Superstring Theory in Melvin Background

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Abstract

In this thesis we discuss various aspects of type II superstring theory in NSNS Melvin background. This model can be described by an exactly solvable conformal field theory. First we show that this background includes the orbifolds $\mathbf{C}/\mathbf{Z}_N$ in type II and type 0 theory. Then we construct the model which corresponds to a higher dimensional Melvin background and show that it can be supersymmetric for the particular values of its parameters. In particular, it includes ALE orbifolds $\mathbf{C}^2/\mathbf{Z}_N$. Next we discuss D-branes in these models by constructing their boundary states. As a result we find two kinds of interesting D-branes, which are T-dual equivalent to each other. One of them wraps the geodesic line of the Melvin background spirally. The other is a bound state of D-branes whose world-volume is given by a topologically trivial torus. The latter is stabilized by the presence of the H-flux and the quantized gauge flux. We also show the relation between these D-branes and fractional D-branes in orbifold theories.

Contents

1 Introduction

String theory often requires the modified notion of geometry. One reason of this is that the string world-sheet has a finite length scale. Another is that there exist various background flux (RR-flux and B-flux) in string theory in addition to the background metric.

In open string theory its geometry has been essentially regarded as a noncommutative one [1]. In the presence of constant B-field this nature is enhanced and we have a noncommutative gauge theory on a D-brane [2]. However, we have limited knowledge of D-branes on curved manifolds with a non-constant B-field[1] , where more general treatment of noncommutative geometry will be required.

Next we would like to consider closed string theory. At present there seems to be no unified idea of closed string geometry even though stringy corrections to standard geometry in particular background such as Calabi-Yau manifolds have been discussed in detail (see e.g.[4]). We can expect that the geometry will be changed in the presence of B-flux or RR-flux[2] . For example, let us consider orbifold theories [6]. The geometry of orbifold in the standard sense possesses singularities. Nevertheless the world-sheet theory remains non-singular because of the B-field on the vanishing cycles [7]. Note that in this example the field strength $H = dB$ is zero and the value of dilaton field is constant.

In this thesis we would like to discuss the geometry of string theory in flux backgrounds whose field strength does not vanish. In order to see how such backgrounds are non-trivial and thus interesting, let us consider D0-branes in the string theory with non-zero H-flux. One may think that D0-branes can be used to probe the geometry. However, in the presence of H-flux the dilaton field takes non-constant values in the spacetime. Thus a D0-brane can stay only at particular points where the dilaton gradient vanishes. This shows that we have a very strange geometry if we use the D0-brane probe in this background. Another intriguing effect of H-flux is the existence of D-branes which wrap topologically trivial cycles. This implies that the notion of a ‘cycle’ in string theory is different from the ordinary one. We will see this phenomenon explicitly in our particular model.

A well-studied example of this kind is the string theory in group manifolds such as

[1] For recent development of noncommutative geometry of D-branes in H-flux see e.g. [3].

[2] A sort of a noncommutativity in closed string theory without any flux was also proposed in [5].

$SU(2)$ WZW-model. This gives a solvable model in the curved background with H-flux and its D-brane spectrum has been discussed [8, 9, 10, 11, 12, 13, 14, 15, 16, 17]. In order to cancel the ghost central charge we should add an extra Liouville field [18, 19, 20] and then the string theory is known to describe the geometry near NS5-branes. The presence of Liouville field shows that the theory is singular and indeed the model is T-dual equivalent to singular ALE space [19, 20]. Thus the computation of various physical quantities sometimes becomes subtle.

Here we would like to consider a smooth curved background with H-flux and investigate string theory in (NSNS) Melvin background [21, 22, 23, 24, 25] including its some generalizations [26, 27] as an example. Originally, Melvin background [28] is a flux tube solution in classical 4D Einstein-Maxwell theory. Its explicit metric and flux are given as follows

$$\begin{aligned} ds^2 &= (1+q^2\rho^2)^2\left(-dt^2+dz^2+d\rho^2\right)+\frac{\rho^2}{(1+q^2\rho^2)^2}d\varphi^2, \\ F_{\rho\varphi} &= \frac{q\rho}{2(1+q^2\rho^2)^2}. \end{aligned} \tag{1.1}$$

What we will consider are flux tube backgrounds in string theory, where we have non-trivial flux of the Kaluza-Klein gauge field and B-field.

The string theory in Melvin background is also interesting for other reasons. First this model is known to be exactly solvable [23, 24]. Thus we can compute the mass spectrum or the partition function exactly. As we will see later, we can also investigate D-branes in this theory by constructing boundary states explicitly.

Another reason is the fact that the magnetic flux break supersymmetry and that the theory generically includes closed string tachyons. Thus it gives a useful example to study closed string tachyon condensation (for recent discussions see e.g [29, 30, 31, 32, 33, 34, 35, 27, 36, 37, 38, 39, 40, 41]).

It is also known that for particular values of parameters this model connects type 0 theory and type II theory [42, 30]. The similar observation can also be applied to string theory in RR Melvin background (fluxbrane) [43, 44, 45, 46, 30, 47, 31, 32, 48, 49, 50, 51, 52, 53, 54, 55], where the flux is due to RR-fields. It is argued that the RR Melvin background in type II string is dual to another RR Melvin background in type 0 string theory [30]. Furthermore, later we will see that the orbifolds in type 0 and type II theory are also included in the 'moduli space' of NSNS Melvin background. In this sense we can view the background as a generalization of orbifold theory.

In this way the string theory in Melvin background presents many interesting aspects and thus we would like to discuss these from the viewpoint of both closed string and open string below.

This thesis is organized as follows. In section two we review RR Melvin background (fluxbrane) in string theory or M-theory. We also give a brief review of type 0 string theory. In section three we review and examine the detailed structure of closed string theory in NSNS Melvin background. This model can be rewritten in terms of free fields. In section four we investigate the relation between the Melvin background and orbifolds $\mathbf{C}/\mathbf{Z}_N$ by using the explicit partition function. We also discuss instability due to closed string tachyons. In section five we construct a higher dimensional generalization of NSNS Melvin background which preserves partial supersymmetry. In section six we study boundary states in this model by using the free field representation and consider the relation to fractional D-branes in orbifold theories. In section seven we discuss the mechanism of stabilization of expanded D-branes which are found in the previous section.

2 Review of String/M-theory in Melvin Background

In this section we would like to review some recent useful results on string theoretic generalizations of the Melvin background (1.1) and their interpretations in M-theory. In order to realize them, we consider the classical solutions of type II supergravity with non-trivial flux and dilaton field. Historically, we would like to refer to [56, 57] for the original literature on the Melvin solution in dilatonic gravity theories and to [43, 44, 45, 46] for important further investigations of this subject.

2.1 Fluxbrane

In this setup there are two possibility of flux: NSNS flux or RR flux. In particular, the latter has been recently called fluxbranes[46, 47, 31, 48]. Even though our main arguments in this thesis are for the former (NSNS Melvin background), we would like to review useful results on fluxbranes first. This is because they give us an important motivation as we will see. The simplest and well-studied example is flux 7-branes (F7-branes) in type IIA string theory [30, 31].

This background is easily constructed if we lift the system to M-theory. Let us consider M-theory compactification on $\mathbf{S}^1$ with the twisted identification

$$(x_{11}, \check{\varphi}) \sim (x_{11} + 2\pi R, \check{\varphi} + 2\pi Rq), \quad (x_{11}, \check{\varphi}) \sim (x_{11}, \check{\varphi} + 2\pi), \tag{2.1}$$

where x_{11} and $(\rho, \check{\varphi})$ stand for the polar coordinate of the circle $\mathbf{S}^1$ (radius R) and its transverse two dimensional plane $\mathbf{R}^2$, respectively. Then we obtain the well-defined coordinate

$$\varphi = \check{\varphi} - qx_{11}, \tag{2.2}$$

which satisfies the usual periodicity. The flat metric in M-theory with this twisted identification (2.1) is rewritten in terms of φ as follows (see fig.1)

$$\begin{aligned} ds_M^2 &= d\rho^2 + \rho^2 d\check{\varphi}^2 + dx_{11}^2 + \sum_{\mu=0}^{7} dx^\mu dx_\mu, \\ &= d\rho^2 + (1+q^2\rho^2)\left(dx_{11} + \frac{q\rho^2}{1+q^2\rho^2}d\varphi\right)^2 + \frac{\rho^2}{1+q^2\rho^2}d\varphi^2 + \sum_{\mu=0}^{7} dx^\mu dx_\mu. \end{aligned} \tag{2.3}$$

By applying the identification of type IIA string as the Kaluza-Klein compactification of M-theory, we obtain the following classical solution in type IIA theory

$$\begin{aligned} ds^2_{IIA} &= \sqrt{1+q^2\rho^2}\left(\sum_{\mu=0}^{7} dx^\mu dx_\mu + d\rho^2\right) + \frac{\rho^2}{\sqrt{1+q^2\rho^2}} d\varphi^2, \\ A_\varphi &= \frac{q\rho^2}{1+q^2\rho^2}, \qquad e^{4/3\phi} = 1+q^2\rho^2, \end{aligned} \tag{2.4}$$

90 000 where A_φ and ϕ denote the RR 1-form and dilaton. This solution represents 7-brane background with RR 1-form flux and is called F7-brane.

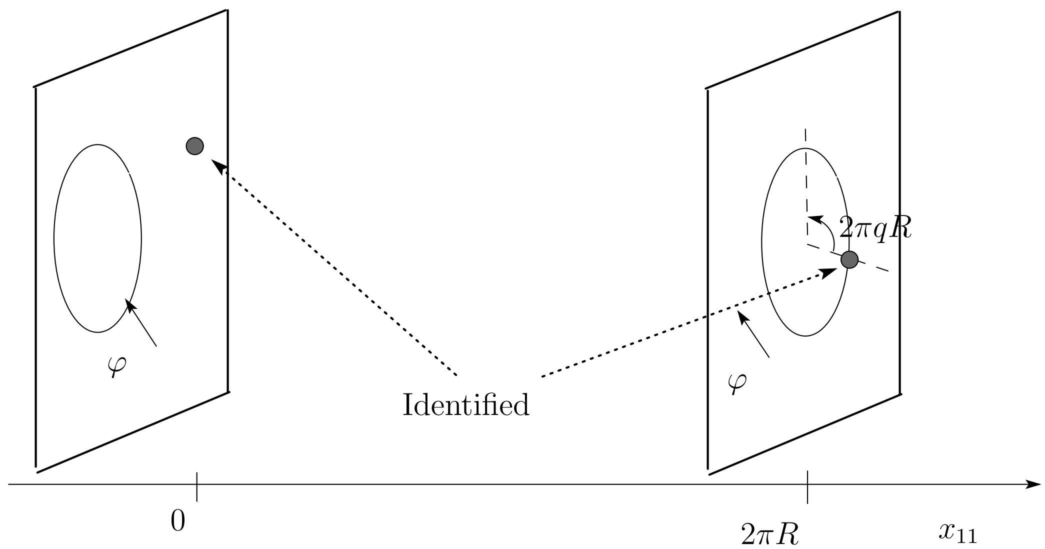

Figure 1: The twisted identification in M-theory.

This new type “brane” in string theory is difficult to get any conformal field theoretic definition in contrast with a D-brane. Instead it is known that we can regard the F7-brane solution as a $\mathrm{D6}-\overline{\mathrm{D6}}$ system in a certain limit[54]. See also [50] for a relation between non-extremal black holes and fluxbranes.

One of the important properties of F7-brane is its non-perturbative structure. The dilaton expectation value of (2.4) shows that the system should be strongly coupled if we go away from the origin. Thus the above classical treatment may be corrected substantially.

Another interesting property is that this background breaks thirty two supersymmetries completely by a rather simple mechanism. This gives the most important motivation to consider it. If a fermion field $\psi(x_{11})$ in this background is moved by the parallel transportation around the circle $\mathbf{S}^1$ once, then it will obtain the phase factor as follows

$$\psi(x_{11} + 2\pi R) = +e^{2i\pi qRJ}\psi(x_{11}) \ \text{ or } \ -e^{2i\pi qRJ}\psi(x_{11}), \tag{2.5}$$

where J is the spin of the fermion in the direction φ. The sign factors of (2.5) represent the two possible spin structures. Since we recover supersymmetric type IIA theory for $q = 0$, we should choose + sign. The above result shows that there is no massless fermion unless $qR \in$ even and thus the supersymmetry is completely broken. This can be regarded as the Scherk-Schwarz mechanism in M-theory. Furthermore, if we take the limit $R \to 0$, then all fermions become infinitely massive and we will obtain a purely bosonic ten dimensional string theory[3] . What kind of string theory is it? Then the reader may think the type 0 string theory [58, 59, 60, 61] as a candidate. Before we see the related conjecture by [42, 30], we would like to review type 0 string theories in the next subsection.

It is also possible to consider fluxbranes in curved spaces [52]. Let us assume the eleven dimensional classical solution $\mathbf{R} \times \mathbf{X}_{10}$, where $\mathbf{X}_{10}$ is any ten dimensional metric with U(1) isometry parameterized by $\check{\varphi}$. The explicit form of the metric is written in the following form

$$ds_M^2 = dx_{11}^2 + f(x)(d\check{\varphi} + f_\mu(x)dx^\mu)^2 + g_{\mu\nu}dx^\mu dx^\nu. \tag{2.6}$$

After the twisted compactification (2.1), we obtain the following F7-brane background,

$$\begin{aligned} ds_{IIA}^2 &= (1 + q^2 f(x))^{-\frac{1}{2}}(d\varphi + f(x)_\mu dx^\mu)^2 + (1 + q^2 f(x))^{\frac{1}{2}} g_{\mu\nu}dx^\mu dx^\nu, \\ A &= \frac{qf(x)}{1 + q^2 f(x)}(d\varphi + f(x)_\mu dx^\mu), \qquad e^{4/3\phi} = 1 + q^2 f(x), \end{aligned} \tag{2.7}$$

Up to now we have discussed F7-branes. We can also consider Fp-branes ($p \neq 7$) which couple to (8-p)-form RR field. Singular solutions of Fp-branes have been obtained exactly and smooth solutions have been analyzed numerically in [47, 31, 48]. Some interesting applications of fluxbrane-like solutions have also been recently discussed. For example, see [48, 49, 51] and [49, 62] for classical solutions of the dielectric-brane [63] and the supertube [64].

2.2 Type 0 String Theory

If we require the world-sheet $\mathcal{N} = (1, 1)$ supersymmetry, well-defined OPEs and modular invariance, we have type IIA(B) string theory or type 0A(B) string theory as

[3] Here we ignore the particular case $3qR \in \mathbf{Z}$.

shown in e.g. [61]. Type 0A(B) string theory [58, 59, 60, 61] is defined by the diagonal GSO projection and consists of the following sectors in the world-sheet theory

$$\begin{aligned} 0\mathrm{A} \quad &: \quad (\mathrm{NS}+,\mathrm{NS}+)\ (\mathrm{NS}-,\mathrm{NS}-)\ (\mathrm{R}+,\mathrm{R}-)\ (\mathrm{R}-,\mathrm{R}+), \\ 0\mathrm{B} \quad &: \quad (\mathrm{NS}+,\mathrm{NS}+)\ (\mathrm{NS}-,\mathrm{NS}-)\ (\mathrm{R}+,\mathrm{R}+)\ (\mathrm{R}-,\mathrm{R}-) \end{aligned} \tag{2.8}$$

where the signs $\pm$ stand for the world-sheet fermion numbers $(-1)^{F_L}$ and $(-1)^{F_R}$. There is T-duality relation between type 0A and 0B theory in the same way as in type II theory.

As we can see from (2.8), the theory contains no fermions and double RR-fields $C_{RR}^{(1)}$ and $C_{RR}^{(2)}$. What is more important is the presence of the oppositely GSO projected sector (NS+, NS+), which includes a tachyon field. Thus it is natural to consider that type 0 string theory is unstable and should decay into tachyon-less string theory. However, even today no complete result on its decay has not been obtained (some discussions will be shown later in this thesis and see also [31, 39]).

We can also regard type 0A(B) string theory as a $\mathbf{Z}_2$ twist of type IIA(B) theory by the operator $(-1)^{F_S}$, where F_S denotes the spacetime fermion number. More generally, we can show the following 'T-duality relation' between type 0 and type II string theory [58, 60, 42]

$$\begin{aligned} \mathrm{IIA(B)\ on\ } \mathbf{S}^1\ (\mathrm{radius\ } 2R) \quad / \quad (-1)^{F_S}\sigma_{1/2} \simeq \\ 0\mathrm{B(A)\ on\ } \mathbf{S}^1\ (\mathrm{radius\ } 1/R)\ /\ (-1)^{F_R}\sigma_{1/2}, \end{aligned} \tag{2.9}$$

where $\sigma_{1/2}$ represents the half shift operator along $\mathbf{S}^1$. This relation can be shown if we write down each spectrum. The spectrum of the IIA theory (radius $2R$) is given by

$$\begin{aligned} \mathit{Untwisted:} \quad & (\mathrm{NS}+,\mathrm{NS}+),\ (\mathrm{R}+,\mathrm{R}-), \quad P_{L,R} = n/R \pm 2wR/\alpha', \\ & (\mathrm{R}+,\mathrm{NS}+),\ (\mathrm{NS}+,\mathrm{R}-), \quad P_{L,R} = (n+1/2)/R \pm 2wR/\alpha', \\ \mathit{Twisted:} \quad & (\mathrm{R}-,\mathrm{NS}-),\ (\mathrm{NS}-,\mathrm{R}+), \quad P_{L,R} = (n+1/2)/R \pm (2w+1)R/\alpha', \\ & (\mathrm{NS}-,\mathrm{NS}-),\ (\mathrm{R}-,\mathrm{R}+), \quad P_{L,R} = n/R \pm (2w+1)R/\alpha', \end{aligned} \tag{2.10}$$

where $P_{L,R}$ denotes the (left-moving and right-moving) momentum along $\mathbf{S}^1$ and we have also defined $n, w \in \mathbf{Z}$. The reason for the appearance of oppositely GSO projected sectors in the twisted sectors is explained by the spectral flow of the boundary condition for fermions (see e.g. [61]). For more detail see the more general arguments by using the partition function in Green-Schwarz formalism in section 3. On the other hand, the spectrum (2.10) can also be interpreted as that in the type 0B theory if we perform the T-duality procedure $P_R \to -P_R$. Similarly we can do the same thing for type IIB and type 0A. In this way we have shown the relation (2.9). It would be also interesting to note that fermions appear in the twisted sector of the type 0 theory.

In this model the limit $R \to 0$ corresponds to the ten dimensional type 0B(A) theory, while the limit $R \to \infty$ to type IIA(B) theory. Thus the model connects type 0 and type II theory. Later we will see a generalization of the interpolation in the examples of string theory on Melvin background (see fig.2).

We would also like to see D-branes in type 0 theory [65, 66, 67], which will be useful for later arguments. Since there are two kinds of RR fields, there are two kinds of Dp-branes for fixed p. We call these an electric Dp-brane and a magnetic Dp-brane. Let us see their boundary states in order to know the coupling of the branes with closed string. A boundary state $|B\rangle$ represents the D-brane boundary condition of the world-sheet from the viewpoint of closed string theory (for a review see [68]). The boundary condition of Dp-branes is defined by

$$\text{Neumann}: \left\{ \begin{array}{ccc} \partial_\tau X^\mu|_{\tau=0}|B\rangle = 0 & \leftrightarrow & (\alpha^\mu_n + \tilde{\alpha}^\mu_{-n})|B\rangle = 0, \\ (\psi^\mu_L - \epsilon\psi^\mu_R)|_{\tau=0}|B\rangle = 0 & \leftrightarrow & (\psi^\mu_r - i\epsilon\tilde{\psi}^\mu_{-r})|B\rangle = 0, \end{array} \right. \tag{2.11}$$

$$\text{Dirichlet}: \left\{ \begin{array}{ccc} \partial_\sigma X^i|_{\tau=0}|B\rangle = 0 & \leftrightarrow & (\alpha^i_n - \tilde{\alpha}^i_{-n})|B\rangle = 0, \\ (\psi^i_L + \epsilon\psi^i_R)|_{\tau=0}|B\rangle = 0 & \leftrightarrow & (\psi^i_r + i\epsilon\tilde{\psi}^i_{-r})|B\rangle = 0, \end{array} \right. \tag{2.12}$$

Here we have assumed the coordinate of its world-volume is given by $x^\mu \quad (0 \leq \mu \leq p)$ and the transverse coordinate by $x^i \quad (p+1 \leq i \leq 9)$. The world-sheet bosonic and (left-moving, right-moving) fermionic fields are denoted by $X^{\mu,i}$ and $\psi^{\mu,i}_{L,R}$, where their left-moving and right-moving oscillators are represented by $\alpha^{\mu,i}_n, \psi^{\mu,i}_r$ and $\tilde{\alpha}^{\mu,i}_n, \tilde{\psi}^{\mu,i}_r$ (for more detail see appendix B.). The sign $\epsilon = \pm$ corresponds to the two different boundary condition for fermions. We also denote the coordinate of world-sheet by (τ, σ). In the flat background it is possible to solve the above condition (2.12) exactly. The explicit expression of this in more general background will be given in section 6. Then we obtain the total boundary state for Dp-branes in type II string theory

$$|\text{typeII Dp}\rangle \;=\; \frac{T_p}{4}\Big((|B,+\rangle_{NSNS} - |B,-\rangle_{NSNS}) + (|B,+\rangle_{RR} + |B,-\rangle_{RR}) \Big), \tag{2.13}$$

where we defined $T_p = \sqrt{\pi}(2\pi\sqrt{\alpha'})^{3-p}$ so that T_p/κ (κ is the gravitational coupling constant) is equal to the tension of type II Dp-brane. This explicit value T_p is computed by requiring that the vacuum cylinder amplitude between two D-branes in open string theory should be equal to that calculated by using the two boundary states. This condition is called Cardy's condition [69] and we will see the explicit calculations later. Note also that the GSO projection $\frac{(1+(-1)^{F_L})(1+(-1)^{F_R})}{4}$ restricts the linear combination of $|B,+\rangle_{NSNS,RR}$ and $|B,-\rangle_{NSNS,RR}$ to the above form and allowed values of p to $p = even$ for type IIA ($p = odd$ for type IIB).

On the other hand, in type 0 string theory we impose the 'diagonal GSO projection' $\frac{1+(-1)^{F_L+F_R}}{2}$ and thus we have the following two different kinds of D-branes

$$\begin{aligned} |\text{electric}\rangle &= \frac{T_p}{2\sqrt{2}}\left(|B,+\rangle_{NSNS} + |B,+\rangle_{RR}\right), \\ |\text{magnetic}\rangle &= \frac{T_p}{2\sqrt{2}}\left(-|B,-\rangle_{NSNS} + |B,-\rangle_{RR}\right). \end{aligned} \tag{2.14}$$

Then we can easily see that the tension of these type 0 D-branes is $1/\sqrt{2}$ times as large as that of type II theory. We can also verify that an electric one couples to the RR-field $C^{(1)}_{RR} + C^{(2)}_{RR}$ and a magnetic one to $C^{(1)}_{RR} - C^{(2)}_{RR}$. Allowed values of p are the same as those in type II theory.

It would be also useful to know about the open string spectrum of these D-branes in type 0 theory. If we consider open string between two D-branes of the same type, then we have only purely bosonic spectrum from the NS-sector with the ordinary GSO projection. However, if we consider two different D-branes, then we have purely fermionic spectrum from the R-sector. Note that this is the only fermionic excitation in type 0 theory which appears perturbatively.

