Noncommutative Solitons and Quasideterminants

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HEP Seminar at Hsinchu on Nov. 11th

Goal

- Extension of all soliton theories and integrable systems to non-commutative (NC) spaces, including the KdV eq. etc.

\[ [x^\mu, x^\nu] = i\theta^{\mu\nu} \]

NC parameter (real const.)
Successful points in NC theories

- Appearance of new physical objects
- Description of real physics (in gauge theory)
- Various successful applications to D-brane dynamics etc.
Ward’s conjecture: Many (perhaps all?) integrable equations are reductions of the ASDYM eqs.

ASDYM eq. is a master eq.!

Solution Generating Techniques

Infinite gauge group

Twistor Theory

Yang’s form

Zakharov

KP

NLS

CBS

mKdV

pKdV

KdV

gauge equiv.

pKdV

N-wave

DS

Ward’s chiral

Boussinesq

sine-Gordon

Liouville

Tzitzeica

Yang’s form

gauge equiv.
NC Ward’s conjecture: Many (perhaps all?) NC integrable eqs are reductions of the NC ASDYM eqs.

New physical objects
Application to string theory
In gauge theory, NC ↔ magnetic fields

Solution Generating Techniques
Infinite gauge group

Today’s talk

In gauge theory, magnetic fields

New physical objects
Application to string theory

Infinite gauge group

Infinite gauge group

Today’s talk

NC Ward’s chiral

Yang’s form

Summarized in [MH NPB 741(06) 368]
2. Review of Soliton Theories

KdV equation: describe shallow water waves

\[ u = 2k^2 \cosh^{-2}(kx - 4k^3 t) \]

\[ \ddot{u} + u''' + 6u'u = 0 \]
Let’s solve it now!

- Hirota’s method [PRL27(1971)1192]

\[
\dot{u} + u''' + 6u'u = 0
\]

\[
u = 2 \partial^2_x \log \tau
\]

\[
\tau \dot{\tau}' - \tau' \dot{\tau} + 3 \tau'' \tau'' - 4 \tau' \tau''' + \tau \tau'''' = 0
\]

- \[\tau = 1 + e^{2(kx - \omega t)}, \quad \omega = 4k^3\]

- \[u = 2k^2 \cosh^{-2}(kx - 4k^3t)\]
2-soliton solution

\[ \tau = 1 + A_1 e^{2\theta_1} + A_2 e^{2\theta_2} + BA_1 A_2 e^{2(\theta_1 + \theta_2)} \]

\[ \theta_i = k_i x - 4k_i^3 t, \quad B = \left( \frac{k_1 - k_2}{k_1 + k_2} \right)^2 \]

