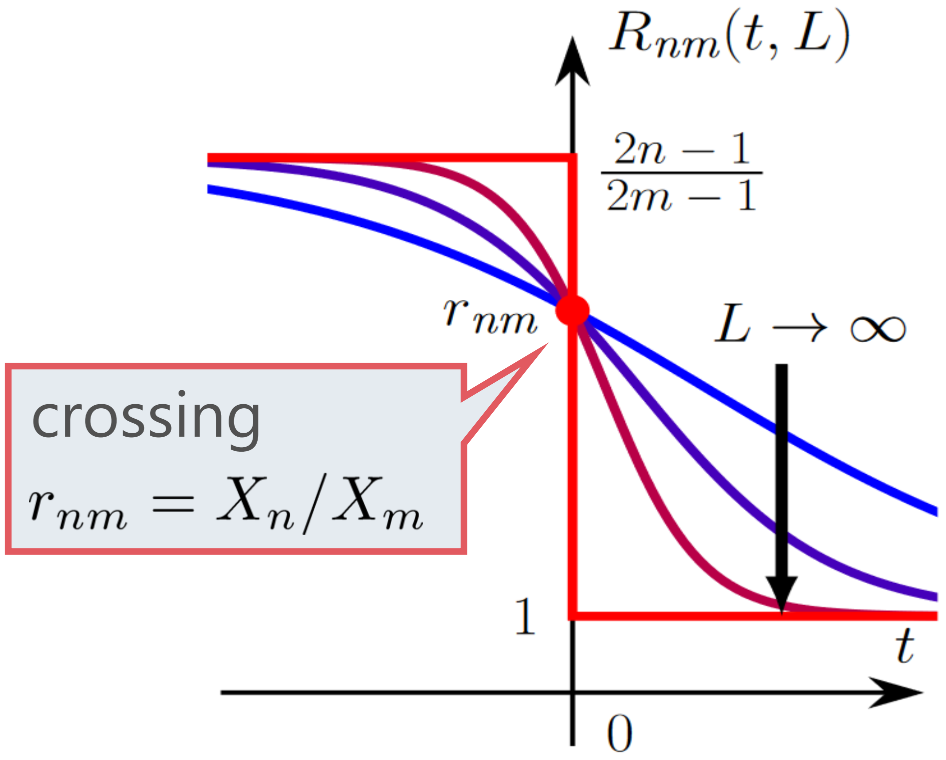
LYZ Ratios $R_{nm}(t) = \frac{h^{(n)}(t)}{h^{(m)}(t)}$

Wada, MK, Kanaya, PRL, '25

$$R_{n1}(t) \xrightarrow[V\to\infty]{} \begin{cases} 2n-1 & t<0 \text{ (1st order)} \\ 1 & t>0 \text{ (crossover)} \end{cases}$$

$$R_{nm}(t) = r_{nm}\left(1 + C_{nm}tL^{y_t} + \mathcal{O}(t^2)\right)$$

near $t = 0$



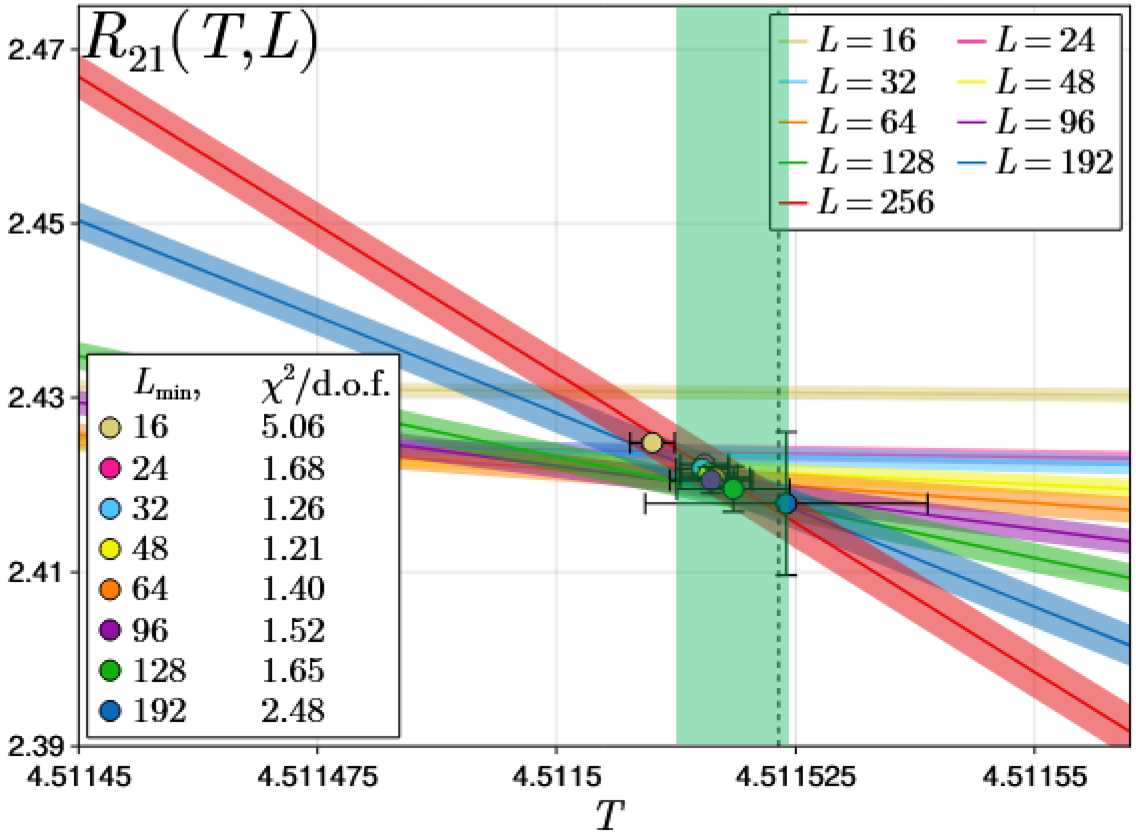
— $R(0)$ is L independent, the universal value.

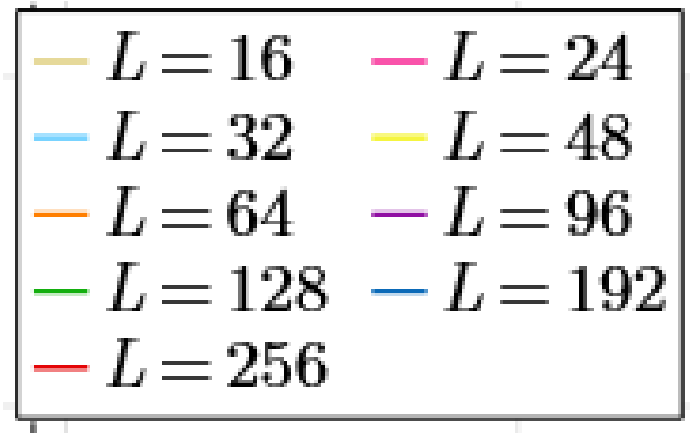
—Intersection point of various L gives the CP.