2.3 Type II/Type 0 duality and Instability

Now we return to the relation between F7-branes and type 0 string theory and explain the conjecture given by [42, 30]. This conjecture is obtained by identifying another spin structure in (2.5) (minus sign) with type 0A theory. For $q' = 0$ (here we add the symbol ' to distinguish type 0 case with type II case), we have the ordinary ten dimensional type 0A theory and for non-zero q' we have a type 0A theory in RR-flux background. Even though the spectrum includes fermions for non-zero R, they will disappear for $R = 0$ (weak coupling limit) and this is consistent with the perturbative spectrum of type 0A theory. In this way we obtain the following novel conjecture [30]

$$\text{type IIA } (q) \simeq \text{type 0A } (q - 1/R). \tag{2.15}$$

If we set $q = 1/R$, the above conjecture (2.15) argues that M-theory compactified on the small circle with the antiperiodic boundary condition for fermions is equivalent to type 0A theory [42]. Furthermore let us compactify the theory on another small circle with the ordinary boundary condition and exchange the two circles ('$9 - 11'$ flip). Then we reproduce the previous result that IIA on the circle twisted by $(-1)^{F_S}\sigma_{1/2}$ leads to type 0B theory in the small radius limit (2.10).

However, there are obviously many problems about this identification (2.15). The background should be treated non-perturbatively because the string coupling is generically strong at large ρ. The analysis of this in the world-sheet theory requires the quantization of string theory with RR flux and is very difficult. Thus the explicit proof of this conjecture seems to be impossible at present. We should also explain the instability of type 0A theory from the viewpoint of type IIA or M-theory. Note also that the relation (2.15) implies that there appear massive fermions non-perturbatively in type 0A theory. In the paper [42] the relation between the fermions and the bound states of D0-branes was discussed. Since the open string between an electric D0-brane and a magnetic D0-brane is purely fermionic, the quantization of the zero-modes gives fermionic excitations in spacetime from the bound state of them.

The instability of this background can be understood in supergravity theory. A kind of deformation of Myers-Perry Kerr instanton in M-theory (5D Kerr black-hole and 6D flat space) represents an instanton which describes the decay of it [45, 30]. There are two such instantons corresponding to type 0A and type IIA. The former is a generalization of the Witten's bubble[4] [70] (Euclidean Schwarzchild black hole). This instanton seems to be interpreted as the existence of closed tachyon in type 0 theory if we take the weak coupling limit. The latter instanton is interpreted[5] as a Kaluza-Klein monopole/anti-monopole pair in M-theory ($\mathrm{D6}-\overline{\mathrm{D6}}$ in type IIA). Because the generation of this instanton reduces the strength of RR-flux, it is natural to speculate that the system will eventually decay into the supersymmetric vacuum (ordinary type IIA theory) [31].

[4] Note that the instanton configuration of the Witten's bubble allows a single spin structure [70] which breaks supersymmetry. Therefore the instability does not occur in the case of type IIA theory.

[5] More precisely, this interpretation is valid if we assume a compactification of the six dimension. In the non-compact case we can instead regard it as a spherical D6-brane [30, 31].

2.4 Supersymmetric Fluxbrane

Let us consider higher dimensional generalizations of the twisted compactification of M-theory [46, 31, 52]. Namely, we can compactify $\mathbf{S}^1$ direction by the following identification

$$(x_{11}, \check{\varphi}_a) \sim (x_{11} + 2\pi R, \check{\varphi}_a + 2\pi R q_a), \quad (x_{11}, \check{\varphi}_a) \sim (x_{11}, \check{\varphi}_a + 2\pi), \tag{2.16}$$

where we have defined $(\rho_a, \check{\varphi}_a), \quad (a = 1, 2, \cdots, n)$ as the polar coordinate of $\mathbf{R}^{2n}$. Then we have the well-defined coordinate $\varphi_a = \check{\varphi}_a - q_a x_{11}$ as before and the eleven dimensional metric is given by

$$ds_M^2 = dx_{11}^2 + \sum_{a=1}^{n} \left(d\rho_a^2 + \rho_a^2 (d\varphi_a + q_a dx_{11})^2 \right) + \sum_{\mu=0}^{9-2n} dx^\mu dx_\mu. \tag{2.17}$$

After the Kaluza-Klein reduction, we have the following type IIA background ('F(9-2n)-brane')

$$\begin{aligned} ds_{IIA}^2 &= \sqrt{\Lambda} \left(\sum_{\mu=0}^{9-2n} dx^\mu dx_\mu + \sum_{a=1}^{n} (d\rho_a^2 + \rho_a^2 \varphi_a^2) \right) - \sum_{a=1}^{n} \frac{q_a^2 \rho_a^4}{\sqrt{\Lambda}} (d\varphi_a)^2, \\ A_{\varphi_a} &= \frac{q_a \rho_a^2}{\Lambda}, \qquad e^{4/3\phi} = \Lambda, \end{aligned} \tag{2.18}$$

where we have defined $\Lambda = 1 + \sum_{a=1}^{n} q_a^2 \rho_a^2$. This kind of background can be more properly said as an intersection of F7-branes [27].

The most important property of these generalized fluxbranes is the fact that we can preserve partial supersymmetry for particular choice of the parameters q_a. For example, if we assume $n = 2$ and $q_1 = \pm q_2$, then we have the background (supersymmetric F5-brane [31]) which preserves half supersymmetry. The reason of this can be understood by considering the boundary condition for fermions like (2.5). In this case we obtain the phase factor $e^{2\pi i R(q_1 J_1 + q_2 J_2)}$ and thus we can preserve half supersymmetry for $q_1 = \pm q_2$. For more comprehensive analysis see [55].

3 Closed Strings in NSNS Melvin Background

The main purpose of this paper is to investigate the string theory in NSNS Melvin Background. As we have seen before, the F7-brane (or equally RR Melvin) background is highly nonperturbative with RR-flux and thus is difficult to investigate the world-sheet sigma model quantitatively. Thus we would like to perform a U-dual transformation of F7-brane so that the string coupling can be small enough and the sigma model includes only NSNS flux. Such a transformation is given by the interchange of x^9 and x^{11} direction (so called $9-11$ flip) in M-theory and is represented in terms of type II string theory as follows.

$$\begin{array}{ccccccc} \text{IIA} & \xrightarrow{\text{T-dual}} & \text{IIB} & \xrightarrow{\text{S-dual}} & \text{IIB} & \xrightarrow{\text{T-dual}} & \text{IIA} \\ \text{RR1-form}(q) & & \text{RR2-form} & & \text{NSNS2-form} & & \text{Metric} \\ \text{NSNS2-form}(\beta) & & \text{Metric} & & \text{Metric} & & \text{NSNS2-form} \end{array}$$

Here we can add the NSNS B-field to the background and we have two magnetic parameters q and β. The target space of this model has the structure of Kaluza-Klein theory and has the topology $\mathrm{M}_3 \times \mathbf{R}^{1,6}$. The three dimensional manifold M_3 is given by $\mathbf{S}^1$ fibration over $\mathbf{R}^2$. We write the coordinate of $\mathbf{R}^2$ and $\mathbf{S}^1$ by ρ, φ (polar coordinate) and y (with radius R). The non-trivial fibration is due to two Kaluza-Klein gauge fields A_φ and B_φ (see (3.1)) which originate from Kaluza-Klein reduction of metric $G_{\varphi y}$ and B-field $B_{\varphi y}$, respectively. In the most of the discussions below, we will neglect the trivial flat part $\mathbf{R}^{1,6}$.

The explicit metric and other NSNS fields before the Kaluza-Klein reduction are given as follows

$$\begin{aligned} ds^2 &= d\rho^2 + \frac{\rho^2}{(1+\beta^2\rho^2)(1+q^2\rho^2)}d\varphi^2 + \frac{1+q^2\rho^2}{1+\beta^2\rho^2}(dy + A_\varphi d\varphi)^2, \\ &= d\rho^2 + \frac{1}{(1+\beta^2\rho^2)}\left(dy^2 + \rho^2(d\varphi + q dy)^2\right), \\ A_\varphi &= \frac{q\rho^2}{1+q^2\rho^2}, \quad B_{\varphi y} \equiv B_\varphi = -\frac{\beta\rho^2}{1+\beta^2\rho^2}, \quad e^{2(\phi-\phi_0)} = \frac{1}{1+\beta^2\rho^2}, \end{aligned} \tag{3.1}$$

where q, β are the magnetic parameters which are proportional to the strength of the two gauge fields and ϕ_0 is the constant value of the dilaton ϕ at $\rho = 0$.

In this section we would like to mainly review and discuss the sigma model of this background (3.1) in type II superstring. Interestingly, this model is exactly solvable as shown in [24, 25] (see also [21, 22, 23] for such a model in bosonic string theory). This gives one interesting example of a solvable string theory in a non-trivial background.

It would also be useful to note that if $\beta = 0$, we get a locally flat metric with the zero curvature tensor

$$ds^2 = d\rho^2 + dy^2 + \rho^2(d\varphi + q dy)^2. \tag{3.2}$$

This background is globally non-trivial because the angle φ is compactified such that its period is 2π. For example, its geodesic lines[6]

$$\varphi + qy = \text{constant}, \tag{3.3}$$

are spiral and do not return to the same point for irrational qR if one goes around the circle $\mathbf{S}^1$. As we will see later in section 6, this geometry rules the D-brane spectrum [71].

3.1 Sigma Model Description and Its Free Field Representation

At first sight the above background (3.1) for general q, β does not seem to be tractable in the description of the two dimensional sigma model. However, with appropriate T-duality transformations which we will review in appendix A (see also [72]) one can solve this sigma model in terms of free fields [23, 24]. In this paper we define the coordinate of world-sheet as $z = \sigma_1 + i\sigma_2$ and the derivatives as $\partial = \frac{1}{2}(\partial_1 - i\partial_2), \bar{\partial} = \frac{1}{2}(\partial_1 + i\partial_2)$.

The sigma model for the background (3.1) is given[7] by (we show only the bosonic part)

$$\begin{aligned} S &= \frac{1}{\pi\alpha'}\int d^2\sigma\Big[\bar{\partial}\rho\partial\rho + \frac{(1+q^2\rho^2)}{(1+\beta^2\rho^2)}(\bar{\partial}Y + \frac{q\rho^2}{1+q^2\rho^2}\bar{\partial}\varphi)(\partial Y + \frac{q\rho^2}{1+q^2\rho^2}\partial\varphi) \\ &\quad + \frac{\rho^2}{(1+\beta^2\rho^2)(1+q^2\rho^2)}\bar{\partial}\varphi\partial\varphi - \frac{\beta\rho^2}{1+\beta^2\rho^2}(\bar{\partial}Y\partial\varphi - \partial Y\bar{\partial}\varphi)\Big] \\ &= \frac{1}{\pi\alpha'}\int d^2\sigma\Big[\bar{\partial}\rho\partial\rho + \bar{\partial}Y\partial Y + \frac{\rho^2}{1+\beta^2\rho^2}(\bar{\partial}\varphi + (q-\beta)\bar{\partial}Y)(\partial\varphi + (q+\beta)\partial Y)\Big], \end{aligned} \tag{3.4}$$

[6] For non-zero β the geodesic lines in (3.1) are given by $\varphi + (q \pm \beta)y =$const.

[7] The background eq.(3.1) satisfies the equation of motion even if we take α' corrections into account [21] as can be shown by using the general arguments in [73].

where we have omitted the term of the dilaton coupling for simplicity. We have also abbreviated the fermion terms since it is easily obtained if we use the superfield. They are easily incorporated if we use the $\mathcal{N}=1$ world-sheet superfield formulation[8] .

First let us perform the T-duality which transforms the field φ into the new one $\tilde{\varphi}$ (for more explanations see the appendix A). The result is given by

$$S = \frac{1}{\pi\alpha'}\int d^2\sigma\Big[\bar{\partial}\rho\partial\rho + (\bar{\partial}Y + \beta\bar{\partial}\tilde{\varphi})(\partial Y + \beta\partial\tilde{\varphi}) + q(\bar{\partial}\tilde{\varphi}\partial Y - \bar{\partial}Y\partial\tilde{\varphi}) + \frac{1}{\rho^2}\bar{\partial}\tilde{\varphi}\partial\tilde{\varphi}\Big]. \tag{3.5}$$

After we define the field Y' by

$$Y' = Y + \beta\tilde{\varphi}, \tag{3.6}$$

we can again take the T-duality along $\tilde{\varphi}$ into φ'. Then we obtain

$$S = \frac{1}{\pi\alpha'}\int d^2\sigma\Big[\bar{\partial}\rho\partial\rho + \bar{\partial}Y'\partial Y' + \rho^2(\bar{\partial}\varphi' + q\bar{\partial}Y')(\partial\varphi' + q\partial Y')\Big]. \tag{3.7}$$

From this expression it is easy to see that one can describe the sigma model by free fields X' and $\bar{X}'$ which are defined by

$$X' = \rho e^{i\varphi''},\ \bar{X}' = \rho e^{-i\varphi''}, \tag{3.8}$$

where

$$\varphi'' = \varphi' + qY'. \tag{3.9}$$

Here we will examine the relation between the free fields $X', \bar{X}'$ and the fields $X = \rho\ e^{i\varphi}, \bar{X} = \rho\ e^{-i\varphi}$ which represent the original plane $\mathbf{R^2} \in \mathrm{M}_3$ in (3.1). Applying the relation (A.6) to the above two different T-duality transformations[9] , we can obtain

$$\partial\varphi = \partial\varphi'' - q\partial Y - \beta\partial Y', \quad \bar{\partial}\varphi = \bar{\partial}\varphi'' - q\bar{\partial}Y + \beta\bar{\partial}Y'. \tag{3.10}$$

This shows that the field φ is rewritten as

$$\varphi(z,\bar{z}) = \varphi''(z,\bar{z}) - qY(z,\bar{z}) + \beta\left(Y'_R(\bar{z}) - Y'_L(z)\right), \tag{3.11}$$

[8] To do this one has only to replace the derivatives $\partial,\ \bar{\partial}$ with $D_\theta = \partial_\theta + \theta\partial,\ D_{\bar{\theta}} = \partial_{\bar{\theta}} + \bar{\theta}\bar{\partial}$ and a bosonic field X with $\mathbf{X}(z,\bar{z}) = X(z,\bar{z}) + i\theta\psi_L(z) + i\bar{\theta}\psi_R(\bar{z}) + \cdots$.

[9] The same result can be obtained by performing the T-duality about Y once if we regard $\check{\varphi} \equiv \varphi + qY$ and Y as fundamental fields in the sigma model. In this subsection we have used the T-duality about φ for the convenience of the explanation.

where $Y'_L(z)$ ($Y'_R(\bar{z})$) is the left(right)-moving part of Y'. Therefore from this the relation between $X', \bar{X}'$ and $X, \bar{X}$ is represented as

$$X(z,\bar{z}) = e^{-iqY(z,\bar{z})+i\beta Y'_R(\bar{z})-i\beta Y'_L(z)} X'(z,\bar{z}). \tag{3.12}$$

It is easy to generalize the above results into those in the supersymmetric case since we can use $\mathcal{N}=1$ world-sheet superfield formulation. Then the above equation (3.12) does hold as a superfield and we can also define the free fields $(\psi'_{L,R}, \bar{\psi}'_{L,R})$ as the partners of $(X', \bar{X}')$.

Since we have the free field representation $(Y', \eta'_{L,R})$, $(X', \psi'_{L,R})$ and $(\bar{X}', \bar{\psi}'_{L,R})$, the quantization of the Melvin background can be performed. Before that, we have to examine the boundary condition of the field Y'. From the relations (A.7) one obtains

$$\begin{aligned} \partial\tilde{\varphi} &= -\rho^2\partial\varphi'' + i\psi'_L\bar{\psi}'_L \equiv i\alpha' j_L, \\ \bar{\partial}\tilde{\varphi} &= \rho^2\bar{\partial}\varphi'' - i\psi'_R\bar{\psi}'_R \equiv -i\alpha' j_R, \end{aligned} \tag{3.13}$$

and the conservation law of the above current $j_{L,R}$ follows directly. Notice also the useful relation

$$\rho^2\partial\varphi'' = \frac{1}{2i}(\bar{X}'\partial X' - X'\partial\bar{X}'). \tag{3.14}$$

Then we can define the angular momentum operators $\hat{J}_L, \hat{J}_R$ in $(X', \bar{X}') \in \mathbf{R^2}$ directions as follows[10]

$$\hat{J}_L = \frac{1}{2\pi i}\oint dz\ j_L(z), \quad \hat{J}_R = -\frac{1}{2\pi i}\oint d\bar{z}\ j_R(\bar{z}). \tag{3.15}$$

Then we can see from (3.13) and (3.15) how the boundary condition of $\tilde{\varphi}$ should be twisted

$$\tilde{\varphi}(\tau, \sigma+2\pi) = \tilde{\varphi}(\tau,\sigma) - 2\pi\alpha'\hat{J}, \tag{3.16}$$

where we have defined the total angular momentum operator as $\hat{J} = \hat{J}_R + \hat{J}_L$ and the new world-sheet coordinates τ, σ as $z = \exp(\tau + i\sigma)$. Moreover notice that the original coordinate Y satisfies

$$Y(\tau, \sigma+2\pi) = Y(\tau,\sigma) + 2\pi R w. \tag{3.17}$$

[10] Note the operator product expansions (OPE) $j_L(z)\partial X'(w) \sim \frac{1}{2(z-w)}\partial X'(w) + \frac{1}{2(z-w)^2}X'(w)$, $j_L(z)\psi'_L(w) \sim \frac{1}{z-w}\psi'_L(w)$, $j_L(z)\bar{\psi}'_L(w) \sim -\frac{1}{z-w}\bar{\psi}'_L(w)$ and similar results for the right-moving sector. Here we have used the relation (3.13), (3.14) and OPE for free fields normalized such that $X'(z)\bar{X}'(w) \sim -\alpha'\ln(z-w)$ and $\psi'_L(z)\bar{\psi}'_L(w) \sim \frac{\alpha'}{z-w}$. Thus we can find that the operators $\psi'_{L,R}$ and $\partial X', \bar{\partial}X'$ have charges $\hat{J}_{L,R}=1$ and on the other hand $\bar{\psi}'_{L,R}$ and $\partial\bar{X}', \bar{\partial}\bar{X}'$ have charges $\hat{J}_{L,R}=-1$.

After all from (3.6), (3.16) and (3.17) the periodicity of the field Y' is given by

$$Y'(\tau,\sigma+2\pi)=Y'(\tau,\sigma)+2\pi Rw-2\pi\alpha'\beta\hat{J}. \tag{3.18}$$

On the other hand, the canonical momentum of Y is

$$\begin{aligned} P_Y &= \frac{1}{2\pi\alpha'}\int d\sigma(q\bar{\partial}\tilde{\varphi}-q\partial\tilde{\varphi}+\partial Y'+\bar{\partial}Y') \\ &= q\hat{J}+\frac{1}{2}(P'_L+P'_R), \end{aligned} \tag{3.19}$$

where the first line is obtained from (3.5) and (3.6). Therefore from the quantization of P_Y as $P_Y=\frac{n}{R}$ ($n\in\mathbf{Z}$) the quantized zero modes of Y' are obtained as follows

$$P'_L+P'_R=2(\frac{n}{R}-q\hat{J}),\quad P'_L-P'_R=2(\frac{Rw}{\alpha'}-\beta\hat{J}). \tag{3.20}$$

Next we turn to the quantization of the free fields $X',\bar{X}'$ and $\psi'_{L,R},\bar{\psi}'_{L,R}$. They obey the following twisted boundary conditions which can be obtained from (3.12), (3.17) and (3.20),

$$\begin{aligned} X'(\tau,\sigma+2\pi) &= e^{2\pi i\gamma}X'(\tau,\sigma), \\ \psi'_L(\tau,\sigma+2\pi) &= e^{2\pi i\gamma}\psi'_L(\tau,\sigma),\quad \psi'_R(\tau,\sigma+2\pi)=e^{2\pi i\gamma}\psi'_R(\tau,\sigma), \end{aligned} \tag{3.21}$$

$$\text{where}\qquad \gamma\equiv qRw+\beta\alpha'(\frac{n}{R}-q\hat{J}). \tag{3.22}$$

Note that there are no zero-modes for $X',\bar{X}'$ if γ is not an integer. This fact will be crucial when we consider D-branes in this model later.

The above boundary conditions are similar to those in orbifold theories and therefore it is straightforward to perform the mode expansion and its canonical quantization. We summarize these results in the appendix B.

3.2 Mass Spectrum

We have explained that the sigma model of the Melvin background can be solved in terms of free fields $(X',\bar{X}',Y')$. Thus it is straightforward to compute the mass spectrum of this model in the NS-R formalism, which can be obtained from $L_0,\tilde{L}_0$ in (B.4). If we define $\hat{N}_{R,L}$ and $\hat{J}_{L,R}$ as the (left or right-moving) occupation number operator and the angular momentum in the φ'' direction both of which include the zero point energy ($\pm\frac{1}{2}$

for NS-sector and 0 for R-sector). For explicit formula see (B.5) and (B.6). Then the result [24, 25] is given by

$$\begin{aligned}\frac{\alpha' M^2}{2} &= \frac{\alpha'}{2R^2}(n - qR\hat{J})^2 + \frac{R^2}{2\alpha'}(w - \frac{\alpha'}{R}\beta\hat{J})^2 + \hat{N}_R + \hat{N}_L - \hat{\gamma}(\hat{J}_R - \hat{J}_L), \\ &\qquad \text{where} \quad \hat{\gamma} \equiv \gamma - [\gamma],\end{aligned} \tag{3.23}$$

with the level matching constraint

$$\hat{N}_R - \hat{N}_L - nw + [\gamma]\hat{J} = 0, \tag{3.24}$$

where $[\gamma]$ denotes the integer part of γ. Moreover the GSO-projection for type II theory restricts the above spectrum, which causes a little subtlety [24, 25, 26]. For $2n \leq \gamma < 2n+1$ ($n \in \mathbf{Z}$) it is the standard type II GSO-projection and the allowed spectra are those which give $\hat{N}_{L,R}$ the integer values for NSNS-sector. However, for $2n+1 \leq \gamma < 2n+2$ it is the reversed one, where $\hat{N}_{L,R}$ takes half-integer values for NSNS-sector. This fact can be seen from the one-loop partition function $Z(R, q, \beta)$ [24, 25, 26] of the Melvin background by comparing the result in the NS-R formalism with that in the Green-Schwarz formalism, where GSO-projection is not needed. In fact the spectrum (3.23) and (3.24) are the same as those obtained in the Green-Schwarz formalism [24, 25, 26] as we will see in the next subsection.

Before we proceed let us see interesting symmetries of the string model. First one is the T-duality symmetry[11] . The spectrum is invariant under the exchange of q,n and R for β,w and $1/R$ and the partition function satisfies

$$Z(R, q, \beta) = Z\left(\frac{\alpha'}{R}, \beta, q\right). \tag{3.25}$$

This represents the T-duality in $\mathbf{S}^1$ direction which interchanges the metric $G_{\varphi y}$ and B-field $B_{\varphi y}$. Another one is the periodicity with respect to q and β

$$Z(R, q, \beta) = Z\left(R,\ q + 2n_1/R,\ \beta + 2n_2 R/\alpha'\right). \tag{3.26}$$

Note that the observed periodicity is consistent with that for F7-brane discussed in section 2.