= A determinant of Wronski matrix (general property of soliton sols.)
````tau-functions````
3. Darboux transforms for NC KdV eq.

- In this section, we give an exact soliton solutions of NC KdV eq. by a Darboux transformation. [Gilson-Nimmo, JPA40(07) 3839, nlin.si/0701027]

- We see that ingredients of quasi-determinants are naturally generated by iteration of the Darboux transformation. (an origin of quasi-determinants)

- We also make a comment on asymptotic behavior of soliton scattering process.
Review of Quasi-determinants

- Quasi-determinants are not just a NC generalization of commutative determinants, but rather related to inverse matrices.

- **[Def1]** For an $n$ by $n$ matrix $X = (x_{ij})$ and the inverse $Y = (y_{ij})$ of $X$, quasi-determinant of $X$ is directly defined by

$$|X|_{ij} = y_{ji}^{-1}$$

$X^{ij}$ : the matrix obtained from $X$ deleting $i$-th row and $j$-th column

- **[Def2]** (Iterative definition)

$$|X|_{ij} = x_{ij} - \sum_{i',j'} x_{ii'}((X^{ij})^{-1})_{i'j'} x_{j'j} = x_{ij} - \sum_{i',j'} x_{ii'}(|X^{ij}|_{j'j})^{-1} x_{j'j}$$

$n+1 \not\equiv n+1$

$n \not\equiv n$
A comment on Def 2

Formula for inverse matrix:

\[
X = \begin{pmatrix}
  A & B \\
  C & d \\
\end{pmatrix} \quad \Rightarrow \\
Y = X^{-1} = \begin{pmatrix}
  A^{-1} + A^{-1}B(d - CA^{-1}B)^{-1}CA^{-1} & -A^{-1}B(d - CA^{-1}B)^{-1} \\
  -(d - CA^{-1}B)^{-1}CA^{-1} & (d - CA^{-1}B)^{-1} \\
\end{pmatrix}
\]

A convenient notation:

\[
|X|_{ij} = \begin{pmatrix}
  \cdots & \cdots & \cdots \\
  \cdots & x_{ij} & \cdots \\
  \cdots & \cdots & \cdots \\
\end{pmatrix}
\]
Examples of quasi-determinants

\[
\begin{align*}
 n = 1 : \quad & |X|_{ij} = x_{ij} \\
n = 2 : \quad & |X|_{11} = \begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{vmatrix} = x_{11} - x_{12} \cdot x_{22}^{-1} \cdot x_{21}, \quad |X|_{12} = \begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{vmatrix} = x_{12} - x_{11} \cdot x_{21}^{-1} \cdot x_{22}, \\
n = 3 : \quad & |X|_{11} = \begin{vmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{vmatrix} = x_{11} - (x_{12}, x_{13}) \begin{pmatrix} x_{22} & x_{23} \\ x_{32} & x_{33} \end{pmatrix}^{-1} \begin{pmatrix} x_{21} \\ x_{31} \end{pmatrix} \\
&= x_{11} - x_{12} \cdot (x_{22} - x_{23} \cdot x_{33}^{-1} \cdot x_{32})^{-1} \cdot x_{21} - x_{13} \cdot (x_{32} - x_{33} \cdot x_{23}^{-1} \cdot x_{22})^{-1} \cdot x_{21} \\
&\quad - x_{12} \cdot (x_{23} - x_{22} \cdot x_{32}^{-1} \cdot x_{33})^{-1} \cdot x_{31} - x_{13} \cdot (x_{33} - x_{32} \cdot x_{22}^{-1} \cdot x_{23})^{-1} \cdot x_{31}
\end{align*}
\]

Note:

\[
X = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad \Rightarrow \quad Y = X^{-1} = \begin{pmatrix} A^{-1} + A^{-1}B(D - CA^{-1}B)^{-1}CA^{-1} & -A^{-1}B(D - CA^{-1}B)^{-1} \\ -(D - CA^{-1}B)^{-1}CA^{-1} & (D - CA^{-1}B)^{-1} \end{pmatrix} \\
= \begin{pmatrix} (A - BD^{-1}C)^{-1} & (C - DB^{-1}A)^{-1} \\ (B - AC^{-1}D)^{-1} & (D - CA^{-1}B)^{-1} \end{pmatrix}
\]
Some identities of quasideterminants

