Soliton theories on noncommutative spaces

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cf. MH,``NC Solitons and D-branes,”
1. Introduction

• Non-Commutative (NC) spaces are defined by noncommutativity of the coordinates:

\[ [x^i, x^j] = i\theta^{ij} \quad \theta^{ij} \quad \text{space-space uncertainty relation} \]

This looks like CCR in QM: \([q, p] = i\hbar\) (``space-space uncertainty relation"")

Resolution of singularity

(\(\rightarrow\) New physical objects)

e.g. resolution of small instanton singularity

(\(\rightarrow\) U(1) instantons)
NC gauge theories \[\rightarrow\] Com. gauge theories in background of (real physics) magnetic fields

\[L = \frac{m}{2} (\dot{x}^2 + \dot{y}^2) + qBxy\dot{y}\]
\[\downarrow \quad m \rightarrow 0\]
\[L = qBxy\dot{y}\]

\[p_y = \frac{\partial L}{\partial \dot{y}} = qBx\]

\[\underbrace{[y, p_y = i]} \quad \rightarrow \quad [x, y] = -i \frac{1}{qB}\]
Gauge theories are realized on D-branes which are solitons in string theories. In this context, (NC) solitons are (lower-dim.) D-branes.

Analysis of NC solitons (easy to treat) → Analysis of D-branes

Various applications
  e.g. confirmation of Sen’s conjecture on decay of D-branes
Plan of this talk

1. Introduction
2. NC gauge theories
3. ADHM construction of (NC) instantons
4. Applications to D-brane dynamics (omitted)

5. NC extension of soliton theories
6. NC Sato’s theories
7. Conservation Laws
8. Exact Solutions and Ward’s conjecture
9. Conclusion and Discussion
2. NC Gauge Theories

Here we discuss NC gauge theory of instantons.
(Ex.) 4-dim. (Euclidean) G=U(N) Yang-Mills theory

• Action

\[ S = -\frac{1}{2} \int d^4x \text{Tr} F_{\mu\nu} F^{\mu\nu} = -\frac{1}{4} \int d^4x \text{Tr} \left( F_{\mu\nu} F_{\mu\nu} + \widetilde{F}_{\mu\nu} \widetilde{F}_{\mu\nu} \right) \]

\[ = -\frac{1}{4} \int d^4x \text{Tr} \left[ \left( F_{\mu\nu} \mp \widetilde{F}_{\mu\nu} \right)^2 \pm 2 F_{\mu\nu} \widetilde{F}_{\mu\nu} \right] \]

\[ = 0 \iff \quad \text{(A)SDYM eq.)} \]

• Eq. Of Motion:

\[ [D^\nu, [D_\nu, D_\mu]] = 0 \]

• BPS eq. (= (A)SDYM eq.)

\[ F_{\mu\nu} = \pm \widetilde{F}_{\mu\nu} \]

\[ \Leftrightarrow F_{\bar{z}_1} + F_{\bar{z}_2} = 0, \quad F_{\bar{z}_1 \bar{z}_2} = 0 \]
(Q) How we get NC version of the theories?

(A) They are obtained from ordinary commutative gauge theories by replacing products of fields with star-products:

\[ f(x)g(x) \rightarrow f(x) \ast g(x) \]

- The star product:

\[
\begin{align*}
 f(x) \ast g(x) &:= f(x) \exp\left(\frac{i}{2} \theta^{ij} \bar{\partial}_i \partial_j \right) g(x) = f(x)g(x) + i \frac{\theta^{ij}}{2} \partial_i f(x) \partial_j g(x) + O(\theta^2) \\
 \end{align*}
\]

\[
 f \ast (g \ast h) = (f \ast g) \ast h \\
 [x^i, x^j]_\ast := x^i \ast x^j - x^j \ast x^i = i\theta^{ij}
\]
(Ex.) 4-dim. NC (Euclidean) G=U(N) Yang-Mills theory
(All products are star products)

• Action
\[
S = -\frac{1}{2} \int d^4 x \text{Tr} \ F_{\mu\nu} \ast F^{\mu\nu} = -\frac{1}{4} \int d^4 x \text{Tr} \left( F_{\mu\nu} \ast F_{\mu\nu} + \tilde{F}_{\mu\nu} \ast \tilde{F}_{\mu\nu} \right)
\]
\[
= -\frac{1}{4} \int d^4 x \text{Tr} \left( F_{\mu\nu} \ast F_{\mu\nu} + \tilde{F}_{\mu\nu} \ast \tilde{F}_{\mu\nu} \right)
= 0 \iff C_2
\]

(F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu])

• Eq. Of Motion:
\[
[D^\nu, [D_\nu, D_\mu]] = 0
\]