[11] Note that this T-duality is involved with $\mathbf{S}^1$ and has nothing to do with those used in section 3.1.

3.3 Green-Schwarz Formulation and Partition Function

In order to compute the one-loop partition function with the correct GSO projection, it is more convenient to use Green-Schwarz formulation [24, 25].

On the world-sheet in the light-cone Green-Schwarz formulation, there are eight (real) bosonic fields ρ, φ, Y, X_i $(i = 2, 3, \cdots, 6)$ and eight left-moving and right-moving fermionic fields S^a_L, S^a_R $(a = 1, 2, \cdots, 8)$. The fermionic fields are divided into two groups $S^r_{L,R}$ and $\bar{S}^r_{L,R}$ $(r = 1, 2, 3, 4)$ according to the angular momenta $\hat{J}_{L,R} = \frac{1}{2}$ and $-\frac{1}{2}$ in the $(X'', \bar{X}'')$ plane, where we defined $U(1)$-charge such that $X = \rho e^{i\varphi}$ and $\bar{X} = \rho e^{-i\varphi}$ also have the charge $\hat{J}_L = \hat{J}_R = 1$ and -1 as in [23, 25].

Now let us see the calculation of the partition function. Introducing the auxiliary vector fields $V, \bar{V}$, we rewrite the world-sheet action as follows[12]

$$
\begin{aligned}
S &= \frac{1}{\pi\alpha'} \int (d\sigma)^2 \Big[\bar{\partial}\rho\partial\rho + (1+\beta^2\rho^2)V\bar{V} + V(\bar{\partial}Y + \beta\rho^2\bar{\partial}(\varphi + qY) + \frac{i\beta}{2}\bar{S}^r_R S^r_R) \\
&\quad -\bar{V}(\partial Y - \beta\rho^2\partial(\varphi + qY) + \frac{i\beta}{2}\bar{S}^r_L S^r_L) + \rho^2\partial(\varphi + qY)\bar{\partial}(\varphi + qY) \\
&\quad + \bar{S}^r_R(\partial + \frac{iq}{2}\partial Y)S^r_R + \bar{S}^r_L(\bar{\partial} - \frac{iq}{2}\bar{\partial}Y)S^r_L\Big] \\
&= \frac{1}{\pi\alpha'} \int (d\sigma)^2 \Big[(\partial + i\beta V + iq\partial Y)X(\bar{\partial} - i\beta\bar{V} - iq\bar{\partial}Y)\bar{X} \\
&\quad + \bar{S}^r_R(\partial + \frac{i\beta}{2}V + \frac{iq}{2}\partial Y)S^r_R + \bar{S}^r_L(\bar{\partial} - \frac{i\beta}{2}\bar{V} - \frac{iq}{2}\bar{\partial}Y)S^r_L \\
&\quad + V\bar{V} - \bar{V}\partial Y + V\bar{\partial}Y\Big]. \qquad (3.27)
\end{aligned}
$$

Here we abbreviate the bosonic parts which come from the trivial directions $\mathbf{R}^{1,6}$. Note that if we neglect the fermionic fields, then we can obtain the (bosonic) sigma model for the Melvin background (3.1) after we integrate out the auxiliary fields V, $\bar{V}$.

Next we would like to integrate out the (non-zero mode part of) field Y. Then the equation of motion (3.27) for Y shows $\bar{\partial}V - \partial\bar{V} = 0$ if we use also other equations of motion. Therefore we can write V as

$$
V = C + \partial\tilde{Y}, \quad \bar{V} = \bar{C} + \bar{\partial}\tilde{Y}, \qquad (3.28)
$$

[12] For the related analysis of the curved backgrounds in Green-Schwarz formalism (light-cone gauge) see [74]. In principle it is possible for the world-sheet action in the formalism to include other terms higher than quartic in the fermions. However, our specific models are expected to have no higher terms since the free field representation is possible as shown by the T-duality. We thank A.A.Tseytlin for showing us this observation.

where C is the constant part; $\tilde{Y}$ is a bosonic field which has no zero-modes (thus we have $\int \partial\tilde{Y} = 0$). Finally we obtain (we show only bosonic parts)

$$\begin{aligned} S &= \frac{1}{\pi\alpha'}\int (d\sigma)^2 \Big[C\bar{C} - \bar{C}\partial Y + C\bar{\partial}Y + \partial\tilde{Y}\bar{\partial}\tilde{Y} \\ &+(\partial + i\beta C + i\beta\partial\tilde{Y} + iq\partial Y)X\cdot(\bar{\partial} - i\beta\bar{C} - i\beta\bar{\partial}\tilde{Y} - iq\bar{\partial}Y)\bar{X} \\ &+\bar{S}^r_R(\partial + i\frac{\beta}{2}C + i\frac{\beta}{2}\partial\tilde{Y} + i\frac{q}{2}\partial Y)S^r_R + \bar{S}^r_L(\bar{\partial} - i\frac{\beta}{2}\bar{C} - i\frac{\beta}{2}\bar{\partial}\tilde{Y} - i\frac{q}{2}\bar{\partial}Y)S^r_L\Big]. \end{aligned} \tag{3.29}$$

Note that the terms which include ∂Y and $\bar{\partial}Y$ in the above action contain only the zero-mode part of Y. This is because we have only integrated out the non-zeromode part of Y to get the result (3.28).

This result (3.29) shows that $X,\ S^r_R$ can also be written in terms of free field $X'',\ S''^r_R$ such that

$$\begin{aligned} X(z,\bar{z}) &= e^{-i\beta(Cz+\bar{C}\bar{z})-i\beta\tilde{Y}-iqY}X''(z,\bar{z}), \\ S^r_R(z,\bar{z}) &= e^{-i\frac{\beta}{2}(Cz+\bar{C}\bar{z})-i\frac{\beta}{2}\tilde{Y}-i\frac{q}{2}Y}S''^r_R(z,\bar{z}), \\ S^r_L(z,\bar{z}) &= e^{i\frac{\beta}{2}(Cz+\bar{C}\bar{z})+i\frac{\beta}{2}\tilde{Y}+i\frac{q}{2}Y}S''^r_L(z,\bar{z}). \end{aligned} \tag{3.30}$$

Note also that the terms $\frac{1}{\pi\alpha'}\int(d\sigma)^2(-\bar{C}\partial Y + c.c.)$ involve only zero-mode parts of Y. Since the field Y has the periodicity $Y \sim Y + 2\pi R$, its zero-mode part is quantized in terms of winding numbers $w, w' \in \mathbf{Z}$ as follows

$$Y(\sigma_1,\sigma_2) = \sigma_1 wR + \sigma_2(w' - w\tau_1)R/\tau_2, \tag{3.31}$$

where the coordinate of the torus (σ_1,σ_2) follows the periodicity

$$(\sigma_1,\sigma_2) \sim (\sigma_1 + 2\pi, \sigma_2) \sim (\sigma_1 + 2\pi\tau_1, \sigma_2 + 2\pi\tau_2). \tag{3.32}$$

Then it is easy to perform the path-integral[13] and the result is as follows [24]

$$\begin{aligned} Z(R,q,\beta) &= (2\pi)^{-7}V_7R(\alpha')^{-5}\int\frac{(d\tau)^2}{(\tau_2)^6}\int(dC)^2\sum_{w,w'\in\mathbf{Z}}\frac{|\theta_1(\frac{\chi}{2}|\tau)|^8}{|\eta(\tau)|^{18}|\theta_1(\chi|\tau)|^2} \\ &\times\exp\left[-\frac{\pi}{\alpha'\tau_2}(4C\bar{C} - 2\bar{C}R(w' - w\tau) + 2CR(w' - w\bar{\tau}))\right], \end{aligned} \tag{3.33}$$

where we have defined

$$\chi = 2\beta C + qR(w' - \tau w), \quad \bar{\chi} = 2\beta\bar{C} + qR(w' - \bar{\tau}w). \tag{3.34}$$

[13] Here we have redefined $C, \bar{C}$ such that $C \to i\bar{C}/\tau_2,\ \ \bar{C} \to -iC/\tau_2$.

The last exponential factor comes from zero modes of Y. The theta-function terms originate from the following path-integral of non-zero modes

$$
\begin{aligned}
\frac{\det'(\partial - \bar{\chi}/(2\tau_2))\det'(\bar{\partial} - \chi/(2\tau_2))}{\det'(\partial)\det'(\bar{\partial})} &= \prod_{(n,n')\neq(0,0)} \frac{(n'-\tau n+\chi)(n'-\bar{\tau}n+\bar{\chi})}{(n'-\tau n)(n'-\bar{\tau}n)} \\
&= e^{\frac{\pi(\chi-\bar{\chi})^2}{2\tau_2}} \left| \frac{\theta_1(\chi|\tau)}{\chi\theta_1'(0|\tau)} \right|^2 . \qquad (3.35)
\end{aligned}
$$

In this way we can obtain the one-loop partition function in the Green-Schwarz formulation. It is easy to check its modular invariance using theta-function formulas (C.2).

Now one may ask how to interpret the above result in the NS-R formulation. If one uses the Jacobi identity (C.4)

$$
\theta_3(0|\tau)^3\theta_3(\chi|\tau) - \theta_2(0|\tau)^3\theta_2(\chi|\tau) - \theta_4(0|\tau)^3\theta_4(\chi|\tau) = 2\theta_1(\frac{\chi}{2}|\tau)^4, \qquad (3.36)
$$

then this explicitly represents the path-integral in the NS-R formulation with type II GSO-projection. The above partition function does not vanish generically and this implies the spacetime supersymmetry breaking as we will discuss later. Note also that the GSO projection in the NS-R formulation is completely determined by the calculation (3.33) in the Green-Schwarz formalism.

Next we would like to relate the previous result to the mass spectrum in the operator formulation. Let us assume that the theta-functions are all expanded such that each term is an eigen state of the angular momentum operators $\hat{J}_R, \hat{J}_L$. Here we have defined the operators $\hat{J}_R, \hat{J}_L$ of the term such that it includes the factor (see (C.1))

$$
e^{2\pi i\chi\hat{J}_R + 2\pi i\bar{\chi}\hat{J}_L}. \qquad (3.37)
$$

Then by using the Poisson resummation formula

$$
\sum_{n\in\mathbf{Z}} \exp(-\pi a n^2 + 2\pi i b n) = \frac{1}{\sqrt{a}} \sum_{m\in\mathbf{Z}} \exp(-\frac{\pi(m-b)^2}{a}) \qquad (3.38)
$$

we can show

$$
\begin{aligned}
&\int (dC)^2 \sum_{w,w'\in\mathbf{Z}} \exp\Big[-\frac{\pi}{\alpha'\tau_2}(4C\bar{C} - 2\bar{C}R(w'-w\tau) + 2CR(w'-w\bar{\tau})) \\
&+2\pi i\chi\hat{J}_R + 2\pi i\bar{\chi}\hat{J}_L + 2\pi i\tau\hat{N}_R - 2\pi i\bar{\tau}\hat{N}_L\Big] \\
&= \frac{\sqrt{(\alpha'\tau_2)^3}}{4R} \sum_{m,w\in\mathbf{Z}} \exp(-\pi\alpha'\tau_2 M^2 + 2\pi\tau_1 i(\hat{N}_R - \hat{N}_L - nw)). \qquad (3.39)
\end{aligned}
$$

Then the mass spectrum M^2 reproduces the previous result (3.23) in the NS-R formulation by shifting the oscillator moding such that $\gamma \rightarrow \hat{\gamma}$.

3.4 Closed String Tachyons in Melvin Background

As we can see the explicit expression of partition function (3.33) and the formula (3.36), the background breaks supersymmetry completely except the trivial case $(qR, \beta\alpha'/R) \in (2\mathbf{Z}, 2\mathbf{Z})$. Note that the particular cases $(qR, \beta\alpha'/R) \in (2\mathbf{Z}+1, 2\mathbf{Z})$ or $(2\mathbf{Z}, 2\mathbf{Z}+1)$ are equivalent to the previous model (2.9), which connects type II and type 0 theory.

The reason of supersymmetry breaking can also be explained in supergravity. To make matters simple let us assume $\beta = 0$. If one goes around the circle $\mathbf{S}^1$, then all of spin 1/2 fermions receive the phase factor $e^{\pm i\pi qR}$ (see (3.30)) as in the previous section. Since this is not equal to one in general, there are no Killing spinor. Thus only for $qR \in 2\mathbf{Z}$ the supersymmetry is preserved.

In this way the string theory in Melvin background is non-supersymmetric and therefore one may expect the existence of closed string tachyons. For simplicity let us assume $\beta = 0$ and examine the tachyons briefly. Consider the mass spectrum (3.23) with zero momentum $n = 0$ and non-zero winding number $0 < qRw < 1$. If we note the relations $\hat{J}_R \leq \hat{N}_R + 1/2$ and $-\hat{N}_L - 1/2 \leq \hat{J}_L$, we can see that in order to realize the smaller $(\text{mass})^2$ we must take $\hat{J}_R = \hat{N}_R + 1/2$ and $\hat{J}_L = -\hat{N}_L - 1/2$. Then by using the level matching condition (3.24) we have $\hat{N}_R = \hat{N}_L (= N)$. Thus the mass spectrum is given by

$$\frac{\alpha' M^2}{2} = 2N + \frac{R^2}{2\alpha'} w^2 - qR(2N+1)w, \tag{3.40}$$

and at least one tachyon appears if

$$q > \frac{4N + w^2 R^2/\alpha'}{(4N+2)wR}. \tag{3.41}$$

Since we have assumed $0 < qRw < 1$ (the other cases can be treated in the same way), we must require $wR < \sqrt{2\alpha'}$. Then the lowest bound of q is realized for $N = 0$ and we have $\alpha' q > R/2$. Thus we can conclude that if we choose the radius so that $R < \sqrt{2\alpha'}$, then the tachyons exist for any sufficient large values of q [24]. For generic values of $(qR, \beta\alpha'/R)$ it can be shown that there exist tachyons for any small values of q and β [24].

In particular if we set $qR = \frac{k}{m}$ $(k \in 2\mathbf{Z}+1,\ m \in \mathbf{Z})$, we have the following mass

$$\frac{\alpha' M^2}{2} = -1 + \frac{m^2 R^2}{2\alpha'} \tag{3.42}$$

corresponding to the lightest state with the winding number $w = m$. When the radius is enough small, this mode becomes tachyonic. This is a example of 'bulk tachyon' field which spreads over the space infinitely because of the existence of zeromodes (see 3.21). However note that for many other values of w we have 'localized tachyons', which is localized at the origin.

Thus the string theory in Melvin background includes various closed string tachyons and the analysis of tachyon condensation will be important. We will return to this point in the last of next section.

4 Melvin Background and Orbifold Theory

Since we have given an exactly solvable string model, we would like to know the relation to more familiar solvable theories. As we will show below, the string theory in NSNS background can be regarded as a generalization of that in orbifold theories. In particular if we take appropriate limits, the model becomes equivalent to two dimensional orbifolds $\mathbf{C}/Z_N$ in type II or type 0 theory [26].

4.1 Orbifold $\mathbf{C}/Z_N$ from Melvin Background

It is interesting to examine the small radius (or large radius in the T-dualized picture) limit of the Melvin backgrounds for various values of parameters $q,\ \beta$ since it is expected to produce some non-trivial decompactified ten-dimensional theories after T-duality. Inspired by this motivation let us consider the limit $R \to 0$ and $\frac{\beta\alpha'}{R} \to 0$ with the rational value $qR = \frac{k}{N}$, where k and N are coprime integers. Note that this limit is T-dual to $R \to \infty$ and $qR \to 0$ with $\frac{\beta\alpha'}{R} = \frac{k}{N}$ by using (3.25). In the former picture the redefined world-sheet fields (3.30) are twisted as follows

$$
\begin{aligned}
&X''(\sigma_1 + 2\pi, \sigma_2) = e^{2\pi i \frac{k}{N} w} X''(\sigma_1, \sigma_2), \\
&S_R''^{r}(\sigma_1 + 2\pi, \sigma_2) = e^{\pi i \frac{k}{N} w} S_R''^{r}(\sigma_1, \sigma_2), \\
&Y(\sigma_1 + 2\pi, \sigma_2) = Y(\sigma_1, \sigma_2) + 2\pi R w.
\end{aligned} \tag{4.1}
$$

Thus we can see that this string model is equivalent to a $\mathbf{Z}_N$ freely acting orbifold[14] if we divide the winding number w into the new winding number $\beta \in \mathbf{Z}$ and the internal quantum number $m = 0, 1, \cdots, N-1$ such that $w = N\beta + m$. The value of m represents the m-th twisted sector of the orbifold theory. Furthermore we can speculate that the limit $R \to 0$ may be related to the orbifold $\mathbf{C}/\mathbf{Z}_N$, where the action $g \in \mathbf{Z}_N$ is defined by the action $g : X'' \to e^{2\pi i \frac{k}{N}} X''$ (equally we can say $g = e^{2\pi i \frac{k}{N} \hat{J}}$). In order to show the exact relation we should investigate the one-loop partition function. The calculation in the Green-Schwarz formalism is useful since in this formalism the flip of GSO-projection is automatically included due to spectral flow[24], which is crucial to determine that the theory is type 0 or type II.

[14] Here 'freely acting' means that the orbifold action does not have any fixed points. For earlier discussions of the related Scherk-Schwarz compactification in string theory see for example [58, 60, 75, 76].

The partition function (3.33) in this limit is given as follows[15] by using the identity (3.36) and the quasi-periodicity of theta-functions (C.3)

$$\lim_{R\to 0} Z(R,q,\beta) = (2\pi)^{-7} V_7 R(\alpha')^{-4} \int \frac{(d\tau)^2}{4(\tau_2)^5} \sum_{l,m=0}^{N-1} \sum_{\alpha,\beta\in\mathbf{Z}} \left(\lim_{R\to 0} e^{-\frac{\pi N^2 R^2}{\alpha'\tau_2}|\alpha-\beta\tau|^2} \right)$$
$$\times \frac{|\theta_3(\nu_{lm}|\tau)\theta_3(\tau)^3 - (-1)^{k\alpha}\theta_2(\nu_{lm}|\tau)\theta_2(\tau)^3 - (-1)^{k\beta}\theta_4(\nu_{lm}|\tau)\theta_4(\tau)^3|^2}{4|\eta(\tau)|^{18}|\theta_1(\nu_{lm}|\tau)|^2}, \tag{4.2}$$

where we have defined $\nu_{lm} = \frac{lk}{N} - \frac{mk}{N}\tau$ and integers $l,\ m,\ \alpha,\ \beta$ are given as $w' = N\alpha + l, w = N\beta + m$ $(l, m = 0, 1, \cdots, N-1)$.

If k is an even integer, then the sign factors in front of the theta-functions are all plus and we obtain

$$\begin{aligned} Z(0,q,\beta) &= V_1 V_7 \int \frac{(d\tau)^2}{4(\tau_2)} (4\pi^2\alpha'\tau_2)^{-4} \\ &\quad \times \sum_{l,m=0}^{N-1} \frac{|\theta_3(\nu_{lm}|\tau)\theta_3(\tau)^3 - \theta_2(\nu_{lm}|\tau)\theta_2(\tau)^3 - \theta_4(\nu_{lm}|\tau)\theta_4(\tau)^3|^2}{4N|\eta(\tau)|^{18}|\theta_1(\nu_{lm}|\tau)|^2}, \end{aligned} \tag{4.3}$$

where the divergent factor V_1 is also given by $V_1 = \lim_{R\to 0} \frac{2\pi\alpha'}{NR}$ and this corresponds to the volume of the noncompact direction. This value of radius $\frac{\alpha'}{NR}$ is consistent with that expected from the boundary condition (4.1) by T-duality. The sums over l and m in this expression (4.3) should be regarded as the $\mathbf{Z}_N$ projection $\frac{1}{N}\sum_{l=0}^{N-1} g^l$ $(g = \exp(2\pi i \frac{k}{N}\hat{J}))$ and the sum over twisted sectors, respectively. Therefore the model in this limit is identified with the orbifold $\mathbf{C}/\mathbf{Z}_N$ [77] in type II string theory. Note that for a fixed value of N the orbifold theories with different even integers k define the same theory. These orbifolds were discussed in the context of closed tachyon condensation [33].

Next let us turn to the case where k is an odd integer. The result is

$$\begin{aligned} Z(0,q,\beta) &= V_1' V_7 \int \frac{(d\tau)^2}{4(\tau_2)} (4\pi^2\alpha'\tau_2)^{-4} \sum_{l,m=0}^{N-1} \\ &\quad \times \frac{|\theta_3(\nu_{lm}|\tau)\theta_3(\tau)^3|^2 + |\theta_2(\nu_{lm}|\tau)\theta_2(\tau)^3|^2 + |\theta_4(\nu_{lm}|\tau)\theta_4(\tau)^3|^2}{2N|\eta(\tau)|^{18}|\theta_1(\nu_{lm}|\tau)|^2}, \end{aligned} \tag{4.4}$$

where we have defined[16] $V_1' = V_1/2$. Thus this model is equivalent to the orbifold $\mathbf{C}/\mathbf{Z}_N$

[15] If lk/N and mk/N are both integers, then $\theta_1(\nu_{lm}|\tau)$ does vanish and the partition function will be divergent. This is due to the appearance of the zero modes of $(X'', \bar{X}'') \in \mathbf{R^2}$ and one should extract this divergence as the volume factor V_2.

[16] The extra factor $1/2$ in comparison with the case of even k is understood if one notes that the 'GSO-projection' in type 0 is the diagonal $\mathbf{Z}_2$ projection $(1 + (-1)^{F_L + F_R})/2$, while in type II theory it is given by the $\mathbf{Z}_2 \times \mathbf{Z}_2$ projection $(1 + (-1)^{F_L})(1 + (-1)^{F_R})/4$.

in type 0 string theory with radius $\frac{\alpha'}{2NR} \to \infty$. The above results are summarized in Fig.2.

This identification can also be seen from the mass spectrum (3.23). In the limit of $R \to 0$ we have the constraint $n = \frac{k}{N}\hat{J}$. This gives the correct $\mathbf{Z}_N$ orbifold projection. The shift of the energy $-\hat{\gamma}(\hat{J}_R - \hat{J}_L) = -(\widehat{\frac{km}{N}})(\hat{J}_R - \hat{J}_L)$ corresponds to the shift of modings in twisted sectors. If k is even, then $\hat{J}$ can be half integer and the NS-R and R-NS sector are allowed (remember that $\hat{J}$ is the total angular momentum.). On the other hand if k is odd, then those sectors are not allowed. This fact gives the difference between type II and type 0 string theory. One can also read off the mass of lightest state for each twisted sectors. The result is given by for even k (type II)

$$\frac{\alpha' M^2}{2} = \begin{cases} -\hat{\mu} & \text{if } [\mu] \in \text{even} \\ \hat{\mu} - 1 & \text{if } [\mu] \in \text{odd} \end{cases}, \tag{4.5}$$

and for odd k (type 0)

$$\frac{\alpha' M^2}{2} = \min\{\hat{\mu} - 1, -\hat{\mu}\}, \tag{4.6}$$

where we have defined $\mu = \frac{km}{N}$. The reason that the mass (4.5) depends on whether $[\mu]$ is even or odd is due to the 'flip' of the sign of GSO projections by the spectral flow [24]. From the above results we can conclude that in the type II orbifolds the tachyon appears in all twisted sectors while in type 0 orbifolds it does in untwisted sectors as well as in twisted sectors.