—Reminiscent of Binder-cumulant analysis

LYZR vs Binder Cumulant in 3d-Ising

LYZR

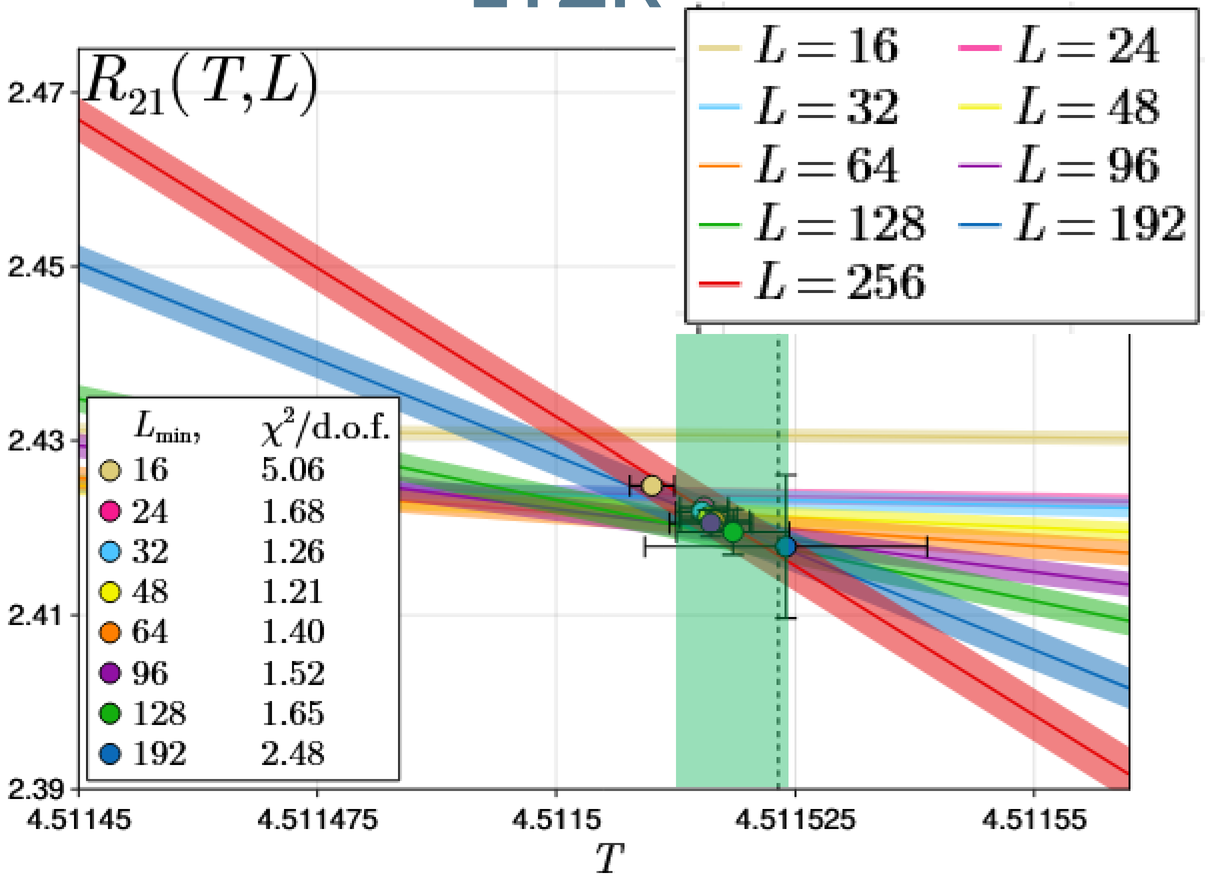




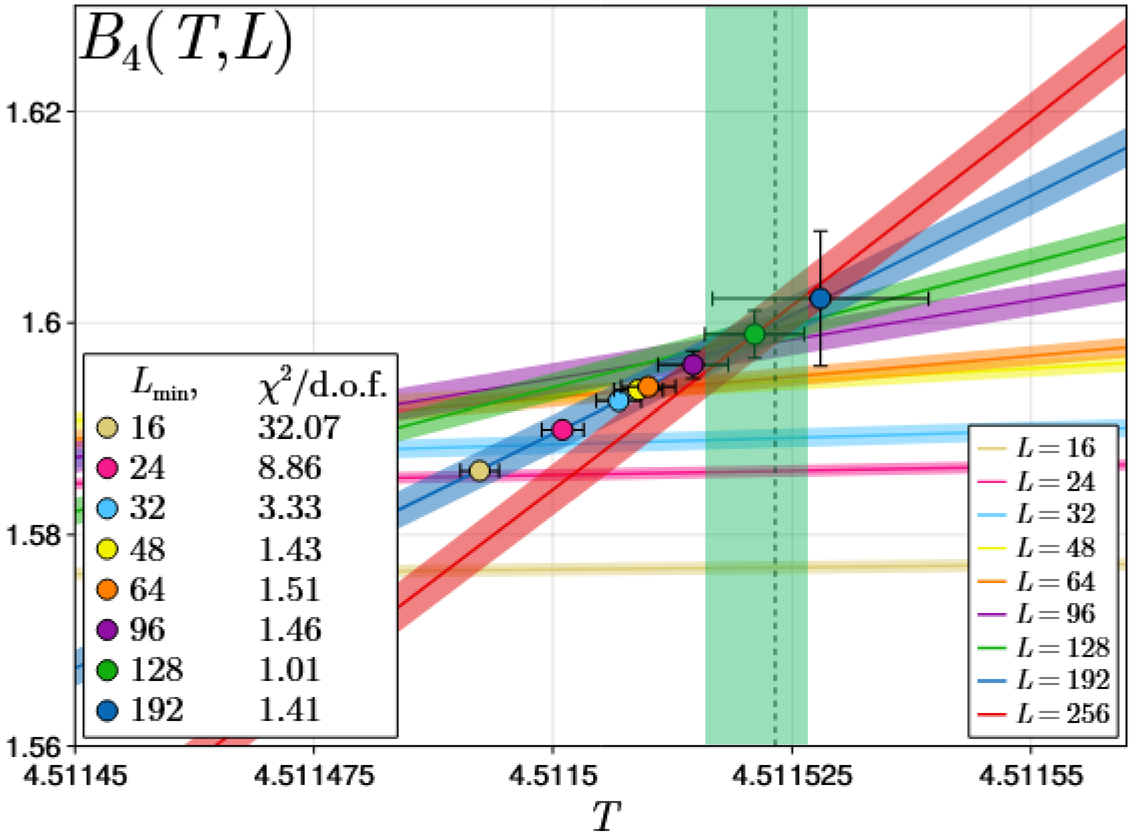
Wada, MK, Kanaya, arXiv:2508.20422

LYZR vs Binder Cumulant in 3d-Ising

LYZR



Binder cumulant

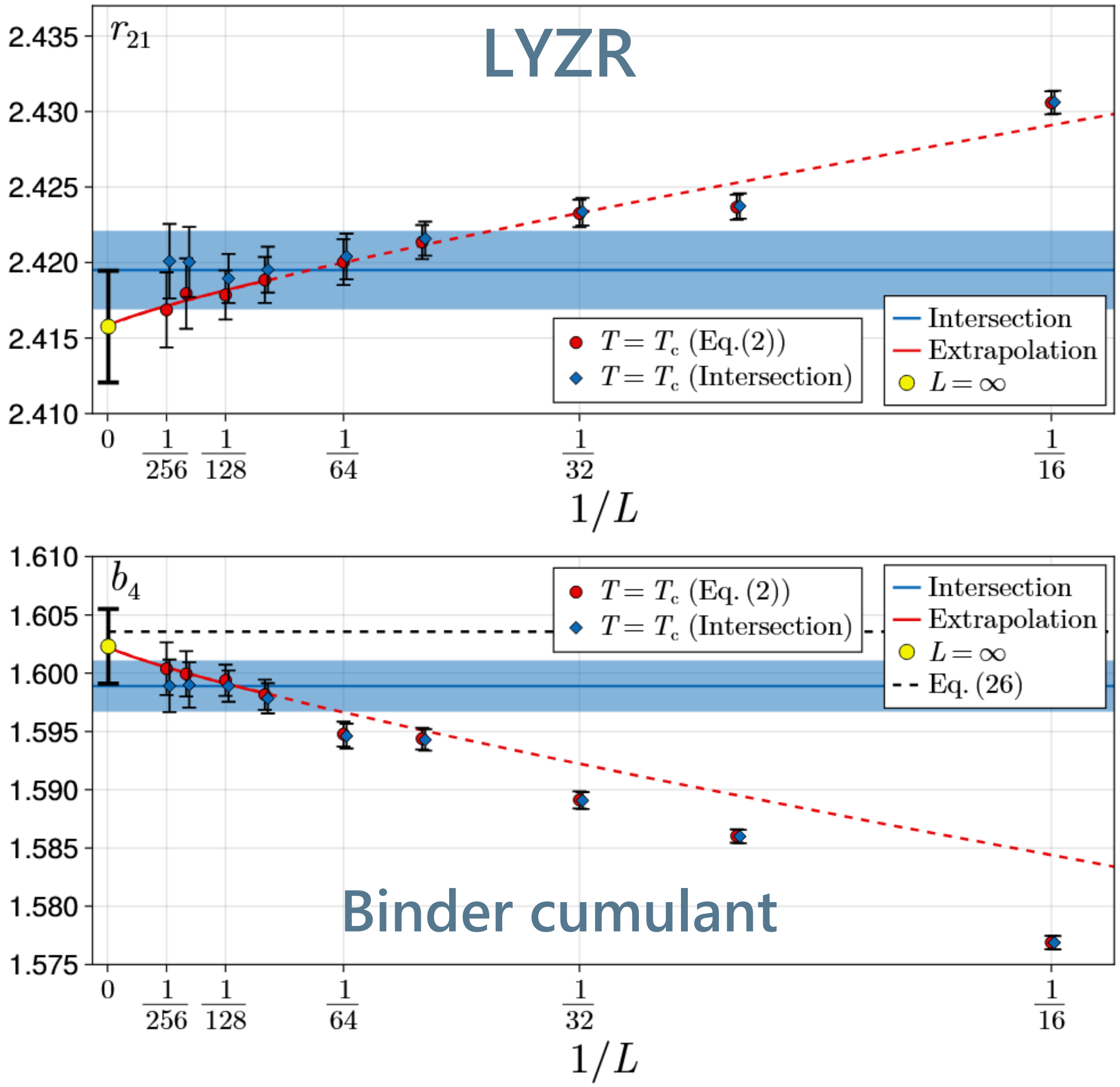


Faster convergence of the violation of FSS in LYZ?

Wada, MK, Kanaya, arXiv:2508.20422

Convergence at $T = T_c$

Wada, MK, Kanaya, arXiv:2508. 20422



Red: T_c of Ferrenberg ('18)
Blue: T_c of intersection point

$$R_{21}(0, L) = r_{21}(1 + cL^{-\omega})$$

In 3d-Ising,
violation of FSS is more quickly suppressed in the LYZR than the Binder cumulant for $L \to \infty$.