- **Homological relation**

\[
\begin{vmatrix}
A & B & C \\
D & f & g \\
E & h & i
\end{vmatrix}
= \begin{vmatrix}
A & B & C \\
D & f & g \\
E & h & i
\end{vmatrix}
- \begin{vmatrix}
A & B & C \\
D & f & g \\
E & h & i
\end{vmatrix}
\]

- **NC Jacobi’s identity**

\[
\begin{vmatrix}
A & B & C \\
D & f & g \\
E & h & i
\end{vmatrix}
= \begin{vmatrix}
A & C \\
E & i
\end{vmatrix}
- \begin{vmatrix}
A & B \\
E & h
\end{vmatrix}
+ \begin{vmatrix}
A & B^{-1} \\
D & f
\end{vmatrix}
- \begin{vmatrix}
A & C \\
D & g
\end{vmatrix}
\]


Lax formalism of commutative KdV eq.

**Linear systems:**

\[ L \psi = (\partial_x^2 + u - \lambda^2) \psi = 0, \]
\[ M \psi = (\partial_t - \partial_x^3 - (3/2)u \partial_x - (3/4)u_x) \psi = 0. \]

**Compatibility condition of the linear system:**

\[ [L, M] = 0 \iff \dot{u} = \frac{1}{4} u_{xxx} + \frac{3}{2} uu_x \]

: KdV equation

Lax pair of KdV
How we get NC version of the theories?

We have only to replace all products of fields in ordinary commutative gauge theories with star-products: \( f(x)g(x) \rightarrow f(x) \ast g(x) \)

The star product: (NC and associative)

\[
f(x) \ast g(x) := f(x) \exp \left( \frac{i}{2} \theta_{\mu \nu} \bar{\partial}_\mu \bar{\partial}_\nu \right) g(x) = f(x)g(x) + i \frac{\theta_{\mu \nu}}{2} \partial_\mu f(x) \partial_\nu g(x) + O(\theta^2)
\]

Note: coordinates and fields themselves are usual c-number functions. But commutator of coordinates becomes…

\[
[x^\mu, x^\nu]_\ast := x^\mu \ast x^\nu - x^\nu \ast x^\mu = i \theta^{\mu \nu}
\]
Lax pair of NC KdV eq.

- Linear systems:
  \[ L \psi = (\partial_x^2 + u - \lambda^2) \psi = 0, \]
  \[ M \psi = (\partial_t - \partial_x^3 - (3/2)u \partial_x - (3/4)u_x) \psi = 0. \]

- Compatibility condition of the linear system:
  \[ [L, M] = 0 \quad \Leftrightarrow \quad \dot{u} = \frac{1}{4} u_{xxx} + \frac{3}{4} (u \ast u_x + u_x \ast u) \]
  :NC KdV equation \( [t, x] = i \theta \)

- Darboux transform for NC KdV [Gilson-Nimmo]

Let us take an eigen function \( W \) of \( L \) and define \( \Phi = W \ast \partial_x W^{-1} \)
Then the following trf. leaves the linear systems as it is

\[
\tilde{L} = \Phi \ast L \ast \Phi^{-1}, \quad \tilde{M} = \Phi \ast M \ast \Phi^{-1}, \quad \tilde{\psi} = \Phi \ast \psi
\]

and

\[
\tilde{u} = u + 2(W_x \ast W^{-1})_x
\]
The Darboux transformation can be iterated

- **Let us take eigen fcns.** \((f_1, \cdots, f_N)\) of \(L\) and define
  \[
  \Phi_i = W_i \ast \partial_x W_i^{-1} = \partial_x - W_{i,x} \ast W_i^{-1} \quad (W_1 \equiv f_1, \Phi_1 = f_1 \ast \partial_x f_1)
  \]
  \[
  W_{i+1} = \Phi_i \ast f_{i+1} = f_{i+1,x} - W_{i,x} \ast W_i^{-1} \ast f_{i+1} \quad (i = 1, 2, 3, \cdots)
  \]
  \[
  \psi_{i+1} = \Phi_i \ast \psi_i = \psi_{i,x} - W_{i,x} \ast W_i^{-1} \ast \psi_i
  \]

- **Iterated Darboux transform for NC KdV**

  The following trf. leaves the linear systems as it is

  \[
  L_{[i+1]} = \Phi_i \ast L_{[i]} \ast \Phi_i^{-1}, \quad M_{[i+1]} = \Phi_i \ast M_{[i]} \ast \Phi_i^{-1}, \quad \psi_{[i+1]} = \Phi_i \ast \psi_{[i]}