Now we would like to consider equivalence between the Melvin background with $qR = \frac{k}{N}, \beta = 0$ for the finite radius and the freely acting orbifold (4.1) in detail. By applying the Poisson resummation on α to the partition function (4.2) without taking the limit we obtain

$$\begin{aligned} & Z(R, q, \beta = 0) \\ & = V_7 \int \frac{(d\tau)^2}{4\tau_2} (4\pi^2\alpha'\tau_2)^{-\frac{7}{2}} \sum_{l,m=1}^{N} \sum_{\tilde{\alpha},\beta \in \mathbf{Z}} \exp\Bigg[-\pi\tau_2 \left\{ \frac{N^2R^2}{2\alpha'}(\beta + \frac{m}{N})^2 + \frac{\alpha'}{2N^2R^2}(\tilde{\alpha} - \frac{\epsilon}{2})^2 \right] \Bigg\} \\ & \quad - 2\pi i\tau_1(\tilde{\alpha} - \frac{\epsilon}{2})(\beta + \frac{m}{N}) + 2\pi i \left\{ \frac{l}{N}(\tilde{\alpha} - \frac{\epsilon}{2}) + \frac{\epsilon'}{2}\beta \right\} \Bigg] \times \frac{|\theta_1\left(\frac{\nu_{lm}}{2}|\tau\right)|^8}{4N|\eta(\tau)|^{18}|\theta_1(\nu_{lm}|\tau)|^2}, \end{aligned} \tag{4.7}$$

where we have defined $\epsilon = 0, 1$ and $\epsilon' = 0, 1$ as follows. In the sector which includes $(-1)^{k\alpha}$ (or $(-1)^{k\beta}$) factor when we expand the theta function term as in (4.2) the value of ϵ (or ϵ') is given by 1. For example, if we assume that k is even, then we always have

$\epsilon = \epsilon' = 0$. From the expression (4.7) it is easy to see that the twisted sectors correspond to the non-zero values of m and the summation over l is regarded as a $\mathbf{Z}_N$ projection as before. Note that in this case we have also nontrivial twist in the $\mathbf{S}^1$ direction at the same time. Thus we can conclude that this special background is equivalent to the $\mathbf{Z}_N$ orbifold IIA(B)$/\sigma_{\frac{1}{N}} \cdot g$ (here we define $g = \exp(2\pi i \frac{k}{N} \hat{J})$) with radius NR (for even k) or $\mathbf{Z}_{2N}$ orbifold IIA(B)$/\sigma_{\frac{1}{2N}} \cdot g$ with radius $2NR$ (for odd k). Here the operators $\sigma_{\frac{1}{N}}$ and $\sigma_{\frac{1}{2N}}$ mean $\frac{1}{N}$ and $\frac{1}{2N}$ shift along $\mathbf{S}^1$. The latter case is T-dual to the $\mathbf{Z}_2 \times \mathbf{Z}_N$ orbifold 0B(A)$/\{(-1)^{F_R} \cdot \sigma_{\frac{1}{2}},\ \tilde{\sigma}_{\frac{1}{N}} \cdot g\}$ with radius $\frac{1}{NR}$, where the operator $\tilde{\sigma}_{\frac{1}{N}}$ is the T-dual to $\sigma_{\frac{1}{N}}$. For the special case $N = k = 1$ these results are reduced to the results in [42, 30] if we note the relation $g^N = (-1)^{F_S}$ for odd k. We summarize these results in Table 1.

k	type II orbifold (radius)	T-dualized orbifold (radius)
even	IIA(B)$/\sigma_{1/N} \cdot g$ $\quad (NR = \frac{N\alpha'}{\tilde{R}})$	IIB(A)$/\tilde{\sigma}_{1/N} \cdot g$ $\quad (\frac{\alpha'}{NR} = \frac{\tilde{R}}{N})$
odd	IIA(B)$/\sigma_{1/2N} \cdot g$ $\quad (2NR = \frac{2N\alpha'}{\tilde{R}})$	0B(A)$/\{(-1)^{F_R} \cdot \sigma_{1/2},\ \tilde{\sigma}_{1/N} \cdot g\}$ $\quad (\frac{\alpha'}{NR} = \frac{\tilde{R}}{N})$

Table 1: The equivalence of the IIA(B) Melvin background for $qR = \frac{k}{N}, \beta = 0$ with freely acting orbifolds. The operators $g = \exp(2\pi i \frac{k}{N} \hat{J})$, $\sigma_{\frac{1}{N}}$ and $\tilde{\sigma}_{\frac{1}{N}}$ represent the projection operator, $\frac{1}{N}$-shift operator in the circle and its dual operator, respectively.

Finally let us discuss the limit $R \to 0$ with irrational values of qR. The mass spectrum (3.23) shows the constraint $n - qR\hat{J} = 0$ again and this is satisfied if $n = \hat{J} = 0$ (thus purely bosonic). Remarkably, we can not divide this system into a kind of a two dimensional orbifold and one dimensional non-compact space. This may also be regarded as a large N limit of $\mathbf{Z}_N$ orbifold if we are reminded that any irrational number can be infinitely approximated by rational numbers. Note that taking this limit needs one extra dimension and the non-trivial background is given by the non-compact three dimensional space. In any case we have to say that there are such unfamiliar non-compact string backgrounds, which interpolate the previous $\mathbf{Z}_N$ orbifolds.

A little analysis of the mass spectrum in this 'irrational' model shows that tachyon fields appear in the sectors $w \neq 0$ and there exist tachyon fields whose mass is given by $-1 < \frac{\alpha' M^2}{2} < -1 + \epsilon$ for any infinitesimal ϵ.

In this way the limits of the Melvin backgrounds depend very sensitively on whether the value of the magnetic flux is rational or irrational. The existence of two remarkably different kinds of limits can also be seen in the D-brane spectrums in such backgrounds as we will discuss later [71].

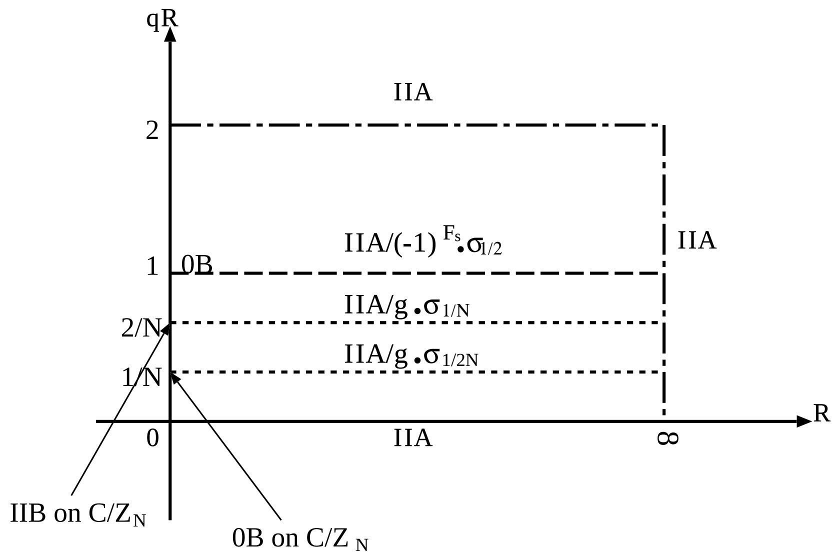

Figure 2: Moduli space of the string models in type IIA Melvin backgrounds with $\beta = 0$.

4.2 Comments on Closed String Tachyon Condensation

Before we finish the investigation of closed strings in Melvin background, we would like to discuss very roughly the instability due to closed string tachyons employing some recently known results. In general, closed tachyon condensation is difficult to analyze unlike open string tachyon condensation [78]. One reason of the difficulty is due to the c-theorem [79]. If we conventionally identify the world-sheet RG flow due to relevant perturbations with tachyon condensation (see e.g. [80]), then the perturbation should reduce the value of central charge c following the c-theorem. Then the ghost anomaly cancellation in the world-sheet theory seems to be violated. A possible resolution of this problem is to consider non-critical string identifying one of the spacetime coordinate as a Liouville field (some relevant discussions see e.g.[81, 29]). However, the complete argument about the decay of type 0 theory or bosonic string due to its bulk tachyon field has not been known [17] .

[17] Here the author would like to point out the difference between tachyon in type 0 theory and in bosonic string. The c-theorem tells us that the flow of the central charge is proportional to the two point function of vertex operators which correspond to the tachyonic perturbation. In type 0 theory we must perform picture change (from -1 to 0 picture) and then we naively find that there is no contribution from the constant tachyon field in the leading order. However in bosonic string we have a non-trivial contribution and we find that the condensation of constant tachyon field reduces the value of the central charge. This viewpoint is closely related to the absence of tachyon potential in the sigma model approach

Another reason will be the intricacy of closed string field theory. For example the closed string field theory of [83] includes infinitely many terms of string fields and seems not so tractable as the open string field theory [1]. Furthermore, we have a doubt that an off-shell closed string theory exists against the holography [84], which relates on-shell closed string theory and off-shell open string theory.

In spite of these difficulties, recently in the papers [33, 36, 37], some results about condensation of 'localized tachyons' in orbifold theories such as $\mathbf{C}/\mathbf{Z}_N$ were obtained. The term 'localized tachyons' means tachyon fields which can take non-zero values only in a very small region $\sim \alpha'$ and they typically appear in twisted sectors. The important fact is that we cannot apply the c-theorem to localized tachyons and the value of c does not change [37, 33] in many examples. In Melvin background we also have localized tachyons[18] . Let us follow the conjecture in [37] that the function g_{cl}, which is an analog of g-function [85] in open string theory, should decrease along the RG flow from UV to IR. The function g_{cl} is defined as the coefficient of density of localized bosonic state $\rho(E)_{localized}$ for high energy E and is equivalently[19] defined by the partition function for localized sectors as follows

$$Z_{localized}(\tau_2 \to 0) \sim g_{cl} \exp(\pi c/6\tau_2). \tag{4.8}$$

In particular in the case of the orbifold $\mathbf{C}/\mathbf{Z}_N$, g_{cl} is computed[20] in [37] and the result is

$$g_{cl} = \frac{1}{N}\sum_{s=1}^{N-1}\frac{1}{(2\sin\pi s/N)^2} = \frac{1}{12}(N-1/N). \tag{4.9}$$

Thus we can speculate that after the tachyon condensation the value of N will decrease [37]. This result is supported by the analysis using D-brane probe [33] and by the analysis of linear sigma model [36]. Another quantity which should decrease after tachyon condensation will be tachyon $(\text{mass})^2$. This is because the leading quantity which determines the stability will be the one-loop amplitude (or cosmological constant). Even though this value is IR divergent in the general parameter region, this divergence can be estimated

(or 'BSFT')[82]

[18] It should also be noted that bulk tachyon fields exist in specific Melvin background as we have observed in (3.42).

[19] Note the relation $Z_{localized} = \int_0^\infty dE\ \rho(E)_{localized}\ e^{-\beta E} \qquad (\beta = \tau_2)$.

[20] Recently the more refined version of g_{cl} has been proposed in [40]. The result will not change substantially if we use this second definition.

by the tachyon $(\text{mass})^2$ whose absolute value is largest. This consideration also leads to the same result as before if we remember the explicit tachyon mass (4.5) as discussed[21] in [26]. We would like to note that in this process which reduces the value of N the topology of the space does change. For example, let us note that the Witten index of the orbifold $\mathbf{C}/\mathbf{Z}_N$, which is given by $\mathrm{tr}_{RR}(-1)^F = N$, actually depends on N.

If we apply this argument to the string theory in the Melvin background $qR = \frac{k}{N}$ and $\beta = 0$, then we have the same value g_{cl} as the above (4.9) as can be seen from the expression (4.7). Thus we can again speculate that after the tachyon condensation the value of N will decrease in Melvin background. This result is consistent with the result obtained in [39] by using the linear sigma model description [36].

Finally we would like to mention the decay processes of the orbifold $\mathbf{C}/\mathbf{Z}_N$ by on-shell deformations. Then we have no tree level potential [22] . If we are reminded of the 'moduli space' (see Fig.2 or Fig.3) of Melvin background, we find many decay routes. The most straightforward way is to decay into the ordinary type II string vacuum by the spontaneous (de)compactification of Y direction. Another possibility is the shift of the magnetic parameter. However the infinitesimal shift will make the value of qR irrational. Since such a background is more tachyonic (see (4.5)) and has an infinite value of g_{cl}, this decay route is not preferable. On the other hand, the decay mode from $\mathbf{C}/\mathbf{Z}_N$ into $\mathbf{C}/\mathbf{Z}_{N-2}$ discussed in [33, 36, 37] is more favorable because the absolute value of tachyon $(\text{mass})^2$ decreases.

[21] Quite recently, a similar analysis was made in more general orbifolds [41].

[22] This viewpoint may be supported by the observed absence of tree level tachyon potential [82, 32].

5 Higher Dimensional Melvin Background and ALE spaces

The closed string backgrounds we have discussed above do not preserve any supersymmetry in general. However, there are many cases where supersymmetric background is favored. For example, if we would like to discuss D-brane charges, it will be more desirable to consider those in supersymmetric backgrounds. As we will see below, we can realize exactly solvable string theories in supersymmetric backgrounds [26] (see also the independent and overlapping paper [27]) if we consider higher dimensional generalizations of NSNS Melvin background (3.1).

In particular these models include the $9-11$ flip of supersymmetric fluxbranes (F5,F3 and F1-brane) discussed in section 2.

The construction of the higher dimensional generalizations is more transparent in Green-Schwarz formulation than in NS-R one. Thus we first study the Green-Schwarz formulation of the sigma model. After that we will return to the NS-R formulation.

5.1 Green-Schwarz Formulation

Let us consider a background of the form $\mathbf{M}_5 \times \mathbf{R}^{1,4}$, where $\mathbf{M}_5$ is a fibration of $\mathbf{S}^1 \ni Y$ over $\mathbf{R}^2 \times \mathbf{R}^2 \ni (X^1, \bar{X}^1) \times (X^2, \bar{X}^2)$. At the sigma model level this is possible if one assumes the higher dimensional generalizations of (3.27):

$$\begin{aligned} S \;=\; & \frac{1}{\pi\alpha'}\int (d\sigma)^2\Big[(\partial + i\beta_1 V + iq_1\partial Y)X^1(\bar{\partial} - i\beta_1\bar{V} - iq_1\bar{\partial}Y)\bar{X}^1 \\ & +(\partial + i\beta_2 V + iq_2\partial Y)X^2(\bar{\partial} - i\beta_2\bar{V} - iq_2\bar{\partial}Y)\bar{X}^2 + V\bar{V} - \bar{V}\partial Y + V\bar{\partial}Y\Big], \end{aligned} \tag{5.1}$$

where we have omitted the fermion terms. The explicit metric of this model is somewhat complicated and is given in the appendix D. The special case $\beta_1 = \beta_2 = 0$ corresponds to the following simplified metric

$$ds^2 = d\rho^2 + dr^2 + \rho^2(d\varphi + q_1 dy)^2 + r^2(d\theta + q_2 dy)^2, \tag{5.2}$$

where we have defined $X^1 = \rho e^{i\varphi}$ and $X^2 = re^{i\theta}$. This background can be regarded as the $9-11$ flip of the F5-brane [31, 52]. The free field representation is again possible almost in the same way as before.

In the Green-Schwarz formulation, the four of eight (light-cone gauge) spinor fields do not suffer from the phase factor when σ_1 is shifted by 2π if $q_1 = q_2, \beta_1 = \beta_2$ or

$q_1 = -q_2, \beta_1 = -\beta_2$. Therefore we can conclude that in these cases half of thirty two supersymmetries are preserved[23] . From the supergravity viewpoint, we can see this as follows. For simplicity, let us set $\beta_1 = \beta_2 = 0$. Then if we go around the circle $\mathbf{S}^1$, the spinor fields obtain the phase $e^{i\pi(\pm q_1 \pm q_2)R}$. Thus if $q_1 = q_2$ or $q_1 = -q_2$, there are sixteen Killing spinors. We would also like to mention that similar arguments can also be generalized into seven or nine dimensional background $\mathbf{M}_7$ and $\mathbf{M}_9$ which are fibrations of $S^1 \ni Y$ over $\mathbf{R}^2 \times \mathbf{R}^2 \times \mathbf{R}^2$ and $\mathbf{R}^2 \times \mathbf{R}^2 \times \mathbf{R}^2 \times \mathbf{R}^2$. The '9-11' flip of these will include the supersymmetric F3-brane, F1-brane (see also [52, 27]).

Another proof of the existence of the supersymmetry is to check the vanishing of the partition function, which is equivalent to the Bose-Fermi degeneracy. In the path-integral formulation of Green-Schwarz string one can compute this as before. The result is given by

$$\begin{aligned} Z(R, q_1, q_2, \beta_1, \beta_2) &= (2\pi)^{-5} V_5 R(\alpha')^{-4} \int \frac{(d\tau)^2}{(\tau_2)^5} \int (dC)^2 \sum_{w,w' \in \mathbf{Z}} \frac{|\theta_1(\frac{\chi_1+\chi_2}{2}|\tau)\theta_1(\frac{\chi_1-\chi_2}{2}|\tau)|^4}{|\eta(\tau)|^{12}|\theta_1(\chi_1|\tau)\theta_1(\chi_2|\tau)|^2} \\ &\quad \times \exp\left[-\frac{\pi}{\alpha'\tau_2}(4C\bar{C} - 2\bar{C}R(w' - w\tau) + 2CR(w' - w\bar{\tau}))\right], \end{aligned} \tag{5.3}$$

where we have defined

$$\chi_1 = 2\beta_1 C + q_1 R(w' - \tau w), \qquad \chi_2 = 2\beta_2 C + q_2 R(w' - \tau w). \tag{5.4}$$

Thus it is easy to see the vanishing of $Z(R, q_1, q_2, \beta_1, \beta_2)$ if $\chi_1 = \chi_2$ or $\chi_1 = -\chi_2$, which is equivalent to $q_1 = q_2, \beta_1 = \beta_2$ or $q_1 = -q_2, \beta_1 = -\beta_2$.

If one wants the partition function in NS-R formalism, then one has only to note the Jacobi identity (C.4) again

$$\begin{aligned} &2\theta_1\left(\chi_1/2 + \chi_2/2|\tau\right)^2 \theta_1\left(\chi_1/2 - \chi_2/2|\tau\right)^2 \\ &= \theta_3(\chi_1|\tau)\theta_3(\chi_2|\tau)\theta_3(0|\tau)^2 - \theta_2(\chi_1|\tau)\theta_2(\chi_2|\tau)\theta_2(0|\tau)^2 - \theta_4(\chi_1|\tau)\theta_4(\chi_2|\tau)\theta_4(0|\tau)^2. \end{aligned} \tag{5.5}$$

Then the above partition function does correctly reproduce the mass spectrum computed in the free field representation in NS-R formulation discussed in the next subsection.

[23] The mechanism of preserving supersymmetry in our model is closely related to the compactified model discussed in [76].

5.2 NS-R Formulation

If we consider the higher dimensional generalization of (3.4) which is consistent with the previous Green-Schwarz formulation (5.1), its world-sheet action is given as follows

$$\begin{aligned} S &= \frac{1}{\pi\alpha'}\int d^2\sigma\Big[\bar{\partial}\rho\partial\rho+\bar{\partial}r\partial r+\rho^2\bar{\partial}\check{\varphi}\partial\check{\varphi}+r^2\bar{\partial}\check{\theta}\partial\check{\theta} \\ &\quad +(1+\beta_1^2\rho^2+\beta_2^2r^2)^{-1}(\bar{\partial}Y+\beta_1\rho^2\bar{\partial}\check{\varphi}+\beta_2r^2\bar{\partial}\check{\theta})(\partial Y-\beta_1\rho^2\partial\check{\varphi}-\beta_2r^2\partial\check{\theta})\Big], \end{aligned} \tag{5.6}$$

where we have defined $\check{\varphi}=\varphi+q_1Y,\ \check{\theta}=\theta+q_2Y$. After T-duality of Y into $\tilde{Y}'$[24] , we obtain

$$S=\frac{1}{\pi\alpha'}\int d^2\sigma\left[\bar{\partial}\rho\partial\rho+\bar{\partial}r\partial r+\bar{\partial}\tilde{Y}'\partial\tilde{Y}'+\rho^2\bar{\partial}\varphi''\partial\varphi''+r^2\bar{\partial}\theta''\partial\theta''\right], \tag{5.7}$$

where $\varphi''\equiv\check{\varphi}-\beta_1\tilde{Y}',\quad \theta''\equiv\check{\theta}-\beta_2\tilde{Y}'$ are the higher dimensional generalization[25] of φ'' which appeared in (3.10). The T-dual of $\tilde{Y}'$ is equivalent to Y' in (3.7). The zero-mode of Y' is quantized in the same way as (3.20)

$$P_L'+P_R'=2\left(\frac{n}{R}-q_1\hat{J}_1-q_2\hat{J}_2\right),\quad P_L'-P_R'=2\left(\frac{Rw}{\alpha'}-\beta_1\hat{J}_1-\beta_2\hat{J}_2\right), \tag{5.8}$$

where the angular momenta $\hat{J}_1=\hat{J}_{1L}+\hat{J}_{1R}$ and $\hat{J}_2=\hat{J}_{2L}+\hat{J}_{2R}$ are defined for each $\mathbf{R}^2$ as done in (3.15).

From (5.7) the bosonic fields $X^{1'}=\rho e^{i\varphi''},\quad X^{2'}=re^{i\theta''}$ and their superpartners $\psi_L^{1'},\psi_R^{1'},\psi_L^{2'},\psi_R^{2'}$ become free fields and they obey the following boundary conditions for each $i=1,2$

$$\begin{aligned} X^{i'}(\tau,\sigma+2\pi) &= e^{2\pi i\gamma_i}X^{i'}(\tau,\sigma), \\ \psi_L^{i'}(\tau,\sigma+2\pi) &= e^{2\pi i\gamma_i}\psi_L^{i'}(\tau,\sigma),\qquad \psi_R^{i'}(\tau,\sigma+2\pi)=e^{2\pi i\gamma_i}\psi_R^{i'}(\tau,\sigma), \end{aligned} \tag{5.9}$$

$$\text{where}\qquad \hat{\gamma}_i\equiv\gamma_i-[\gamma_i],\quad \gamma_i\equiv q_iRw+\beta_i\alpha'(\frac{n}{R}-q_1\hat{J}_1-q_2\hat{J}_2). \tag{5.10}$$

Then we can obtain the mass spectrum as follows

$$\begin{aligned} \frac{\alpha'M^2}{2} &= \frac{\alpha'}{2R^2}(n-q_1R\hat{J}_1-q_2R\hat{J}_2)^2+\frac{R^2}{2\alpha'}(w-\frac{\alpha'}{R}\beta_1\hat{J}_1-\frac{\alpha'}{R}\beta_2\hat{J}_2)^2 \\ &\quad +\hat{N}_R+\hat{N}_L-\sum_{i=1}^{2}\hat{\gamma}_i(\hat{J}_{Ri}-\hat{J}_{Li}), \end{aligned} \tag{5.11}$$

---

[24] Here we take the different process of T-duality from that in section 2.