$$Z = Z_{\text{sing}} \times Z_{\text{reg}}$$

CP in a General System

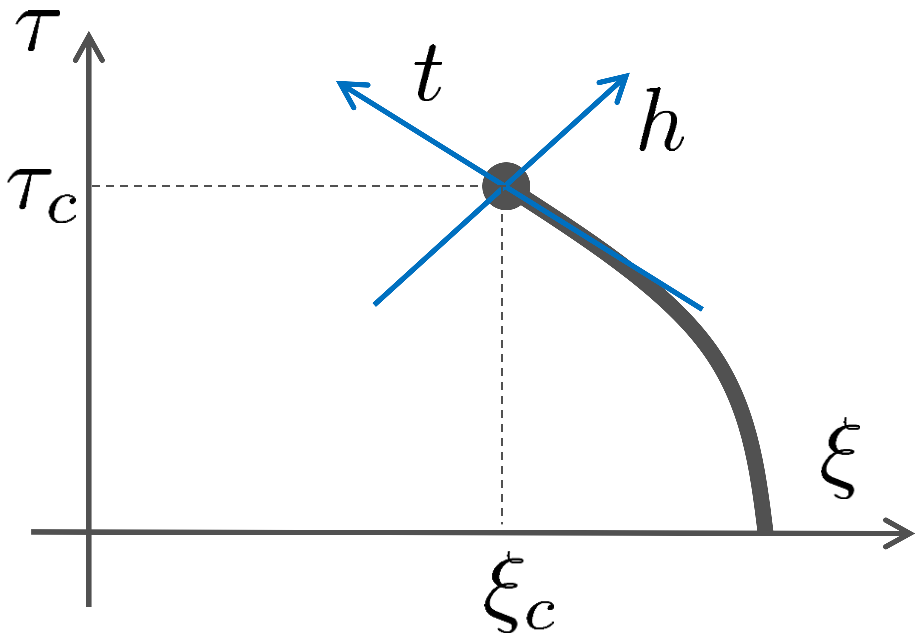
Wada, MK, Kanaya, PRL ('25)

$$\bar{y} = y_t - y_h = -0.894$$

- CP on a $\tau - \xi$ plane
- **LYZ on the complex ξ plane**

$$\begin{pmatrix} t \\ h \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} \tau - \tau_c \\ \xi - \xi_c \end{pmatrix} = A \begin{pmatrix} \delta\tau \\ \delta\xi \end{pmatrix}$$

$$L^{y_h} h^{(n)}(t) \simeq X_n + Y_n L^{y_t} t$$



$$Z_{\rm sing}(\tau, \xi) = \tilde{Z}_{\rm Ising}\big(t(\tau, \xi), \xi(\tau, \xi)\big)$$

CP in a General System

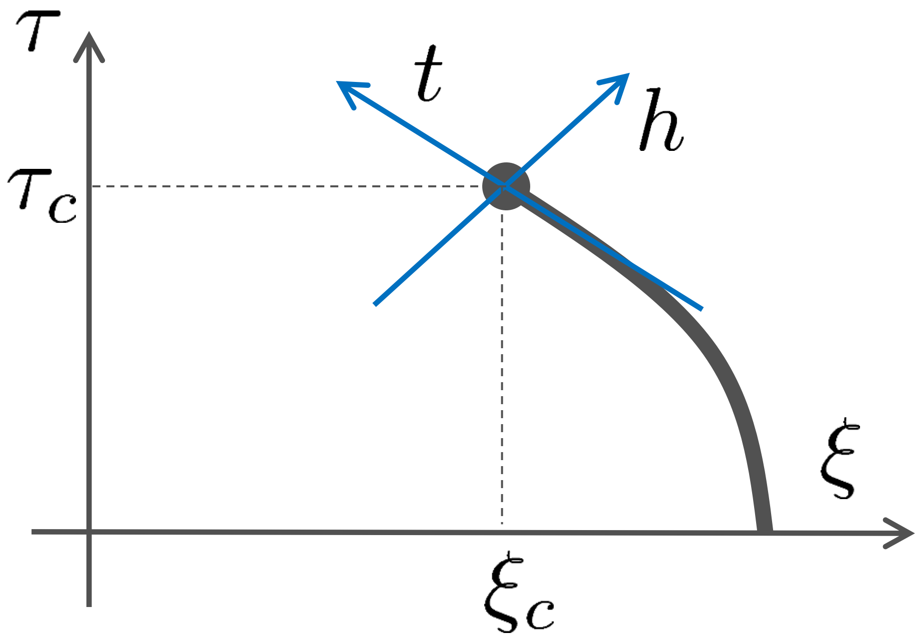
Wada, MK, Kanaya, PRL ('25)