  \]

  \[
  (L_{[1]}, M_{[1]}, \psi_{[1]}) \xrightarrow{\Phi_1} (L_{[2]}, M_{[2]}, \psi_{[2]}) \xrightarrow{\Phi_2} \cdots
  \]

  In fact, \((W_i, \psi_i)\) are quasi-determinants of Wronskian matrices!

and

\[
L_{[N+1]} = u + 2 \sum_{i=1}^{N} (W_{i,x} \ast W_i^{-1})_x \quad (\theta \to 0 \to u + 2 \partial_x^2 \log W(f_1, \cdots, f_N))
\]
The Darboux transformation can be iterated

- **Let us take eigen fcns.** \((f_1, \cdots, f_N)\) **of** \(L\) **and define**
  \[
  \Phi_i = W_i * \partial_x W_i^{-1} = \partial_x - W_{i,x} * W_i^{-1} \quad (W_1 \equiv f_1, \Phi_1 = f_1 * \partial_x f_1)
  \]
  \[
  W_{i+1} = \Phi_i * f_{i+1} = f_{i+1,x} - W_{i,x} * W_i^{-1} * f_{i+1} \quad (i = 1, 2, 3, \cdots)
  \]
  \[
  \psi_{i+1} = \Phi_i * \psi_i = \psi_{i,x} - W_{i,x} * W_i^{-1} * \psi_i
  \]

- **Examples**
  \[
  W_1 \equiv f_1
  \]
  \[
  W_2 = f_{2,x} - W_{1,x} * W_1^{-1} * f_2 = \begin{vmatrix} f_1 & f_2 \\ f'_1 & f'_2 \end{vmatrix}
  \]
  **Q-det !**
  \[
  W_3 = f_{3,x} - W_{2,x} * W_2^{-1} * f_3 = \begin{vmatrix} f_1 & f_2 & f_3 \\ f'_1 & f'_2 & f'_3 \\ f''_1 & f''_2 & f''_3 \end{vmatrix}
  \]
  **Q-det !

\[
 u_{[N+1]} = u + 2 \sum_{i=1}^{N} (W_{i,x} * W_i^{-1})_x \xrightarrow[\theta \to 0]{\partial_x} u + 2 \partial^2_x \log W(f_1, \cdots, f_N)
\]
Exact $N$-soliton solutions of the NC KdV eq.

$$u = 2\partial_x \sum_{i=1}^{N} (\partial_x W_i) * W_{i}^{-1}$$

$$\left(\theta \rightarrow 0 \right) \quad \partial_x^2 \log \det W(f_1, \ldots, f_N)$$

$$W_i := \left| W(f_1, \ldots, f_i) \right|_{i,i}$$

$$f_i = \exp \left( \xi(x, \lambda_i) \right) + a_i \exp \left( -\xi(x, \lambda_i) \right)$$

$$\xi(x, t, \lambda) = x_1 \lambda + t \lambda_i^3$$

$$(L * f_i = (\partial_x^2 - \lambda^2) f_i = 0, M * f_i = (\partial_t - \partial_x^3) f_i = 0)$$

$$W(f_1, f_2, \ldots, f_m) = \begin{bmatrix} f_1 & f_2 & \cdots & f_m \\ \partial_x f_1 & \partial_x f_2 & \cdots & \partial_x f_m \\ \vdots & \vdots & \ddots & \vdots \\ \partial_x^{m-1} f_1 & \partial_x^{m-1} f_2 & \cdots & \partial_x^{m-1} f_m \end{bmatrix}$$
Scattering process of the N-soliton solutions

- We have found exact N-soliton solutions for the NC KdV eq.
- Physical interpretations are non-trivial because when \( f(x), g(x) \) are real, \( f(x) * g(x) \) is not in general.
- However, the solutions could be real in some cases.
  - (i) 1-soliton solutions are all the same as commutative ones because of
    \[
    f(x - vt) * g(x - vt) = f(x - vt)g(x - vt)
    \]
  - (ii) In asymptotic region, configurations of multi-soliton solutions could be real in soliton scatterings and the same as commutative ones.

MH JHEP [hep-th/0610006]
2-soliton solution of NC KdV

each packet has the configuration:

\[ u^{(i)} = 2k_i^2 \cosh^{-2}(k_i x - 4k_i^3 t), \quad v_i = 4k_i^2, \quad h_i = 2k_i^2 \]
4. Toward NC Sato’s Theory

Sato’s Theory: one of the most beautiful theory of solitons

- Based on the existence of hierarchies and tau-functions

\[
\begin{align*}