[25] This correspondence can be shown if one notes that the relations (A.6) are rewritten as $-\partial\tilde{Y}'=\partial Y-\beta_1\rho^2\partial\varphi''-\beta_2r^2\partial\theta''$ and $\bar{\partial}\tilde{Y}'=\bar{\partial}Y+\beta_1\rho^2\bar{\partial}\varphi''+\beta_2r^2\bar{\partial}\theta''$.

with the level matching constraint

$$\hat{N}_R - \hat{N}_L - nw + \sum_{i=1}^{2}[\gamma_i]\hat{J}_i = 0, \tag{5.12}$$

where $\hat{N}_{L,R}$ is defined in the same way as (B.5). From the above expression we can find the T-duality symmetry $q_i \leftrightarrow \beta_i,\ R \leftrightarrow \frac{\alpha'}{R}$ if $q_1\beta_2 = q_2\beta_1$. Note that the supersymmetric model ($q_1 = \pm q_2,\ \beta_1 = \pm\beta_2$) satisfies this condition.

In the supersymmetric case we have found the Bose-Fermi degeneracy. Therefore this system should have no tachyons. Let us show this explicitly using the mass spectrum (5.10). We can assume $0 < \gamma_1 = \gamma_2 < 1$ without any loss of generality since there are no flip of GSO-projection if $\gamma_1 = \gamma_2$. Taking the GSO-projection into consideration, we obtain the inequalities $\hat{J}_{1R} + \hat{J}_{2R} \leq \hat{N}_R$ and $-\hat{N}_L \leq \hat{J}_{1L} + \hat{J}_{2L}$. Then it is easy to see in the NSNS sector

$$\frac{\alpha' M^2}{2} = (\hat{N}_R - \gamma\hat{J}_{1R} - \gamma\hat{J}_{2R}) + (\hat{N}_L + \gamma_1\hat{J}_{1L} + \gamma_2\hat{J}_{2L}) + \frac{\alpha'}{4}(P_R^2 + P_L^2)\ \geq\ 0. \tag{5.13}$$

It is also easy to see from the spectrum (5.13) that if $n_1 + n_2,\ n_1 - n_2 \in 2\mathbf{Z}$, we obtain the following periodicity

$$Z(R, q_1, q_2, \beta_1, \beta_2) = Z(R, q_1 + \frac{n_1}{R}, q_2 + \frac{n_2}{R}, \beta_1, \beta_2), \tag{5.14}$$

and the periodicity for β_i can be also obtained by the T-duality.

5.3 ALE Orbifold from Higher Dimensional Melvin Background

Now we would like to discuss the relation between the above models and orbifolds. We consider both supersymmetric and non-supersymmetric cases. Let us take the limit $R \to 0$ with $\frac{\beta_1\alpha'}{R} \to 0$ and $\frac{\beta_2\alpha'}{R} \to 0$, and assume that q_1R and q_2R are fractional. We can write[26] them as $q_1R = \frac{k_1}{N}$ and $q_2R = \frac{k_2}{N}$. Then the partition function in this limit becomes as in the previous calculations

$$\lim_{R\to 0} Z(R, q_1, q_2, \beta_1, \beta_2) = (2\pi)^{-5}V_5R(\alpha')^{-3}\int\frac{(d\tau)^2}{16(\tau_2)^4}\left(\lim_{R\to 0}\sum_{\alpha,\beta\in\mathbf{Z}} e^{-\frac{\pi N^2R^2}{\alpha'\tau_2}|\alpha-\beta\tau|^2}\right)|\eta(\tau)|^{-12}$$
$$\times\sum_{l,m=0}^{N-1}\Big|\theta_3(\nu^1_{l,m}|\tau)\theta_3(\nu^2_{l,m}|\tau)\theta_3(\tau)^2 - (-1)^{(k_1+k_2)\alpha}\theta_2(\nu^1_{l,m}|\tau)\theta_2(\nu^2_{l,m}|\tau)\theta_2(\tau)^2$$

[26] Here we assume that there is no positive integer other than one which divides all of the three integers N, k_1 and k_2.

$$-(-1)^{(k_1+k_2)\beta}\theta_4(\nu^1_{l,m}|\tau)\theta_4(\nu^2_{l,m}|\tau)\theta_4(\tau)^2\Big|^2\cdot\Big|\theta_1(\nu^1_{l,m}|\tau)\theta_1(\nu^2_{l,m}|\tau)\Big|^{-2}, \tag{5.15}$$

where we have defined $\nu^1_{l,m}=\frac{k_1}{N}(l-m\tau)$ and $\nu^2_{l,m}=\frac{k_2}{N}(l-m\tau)$.

If k_1+k_2 is even, then we get the result

$$Z(0,q_1,q_2,\beta_1,\beta_2)=V_1V_5\int\frac{(d\tau)^2}{4\tau_2}(4\pi^2\alpha'\tau_2)^{-3}\sum_{l,m=0}^{N-1}$$
$$\times\frac{|\theta_3(\nu^1_{l,m}|\tau)\theta_3(\nu^2_{l,m}|\tau)\theta_3(\tau)^2-\theta_2(\nu^1_{l,m}|\tau)\theta_2(\nu^2_{l,m}|\tau)\theta_2(\tau)^2-\theta_4(\nu^1_{l,m}|\tau)\theta_4(\nu^2_{l,m}|\tau)\theta_4(\tau)^2|^2}{4N\ |\eta(\tau)|^{12}|\theta_1(\nu^1_{l,m}|\tau)\theta_1(\nu^2_{l,m}|\tau)|^2}. \tag{5.16}$$

Thus we have the abelian non-compact four dimensional orbifolds $\mathbf{C}^2/\mathbf{Z}_N$ in type II string theory. The $\mathbf{Z}_N$ action is defined as follows

$$g\in\mathbf{Z}_N:\ (X^{1'},X^{2'})\rightarrow(e^{2\pi i\frac{k_1}{N}}X^{1'},e^{2\pi i\frac{k_2}{N}}X^{2'}). \tag{5.17}$$

These include both the supersymmetric and non-supersymmetric orbifolds. The former correspond to the values $k_1=\pm k_2$ and it is easy to see that for fixed N the partition functions (5.16) for each k_1,k_2 give the same value. This represents the A_{N-1}-type ALE space (for a review see [86]) in the orbifold limit.

The other orbifolds are all non-supersymmetric and the tachyon can appear only in the twisted sectors. These include the examples discussed in [33], where the specific non-supersymmetric orbifolds are argued to decay into ALE spaces. Our results show that both such non-supersymmetric orbifolds and supersymmetric ALE orbifolds (including the type II string in flat space $N=1$) are connected in the 'moduli space' of solvable superstring models if we compactify one direction.

Next we turn to the case where k_1+k_2 is odd. The partition function is given by

$$Z(0,q_1,q_2,\beta_1,\beta_2)=V_1'V_7\int\frac{(d\tau)^2}{4\tau_2}(4\pi^2\alpha'\tau_2)^{-3}\sum_{l,m=0}^{N-1}$$
$$\times\frac{|\theta_3(\nu^1_{l,m}|\tau)\theta_3(\nu^2_{l,m}|\tau)\theta_3(\tau)^2|^2+|\theta_2(\nu^1_{l,m}|\tau)\theta_2(\nu^2_{l,m}|\tau)\theta_2(\tau)^2|^2+|\theta_4(\nu^1_{l,m}|\tau)\theta_4(\nu^2_{l,m}|\tau)\theta_4(\tau)^2|^2}{2N|\eta(\tau)|^{12}|\theta_1(\nu^1_{l,m}|\tau)\theta_1(\nu^2_{l,m}|\tau)|^2}. \tag{5.18}$$

This explicitly shows that the systems now considered are equivalent to the non-compact four dimensional orbifolds $\mathbf{C}^2/\mathbf{Z}_N$ in type 0 string theory[27] . The above result

[27] Such orbifolds were considered in the context of D-branes in [67].

shows that orbifolds in type 0 theory are connected to those in type II theory. In particular, this shows that various orbifolds in type 0 theory can be regarded as non-supersymmetric backgrounds in type II string theory.

In this way we have shown that the various orbifolds both non-supersymmetric and supersymmetric are included in the 'moduli space' of the higher dimensional Melvin background (see Fig.3).

Then let us discuss the tachyons in these orbifolds. The mass of the lightest state is given by for even $k_1 + k_2$ (type II)

$$\frac{\alpha' M^2}{2} = \begin{cases} -|\hat{\mu_1} - \hat{\mu_2}| & \text{if} \quad ([\mu_1],[\mu_2]) \in \text{(even,even) or (odd,odd)} \\ \hat{\mu_1} + \hat{\mu_2} - 1 & \text{if} \quad ([\mu_1],[\mu_2]) \in \text{(even,odd) or (odd,even)} \end{cases}, \tag{5.19}$$

and for odd $k_1 + k_2$ (type 0)

$$\frac{\alpha' M^2}{2} = \min\{\hat{\mu_1} + \hat{\mu_2} - 1, -|\hat{\mu_1} - \hat{\mu_2}|\}, \tag{5.20}$$

where we have defined $\mu_1 = \frac{k_1 m}{N}$ and $\mu_2 = \frac{k_2 m}{N}$. These results show that in both orbifolds some of the twisted sectors will contain tachyon and in type 0 orbifolds tachyon also appears in the untwisted sector. The type II string theory on ALE orbifolds $k_1 = \pm k_2$ are only examples of tachyon less orbifolds.

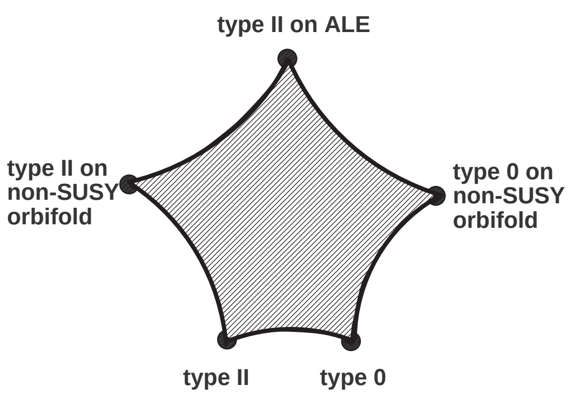

Figure 3: Moduli space of the string models in the higher dimensional Melvin background.

Finally, if we turn to the small radius limit with irrational values of $q_1 R$ and $q_2 R$, we obtain the 'irrationally orbifolded' noncompact space (or a kind of a 'large N limit of the orbifolds $\mathbf{C}^2/\mathbf{Z}_\infty$') as in the original Melvin background. For specific values $q_1 = \pm q_2$ this background preserves the half of thirty two supersymmetries and is connected to ALE spaces.

It will be also interesting to consider the brane picture of the general supersymmetric backgrounds with $\beta_{1,2} \neq 0$, which include the previous orbifolds as special examples by T-duality. These models have the non-trivial H-flux and dilaton gradient. We left its relation to NS5-branes as a future problem.

Even though the examples we have examined are two and four dimensional orbifolds $\mathbf{C}/\mathbf{Z}_N,\ \mathbf{C}^2/\mathbf{Z}_N$, our results will be easily generalized into much higher dimensional orbifolds $\mathbf{C}^n/\mathbf{Z}_N,\ n = 3, 4$. Other abelian orbifolds such as $\mathbf{C}^n/\mathbf{Z}_{N_1} \times \mathbf{Z}_{N_2} \times \cdots \times \mathbf{Z}_{N_K}$ can also be obtained if we replace $\mathbf{S}^1$ with K dimensional tori. Non-abelian orbifolds (e.g. D,E type) will deserve future study.

6 Boundary States in Melvin Background

In this section we consider D-branes in Melvin background (3.1) from the viewpoint of the world-sheet theory following the paper [71] (see also the independent and overlapped paper [87]). In the previous section we have seen that the nonlinear sigma model in the Melvin background can be exactly solved. By applying the T-duality in the curved space and by redefining appropriate target space variables, the nonlinear sigma model can be rewritten by the free fields with the nontrivial boundary conditions, and we can quantize this action in the same way as that in the flat space.

With regard to open strings the quantization process can be also performed by the usual method. By changing the boundary conditions of open strings we can obtain the various D-branes[28] in the Melvin background, while the several constraints which can not be seen in the flat space arise due to the nontriviality of the Melvin geometry. In this section we define a Dp-brane as the D-brane which has $p+1$ Neumann boundary conditions and $(9-p)$ Dirichlet boundary conditions in terms of the free fields $(X', \bar{X}', Y')$, not the original fields $(X, \bar{X}, Y)$ (see Table 2). Then we will find two types of D-branes. One is pinned at the origin $\rho = 0$, while the other is movable.

The interpretation of such a D-brane in the original coordinate $(X, \bar{X}, Y)$ of the Melvin background (3.1) is very nontrivial (see Table 3 in the next section). We will find several kinds of D-branes which wrap the nontrivial 'cycle' in the Melvin geometry. Note that for some of them the meaning of 'cycle' is different from the ordinary one due to the presence of H-flux. Such an example will be explained by the phenomenon of flux stabilization later in the next section.

There is another motivation to consider D-branes in the Melvin background. As we have seen in the previous section, by tuning magnetic parameters qR and $\frac{\beta\alpha'}{R}$ such that they take fractional values, the various ten dimensional backgrounds (type II, type 0 in flat space and their abelian orbifolds $\mathbf{C}/\mathbf{Z}_N$) can be realized [26]. Therefore, if we consider D-branes in the Melvin background, we can understand how the D-branes in the above various backgrounds are connected with each other.

In orbifold theories there exist two types of D-branes which are called the fractional D-brane and the bulk D-brane [89, 90, 91] (for a brief review see the appendix F of this

[28] For $\beta = 0$ case the D-brane systems we discuss below are closely related to the D-branes in toroidal compactification of freely acting orbifolds [88].

thesis). What we will find is that the fractional D-branes and bulk D-branes correspond to the pinned and movable D-branes which we have mentioned, respectively. Moreover, these D-branes prove to naturally correspond not only to the D-branes in type II theory in flat space but also to a system of an electric D-brane and a magnetic D-brane in type 0 theory (for D-brane in type 0 string see the review in section 2.2). We can verify this identification by the calculation in the boundary state formalism.

6.1 Boundary State in Melvin Background

In order to investigate D-branes in general conformal field theories, it is convenient to use the boundary state formalism. The boundary state is one way representing D-branes in the closed string Hilbert space and thus from this we can investigate the interactions between two D-branes and between a D-brane and closed strings as we have already seen in section 2.2.

From previous sections we know that the action for the Melvin background reduces to that written by free fields. Therefore the construction of the boundary state is similar to that in the flat space (see [68] and references there in), or more precisely to that in orbifold theories [67, 92, 93, 94] even though the D-branes which we will construct have various new intriguing structures. Below we will omit the trivial oscillators in $\mathbf{R}^{1,6}$ directions on the world-sheet. One can use either the light-cone or covariant formulation.

In the compactified direction Y', the usual Neumann and Dirichlet boundary conditions are both allowed

$$\text{Neumann}: \quad \begin{cases} \partial_\tau Y'|_{\tau=0}|B\rangle = 0 & \leftrightarrow \quad (\frac{n}{R} - q\hat{J})|B\rangle = 0,\ (\beta_m + \tilde{\beta}_{-m})|B\rangle = 0, \\ (\eta'_L - \epsilon\eta'_R)|_{\tau=0}|B\rangle = 0 & \leftrightarrow \quad (\eta_r - i\epsilon\tilde{\eta}_{-r})|B\rangle = 0, \end{cases} \tag{6.1}$$

$$\text{Dirichlet}: \quad \begin{cases} \partial_\sigma Y'|_{\tau=0}|B\rangle = 0 & \leftrightarrow \quad (\frac{Rw}{\alpha'} - \beta\hat{J})|B\rangle = 0,\ (\beta_m - \tilde{\beta}_{-m})|B\rangle = 0, \\ (\eta'_L + \epsilon\eta'_R)|_{\tau=0}|B\rangle = 0 & \leftrightarrow \quad (\eta_r + i\epsilon\tilde{\eta}_{-r})|B\rangle = 0, \end{cases} \tag{6.2}$$

where $|B\rangle$ is the boundary state and the parameter ϵ takes the values ± 1 which come from open string boundary conditions for world-sheet fermions. Note that the zero mode for Y' is given by (3.20). For the directions $X', \bar{X}'$ the allowed boundary conditions are

$$\text{Neumann-Neumann}: \quad \begin{cases} \partial_\tau X'|_{\tau=0}|B\rangle = 0 & \leftrightarrow \quad (\alpha_{m-\gamma} + \tilde{\alpha}_{-m+\gamma})|B\rangle = 0, \\ \partial_\tau \bar{X}'|_{\tau=0}|B\rangle = 0 & \leftrightarrow \quad (\bar{\alpha}_{m+\gamma} + \tilde{\bar{\alpha}}_{-m-\gamma})|B\rangle = 0, \\ (\psi'_L - \epsilon\psi'_R)|_{\tau=0}|B\rangle = 0 & \leftrightarrow \quad (\psi_{r-\gamma} - i\epsilon\tilde{\psi}_{-r+\gamma})|B\rangle = 0, \\ (\bar{\psi}'_L - \epsilon\bar{\psi}'_R)|_{\tau=0}|B\rangle = 0 & \leftrightarrow \quad (\bar{\psi}_{r+\gamma} - i\epsilon\tilde{\bar{\psi}}_{-r-\gamma})|B\rangle = 0, \end{cases} \tag{6.3}$$

$$
\text{Dirichlet-Dirichlet}: \begin{cases} \partial_\sigma X'|_{\tau=0}|B\rangle = 0 & \leftrightarrow \quad (\alpha_{m-\gamma} - \tilde{\alpha}_{-m+\gamma})|B\rangle = 0, \\ \partial_\sigma \bar{X}'|_{\tau=0}|B\rangle = 0 & \leftrightarrow \quad (\bar{\alpha}_{m+\gamma} - \tilde{\bar{\alpha}}_{-m-\gamma})|B\rangle = 0, \\ (\psi'_L + \epsilon\psi'_R)|_{\tau=0}|B\rangle = 0 & \leftrightarrow \quad (\psi_{r-\gamma} + i\epsilon\tilde{\psi}_{-r+\gamma})|B\rangle = 0, \\ (\bar{\psi}'_L + \epsilon\bar{\psi}'_R)|_{\tau=0}|B\rangle = 0 & \leftrightarrow \quad (\bar{\psi}_{r+\gamma} + i\epsilon\tilde{\bar{\psi}}_{-r-\gamma})|B\rangle = 0, \end{cases} \tag{6.4}
$$

where γ is given by (3.22)[29] . Here we have to note that the Neumann-Dirichlet or Dirichlet-Neumann boundary conditions can be defined only if γ takes integer or half-integer values. In this thesis we will not consider these cases in detail, though we will mention them at the end of next subsection.

We would also like to stress that in the above arguments we have defined the Dirichlet or Neumann boundary conditions with respect to the free fields $(X', \bar{X}', Y')$. These boundary conditions are not always equivalent to the Dirichlet or Neumann boundary conditions with respect to the original fields $(X, \bar{X}, Y)$ in the Melvin sigma model (3.4) as we will see later.

From these conditions and from (B.3) and (B.4) we can verify that the boundary state defined by (6.1) $\sim$ (6.4) satisfies the $\mathcal{N} = 1$ superconformal invariance

$$
\begin{aligned}
&(L_m - \tilde{L}_{-m})|B\rangle = 0, \\
&(G_r + i\epsilon\tilde{G}_{-r})|B\rangle = 0.
\end{aligned} \tag{6.5}
$$

Moreover from (6.3) and (6.4) we can verify

$$
\hat{J}|B\rangle = (\hat{J}_L + \hat{J}_R)|B\rangle = 0, \tag{6.6}
$$

where the mode expansions of $\hat{J}_L$ and $\hat{J}_R$ are given by (B.6). This shows that these D-branes (NN or DD boundary condition) preserve the rotational symmetry on the plane $\mathbf{R}^2$ as expected[30] .

[29] At this stage we can take the more general boundary conditions for complex fermions (ϵ can take U(1) complex values and can be unequal to ϵ in (6.1) and (6.2)). However, if we consider the $\mathcal{N} = 1$ superconformal invariance (6.5), such boundary conditions are not allowed.

[30] More precisely, one should take into account the bosonic zero-mode contribution to the angular momentum $\hat{J}_0 = i\sqrt{\frac{2}{\alpha'}}(x_0\bar{\alpha}_0 - \bar{x}_0\alpha_0)$ if γ takes an integer value. However, we can neglect this if the D-brane obeys the Neumann-Neumann (NN) boundary condition for $(X', \bar{X}')$ or if the D-brane obeys the Dirichlet-Dirichlet (DD) boundary condition located at $\rho = 0$. Even if we consider a D-brane with the DD boundary condition and move it away from the origin $\rho = 0$, we can choose n or w such that $(\frac{n}{R} - q\hat{J}_0)|B\rangle = 0$ or $(\frac{Rw}{\alpha'} - \beta\hat{J}_0)|B\rangle = 0$, respectively. We will return to this point in section 6.2.

Now we can write down the boundary state explicitly. From (6.1) $\sim$ (6.4) its explicit form is almost the same as that in the flat space [68], except the shift of oscillator indices by $\hat{\gamma}$ and the GSO-projection. For example, the NSNS sector of the boundary state for a D0-brane at $\rho = 0$ is given by

$$
\begin{aligned}
|B,\gamma,\epsilon\rangle_{NSNS} = \exp\left[\sum_{m=1}^{\infty} m^{-1}\beta_{-m}\tilde{\beta}_{-m}\right]|n,0\rangle \otimes \exp\left[-i\epsilon\sum_{r=1/2}^{\infty}\eta_{-r}\tilde{\eta}_{-r}\right]|0\rangle \\
\otimes \exp\left[\sum_{m=0}^{\infty}(m+\hat{\gamma})^{-1}\alpha_{-m-\hat{\gamma}}\tilde{\bar{\alpha}}_{-m-\hat{\gamma}} + \sum_{m=1}^{\infty}(m-\hat{\gamma})^{-1}\bar{\alpha}_{-m+\hat{\gamma}}\tilde{\alpha}_{-m+\hat{\gamma}}\right]|0\rangle_{\hat{\gamma}} \\
\otimes \exp\left[-i\epsilon\sum_{r=1/2}^{\infty}\{\psi_{-r-\hat{\gamma}}\tilde{\bar{\psi}}_{-r-\hat{\gamma}} + \bar{\psi}_{-r+\hat{\gamma}}\tilde{\psi}_{-r+\hat{\gamma}}\}\right]|0\rangle_{\hat{\gamma}+\frac{1}{2}}, \qquad (6.7)
\end{aligned}
$$

where γ is equal to $\frac{\beta\alpha' n}{R}$ ($n \in \mathbf{Z}$) as can be seen from (3.22) and (6.2), and $\hat{\gamma}$ is defined in (3.23)[31] . In the above expression we have suppressed the trivial part which comes from the other directions than $X', \bar{X}'$ and Y'. Boundary states for D-branes which obey other boundary conditions can be obtained similarly. The total boundary state is given by $|B\rangle = |B\rangle_{NSNS} \pm |B\rangle_{RR}$. The plus (minus) sign corresponds to a D-brane (an anti D-brane).