• BPS eq. (=NC (A)SDYM eq.)
\[
F_{\mu\nu} = \pm \tilde{F}_{\mu\nu}
\]
\[
(\iff F_{z_1 \bar{z}_1} + F_{z_2 \bar{z}_2} = 0, \quad F_{z_1 \bar{z}_2} = 0)
\]

\[
\therefore \quad U(1) \cong U(\infty)
\]
3. ADHM construction of (NC) instantons

\[ [B_1, B_1^*] + [B_2, B_2^*] + I I^* - J^* J = 0 \]
\[ [B_1, B_2] + I J = 0 \]

\( \mathbf{B}_{1,2} : k \times k, \quad I : k \times N, \quad J : N \times k \)

\( A_\mu : N \times N \)

\[ F_{z_1 \bar{z}_1} + F_{z_2 \bar{z}_2} = 0 \]
\[ F_{z_1 \bar{z}_2} = 0 \]
ADHM construction of BPST instanton (N=2,k=1)

\[ [B_1, B_1^*] + [B_2, B_2^*] + I I^* - J^* J = 0 \]
\[ [B_1, B_2] + IJ = 0 \]

\[ B_{1,2} = \alpha_{1,2}, \quad I = (\rho,0), \quad J = \begin{pmatrix} 0 \\ \rho \end{pmatrix} \]

\[ A_\mu = \frac{i(x - b)^\nu \eta^{(-)}_{\mu\nu}}{(x - b)^2 + \rho^2}, \quad F_{\mu\nu} = \frac{2i\rho^2}{((x - b)^2 + \rho^2)^2} \eta^{(-)}_{\mu\nu} \]

\[ F_{\bar{z}_1 z_1} + F_{\bar{z}_2 z_2} = 0 \]
\[ F_{\bar{z}_1 \bar{z}_2} = 0 \]
ADHM construction of NC BPST instanton
(N=2, k=1)

\[
\begin{align*}
[B_1, B_1^*] + [B_2, B_2^*] + I I^* - J^* J &= \zeta \\
[B_1, B_2] + IJ &= 0
\end{align*}
\]

\[
B_{1,2} = \alpha_{1,2}, \quad I = (\sqrt{\rho^2 + \zeta}, 0), \quad J = \begin{pmatrix} 0 \\ \rho \end{pmatrix}
\]

\[
A_\mu, F_{\mu\nu}
\]

\[
F_{\bar{z}_1} + F_{\bar{z}_2} = 0
\]

\[
F_{\bar{z}_1} = 0
\]
## 5. NC Extension of Soliton Theories

<table>
<thead>
<tr>
<th></th>
<th>Anti-Self-Dual Yang-Mills eq. (instantons) $F_{\mu\nu} = -\widetilde{F}_{\mu\nu}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>Bogomol’nyi eq. (monopoles)</td>
</tr>
<tr>
<td>2</td>
<td>KP eq. BCS eq. DS eq. …</td>
</tr>
<tr>
<td>1</td>
<td>KdV eq. Boussinesq eq. NLS eq. Burgers eq. sine-Gordon eq. Sawada-Kotera eq</td>
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</tbody>
</table>
Ward’s observation:
Almost all integrable equations are reductions of the ASDYM eqs.

AsDYM eq.  ↓  Reductions
                     KP eq.  BCS eq.
                     KdV eq.  Boussinesq eq.
                     NLS eq.  mKdV eq.
                     sine-Gordon eq.  Burgers eq. …
                     (Almost all ! )
NC Ward’s observation: Almost all NC integrable equations are reductions of the NC ASDYM eqs.

NC ASDYM eq.  ↓  Reductions

NC KP eq.  NC BCS eq.
NC KdV eq.  NC Boussinesq eq.
NC NLS eq.  NC mKdV eq.
NC sine-Gordon eq.  NC Burgers eq. …

(Almost all !?)

Successful

Successful?
6. NC Sato’s Theories

• Sato’s Theory: one of the most beautiful theory of solitons
  – Based on the existence of hierarchies and tau-functions

• Sato’s theory reveals essential aspects of solitons:
  – Construction of exact solutions
  – Structure of solution spaces
  – Infinite conserved quantities
  – Hidden infinite-dim. Symmetry

Let’s discuss NC extension of Sato’s theory
Derivation of soliton equations

• Prepare a Lax operator which is a pseudo-differential operator

\[ L := \partial_x + u_2 \partial_x^{-1} + u_3 \partial_x^{-2} + u_4 \partial_x^{-3} + \cdots \]

\[ u_k = u_k(x^1, x^2, x^3, \cdots) \]