$\bar{y} = y_t - y_h = -0.894$

- CP on a $\tau - \xi$ plane
- **LYZ on the complex ξ plane**

$$\begin{pmatrix} t \\ h \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} \tau - \tau_c \\ \xi - \xi_c \end{pmatrix} = A \begin{pmatrix} \delta\tau \\ \delta\xi \end{pmatrix}$$

$$L^{y_h} h^{(n)}(t) \simeq X_n + Y_n L^{y_t} t$$



$$\begin{cases} \xi_R^{(n)} = \xi_c - \dfrac{a_{21}}{a_{22}}\delta\tau + \mathcal{O}(L^{2\bar{y}}) \\ \xi_I^{(n)} = \dfrac{X_n}{a_{22}} L^{-y_h} + \dfrac{\det AY_n}{a_{22}^2}\delta\tau L^{\bar{y}} + \mathcal{O}(L^{2\bar{y}}) \end{cases}$$

$L \to \infty$

generalization to finite V

LY Edge Singularity

$$\begin{cases} \mathrm{Re}\xi_{\mathrm{LYES}} \simeq c_1 \tau \\ \mathrm{Im}\xi_{\mathrm{LYES}} \simeq c_2 \tau^{\beta\delta} \end{cases}$$

Stephanov, 2006

LYZ Ratios for General CP

Wada, MK, Kanaya, PRL ('25)

$\bar{y} = y_t - y_h = -0.894$

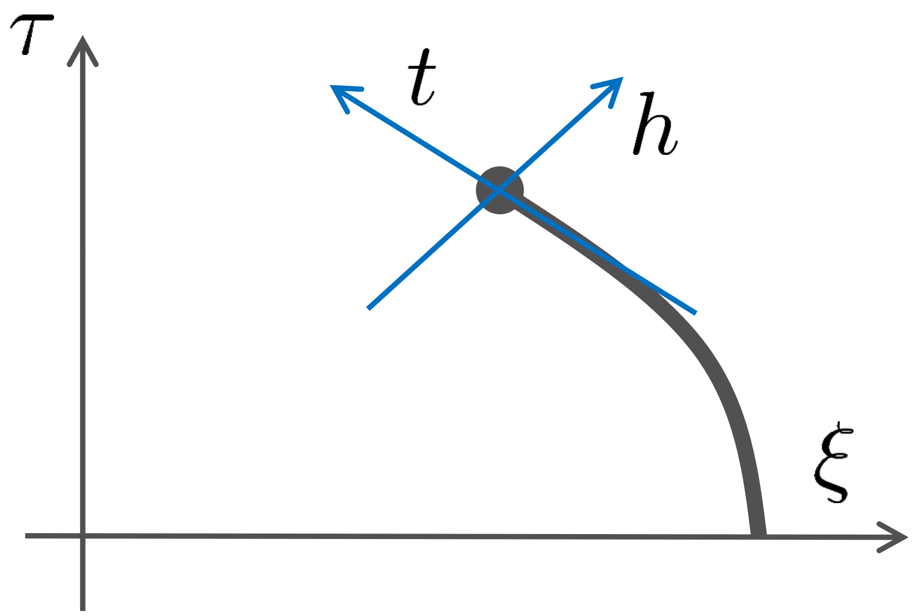
LYZ Ratio

$$R_{nm}(\tau) = \frac{\xi_{\mathrm{I}}^{(n)}(\tau)}{\xi_{\mathrm{I}}^{(m)}(\tau)} = r_{nm}\left(1 + CL^{y_t}\delta\tau + \mathcal{O}(\delta\tau^2)\right)\left(1 + DL^{2\bar{y}} + \mathcal{O}(L^{4\bar{y}})\right)$$

nonzero for $a_{12} \neq 0$

$$r_{nm} = \frac{X_n}{X_m}, \quad C = \frac{\det A}{a_{22}}\left(\frac{Y_2}{X_2} - \frac{Y_1}{X_1}\right), \quad D = \frac{a_{12}^2}{a_{22}^2}(Y_1^2 - Y_2^2)$$

$$\begin{pmatrix} t \\ h \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} \tau - \tau_c \\ \xi - \xi_c \end{pmatrix} = A\begin{pmatrix} \delta\tau \\ \delta\xi \end{pmatrix}$$



r_{nm} are universal constants specific to universality classes.

Wada, MK, Kanaya, PRL ('25)

$\bar{y} = y_t - y_h = -0.894$

LYZ Ratios vs Binder Cumulant

LYZ Ratio

$$R_{nm}(\tau) = \frac{\xi_{\rm I}^{(n)}(\tau)}{\xi_{\rm I}^{(m)}(\tau)} = r_{nm}\left(1 + CL^{y_t}\delta\tau + \mathcal{O}(\delta\tau^2)\right)\left(1 + DL^{2\bar{y}} + \mathcal{O}(L^{4\bar{y}})\right)$$

nonzero for $a_{12} \neq 0$

$$r_{nm} = \frac{X_n}{X_m}, \quad C = \frac{\det A}{a_{22}}\left(\frac{Y_2}{X_2} - \frac{Y_1}{X_1}\right), \quad D = \frac{a_{12}^2}{a_{22}^2}(Y_1^2 - Y_2^2)$$

Binder cumulant

Jin+, PRD86, 2017

$$B_4(t) = b_4\left(1 + c\tau L^{y_t} + \mathcal{O}(t^2)\right)\left(1 + dL^{\bar{y}} + \mathcal{O}(L^{2\bar{y}})\right)$$

nonzero for $a_{12} \neq 0$

r_{nm} are universal constants specific to universality classes.

Deviation at $t = 0$ due to $a_{12} \neq 0$ converges faster in LYZ ratio.