Next we have to consider the closed string GSO-projection. This is somewhat nontrivial because as we said in the lines below (3.24) the GSO-projection for $2n \le \gamma < 2n+1$ ($n \in \mathbf{Z}$) is the usual projection for type II theory, while for $2n+1 \le \gamma < 2n+2$ ($n \in \mathbf{Z}$) it is the projection with an additional minus sign. Thus, the GSO-invariant boundary state is represented by

$$
\begin{aligned}
|B\rangle_{NSNS} &= \frac{1}{2\pi R}\sum_{n=-\infty}^{\infty} f_\gamma \, \frac{1+(-1)^{[\gamma]}(-1)^{F_L}}{2}\frac{1+(-1)^{[\gamma]}(-1)^{F_R}}{2}|B,\gamma,+\rangle_{NSNS}, \\
&= \frac{1}{2(2\pi R)}\sum_{n=-\infty}^{\infty} f_\gamma\left[|B,\gamma,+\rangle_{NSNS} - (-1)^{[\gamma]}|B,\gamma,-\rangle_{NSNS}\right], \\
|B\rangle_{RR} &= \frac{1}{2\pi R}\sum_{n=-\infty}^{\infty} f_\gamma \, \frac{1+(-1)^{[\gamma]}(-1)^{F_L}}{2}\frac{1\mp(-1)^{[\gamma]}(-1)^{F_R}}{2}|B,\gamma,+\rangle_{RR}, \\
&= \frac{1}{2(2\pi R)}\sum_{n=-\infty}^{\infty} f_\gamma\left[|B,\gamma,+\rangle_{RR} + (-1)^{[\gamma]}|B,\gamma,-\rangle_{RR}\right], \quad \left(\mp = \begin{cases} - & \text{for IIA,} \\ + & \text{for IIB,}\end{cases}\right)
\end{aligned}
$$
(6.8)

[31] We have assumed the specific range $0 < \hat{\gamma} < 1/2$. The extension of this expression to the other range of $\hat{\gamma}$ and to RR-sector is straightforward.

where $[\gamma]$ is the Gauss symbol which picks up the maximal integer part of γ, and constants f_γ in (6.8) are determined by the Cardy's condition (or open-closed duality) [69]. This is the consistency condition that the vacuum amplitude between two D-branes computed in the closed string sector by using the boundary state should be equal to the cylinder amplitude of open string.

Therefore we would like to calculate the vacuum amplitude by using the above boundary state. For a D0-brane we obtain the following result[32] by using the explicit form of L_0 and $\tilde{L}_0$ in (B.4) for the closed string propagator

$$\Delta = \frac{1}{2}\int_0^\infty ds\ e^{-s(L_0+\tilde{L}_0)}, \tag{6.9}$$

and employing the quasi periodicity of theta functions (C.3)

$$\begin{aligned} A &= \langle B|\Delta|B\rangle \\ &= \frac{\alpha'}{8\pi R}V_0\sum_{\gamma\in\mathbf{Z}}\ |f_\gamma|^2\ \int_0^\infty ds\ (2\pi\alpha' s)^{-4}\exp\left(-\frac{s\alpha' n^2}{2R^2}\right)(\eta(\tau))^{-12} \\ &\qquad \times[(\theta_3(0|\tau))^4-(-1)^\gamma(\theta_4(0|\tau))^4-(\theta_2(0|\tau))^4] \\ &\quad +\frac{i\alpha'}{8\pi R}V_0\sum_{\gamma\notin\mathbf{Z}}\ (-1)^{[\gamma]}\ |f_\gamma|^2\ \int_0^\infty ds\ (2\pi\alpha' s)^{-3}\exp\left(-\frac{s\alpha' n^2}{2R^2}\right)(\eta(\tau))^{-9}(\theta_1(\nu|\tau))^{-1} \\ &\qquad \times[(\theta_3(0|\tau))^3(\theta_3(\nu|\tau))-(\theta_4(0|\tau))^3(\theta_4(\nu|\tau))-(\theta_2(0|\tau))^3(\theta_2(\nu|\tau))], \end{aligned} \tag{6.10}$$

where $\tau=\frac{is}{\pi}, \nu=\frac{i\gamma s}{\pi}$. The volume factor for the time-direction is denoted by V_0. By replacing s with $\frac{\pi}{t}$ and using the modular transformations for theta functions (C.2) we can obtain the following result

$$\begin{aligned} A &= \frac{\pi\alpha'}{(8\pi R)(2\pi^2\alpha')^4}V_0\sum_{\gamma\in\mathbf{Z}}\ |f_\gamma|^2\ \int_0^\infty \frac{dt}{t^2}\ \exp\left(-\frac{\pi\alpha' n^2}{2R^2 t}\right)(\eta(it))^{-12} \\ &\qquad \times[(\theta_3(0|it))^4-(-1)^\gamma(\theta_2(0|it))^4-(\theta_4(0|it))^4] \\ &\quad +\frac{\pi\alpha'}{(8\pi R)(2\pi^2\alpha')^3}V_0\sum_{\gamma\notin\mathbf{Z}}\ (-1)^{[\gamma]}\ |f_\gamma|^2\ \int_0^\infty \frac{dt}{t^2}\ \exp\left(-\frac{\pi\alpha' n^2}{2R^2 t}\right)(\eta(it))^{-9}(\theta_1(\gamma|it))^{-1} \\ &\qquad \times[(\theta_3(0|it))^3(\theta_3(\gamma|it))-(\theta_2(0|it))^3(\theta_2(\gamma|it))-(\theta_4(0|it))^3(\theta_4(\gamma|it))]. \end{aligned} \tag{6.11}$$

[32] Note that we have divided γ into the integer part and the other part, because if γ takes an integer value the naive calculation of the vacuum amplitude by using (6.7) diverges. This can be seen in the third line of (6.10) if we notice the relation $\theta_1(\nu|\tau)=0$ ($\gamma\in\mathbf{Z}$) by using (C.3). This divergence is due to the reappearance of the zero modes of $X',\bar{X}'$, which can be seen from (3.21) and (B.1). Therefore, the expression of the boundary state such as (6.7) is not correct for integer γ, and we have to redefine its bosonic part for $X',\bar{X}'$.

On the other hand the vacuum amplitude can be obtained also from the open string one-loop calculation

$$Z_O = \int_0^\infty \frac{dt}{2t} \mathrm{Tr}_{NS-R}\left[\frac{1+(-1)^F}{2} q^{H_O}\right] \qquad (q \equiv e^{-2\pi t}), \tag{6.12}$$

where we have defined $\mathrm{Tr}_{NS-R} = \mathrm{Tr}_{NS} - \mathrm{Tr}_R$, where the trace includes the factor two due to the Chan-Paton factor. The operator H_O denotes the open string Hamiltonian

$$H_O = \alpha' p^2 + \alpha' \left(\frac{Rw}{\alpha'} - \beta \hat{J}\right)^2 + \hat{N}, \tag{6.13}$$

where $\hat{N}$ and $\hat{J}$ represent the occupation number operator including the zero point energy ($-1/2$ for NS-sector and 0 for R-sector) and the angular momentum generator in $\mathbf{R}^2$ plane both of which are the open string analog of (B.5) and (B.6), respectively. The shift of the winding mode is obtained in the same way as in closed string theory (3.23). The different points from closed string spectrum is that the indices of modes for $X', \bar{X}', Y'$ and their superpartners take the integer or half-integer (for NS-sector) values because the boundary conditions of open strings obey usual Neumann or Dirichlet conditions, not twisted ones (3.21). The mode expansion of $\hat{J}$ by open string NS-modes is

$$\hat{J} = \frac{i}{\sqrt{2\alpha'}}(x_0\bar{\alpha}_0 - \bar{x}_0\alpha_0) + \sum_{n=1}^{\infty}\frac{1}{n}(\alpha_{-n}\bar{\alpha}_n - \bar{\alpha}_{-n}\alpha_n) + \sum_{r=1/2}^{\infty}(\psi_{-r}\bar{\psi}_r - \bar{\psi}_{-r}\psi_r), \tag{6.14}$$

where x_0 and $\bar{x}_0$ are the zero modes for X' and $\bar{X}'$, respectively[33] . For R-sector the indices of ψ and $\bar{\psi}$ run integer values, and its zero mode contribution $\frac{1}{2}[\psi_0, \bar{\psi}_0]$ should be added. Therefore we can see that the eigenvalues of $\hat{J}$ take integer values for NS-sector and half-integer values for R-sector being consistent with the spin-statistics relation.

Now let us apply the Cardy's condition [69]. By using the Poisson resummation formula the open string amplitude (6.12) becomes

$$\begin{aligned} A &= \frac{2\pi\alpha'}{R}\int_0^\infty \frac{dt}{2t}(8\pi^2\alpha' t)^{-\frac{1}{2}} \\ &\quad \times \mathrm{Tr}_{NS-R}\left[\frac{1+(-1)^F}{2}\, q^{\alpha' p^2 + \hat{N}} \sum_{n=-\infty}^{\infty} \exp\left(-\frac{\pi\alpha'}{2R^2 t}n^2 + 2\pi i \frac{\beta\alpha'}{R} n\hat{J}\right)\right]. \end{aligned} \tag{6.15}$$

By requiring the equality between (6.11) and (6.15) we obtain

$$f_\gamma = \begin{cases} \frac{1}{2}T_0 \quad (\gamma \in \mathbf{Z}), \\ \frac{1}{\sqrt{2}}\left(\frac{|\sin\pi\gamma|}{2\pi^2\alpha'}\right)^{\frac{1}{2}} T_0 \quad (\gamma \notin \mathbf{Z}), \end{cases} \tag{6.16}$$

[33] Especially for D0-branes which we are considering here the orbital angular momentum part $\hat{J}_0 = \frac{i}{\sqrt{2\alpha'}}(x_0\bar{\alpha}_0 - \bar{x}_0\alpha_0)$ should be neglected due to the absence of the open string zero mode.

where we have defined $T_p = \sqrt{\pi}(2\pi\sqrt{\alpha'})^{3-p}$ and T_p/κ (κ is the gravitational coupling constant) is equal to the tension of an ordinary type II Dp-brane in flat space.

For more general Dp-branes the computations can be performed in the same way. Its open string Hamiltonian is the same as in (6.13) except the reappearance of the zero modes for D-branes which obey the Neumann-Neumann boundary condition in $X', \bar{X}'$ directions. In this case the nontrivial relation is the trace formula which comes from the open string zero modes of X' and $\bar{X}'$ as follows

$$\mathrm{Tr}\ \exp\left[-2\pi\alpha' t p^2 + 2\pi i\gamma \hat{J}_0\right] = \begin{cases} (2\sin\pi\gamma)^{-2} & (\gamma \notin \mathbf{Z}), \\ V_2\ (8\pi^2\alpha' t)^{-1} & (\gamma \in \mathbf{Z}), \end{cases}$$

where $\hat{J}_0 = \frac{i}{\sqrt{2\alpha'}}(x_0\bar{\alpha}_0 - \bar{x}_0\alpha_0)$ is the orbital angular momentum, and V_2 is the volume factor for $(X', \bar{X}') \in \mathbf{R^2}$ plane. Such a trace is familiar in orbifold theories (see for example [95]). Then the value of f_γ is given by

$$f_\gamma = \begin{cases} \frac{1}{2}T_p \quad (\gamma \in \mathbf{Z}), \\ \frac{1}{\sqrt{2}}\left(\frac{|\sin\pi\gamma|}{2\pi^2\alpha'}\right)^{\mp\frac{1}{2}}T_p \quad (\gamma \notin \mathbf{Z}). \end{cases} \tag{6.17}$$

The sign factors $\mp$ take $-$ for Dp-branes with the Neumann-Neumann boundary condition for $X, \bar{X}'$ directions, $+$ for Dp-branes with the Dirichlet-Dirichlet boundary condition for those directions.

The above results for $\gamma \in \mathbf{Z}$ show that a Dp-brane in the Melvin background has the same tension as that in the flat space. We would also like to note that no open string tachyonic modes appear on these Dp-branes in contrast with the closed string theory.

6.2 Structure of D-branes for the Rational Parameters

As we have said in section two, the nature of the Melvin background depends sensitively on whether the (dimensionless) magnetic parameters qR and $\frac{\beta\alpha'}{R}$ take the rational or irrational values. Especially in the former case with $qR = \frac{k}{N}$, $\beta = 0$ (or $\frac{\beta\alpha'}{R} = \frac{k}{N}$, $q = 0$) this background is equivalent to the freely acting orbifold[34] , and under the limit $R \to 0$ ($R \to \infty$) it is reduced to the abelian orbifold $\mathbf{C}/\mathbf{Z}_N$ in type II (for even k) or in type 0 (for odd k) [26].

[34] The discussion includes the $N = 1$ case which is equivalent to the type 0 theory with $\mathbf{Z}_2$ twist $(-1)^{F_R} \cdot \sigma$ [42]. For earlier discussions on the D-branes in this special case see [96, 97].

In this section we consider D-branes in the Melvin background with the rational parameters[35] . As we will see below even for the finite radius a single D-brane and a system of N D-branes have similar properties to a fractional D-brane and a bulk D-brane in orbifold theories, respectively. For a brief review of D-brane in orbifold theories see the appendix F. Therefore, from now on, we will often call these two types of D-branes in the Melvin background the fractional D-brane and the bulk D-brane.

6.2.1 Fractional D0-branes

Let us first discuss a D0-brane, which has the Dirichlet boundary condition in both $\mathbf{R}^2$ and $\mathbf{S}^1$ direction. As can be seen from its boundary state (6.7), its behavior depends only on $\gamma = \frac{\beta\alpha'}{R}n$. Moreover we can find in (6.7) that there are no zero-modes in $X', \bar{X}'$ directions unless γ takes an integer value. We can equally say that this D0-brane can not leave from $X' = \bar{X}' = 0$ unless $\frac{\beta\alpha'}{R} \in \mathbf{Z}$, and thus the D0-brane is expected to become a fractional D0-brane [89, 90, 91, 92] on the orbifolds $\mathbf{C}/\mathbf{Z}_N$ in the limit $R \to \infty$, $\frac{\beta\alpha'}{R} = \frac{k}{N}$ and $qR \to 0$. To verify this we begin with the analysis of this orbifold limit.

First we take this limit for the boundary state (6.8). Here we reparameterize the momentum number as $n = N\alpha + l \quad (\alpha \in \mathbf{Z},\ l = 0, 1, \cdots, N-1)$. Here note that the radius from the viewpoint of the orbifold theories is given by R/N not R (see Table 1 and (6.18)). In the limit the NSNS-sector of the boundary state (6.8) becomes

$$
\lim_{R\to\infty} \frac{1}{2(2\pi R)} \sum_{l=0}^{N-1} \sum_{\alpha\in\mathbf{Z}} e^{i\frac{y'}{R}(N\alpha+l)} f_\gamma \left[|B, \frac{kl}{N}, +\rangle_{NSNS} - (-1)^{k\alpha+[kl/N]} |B, \frac{kl}{N}, -\rangle_{NSNS} \right]
$$

$$
= \begin{cases} \dfrac{\delta(y')}{2} \displaystyle\sum_{l=0}^{N-1} f'_l \left[|B, \frac{kl}{N}, +\rangle_{NSNS} - (-1)^{[\frac{kl}{N}]} |B, \frac{kl}{N}, -\rangle_{NSNS} \right] & \text{(even } k\text{)}, \\ \dfrac{\delta(y')}{2} \displaystyle\sum_{l=0}^{N-1} f'_l\, |B, \frac{kl}{N}, +\rangle_{NSNS} - \dfrac{\delta(y' - \frac{\pi R}{N})}{2} \displaystyle\sum_{l=0}^{N-1} (-1)^{[\frac{kl}{N}]} f'_l\, |B, \frac{kl}{N}, -\rangle_{NSNS} & \text{(odd } k\text{)}, \end{cases} \tag{6.18}
$$

$$
\text{where} \qquad f'_l = \begin{cases} \frac{1}{2N} T_0 & (l = 0), \\ \frac{1}{\sqrt{2}N} \left(\frac{|\sin \pi kl/N|}{2\pi^2\alpha'}\right)^{\frac{1}{2}} T_0 & (l \neq 0), \end{cases} \tag{6.19}
$$

and the RR-sector of the boundary state can be obtained in the same way. Note that here we extract the α dependent factor from the boundary state (6.7) which comes only from the zero mode contribution of Y' as $|n, 0\rangle = \exp\left(i\frac{n}{R}y'\right)|0\rangle$.

[35] Note that our open string results here hold if either qR or $\frac{\beta\alpha'}{R}$ is rational and another is arbitrary. In general the closed string theory with these values can not be identified with the freely acting orbifold[26].

For even k this is just the boundary state for a fractional D0-brane in the type II orbifold $\mathbf{C}/\mathbf{Z}_N$ [36] . Indeed we can identify the summation over l as the contribution from one untwisted sector and $N-1$ twisted sectors. Moreover the coefficient f'_l in (6.19) is $1/N$ times of f_γ in (6.16) and this shows the fractional nature of this D-brane explicitly.

For odd k the boundary state is a little more complicated than for even k. The most important result is that two kinds of D-branes appear each at $y'=0$ (the first term in (6.18)) and at $y'=\pi R/N \to \infty$ (the second term). In this case we have observed that the closed string theory approaches the type 0 orbifold in the limit. Then let us remember the result of D-branes in type 0 theory [65, 66, 42, 67] reviewed in section 2. There are two kinds of D-branes: electric D-branes and magnetic D-branes. This is because we have twice as many RR-fields in type 0 theory as in type II theory. Now we can understand our result (6.18). The former corresponds to an electric fractional D0-brane and the latter a magnetic fractional D0-brane in the orbifold $\mathbf{C}/\mathbf{Z}_N$ of type 0 theory (see Fig.4). For example, let us set $N=1$ for simplicity. Then the untwisted sector ($l=0$) only remains, and if we divide it into one with $\delta(y')$ and another with $\delta(y'-\pi R)$ those boundary states are given by as follows

$$\begin{aligned} |\text{electric}\rangle &= \frac{T_0}{4}\left(|B,0,+\rangle_{NSNS}+|B,0,+\rangle_{RR}\right), \\ |\text{magnetic}\rangle &= \frac{T_0}{4}\left(-|B,0,-\rangle_{NSNS}+|B,0,-\rangle_{RR}\right). \end{aligned} \tag{6.20}$$

These are just the boundary states for an electric D0-brane and a magnetic D0-brane in the type 0 theory[37] (2.14).

The identification with fractional D-branes can also be shown explicitly by examining the vacuum amplitude (6.15). By taking the orbifold limit, the summation part of NS-sector in (6.15) becomes

$$\lim_{R\to\infty}\sum_{l=0}^{N-1}\sum_{\alpha\in\mathbf{Z}}\exp\left[-\frac{\pi\alpha'}{2R^2t}(N\alpha+l)^2+2\pi i\frac{k}{N}(N\alpha+l)\hat{J}\right],$$

[36] Note that in (6.18) we have not used the necessary condition $qR\to 0$ to realize the orbifolds $\mathbf{C}/\mathbf{Z}_N$ since the boundary state (6.18) does not depend on the value of q. This is different from the closed string theory which depends on both of the parameters q,β.

[37] Strictly speaking the coefficient of the boundary state is a little different. The correct coefficient of the boundary state for a type 0 D-brane is $\sqrt{2}$ times as large as one in (6.20). This mismatch is due to the same reason as the appearance of the factor $1/2$ of V'_1 in (4.4). Therefore the most simplified way to understand the coefficient $\sqrt{2}$ is to regard the electric and magnetic D-brane on the circle of radius R as an electric (or magnetic) D-brane on the circle of radius $R/2$.

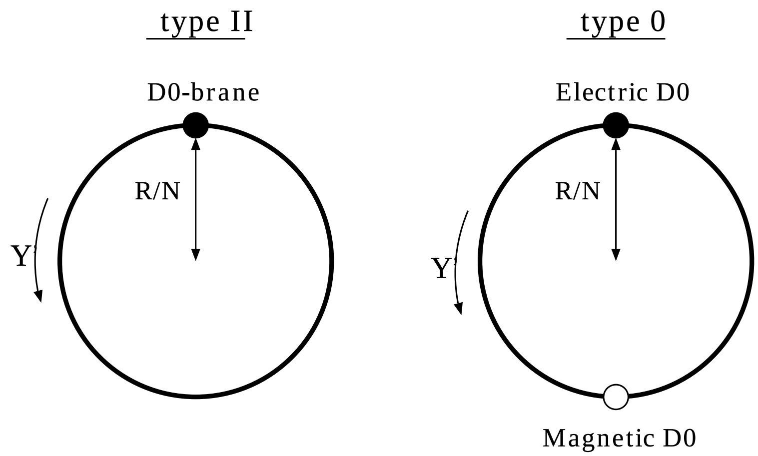

Figure 4: D0-branes

$$= R\int_{-\infty}^{\infty} dp\ \exp\left(-\frac{\pi\alpha' N^2}{2t}p^2\right)\sum_{l=0}^{N-1}\exp\left[2\pi i\frac{k}{N}l\hat{J}\right]=\frac{R}{2\pi\alpha'}(8\pi^2\alpha' t)^{\frac{1}{2}}\frac{\sum_{l=0}^{N-1}\ g^l}{N}. \quad (6.21)$$

where $g=\exp[2\pi i\frac{k}{N}\hat{J}]$. Here we have used the fact that for NS-sector the eigenvalues of $\hat{J}$ take integers. On the other hand for R-sector $\hat{J}$ takes the half-integers, and the phase factor $\exp\left(2\pi i k\alpha\hat{J}\right)=(-1)^{k\alpha}$ appears in the summation $\sum_{\alpha\in\mathbf{Z}}$. From this we can see that the amplitude in the R-sector becomes zero for odd k. Therefore the vacuum amplitude (6.15) under the orbifold limit becomes

$$\begin{aligned}
&\int_0^\infty \frac{dt}{2t}\mathrm{Tr}_{NS-R}\left[\frac{1+(-1)^F}{2}\frac{\sum_{l=0}^{N-1}\ g^l}{N}\ q^{H'_O}\right] \qquad (\text{for even } k),\\
&\int_0^\infty \frac{dt}{2t}\mathrm{Tr}_{NS}\left[\frac{1+(-1)^F}{2}\frac{\sum_{l=0}^{N-1}\ g^l}{N}\ q^{H'_O}\right] \qquad (\text{for odd } k),
\end{aligned} \quad (6.22)$$

where $H'_O = \alpha' p^2+\hat{N}$ is the open string Hamiltonian. This is just the open string amplitude of a fractional D0-brane in type II for even k and that in type 0 for odd k. Note the absence of R-sector for the type 0 reviewed in section 2. Although the R-sector in type 0 can appear from the open strings between an electric D-brane and a magnetic D-brane, these two D-branes in (6.18) are infinitely far away from each other and its spectrum is neglected.

Let us return to the case of the finite radius R. From the above arguments the D-branes are expected to be similar to the fractional D0-branes in the orbifold theories. Indeed it is easy to see that a single D0-brane should be stuck at the point (X'=$\bar{X}'$=0) since there are no zero-modes of $X',\bar{X}'$ for non-integer $\gamma=\frac{\beta\alpha'}{R}n=\frac{k}{N}n$ (see (3.21), (B.1)).