\text{A set of infinite soliton equations} \\
\text{(in terms of } u \text{)}
\end{align*}
\]

\[
\begin{align*}
\text{A set of infinite bilinear equations} \\
\text{(in terms of } \tau \text{)}
\end{align*}
\]

\[
u = 2 \frac{\partial^2}{\partial x^2} \log \tau
\]

Infinite evolution eqs. whose flows are all commuting

\[\downarrow\]

Infinite conserved quantities

Plucker embedding maps which define an infinite-dim. Grassmann manifold. (=the solution space)

\[\downarrow\]

Infinite dimensional symmetry
Derivation of soliton equations

- Prepare a Lax operator which is a pseudo-differential operator

\[ L := \partial_x + u_2 \partial_x^{-1} + u_3 \partial_x^{-2} + u_4 \partial_x^{-3} + \cdots \]

- Introduce a differential operator

\[ B_m := (L \ast \cdots \ast L)_{\geq 0} \]

- Define NC (KP) hierarchy:

\[ \frac{\partial L}{\partial x^m} = [B_m, L]* \]

\[ \partial_m u_2 \partial_x^{-1} + \]
\[ \partial_m u_3 \partial_x^{-2} + \]
\[ \partial_m u_4 \partial_x^{-3} + \cdots \]

\[ f_m^2(u) \partial_x^{-1} + \]
\[ f_m^3(u) \partial_x^{-2} + \]
\[ f_m^4(u) \partial_x^{-3} + \cdots \]

\[ u_k = u_k(x^1, x^2, x^3, \cdots) \]

\[ [x^i, x^j] = i \theta^{ij} \]
Negative powers of differential operators

\[ \partial^n_x \circ f := \sum_{j=0}^{\infty} \binom{n}{j} (\partial_x^j f) \partial_x^{n-j} \]

\[ \frac{n(n-1)(n-2)\cdots(n-(j-1))}{j(j-1)(j-2)\cdots1} \]

\[ \partial^3_x \circ f = f\partial^3_x + 3f\partial^2_x + 3f''\partial^1_x + f''' \]

\[ \partial^2_x \circ f = f\partial^2_x + 2f\partial_x + f'' \]

\[ \partial^{-1}_x \circ f = f\partial^{-1}_x - f\partial^{-2}_x + f''\partial^{-3}_x - \cdots \]

\[ \partial^{-2}_x \circ f = f\partial^{-2}_x - 2f\partial^{-3}_x + 3f''\partial^{-4}_x - \cdots \]
Closer look at NC KP hierarchy

\[ \partial_x^{-1} u_2 = 2u'_3 + u''_2 \]

\[ \partial_x^{-2} u_3 = 2u'_4 + u''_3 + 2u_2 * u' + 2[u_2, u_3] \]

\[ \partial_x^{-3} u_4 = 2u'_5 + u''_4 + 4u_3 * u'_2 - 2u_2 * u''_2 + 2[u_2, u_4] \]

\[ \vdots \]

\[ u_x := \frac{\partial u}{\partial x} \]

\[ \partial_x^{-1} := \int^x dx' \]

\[ \partial_x^{-1} u_2 = u'' + 3u''_3 + 3u''_4 + 3u'_2 * u_2 + 3u_2 * u'_2 \]

\[ u_t = \frac{1}{4} u_{xxx} + \frac{3}{4} (u_x * u + u * u_x) + \frac{3}{4} \partial_x^{-1} u_{yy} + \frac{3}{4} [u, \partial_x^{-1} u_{yy}] \]

\[ u = u(x^1, x^2, x^3, \cdots) \]

\[ x \ y \ t \]
*(KP hierarchy) \rightarrow (various hierarchies.)*

*(Ex.*) KdV hierarchy

Reduction condition

\[ L^2 = B_2 \left( =: \partial_x^2 + u \right) \]

gives rise to NC KdV hierarchy

which includes \((1+1)\)-dim. NC KdV eq.:

\[ u_t = \frac{1}{4} u_{xxx} + \frac{3}{4} (u_x u + u u_x) \]

\[ \frac{\partial u}{\partial x_{2N}} = 0 \]

\[ u(x^1, x^2, x^3, x^4, x^5, \ldots) \]

\[ x, y, t \]

\[ u(x^1, x^3, x^5, \ldots) \]

\[ x, t \]
i-reduction of NC KP hierarchy yields wide class of other NC (GD) hierarchies

- No-reduction $\Rightarrow$ NC KP
  \[(x, y, t) = (x^1, x^2, x^3)\]

- 2-reduction $\Rightarrow$ NC KdV
  \[(x, t) = (x^1, x^3)\]

- 3-reduction $\Rightarrow$ NC Boussinesq