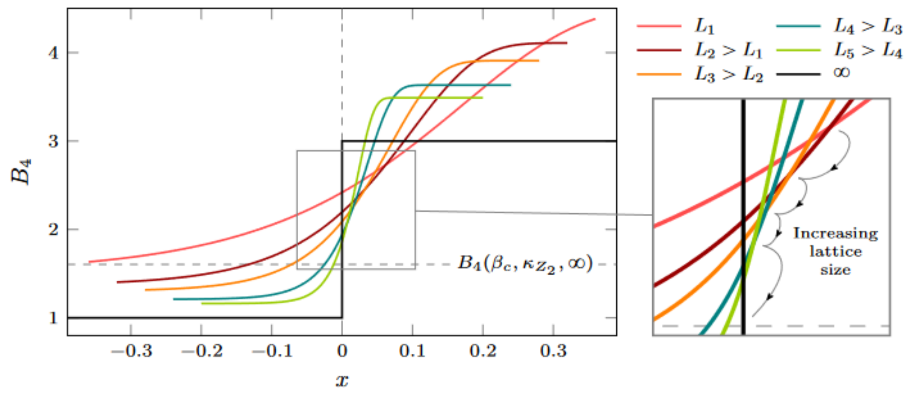
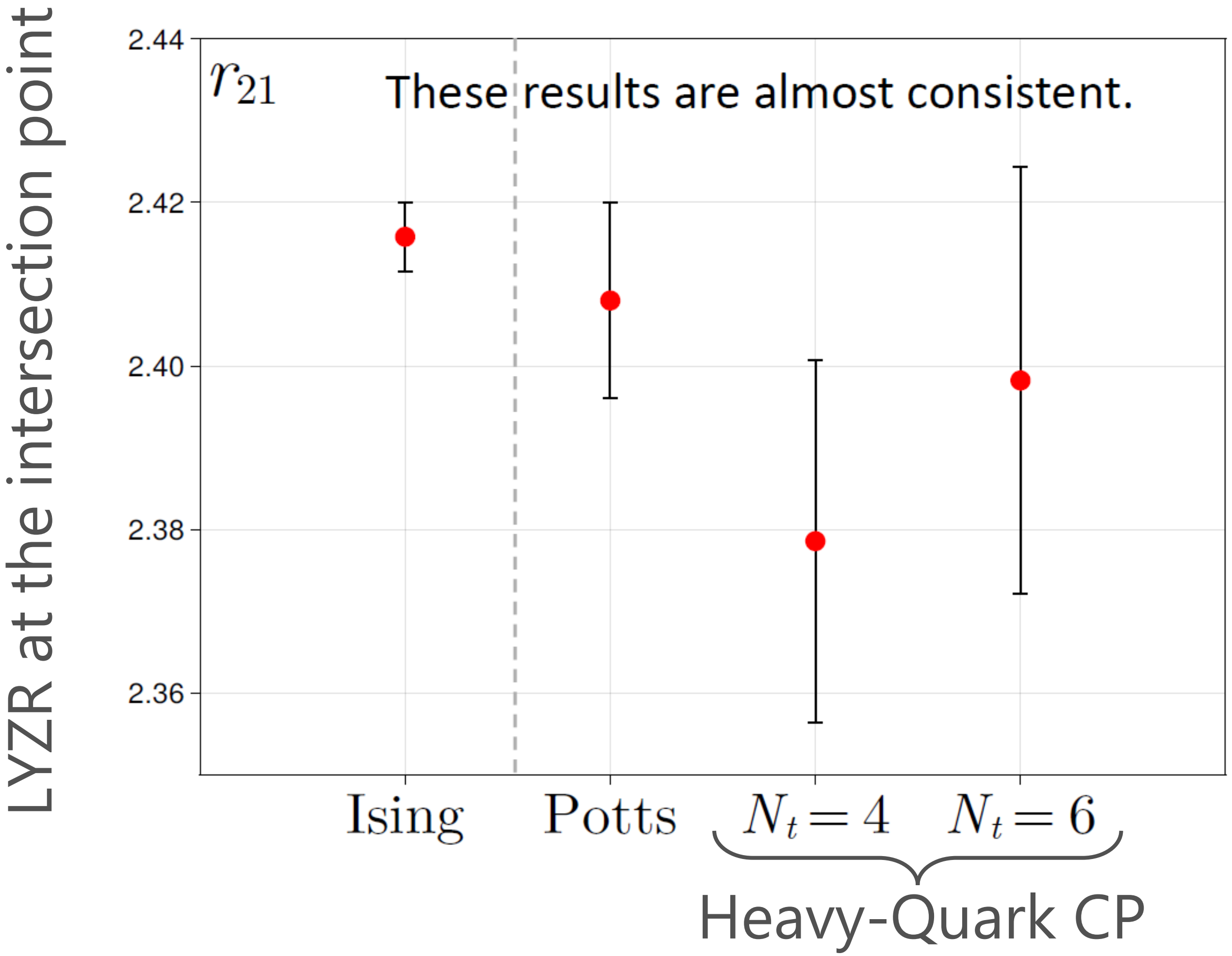


fig from Cuteri+, PRD ('21)