Up to now we have not considered the $U(1)$ phase in the coefficient f_γ (6.16) of the boundary state (6.8). We can consider the freedom of the translation of D0-branes in the compactified Y' direction and this effect can be included in the boundary state (6.8) by replacing f_γ with $f_\gamma \exp{(i\frac{n}{R}y'_0)}$ $(0 \leq y'_0 < 2\pi R)$. To consider its meaning in the orbifold picture, let us again regard the radius of $\mathbf{S^1}$ as $\frac{R}{N}$ and reparameterize y'_0 as $y'_0 = \tilde{y}'_0 + \frac{2\pi R}{N}a$ $(0 \leq \tilde{y}'_0 < \frac{2\pi R}{N},\ a = 0, 1, \cdots, N-1)$. Then its boundary state is written by

$$|B_a, \epsilon\rangle_{NSNS,RR} = \sum_{n\in\mathbf{Z}} e^{i\frac{n}{R}\tilde{y}'_0 + 2\pi i\frac{a}{N}n}|B, \gamma, \epsilon\rangle_{NSNS,RR} \qquad (\epsilon = \pm 1). \tag{6.23}$$

If we take the orbifold limit $R \to \infty$, the boundary state (6.23) becomes (6.18) except that $\delta(y')$ and f'_l are replaced by $\delta(y' - \tilde{y}'_0)$ and $f'_l \exp{(2\pi i\frac{a}{N}l)}$. Then if we remember that the fractional D-branes in an orbifold theory are labeled by irreducible representations of its discrete group [89, 91] (see appendix F), we can see that these boundary states $(a = 0, 1 \cdots N-1)$ represent the N types of fractional D0-branes in the orbifold $\mathbf{C}/\mathbf{Z}_N$. Moreover, we can see that the translation of the D0-brane by $\frac{2\pi R}{N}$ in the Y' direction is equivalent to changing the types of the fractional D0-brane in the orbifold picture since this manipulation is equivalent to $a \to a+1$ in (6.23). This can be equally said that in the orbifold picture with the radius $\frac{R}{N}$ the N types of fractional D0-branes are put at the same point $Y' = \tilde{y}'_0$, and receive the monodromy to change their types into each other if they go around $\mathbf{S^1}$ (see Fig.5).

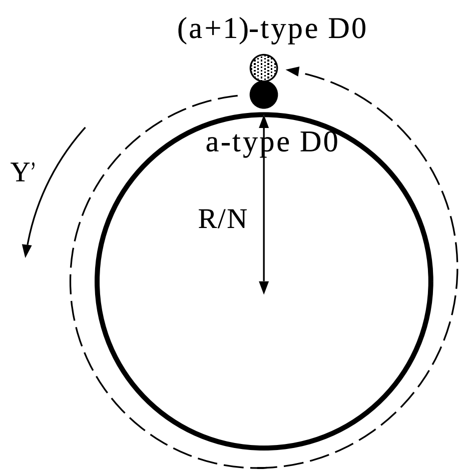

Figure 5: A fractional D0-brane receives the monodromy which changes its type.

Let us see the mass spectrum of open strings between a a-type and a b-type D0-brane (we set $\tilde{y}'_0 = 0$ for simplicity)

$$\alpha' M^2 = \frac{R^2}{N^2\alpha'}\Big(Nw - (k\hat{J} - a + b)\Big)^2 + \hat{N}. \tag{6.24}$$

For odd k the energy in NS-sector due to winding modes in (6.24) can vanish for the appropriate values of w and $\hat{J}$, while for R-sector it cannot because $\hat{J}$ takes the half-integer values and there remains non-zero minimal energy $(\frac{R}{2N\alpha'})^2$. This implies that a something like a system of an electric and a magnetic D0-brane exists in the Melvin background, being the finite distance $\frac{\pi R}{N}$ away from each other as we have already speculated from (6.18) in the limit (see also Fig.4).

Finally we would like to mention another orbifold limit $R \to 0$ with $qR = \frac{k}{N}$ and $\beta = 0$. In this case we again obtain the orbifold $\mathbf{C}/\mathbf{Z}_N$. However, the boundary state (6.7) represents a bulk D0-brane in this orbifold limit since it depends only on β. Thus one may ask whether there exists a D0-brane for finite R which is reduced to a fractional D0-brane in the limit $R \to 0$. Such a D0-brane, if it exists, should have a non-zero winding number $w \neq 0$ and violates the Dirichlet boundary condition (6.2) for Y'. Therefore we must consider the boundary state which breaks the $U(1)$ current algebra symmetry. We will leave this as a future problem.

6.2.2 Bulk D0-branes

The fractional D0-branes are the most fundamental D0-branes in orbifold theories, and other D0-branes can be constructed by the linear combinations of them. In general these D0-branes can not leave from the fixed point $\rho = 0$, while if we collect N different types of fractional D0-branes they can move as a unit such that $\rho \neq 0$. The latter can be regarded as another type of the D0-brane in the orbifold theories which is called a bulk D0-brane, and it is known that the Chan-Paton bundle on this D0-brane obeys the regular representation for the discrete group $\mathbf{Z}_N$ [89].

Then it is natural to ask if such a D0-brane exists for the finite R. The answer is yes if the parameter $\frac{\beta\alpha'}{R}$ is rational, and the bulk D0-brane in the Melvin background is defined as a bound state of N different fractional D-branes whose positions are at N different points $Y' = 0, \frac{2\pi R}{N}, \cdots, \frac{2\pi R(N-1)}{N}$. Its boundary state is defined as follows

$$|B_{bulk}, \epsilon\rangle_{NSNS,RR} \;=\; \sum_{a=0}^{N-1} |B_a, \epsilon\rangle_{NSNS,RR}, \tag{6.25}$$

where $|B_a,\epsilon\rangle_{NSNS,RR}$ is given by (6.23) (here we set $\tilde{y}'_0=0$). The explicit form of the boundary state is obtained in the same way as (6.18) and the result is

$$|B_{bulk},\epsilon\rangle_{NSNS,RR}=\frac{T_0/2}{2(2\pi R/N)}\sum_{\alpha\in\mathbf{Z}}e^{i\frac{Ny'}{R}\alpha}\Big[|B,0,+\rangle_{NSNS,RR}-(-1)^{k\alpha}|B,0,-\rangle_{NSNS,RR}\Big]. \quad (6.26)$$

We can see that its boundary state is exactly the same form as that for a usual D-brane in type II string (for even k) or a system of an electric and a magnetic D-brane away from each other by $\frac{\pi R}{N}$ in type 0 string (for odd k) on $\mathbf{S^1}$ with the finite radius $\frac{R}{N}$ (see Fig.4 again). Note that this boundary state does not have the twisted sectors $l\neq 0$ but picks up only the untwisted sector $l=0$ (or $n=N\alpha$ ($\alpha\in\mathbf{Z}$)). Thus the bosonic zero-modes $x_0,\bar{x}_0$ indeed exist and the D0-brane can move around such that $\rho\neq 0$. To complete this argument one should also examine the first condition in (6.2) carefully since the boundary state with $\rho\neq 0$ has a non-zero orbital angular momentum $\hat{J}_0$ in the zero-mode part. The condition, which is equivalent to $(w-\frac{k}{N}\hat{J}_0)|B\rangle=0$, can be satisfied if $\hat{J}_0$ is a multiple of N. This requires that a bulk D-brane should consist of N fractional D-branes which are located at the N different points $X'=\rho\ e^{i(\theta+2\pi ka/N)}$, $(a=0,1,\cdots,N-1)$ on the plane, where θ is an arbitrary constant (see the most left figure in Fig.7). In this way we have shown that a bulk D0-brane exists if $\frac{\beta\alpha'}{R}\in\mathbf{Q}$ for any values of parameters[38] q and R.

Next we consider its vacuum amplitude. The result is N times as large as (6.12) except that the second term of the open string Hamiltonian (6.13) is modified as follows

$$\left(\frac{R(w-k\hat{J})}{N\alpha'}\right)^2\rightarrow\begin{cases}(\frac{Rw}{N\alpha'})^2 & \text{(for NS-sector and R-sector with even } k\text{)},\\ \left(\frac{R(w+1/2)}{N\alpha'}\right)^2 & \text{(for R-sector with odd } k\text{)}.\end{cases}$$

This is because the angular momentum operator $\hat{J}$ takes integer (half-integer) values in NS-sector (R-sector) and we can erase the effect of $\hat{J}$ by the shift of w. If we take the orbifold limit $R\rightarrow\infty$, the vacuum amplitude is N times as large as (6.22) without the orbifold projection $\sum_{l=0}^{N-1}g^l/N$.

[38] If the value of q is non-zero, the vacuum amplitude with non-zero ρ includes an extra term which depends on q. This was shown in the case $\beta=0$ ($N=1$) in the paper [87]. In the case of rational $\frac{\beta\alpha'}{R}=\frac{k}{N}$ the open string Hamiltonian includes the extra contributions $\Delta H_O=\frac{\rho^2}{\pi^2\alpha'}\sin^2(\pi qRw/N+\pi ka/N)$. This deviation represents the twisted identification due to non-zero q and can be understood from the open string picture.

6.2.3 D1-branes Wrapped on $\mathbf{S}^1$

As we have seen, D0-branes in the Melvin background with the rational parameter are very similar to those in $\mathbf{Z}_N$ orbifolds. On the other hand, a D1-brane wrapped on $\mathbf{S}^1$ has an interesting structure which can not be explained intuitively from the viewpoint of orbifold theories even though a D1-brane is formally transformed into a D0-brane by T-duality.

Here we consider the Melvin background with the rational parameter $qR = \frac{k}{N}$. The boundary state of a D1-brane can be constructed in the same way as that of a D0-brane. Then a single D1-brane is again pinned at the origin $\rho = 0$ (fixed point) in (3.1). However if we consider N D1-branes so that boundary state includes only the restricted winding sectors $w \in N\mathbf{Z}$, this system ('bulk' D1-brane) can move around on the plane $\mathbf{R^2}$ in the same way as in the previous case of D0-branes.

This behavior can be also explained geometrically as follows. Let us set $\beta = 0$ for simplicity and assume that a single D1-brane is placed at $\rho \neq 0$. Though this D1-brane obeys the Dirichlet boundary condition along $(X', \bar{X}') \in \mathbf{R}^2$, in the original coordinate $(X, \bar{X})$ it is rotated by the angle $\frac{2\pi k}{N}$ if it goes around $\mathbf{S}^1$ once, as shown in (3.12). Thus it should wind N times around $\mathbf{S}^1$ in order to move around on the $\mathbf{R^2}$ plane (see Fig.6). It is also useful to note that the geodesic lines along $\mathbf{S}^1$ are given by $\varphi + qy = \text{const.}$ from (3.2) and agree with the world-volume of the D1-brane. This is a good tendency since it will minimize the mass of the D-brane. More detailed treatment of boundary condition for non-zero β will be given in the next section (see (7.17)).

The analysis of the orbifold limit $R \to 0$ with $qR = \frac{k}{N}$ and $\frac{\beta\alpha'}{R} \to 0$ can be performed in the same way as before. A single D1-brane (below we reuse the coordinate $(X', \bar{X}', Y')$) which is stuck at the fixed point is identical to a fractional D0-brane in the limit by the T-duality $R \to 1/R$. On the other hand, the D1-brane winding N times around $\mathbf{S}^1$ corresponds to a bulk D0-brane which is made of N fractional D0-branes in the orbifold $\mathbf{C}/\mathbf{Z}_N$.

Let us consider the vacuum amplitude for a 'bulk' D1-brane. The momentum part of the open string Hamiltonian (the T-dual picture of (6.13)) is given by

$$\left(\frac{n - k\hat{J}}{NR}\right)^2 \to \begin{cases} (\frac{n}{NR})^2 & \text{(for NS-sector and R-sector with even } k), \\ (\frac{n+1/2}{NR})^2 & \text{(for R-sector with odd } k). \end{cases}$$

Note that for odd k case the obtained D1-brane is really a system which consists of an electric ‘bulk’ D1-brane and a magnetic ‘bulk’ D1-brane. They wind N times around $\mathbf{S}^1$ and there is also a $\mathbf{Z}_2$ Wilson line on either D1-brane.

In this way we have found that a D1-brane wrapped on $\mathbf{S}^1$ in the Melvin background has the spiral structure (see Fig.6). This result is consistent with the notion of fractional D-branes by T-duality.

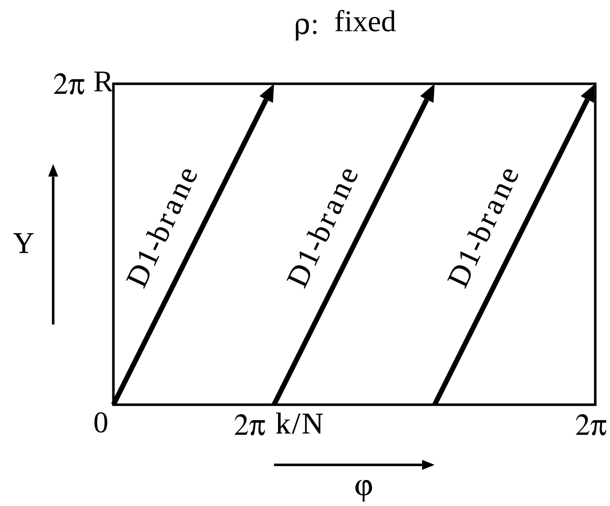

Figure 6: Spiral D1-branes

6.2.4 Other Dp-branes

The analysis of other D-branes can be done in the same way. We can assume that the boundary condition of D-branes with respect to the seven dimensional flat spacetime $\mathbf{R}^{1,6}$ is Dirichlet. Then both a D3-brane and a D2-brane which are extended in the $X', \bar{X}'$ directions are also allowed (see the Table 2). Each of these becomes a ‘fractional’[39] D2-brane on the space $\mathbf{C}/\mathbf{Z}_N$ in the orbifold limit (for D3-branes we take a T-duality transformation in the Y' direction). This kind of D2-brane has the same tension as an ordinary D2-brane almost in the same way as in orbifold theories [95]. If we prepare N pairs of these branes, then we have a ‘bulk D-brane’ as before.

[39] Usually in orbifold theories the Dp-brane ($p \geq 1$) is not called fractional because its tension is not divided by N. However in this thesis we denote ‘fractional’ Dp-branes as the ones which have the open string vacuum amplitude with the orbifold projection in the orbifold limit such as (6.22) in order to distinguish this from the ‘bulk’ Dp-brane without the orbifold projection.

For the rational case there exists another D-brane which obeys the Neumann-Dirichlet or Dirichlet-Neumann boundary conditions for $X', \bar{X}'$ as we mentioned below (6.4). For bulk D-branes we can define such boundary conditions because on such D-branes the parameter γ which appears in the boundary states takes an integer value[40] . Note that these D-branes can not be defined as bound systems of N kinds of fractional D-branes such as (6.25).

In this way we have obtained the several D-branes in Melvin background, thus we summarize these results in the Table 2.

Y'	$X', \bar{X}'$	Existence	Mobility	Tension	Form	Orbifold Limit
D	DD	all	Pinned	T_0	a D0-brane	fractional D0
D	DD	$\frac{\beta\alpha'}{R} = \frac{k}{N}$	Movable	NT_0	N D0-branes	bulk D0
D	DN	$\frac{\beta\alpha'}{R} = \frac{k}{N}$	Movable	NT_1	N D1-branes	bulk D1
D	NN	all	—	T_2	a D2-brane	'fractional' D2
N	DD	all	Pinned	T_1	a D1-brane	fractional D0
N	DD	$qR = \frac{l}{M}$	Movable	MT_1	M D1-branes	bulk D0
N	DN	$qR = \frac{l}{M}$	Movable	MT_2	M D2-branes	bulk D1
N	NN	all	—	T_3	a D3-brane	'fractional' D2

Table 2: D-brane spectrum in the Melvin background. Here we define the boundary condition of open strings in terms of free fields $(X', \bar{X}', Y')$. Each pair of integers (k, N) and (l, M) is coprime. Note that for odd k we regard a Dp in the above as a bound state of an electric Dp and a magnetic Dp.

6.3 Structure of D-branes for the Irrational Parameters

As we have already mentioned, the D-brane spectrum for the irrational parameters is remarkably different from the previous case of rational parameters. A single D-brane can exist only at the origin $\rho = 0$ in (3.1) as before. However if one wants to put a system of D-branes at the other points $\rho \neq 0$, then one must prepare infinitely many D-branes, which can be verified by calculating its tension in the same way as before. This

[40] Remember that bulk D-branes do not have twisted sectors.

fact matches the intuitive picture of the irrational case as a large N limit of the orbifold $\mathbf{C}/\mathbf{Z}_N$ [26]. For example, let us consider a D1-brane wrapped on $\mathbf{S}^1$. In the irrational case the D1-brane along the geodesic line cannot return to the original point even if it goes around $\mathbf{S}^1$ arbitrary times. Thus the system will be a sort of a 'foliation' of the cylinder $\rho =$ const. and may be like a 2-brane.

Even though we cannot answer whether such systems can really exist, we can say that the D-brane spectrum for the irrational case is more restricted than that for the rational case. This will be an example of the geometry in string theory which is quite different from ones in the ordinary mathematics.

6.4 Comments on World-Volume Theory

In the above we have constructed boundary states of various D-branes and have seen their geometric interpretations. Here we would like to briefly discuss the world-volume theory on these D-branes.

First note that the boundary states which have the Neumann (or Dirichlet) boundary condition in the Y' direction depend only on the parameter qR (or $\frac{\beta\alpha'}{R}$) and not on another. This fact is the reason why a D-brane in the background where one of the parameters is rational is treated as if it was a fractional D-brane, even though this background is not always equivalent to any freely acting orbifolds.

This fact will also lead to the intriguing property that the mass spectrum of open string which obeys the Dirichlet boundary condition along $\mathbf{S}^1$ can show the Bose-Fermi degeneracy $Z_O = 0$ for $\beta = 0$ and $q \neq 0$ even if the supersymmetry is completely broken in the closed string sector[41] . This mysterious phenomenon can be a relic of the supersymmetry in type II theory, which is spontaneously broken [58] by the non-zero value of q. In other words the Bose-Fermi degeneracy only on the point-like D-brane along $\mathbf{S}^1$ implies the remaining local supersymmetry, while the degeneracy does not occur on the D-brane wrapped on $\mathbf{S}^1$ because of the lack of global supersymmetry.

Next let us consider the relation to the quiver gauge theory [89]. In particular we take a D0-brane as an example. Let us remember the open string mass spectrum (6.24). If

[41] Note also that this fact may be correct only up to the one-loop order on the string coupling constant. This is because the closed one-loop (open two loop) amplitude between two D-branes contains the two point correlation function on the torus, and in this amplitude both parameters qR and $\frac{\beta\alpha'}{R}$ appear (see [24, 26]).

we take the orbifold limit $R \to \infty$, then it is easy to see that the $\mathbf{Z}_N$ projection of quiver theory

$$k\hat{J} - a + b = Nw \in N\mathbf{Z}, \tag{6.27}$$

appears, which restricts the spectrum to g $(= \exp[2\pi i \frac{k}{N}\hat{J}]) = \exp[2\pi i \frac{a-b}{N}]$. Thus the world-volume theory on D0-branes in the Melvin background can be regarded as a deformation of the quiver theory whose $\mathbf{Z}_N$ projection is softened. One can also see that the opposite limit $R \to 0$ leads to the mass spectrum of an ordinary D0-brane in type II theory. Thus for a finite radius the world-volume theory is regarded as an interpolation between them. In particular the massless fields are the same as those of the quiver gauge theory of the orbifold $\mathbf{C}/\mathbf{Z}_N$.

6.5 D-branes in Supersymmetric Higher Dimensional Models

The closed string backgrounds we have discussed above do not preserve any supersymmetry in general. If we would like to discuss D-brane charges, it will be more desirable to consider those in supersymmetric backgrounds. Therefore in this subsection we investigate D-branes in the higher dimensional generalization of the Melvin model discussed in section 5, which includes the supersymmetric background [26, 27]. This model has a background of the form $\mathbf{M}_5 \times \mathbf{R}^{1,4}$, where $\mathbf{M}_5$ is a $\mathbf{S}^1 \ni Y$ fibration over $\mathbf{R}^2 \times \mathbf{R}^2 \ni (X^1, \bar{X}^1, X^2, \bar{X}^2)$.

We can consider D-branes in the higher dimensional Melvin model in the same way as before. This is because the action is again rewritten by free fields and we can construct the boundary states of D-branes. The boundary condition in the $\mathbf{S}^1$ direction is the same as (6.1) and (6.2) except that the zero-mode part is changed as follows

$$\begin{aligned}
&\text{Neumann:} \quad \left(\frac{n}{R} - q_1\hat{J}_1 - q_2\hat{J}_2\right)|B\rangle = 0, \\
&\text{Dirichlet:} \quad \left(\frac{Rw}{\alpha'} - \beta_1\hat{J}_1 - \beta_2\hat{J}_2\right)|B\rangle = 0.
\end{aligned} \tag{6.28}$$

In the $\mathbf{R}^2 \times \mathbf{R}^2$ direction we have the trivial extension of (6.3) and (6.4). The explicit forms of boundary states are almost the same as in (6.7) and (6.8).

By using such boundary states we can obtain the vacuum amplitude, for example, for a D0-brane as follows

$$
\begin{aligned}
A &= \frac{\pi\alpha' V_0}{8\pi R}\int_0^\infty \frac{dt}{t^2}\times\Bigg[\Big\{\sum_{\gamma_1\in\mathbf{Z},\gamma_2\in\mathbf{Z}}\frac{|f_\gamma|^2}{(2\pi^2\alpha')^4}(\eta(it))^{-12} \\
&\qquad + \sum_{\gamma_1\notin\mathbf{Z},\gamma_2\in\mathbf{Z}}\frac{(-1)^{[\gamma_1]}|f_\gamma|^2}{(2\pi^2\alpha')^3}(\eta(it))^{-9}(\theta_1(\gamma_1|it))^{-1} \\
&\qquad + \sum_{\gamma_1\in\mathbf{Z},\gamma_2\notin\mathbf{Z}}\frac{(-1)^{[\gamma_2]}|f_\gamma|^2}{(2\pi^2\alpha')^3}(\eta(it))^{-9}(\theta_1(\gamma_2|it))^{-1} \\
&\qquad + \sum_{\gamma_1\notin\mathbf{Z},\gamma_2\notin\mathbf{Z}}\frac{(-1)^{[\gamma_1]+[\gamma_2]}|f_\gamma|^2}{(2\pi^2\alpha')^2}(\eta(it))^{-6}(\theta_1(\gamma_1|it))^{-1}(\theta_1(\gamma_2|it))^{-1}\Big\} \\
&\qquad \times\exp\left(-\frac{\pi\alpha'}{2R^2 t}n^2\right)(\theta_1(\frac{\gamma_1+\gamma_2}{2}|it))^2(\theta_1(\frac{\gamma_1-\gamma_2}{2}|it))^2\Bigg], \qquad (6.29)
\end{aligned}
$$

where $\gamma_i = \frac{\beta_i\alpha'}{R}n$, and the summations should be performed about $n\in\mathbf{Z}$ such that the conditions indicated below the symbol $\sum$ are satisfied. Note also that we have transformed the result obtained in the NS-R formulation into that in the Green-Schwarz formulation using the formula (C.4).