• Introduce a differential operator

\[ B_m := (L \ast \cdots \ast L) \geq 0 \]

\[ m \text{times} \]

• Define NC (KP) hierarchy equation:

\[ \frac{\partial L}{\partial x^m} = [B_m, L]_* \]

\[ \partial_m u_2 \partial_x^{-1} + \]

\[ \partial_m u_3 \partial_x^{-2} + \]

\[ \partial_m u_4 \partial_x^{-3} + \cdots \]

\[ F_m^2(u) \partial_x^{-1} + \]

\[ F_m^3(u) \partial_x^{-2} + \]

\[ F_m^4(u) \partial_x^{-3} + \cdots \]
Negative powers of differential operators

\[ \partial_x^n \circ f := \sum_{j=0}^{\infty} \binom{n}{j} (\partial_x^j f) \partial_x^{n-j} \]

\[ \frac{n(n-1)(n-2)\cdots(n-(j-1))}{j(j-1)(j-2)\cdots1} \]

\[ \partial_x^3 \circ f = f\partial_x^3 + 3f\partial_x^2 + 3f''\partial_x + f''' \]
\[ \partial_x^2 \circ f = f\partial_x^2 + 2f\partial_x + f' \]
\[ \partial_x^{-1} \circ f = f\partial_x^{-1} - f\partial_x^{-2} + f''\partial_x^{-3} - \cdots \]
\[ \partial_x^{-2} \circ f = f\partial_x^{-2} - 2f\partial_x^{-3} + 3f''\partial_x^{-4} - \cdots \]
Closer look at NC (KP) hierarchy

\[ \partial^{-1}_x u_2 = 2u'_3 + u''_2 \]

\[ \partial^{-2}_x u_3 = 2u'_4 + u''_3 + 2u_2 * u'_2 + 2[u_2, u_3]. \]

\[ \partial^{-3}_x u_4 = 2u'_5 + u'''_4 + 4u_3 * u'_2 - 2u_2 * u''_2 + 2[u_2, u_4]. \]

\[ \vdots \]

\[ \partial^{-1}_x u_2 = u''_2 + 3u'_3 + 3u''_4 + 3u'_2 * u_2 + 3u_2 * u'_2 \]

\[ \vdots \]

\[ u_t = \frac{1}{4} u_{xxx} + \frac{3}{4} (u_x * u + u * u_x) + \frac{3}{4} \partial^{-1}_x u_{yy} + \frac{3}{4} [u, \partial^{-1}_x u_{yy}]. \]

\[ u = u(x^1, x^2, x^3, \cdots) \]

\[ x \quad y \quad t \]
• (Ex.) KdV hierarchy

Reduction condition
\[ L^2 = B_2 (\equiv \partial_x^2 + u) \]
gives rise to NC KdV hierarchy which includes (1+1)-dim. NC KdV eq.:

\[ u_t = \frac{1}{4} u_{xxx} + \frac{3}{4} (u_x * u + u * u_x) \]

\[ \frac{\partial u}{\partial x_{2N}} = 0 \]

\[ u(x^1, x^2, x^3, x^4, x^5, \ldots) \]
\[ x \quad y \quad t \]

\[ u(x^1, x^3, x^5, \ldots) \]
\[ x \quad t \]
I-reduction of NC KP hierarchy yields wide class of other NC hierarchies

- No-reduction $\rightarrow$ NC KP
  \[ (x, y, t) = (x^1, x^2, x^3) \]
- 2-reduction $\rightarrow$ NC KdV
  \[ (x, t) = (x^1, x^3) \]
- 3-reduction $\rightarrow$ NC Boussinesq
  \[ (x, t) = (x^1, x^2) \]
- 4-reduction $\rightarrow$ NC Coupled KdV
- 5-reduction $\rightarrow$ ...
- 3-reduction of BKP $\rightarrow$ NC Sawada-Kotera
- 2-reduction of mKP $\rightarrow$ NC mKdV
- Special 1-reduction of mKP $\rightarrow$ NC Burgers
- ...
7. Conservation Laws

- Conservation laws: \( \partial_t \sigma = \partial_i J^i \)

\[
Q := \int_{\text{space}} dx \sigma
\]

\[
\therefore \partial_t Q = \int_{\text{space}} dx \partial_t \sigma = \int_{\text{spatial infinity}} dS_i J^i = 0
\]