LYZR in Various Models



CPs in 3d-Z(2) universality class

—**Ising**: 2508.20422
—**Potts**: PRL, '25
—**HQ-CP**: 2501.18904

Consistent within statistics ➤ Confirmation of universality

Single-LYZ Method

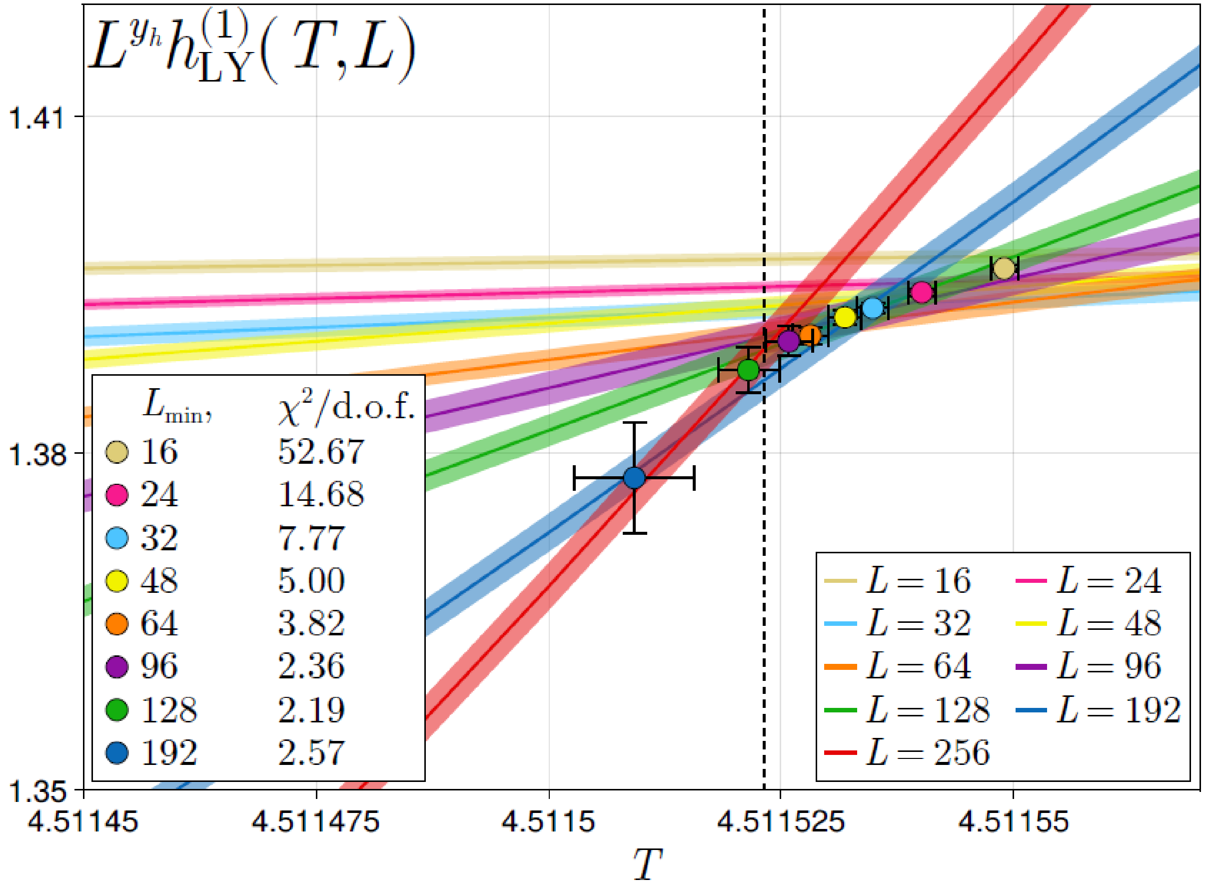
Wada, MK, Kanaya, arXiv:2508. 20422

$$L^{y_h} h^{(n)} = \tilde{h}^{(n)}_{\mathrm{LY}}(L^{y_t} t)$$

$$L^{y_h} h^{(n)} = X_n + Y_n L^{y_t} t + \mathcal{O}(t^2)$$

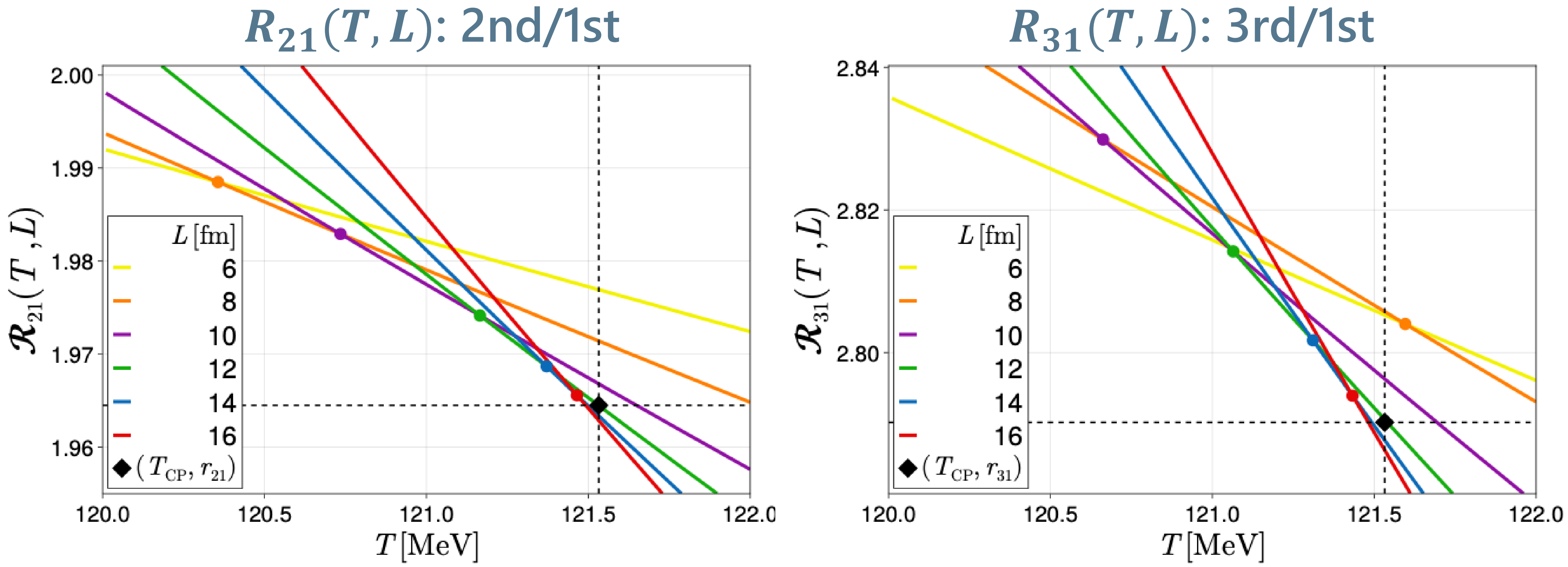
$L^{y_h} h^n(t)$ for various L
intersect at the CP!

LYZ in 3d-Ising



Assuming the universality class and the values of y_t, y_h,
the intersection analysis can be performed only with a single LYZ.

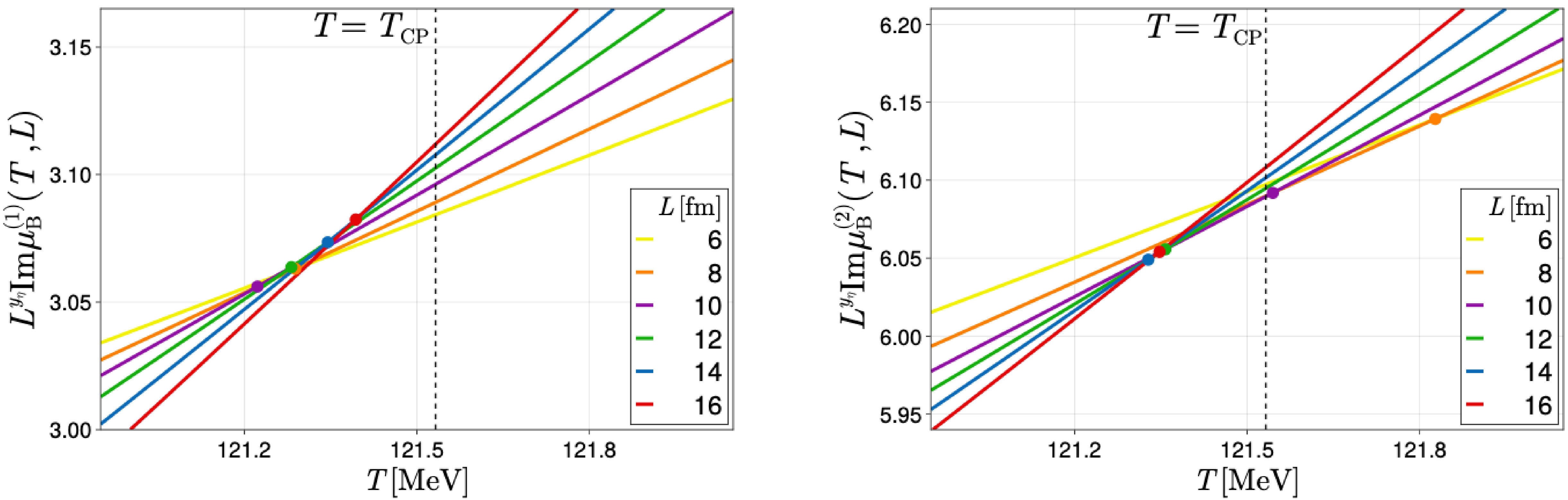
LYZR in Linear Sigma Model



— Intersection point converges toward T_c for $L \to \infty$.

— Good accuracy for $L > 10$ fm.