  \[(x, t) = (x^1, x^2)\]

- 4-reduction $\Rightarrow$ NC Coupled KdV

- 5-reduction $\Rightarrow$ ...

- 3-reduction of BKP $\Rightarrow$ NC Sawada-Kotera

- 2-reduction of mKP $\Rightarrow$ NC mKdV

- Special 1-reduction of mKP $\Rightarrow$ NC Burgers

- ...
5. Conservation Laws

Conservation laws:
\[ \partial_t \sigma = \partial_i J^i \]
\[ Q := \int_{\text{space}} dx \sigma \]
\[ \therefore \partial_t Q = \int_{\text{space}} dx \partial_t \sigma = \int_{\text{spatial infinity}} dS_i J^i = 0 \]

Conservation laws for the hierarchies

\[ \partial^m \text{res}_{-1} L^n = \partial_x J + \theta^{ij} \partial_j \Xi_i \]

\[ \text{res}_{-r} L^n : \partial_{x}^{-r} L^n \]

\[ t \equiv x^m \]

\[ \partial_j \]

\[ \rightarrow \]
Infinite conserved densities for the NC soliton eqs. \( (n=1,2,\ldots, \Box) \)

\[
\sigma_n = \text{res}_{-1} L^n + \theta^{im} \sum_{k=0}^{m-1} \sum_{l=0}^{k} \left( \binom{k}{l} \right) \left( \partial_x^{k-l} \text{res}_{-(l+1)} L^n \right) \diamond (\partial_i \text{res}_k L^m)
\]

\[
t \equiv x^m \quad \text{res}_r L^n : \quad \Box_{\text{res}} L^n \quad \Box \quad L^n
\]

\[
f(x) \diamond g(x) := f(x) \left( \sum_{s=0}^{\infty} \frac{(-1)^s}{(2s+1)!} \left( \frac{1}{2} \theta^{ij} \bar{\partial}_i \bar{\partial}_j \right)^{2s} \right) g(x)
\]

This suggests infinite-dimensional symmetries would be hidden.
We can calculate the explicit forms of conserved densities for the wide class of NC soliton equations. (existence of negative power of derivatives is crucial!)

- **Space-Space noncommutativity:**
  
  NC deformation is slight: \( \sigma_n = \text{res}_{-1}L^n \)

  involutive (integrable in Liouville’s sense)

- **Space-time noncommutativity**

  NC deformation is drastical:

  - Example: NC KP and KdV equations \( ([t, x] = i\theta) \)

  \[
  \sigma_n = \text{res}_{-1}L^n - 3\theta((\text{res}_{-1}L^n) \hat{\wedge} u_3' + (\text{res}_{-2}L^n) \hat{\wedge} u_2')
  \]
6. Conclusion and Discussion

In every situation, Quasideterminants play important roles!

- NC Twistor Theory, Solution Generating Techniques
- NC DS
- NC KP
- NC Zakharov
- NC KdV
- NC NLS
- NC Boussinesq
- NC CBS
- NC mKdV
- NC pKdV
- NC NLS
- NC N-wave
- NC (affine) Toda
- NC sine-Gordon
- NC Liouville
- NC Tzitzeica

Today’s talk

Friday’s talk

MH[hep-th/0507112]

Summarized in [MH NPB 741(06) 368]

Very HOT!
NC Ward’s conjecture (NC KdV eq.)

- Reduced ASDYM eq.: $x^\mu \to (t, x)$

1. $B' = 0$
2. $C' + \dot{A} + [A, C]_* = 0$
3. $A' - \dot{B} + [C, B]_* = 0$

$$A = \begin{pmatrix} q & -1 \\ q' + q^2 & -q \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

$$C = \begin{pmatrix} \frac{1}{2} q'' + q' \ast q & -q' \\ f(q, q', q'', q''') & -\frac{1}{2} q'' - q \ast q' \end{pmatrix}$$

$$(ii) \implies \begin{pmatrix} \oplus & 0 \\ \otimes & -\oplus \end{pmatrix} = 0 \implies \dot{q} = \frac{1}{4} q'''' + \frac{3}{4} q' \ast q'$$

$u = q' \to \theta \to 0$

$A, B, C \in \text{gl}(2) \xrightarrow{\theta \to 0} \text{sl}(2)$