Conservation laws for the hierarchies

\[
\partial_m \text{res}_{-1} L^n = \partial_x J + [A, B]_* = \partial_x J + \Theta^{ij} \partial_j \Xi_i
\]

\[
\partial_j \text{res}_{-r} L^n
\]
Hot (old?) Results

Infinite conserved densities for NC hierarchy eqs. \( (n=1,2,...) \)

\[
\sigma = res_{-1} L^n + \theta^{im} \sum_{k=0}^{m-1} \sum_{l=0}^{k} (-1)^{k-l} \binom{k}{l} (res_{-(l+1)} L^n) \bigotimes (\partial_{x} \partial_{x}^{k-l} res_{k} L^m)
\]

\[t \equiv x^m \quad res_{r} L^n : \quad \bigotimes \partial_{x}^{r} \bigotimes L^n\]

\[
f(x) \bigotimes g(x) := f(x) \left( \sum_{s=0}^{\infty} \frac{(-1)^s}{(2s+1)!} \left( \frac{1}{2} \theta^{ij} \partial_{i} \partial_{j} \right)^{2s} \right) g(x)
\]
We can calculate the explicit forms of conserved densities for the wide class of NC soliton equations.

- **Space-Space noncommutativity:**
  NC deformation is slight: \[ \sigma = \text{res}_1 L^n \]

- **Space-time noncommutativity**
  NC deformation is drastical:
  - Example: NC KP and KdV equations \( ([t, x] = i\theta) \)
    \[ \sigma = \text{res}_{-1} L^n - 3\theta (\text{res}_{-1} L^n) \wedge u'_3 + (\text{res}_{-2} L^n) \wedge u'_2 ) \]
8. Exact Solutions and Ward’s conjecture

• We have found exact N-soliton solutions for the wide class of NC hierarchies.
• 1-soliton solutions are all the same as commutative ones because of
\[ f(x - vt)^* g(x - vt) = f(x - vt) g(x - vt) \]
• Multi-soliton solutions behave in almost the same way as commutative ones except for phase shifts.
• Noncommutatitivity might affect the phase shifts
NC Burgers hierarchy

- NC (1+1)-dim. Burgers equation:
  \[ \dot{u} = u'' + 2u \cdot u' \]
  \[ u = \tau^{-1} \cdot \tau' \quad (\theta \to 0 \rightarrow \partial_x \log \tau) \]
  \[ \dot{\tau} = \tau'' \]
  \[ \tau = 1 + \sum_{l=1}^{N} e^{k_l^2 t} \cdot e^{\pm k_l x} = 1 + \sum_{l=1}^{N} e^{\frac{i}{2} k_l^3 \theta} e^{k_l^2 t \pm k_l x} \]
NC Ward’s observation (NC NLS eq.)

- Reduced ASDYM eq.: \( x^\mu \rightarrow (t, x) \)
  
  (i) \( B' = 0 \)
  
  (ii) \( C' - \dot{A} + [A, C]_* = 0 \)
  
  (iii) \( A' - \dot{B} + [C, B]_* = 0 \)

\[
A = \begin{pmatrix} 0 & q \\ -\bar{q} & 0 \end{pmatrix}, \quad B = \frac{i}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad C = -i \begin{pmatrix} q^* \bar{q} & q' \\ q' & -\bar{q}^* q \end{pmatrix}
\]

(ii) \( \Rightarrow \)

\[
\begin{pmatrix} 0 & i\dot{q} - q'' - 2q^*\bar{q}^* q \\ i\ddot{q} + q''' + 2\bar{q}^* q^* \bar{q} & 0 \end{pmatrix} = 0
\]

\( i\dot{q} = q'' + 2q^*\bar{q}^* q \)

\( A, B, C \in gl(2, C) \xrightarrow{\theta \to 0} sl(2, C) \)
NC Ward’s observation (NC Burgers eq.)

- Reduced ASDYM eq.: $x^\mu \rightarrow (t, x) \quad G=U(1)$

(i) $\dot{A} + [B, A]_* = 0$

(ii) $\dot{C} - B' + [B, C]_* = 0$

\[ \Rightarrow \quad \dot{u} = u'' + 2u' \ast u \]
9. Conclusion and Discussion

• In this talk, we discussed
  – NC instantons where we saw that resolution of singularities yields new physical objects
  – NC soliton theories motivated by Ward’s conjecture where we proved existence of infinite conserved quantities and exact multi-soliton solutions which would suggest hidden infinite-dim. symmetry.
  → New study area of integrable systems and geometry
Further directions

• Completion of NC Sato’s theory
  – Theory of tau-functions → hidden symmetry
    (deformed affine Lie algebras?)
  – Geometrical descriptions from NC extension of the theories of Krichever, Mulase and Segal-Wilson and so on.

• Confirmation of NC Ward’s conjecture
  – NC twistor theory
    – D-brane interpretations → physical meanings

• Foundation of Hamiltonian formalism with space-time noncommutativity
  – Initial value problems, Liouville’s theorem, Noether’s thm,…