Single Scaled LYZ

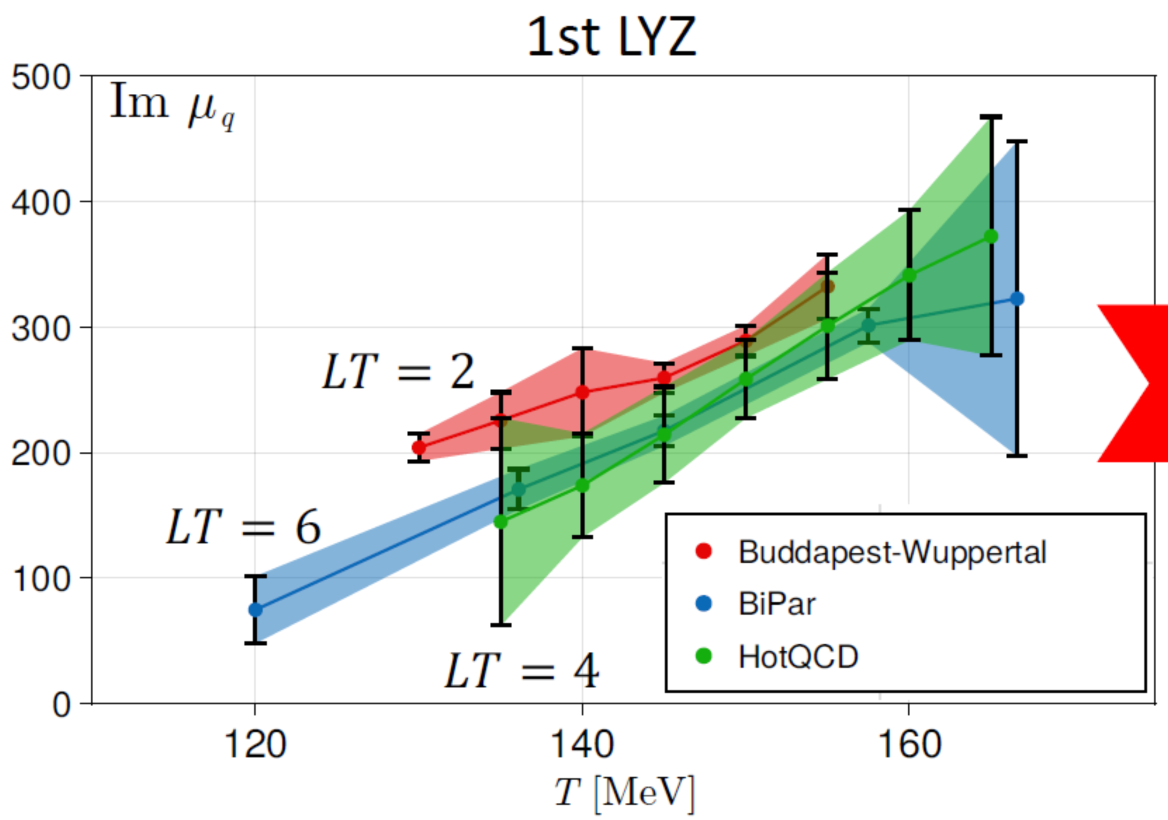


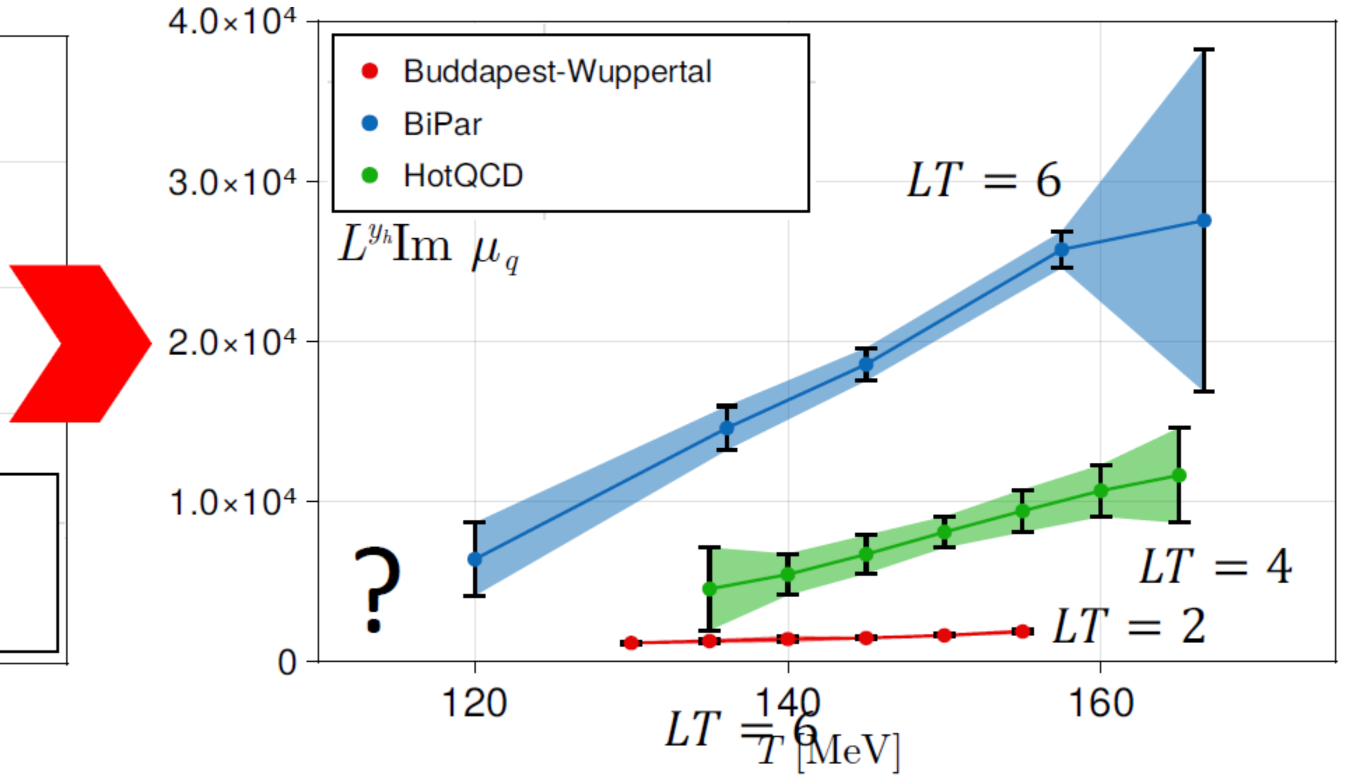
— Intersection point converges toward T_c for $L \to \infty$.

— Good accuracy for $L > 10$ fm.

Where is the QCD Critical Point?

Single-LYZ analysis for 2+1 flavor QCD





Intersection around $T \simeq 100$ MeV? Yet lower T data are mandatory.

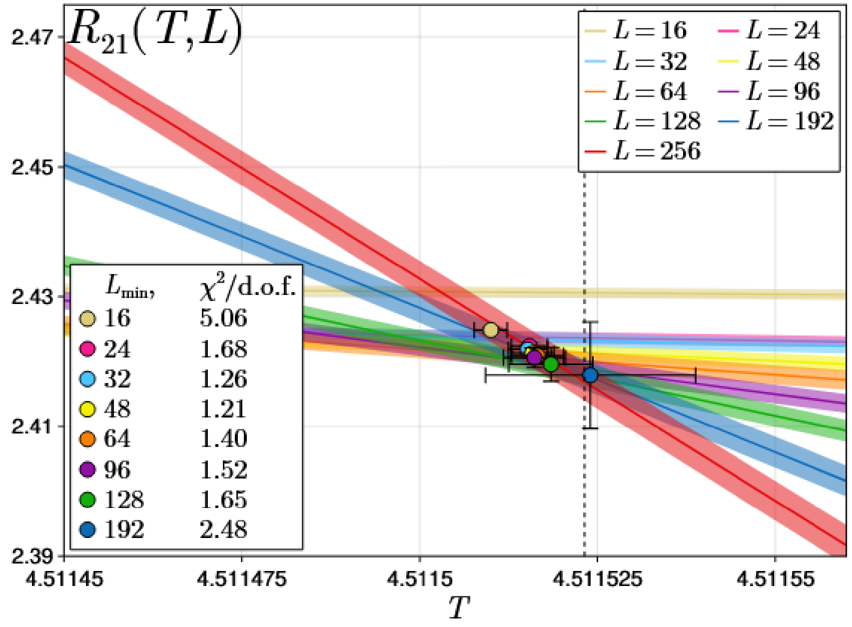
Summary

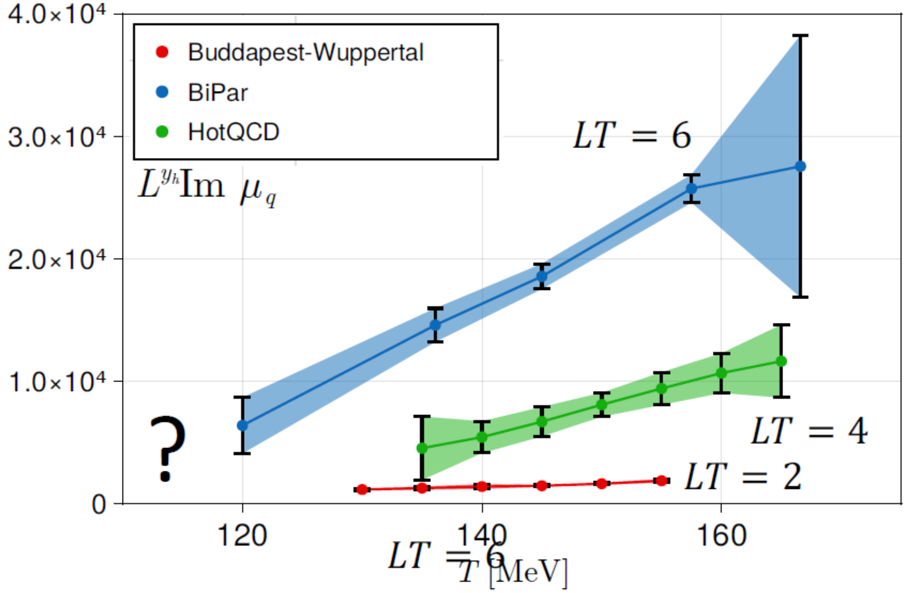
Using a new mean-field approach incorporating finite-size effects, we investigated the LYZs at nonzero T, μ_B in the linear-sigma model.

The study of LYZs and its relation to LYES are investigated.
Large deviation at LT values employes in lattice simulations.
The LYZR method works well.

Outlook

— Roverge-Weiss symmetry/transition
— Better interpretation of lattice results





Outlook

- Roverge-Weiss symmetry/transition
- Better interpretation of lattice results

Convergence of LYZR

The origin of Higher order terms

◆ Parameter mixing

$$\begin{pmatrix} \tau \\ \eta \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} T - T_{\mathrm{CP}} \\ \mu_{\mathrm{B}} - \mu_{\mathrm{CP}} \end{pmatrix}$$

Lowest order

LYZR : $2\bar{y} = 2(y_\tau - y_\eta) = -3/2$

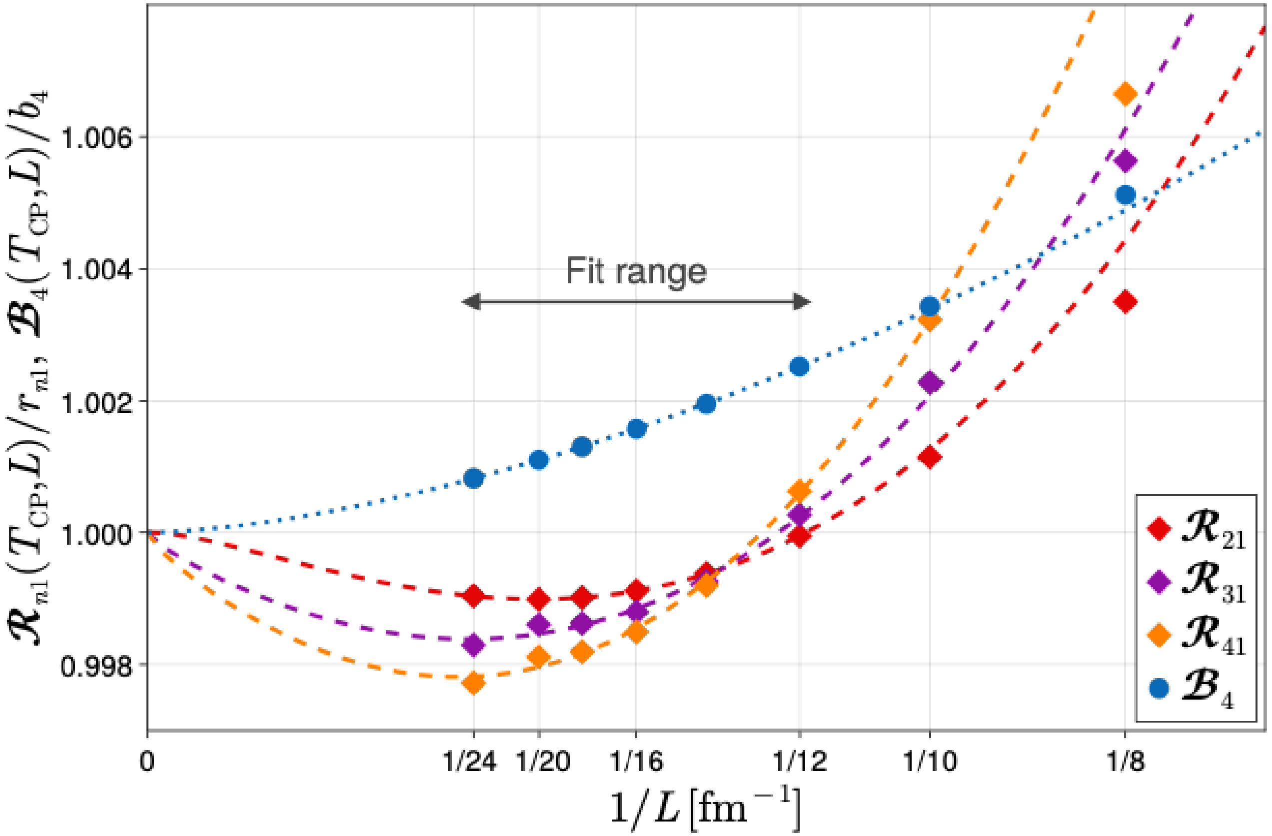
Binder : $\bar{y} = 2(y_\tau - y_\eta) = -3/4$

◆ Irrelevant operator

$$U'_{\mathrm{Landau}}(\bar{\phi}; \tau, \eta, \{\lambda\}) = U'_{\mathrm{Landau}}(\bar{\phi}; \tau, \eta, \lambda_5, \lambda_6, \ldots)$$

$$\equiv U_{\mathrm{Landau}}(\bar{\phi}; \tau, \eta) + \sum_{i=5}^{N} \lambda_i \bar{\phi}^i$$

Exponent : $y_5 = -3/4,\ y_6 = -3/2 \ldots$



Fitting function $(A, a) = (\mathcal{R}_{n1}, r_{n1}), (\mathcal{B}_4, b_4)$

$$\frac{A(T_{\mathrm{CP}}, L)}{a} = 1 + w_1 L^{-3/4} + w_2 L^{-3/2} + w_3 L^{-9/4}$$