Such vacuum amplitude should be consistent with the Cardy's condition and this determines the coefficient f_γ which appears as in (6.8). The result for a general Dp-brane is given by

$$
f_\gamma = \begin{cases}
\frac{1}{2}T_p & (\gamma_1\in\mathbf{Z},\gamma_2\in\mathbf{Z}), \\
\frac{1}{\sqrt{2}}(\frac{|\sin\pi\gamma_1|}{2\pi^2\alpha'})^{\mp\frac{1}{2}}T_p & (\gamma_1\notin\mathbf{Z},\gamma_2\in\mathbf{Z}), \\
\frac{1}{\sqrt{2}}(\frac{|\sin\pi\gamma_2|}{2\pi^2\alpha'})^{\mp\frac{1}{2}}T_p & (\gamma_1\in\mathbf{Z},\gamma_2\notin\mathbf{Z}), \\
(\frac{|\sin\pi\gamma_1|}{2\pi^2\alpha'})^{\mp\frac{1}{2}}(\frac{|\sin\pi\gamma_2|}{2\pi^2\alpha'})^{\mp\frac{1}{2}}T_p & (\gamma_1\notin\mathbf{Z},\gamma_2\notin\mathbf{Z}),
\end{cases} \qquad (6.30)
$$

where the sign factors $\mp$ in the exponent of $|\sin\pi\gamma_i|$ $(i=1,2)$ take $-$ for D-branes with Neumann-Neumann boundary condition, $+$ for D-branes with Dirichlet-Dirichlet boundary conditions for $X^{i'},\bar{X}^{i'}$ directions. In the Neumann-Dirichlet case one can define the boundary state in the same way only if γ is an integer or a half-integer. The above result (6.30) shows that a Dp-brane in the background has the ordinary tension T_p.

The structures of these D-branes are almost similar to those in the original Melvin background discussed in section 3. However, we notice a remarkable property in this case: under the condition that the supersymmetry in the closed string theory is preserved ($\beta_1=\pm\beta_2$ and $q_1=\pm q_2$), the open string one-loop amplitude (6.29) vanishes. Thus the D-branes in this special background are stable BPS objects. In the orbifold limit[42]

[42] Equivalently one can take the limit $R\to 0,\ q_1R=\pm q_2R=\frac{k}{N},\ \beta_1=\beta_2=0$ by T-duality.

$R \to \infty$ with

$$\frac{\beta_1 \alpha'}{R} = \pm \frac{\beta_2 \alpha'}{R} = \frac{k}{N}, \quad q_1 = q_2 = 0, \tag{6.31}$$

these D-branes are identified with BPS fractional D-branes [89, 91] in the supersymmetric ALE orbifolds $\mathbf{C}^2/\mathbf{Z}_N$. More generally, if $\frac{\beta_i \alpha'}{R} = \frac{k_i}{N}$ $(i = 1, 2)$ and if $k_1 + k_2$ is even, the D-brane in the limits becomes a fractional D-brane in type II (not necessarily supersymmetric) orbifolds $\mathbf{C}^2/\mathbf{Z}_N$. On the other hand, if $k_1 + k_2$ is odd, it is divided into an electric and a magnetic fractional D-brane [67] in type 0 orbifolds $\mathbf{C}^2/\mathbf{Z}_N$.

According to the parameter values $q_i R$ and $\frac{\beta_i \alpha'}{R}$ $(i = 1, 2)$ the D-brane spectrum dramatically changes again. For the irrational values only the D-branes which are stuck at the fixed point are allowed, while for the rational values there also exist the D-branes which can move around. The latter are made of N 'fractional' D-branes, which can be seen as a generalization of a bulk D-brane in the orbifold theory $\mathbf{C}^2/\mathbf{Z}_N$ [89]. The detailed arguments of these D-branes are almost the same as before and we omit this.

6.6 Comments on D-brane Charges

It is known that D-brane charges in type IIB and IIA theory are generally classified by K-theory $K^0(X)$ and $K^1(X)$ [98, 99], where X represents the manifold of the spacetime considered. First we assume that the parameters are given by (6.31). Then it is easy to see that the number of massless RR-fields are given by one for finite R and by N for the orbifold limit $R \to \infty$. In the orbifold language each of these N RR-fields comes from one untwisted sector and $N-1$ twisted sectors. Naively one may think that the number of different types of Dp-branes should be given by N in accordance with the N-types of boundary states (6.23). However, one should remember that for finite R the type of a fractional D-brane changes if it goes around the compactified circle. This shows that the types of fractional D-branes (or twisted RR-charges) are not preserved unless we take the limit $R \to \infty$. Thus we conclude that the D-brane charge or equally the corresponding K-group[43] should be given by the rather trivial result $K^0(M_5) = K^1(M_5) = \mathbf{Z}$ for finite R. On the other hand if we take the orbifold limit $R \to \infty$, then the K-group should be given by the equivariant K-theory $K^0_{\mathbf{Z}_N}(\mathbf{R}^4) = \mathbf{Z}^N$ [98, 99, 100, 101], which corresponds to the N-types of fractional D-branes on the orbifold $\mathbf{C}^2/\mathbf{Z}_N$. Thus the spectrum of D-brane

[43] The T-duality of the Melvin background is represented in terms of K-group as $K^0(M_5) = K^1(\hat{M}_5)$, where $\hat{M}_5$ is the T-dual of the five dimensional Melvin background M_5.

charges does not appear to be continuously connected in the orbifold limit even though the masses of twisted RR-fields which couple D-branes continuously change. It would be interesting for the above facts to be understood from the viewpoint of the K-theory with H-flux like the twisted K-theory $K_H(X)$ discussed in [15, 17].

Next we would like to consider the case where $\frac{\beta_i \alpha'}{R}$ is irrational with $q_1 = q_2 = 0$. Then we find that in the limit $R \to \infty$ (a 'large N limit of the orbifold' [26]) there seems to be infinitely many massless RR-fields absorbing the zero-mode along the circle $\mathbf{S}^1$. Thus we may have the 'K-group' $K^0(M_5) = K^1(M_5) = \mathbf{Z}^\infty$ in this limit even though for the irrational parameters the space M_5 is no longer a smooth manifold nor a 'good' singular manifold like ordinary orbifolds.

It may not be so nonsensical even to ask whether one can explain D-brane charges in type 0 theory from the viewpoint of K-theory if one remembers that the type II Melvin background includes type 0 theories in the appropriate limits.

7 Flux Stabilization of D-branes in Melvin Background

As we have discussed in the previous section, the boundary states can be exactly computed in the free field theory. Since we are interested in the string theory in Melvin background, we would like to understand the geometrical aspects of D-branes which are represented by these boundary states, from the viewpoint of Melvin background. For a D1-brane in the free field theory this can be rather easy task as we have seen by intuitive arguments in section 6.2.3. This D-brane is regarded as a spiral D1-brane in the sigma model of Melvin background.

On the other hand, if we consider a bulk D0-brane discussed in section 6.2.2, this geometrical picture is very non-trivial. First let us see how this D0-brane looks like in terms of the original Melvin coordinate $(X, \bar{X}, Y)$ in a heuristic way. Remember that the coordinates $(X', \bar{X}', Y')$ and $(X, \bar{X}, Y)$ are related with each other by performing the T-duality twice as $\varphi'' \to \tilde{\varphi} \to \varphi$ (see section 3.1). After the first T-duality procedure on φ'', the D0-brane located at $Y' =$ fixed, $\rho =$ fixed$(\neq 0)$ and $\varphi'' =$ fixed will be changed into a D1-brane wrapped on the circle $0 \leq \tilde{\varphi} < 2\pi\alpha'$. If we see this in terms of $(Y, \tilde{\varphi})$, the D1-brane can be viewed as a spiral D-string wrapped N times on the circle $0 \leq \tilde{\varphi} < 2\pi\alpha'$ and k times on the circle $0 \leq Y < 2\pi R$ because of the relation $Y = Y' - \beta\tilde{\varphi}$ in (3.6) . If we take the second T-duality on $\tilde{\varphi}$, then we obtain a D2-brane wrapped on the torus $\mathbf{T}^2 \ni (Y, \varphi)$ (see Fig.7). This D2-brane should be regarded as a bound state of k D2-branes and N D0-branes as can be seen from the winding numbers of the spiral D-string.

In this way we have obtained a configuration of a D2-D0 bound state wrapping on a torus which is topologically trivial. Furthermore, in the supersymmetric generalized Melvin background discussed in section 5 we have a BPS D-brane with almost the same property. Therefore one may wonder why such a non-topologically expanded D-brane remains stable. We will answer this question below [102]. We will also briefly summarize the interpretation of all other kinds of D-branes in Melvin background.

7.1 Flux Stabilization of D-branes in Melvin Background

Let us consider what kind of D-branes can exist in the Melvin background (3.1) before we concentrate on the specific kind of D-branes. In the most part of this section we

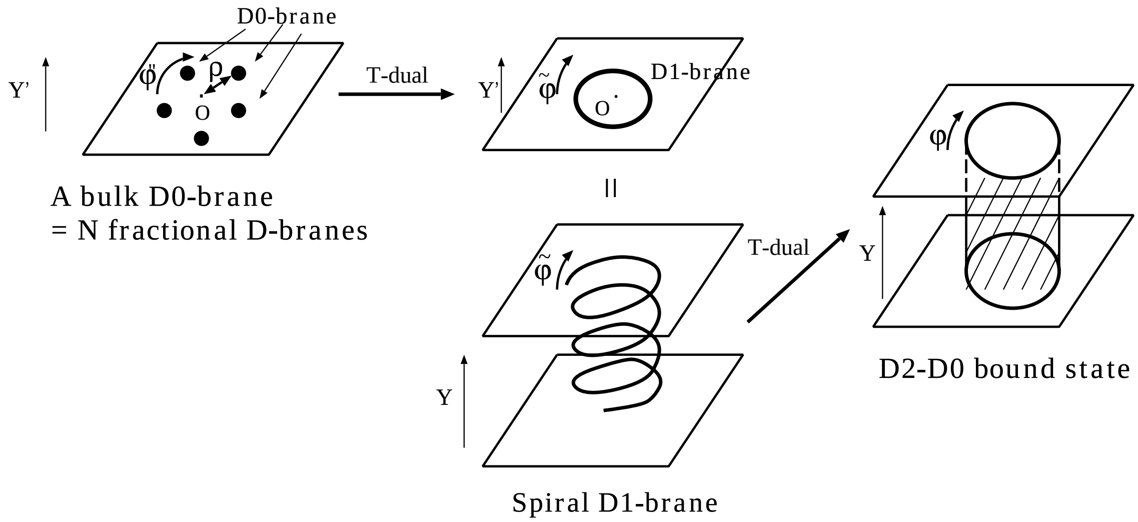

Figure 7: A bulk D0-brane can be regarded as a D2-D0 bound state in the Melvin background (3.1).

assume that the values of the magnetic parameters are rational such that

$$\frac{\beta\alpha'}{R} = \frac{k}{N}, \quad qR = \frac{l}{M}, \tag{7.1}$$

where (k, N) and (l, M) are pairs of coprime integers. In particular we are interested in those D-branes which are localized in the ρ direction. Thus we assume that D-branes obey the Dirichlet boundary condition along ρ. The other D-branes can also be investigated in the same way as the arguments below and we summarize the results in Table 3.

First let us discuss a D0-brane. If we put it in the Melvin background, then we can see that it can exist only at the origin $\rho = 0$ because of the non-trivial ρ dependence of the dilaton ϕ in (3.1). This is easily understood if we remember the value of D0-brane mass $\mathrm{M}_{D0} = e^{-\phi}(\alpha')^{-\frac{1}{2}}$, which takes its minimum value at $\rho = 0$. This somewhat strange fact that a D0-brane cannot exist all points in the spacetime is typical in the presence of H-flux. This is because the H-flux does not generally allow constant dilaton configurations due to one of the equation of motions (see also appendix D)

$$-\frac{1}{2}\nabla^2\phi + \nabla_\mu\phi\nabla^\mu\phi - \frac{1}{24}H_{\mu\nu\rho}H^{\mu\nu\rho} = 0. \tag{7.2}$$

Next we consider a D2-brane whose world-volume is the torus $\rho = \text{const.}, 0 \leq \varphi < 2\pi,\ 0 \leq y < 2\pi R$. However, it is easy to see that the mass of it is proportional to ρ (set F to zero in (7.3)) as in the flat space. Thus it should be squashed and cannot exist.

In this way we have observed that any D2-branes and D0-branes cannot exist at $\rho \neq 0$. Then what happens if we consider D2-D0 bound states? We start with a D2-D0 bound state which is made of p D2-branes and q D0-branes (p and q are coprime) and assume that its world-volume is the same torus. The mass of this object is given by

$$\begin{aligned} \mathrm{M}_{p,q} &= \frac{e^{-\phi}}{4\pi^2(\alpha')^{\frac{3}{2}}}\int dy\ d\varphi \mathrm{Tr}\ \sqrt{\det(G+B+F)} \\ &= \frac{e^{-\phi_0}pR}{(\alpha')^{\frac{3}{2}}}\sqrt{(F\beta-1)^2\rho^2+F^2}, \end{aligned} \tag{7.3}$$

where F is the constant flux which generates q D0-branes and is quantized as usual[44]

$$F = \alpha' R^{-1}\ \frac{q}{p}\ \ . \tag{7.4}$$

In order for this D-brane to exist at $\rho \neq 0$, the ρ dependence of the energy should disappear and we have the constraint $F = 1/\beta$. Since we assume the rational cases (7.1), this can be satisfied if $p = k,\ q = N$. Furthermore the mass of the object for this particular value of flux is given by

$$\mathrm{M}_{k,N} \ = \ N\ \mathrm{T}_0 \quad (\mathrm{T}_0 = e^{-\phi_0}(\alpha')^{-\frac{1}{2}}), \tag{7.5}$$

where T_0 is the mass of a D0-brane at $\rho = 0$. This result (7.5) tells us an interesting fact that the D2-brane part of the mass $\mathrm{M}_{k,N}$ is effectively zero (so called tensionless brane). This is the reason why such an expanded D-brane is allowed which does not wrap any nontrivial cycles. This mechanism may be related to the stabilization of spherical D2-branes in SU(2) WZW-model (NS5-brane background) [10, 11, 12, 16, 103].

It is also interesting to examine the limit $\rho \to 0$. Since the net D2-brane charge is zero for this torus configuration, we have only N D0-branes localized at $\rho = 0$. This is similar to the decay of a $\mathrm{D2}-\overline{\mathrm{D2}}$ system due to tachyon condensation [78]. However, note that our process $\rho \to 0$ is an exactly marginal deformation of boundary conformal field theory as we will see. If we say these results in the opposite way, N D0-branes with the torus D2-brane can leave from the origin $\rho = 0$. This behavior is very similar to that of fractional D-branes in $\mathbf{Z}_N$ orbifolds [89, 91]. Indeed as we has been already seen before by a heuristic argument of T-duality transformations (see Fig.7), we can identify the D2-D0 bound state in the original coordinate system (ρ, φ, Y) with the system of N different fractional D-branes in the other coordinate system (ρ, φ'', Y') (see Fig.8). Let us see this correspondence more explicitly.

[44] Note that this value F is determined by the quantization law $\frac{1}{4\pi^2\alpha'}\int \mathrm{Tr}\ F = q \in \mathbf{Z}$.

Free field	(Y', ρ, φ'')	Melvin (ρ, φ, Y)	Tension
D0*	DDD	D0 fixed at $\rho = 0$	T_0
D0	DDD	D2-D0 bound state $(\rho, Y = \text{fixed})$	NT_0
D1	DND	D3-D1 bound state	NT_1
D2*	DNN	D2 $(Y = \text{fixed})$	T_2
D1*	NDD	D1 fixed at $\rho = 0$	T_1
D1	NDD	Spiral D1 $(\rho,\ \varphi + qY = \text{fixed})$	MT_1
D2	NND	Spiral D2 $(\varphi + qY = \text{fixed})$	MT_2
D3*	NNN	D3	T_3

Table 3: D-brane spectrum in the Melvin background with rational values of parameters $\beta\alpha'/R = k/N, qR = l/M$. We show how the D-branes defined in the free field representation (Y', ρ, φ'') correspond to those in the original Melvin background. In the above table the Neumann and Dirichlet boundary condition are denoted by N and D. The tension T_p represents that of the standard Dp-brane. The D-branes marked by * are regarded as fractional D-branes and even for irrational case they have finite tensions, while others have infinite tensions for irrational case.

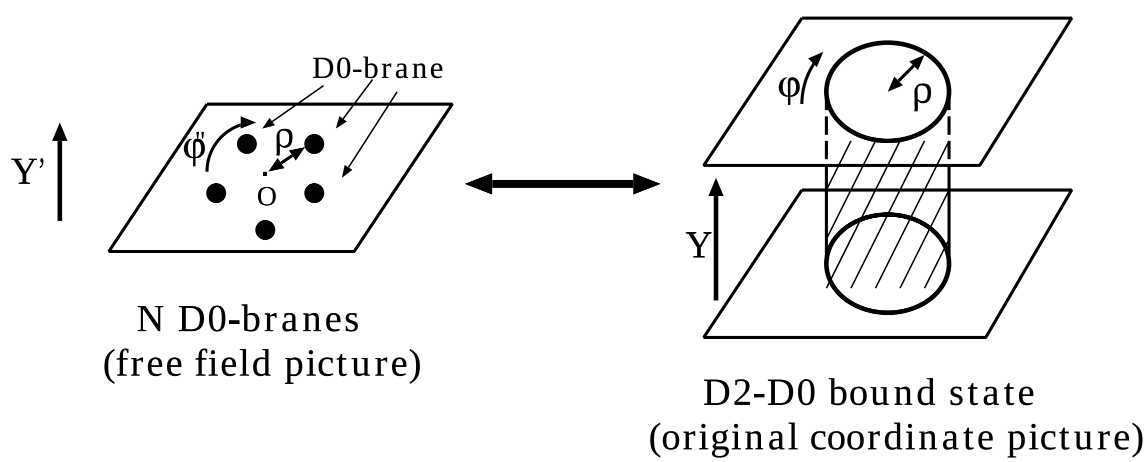

Figure 8: The equivalence between N D0-branes in the free field theory and a D2-D0 bound state in the original sigma model of the Melvin background.

The relation between the free fields and world-sheet fields in the original NSNS Melvin background is given as follows

$$\begin{aligned}
(1+\beta^2\rho^2)\partial\varphi'' &= \partial(\varphi+qY)+\beta\partial Y, \\
(1+\beta^2\rho^2)\bar{\partial}\varphi'' &= \bar{\partial}(\varphi+qY)-\beta\bar{\partial}Y, \\
(1+\beta^2\rho^2)\partial Y' &= \partial Y-\beta\rho^2\partial(\varphi+qY), \\
(1+\beta^2\rho^2)\bar{\partial}Y' &= \bar{\partial}Y+\beta\rho^2\bar{\partial}(\varphi+qY).
\end{aligned} \tag{7.6}$$

These are derived from the relations in the appendix E, which represent the T-dual transformations. The Dirichlet boundary conditions of D0-branes are ρ =const. and

$$\partial_2\varphi''=0, \quad \partial_2 Y'=0, \tag{7.7}$$

at $\sigma_1=0,\pi$ (from now on we will define the boundary conditions in the open string picture). If we rewrite the above equations from the viewpoint of the original Melvin sigma model by using (7.6), they become ρ =const. and

$$\begin{aligned}
&i\partial_2(\varphi+qY)-\beta\partial_1 Y=0, \\
&i\partial_2 Y+\beta\rho^2\partial_1(\varphi+qY)=0.
\end{aligned} \tag{7.8}$$

Thus we have obtained mixed Neumann-Dirichlet boundary conditions. By comparing this result (7.8) with the general formula of the boundary condition

$$G_{\mu\nu}\partial_1 X^\nu+i(B_{\mu\nu}+F_{\mu\nu})\partial_2 X^\nu=0, \tag{7.9}$$

where X^μ denotes the world-sheet field, we obtain the non-trivial value of the flux

$$F=F_{\varphi y}=\frac{1}{\beta}=\alpha' R^{-1}\ \frac{N}{k}. \tag{7.10}$$

This value does match with the previous value (7.4) if we set $p=k,\ q=N$. Therefore we can conclude that N D0-branes at $\rho\neq 0$ (a bulk D0-brane) in the free field picture in (ρ,φ'',Y') is T-dual equivalent to a bound state of N D0-branes and k D2-branes wrapping around the torus (φ,Y) with $\rho\neq 0$ in the original coordinate picture. This shows that expanding the D2-brane corresponds to moving the fractional D0-branes and thus this is an exactly marginal deformation of boundary conformal field theory. It would be also interesting that the quantization of flux F requires the rational values of $\frac{\beta\alpha'}{R}$. In the irrational cases we will have to require $N \to \infty$ in order to move D0-branes, and the bound state becomes infinitely massive.

Let us comment on the world-volume theory on a D2-D0 bound state. Because of the presence of B-flux it becomes a noncommutative theory [104, 2]. Following the prescription [2], it is easy to see that the noncommutativity θ of noncommutative torus $\mathcal{A}_\theta$ is exactly given for any value of ρ as follows

$$\theta = \frac{\beta\alpha'}{R} = \frac{k}{N} \in \mathbf{Q}. \tag{7.11}$$

This shows that it is identified with the fuzzy torus which allows finite dimensional representations.

Such a noncommutativity can be seen more explicitly from the analysis of the non-abelian Dirac-Born-Infield (DBI) action of N D0-branes. The non-abelian DBI action is already proposed in [105, 63] where its form was determined from the T-duality covariance of DBI action. Especially the action of N D0-branes is written by

$$S_{DBI} = -T_0 \int dt \ \mathrm{STr}\left[e^{-\phi}\sqrt{-P\{E_{00} + E_{0i}(Q^{-1i}{}_j - \delta^i{}_j)E^{jk}E_{k0}\}\det(Q^i{}_j)}\right], \tag{7.12}$$

where we defined $E_{\mu\nu}$ and $Q^i{}_j$ as

$$\begin{aligned} E_{\mu\nu} &= G_{\mu\nu} + B_{\mu\nu} \quad (\mu, \nu = 0, \cdots, 9), \\ Q^i{}_j &= \delta^i{}_j + i[\Phi^i, \ \Phi^k]E_{kj} \quad (i, j, k = 1, \cdots, 9), \end{aligned} \tag{7.13}$$

and STr and P denote the symmetrized trace and the pull back onto the D0-brane world-volume, respectively. Here we do not consider the time dependence of fields, thus we can set $\frac{d\Phi^i}{dt}$ to zero. Moreover since the transverse fluctuations for $\mathbf{R^{1,6}}$ directions are irrelevant in this situation, we can also set such fields to zero. After all, the potential part of the action becomes the following form

$$V = T_0 e^{-\phi_0}\mathrm{STr}\left[\sqrt{1 + \rho^2\left\{i[\Phi^\varphi, \ \Phi^Y] + \beta\right\}^2}\right]. \tag{7.14}$$

Here we set Φ^ρ to ρ because we want to consider the expanding of D0-branes into the torus form of D2-branes with constant ρ $(\neq 0)$. From the above equation we can see that the potential is always greater than $NT_0e^{-\phi_0}$, which is the mass of N D0-branes. To realize the lowest limit of the potential the condition $[\Phi^\varphi, \ \Phi^Y] = i\beta$ should be needed. If we normalize coordinates as $\Phi^Y \to \frac{R}{2\pi\alpha'}\Phi^Y$ to make them dimensionless and to set these periodicity to 2π, then we can get the following relation

$$[\Phi^\varphi, \ \Phi^Y] = 2\pi i\theta. \tag{7.15}$$

This is exactly the algebra of noncommutative torus. However this relation holds only for the infinite dimensional representation of Φ^{φ} and Φ^{Y}, while here we consider the finite dimensional $(N \times N)$ representation. We can approximate by using N dimensional fuzzy torus algebra, which is generated by $e^{i\Phi^{\varphi}} = U$ and $e^{i\Phi^{Y}} = V$ with the relation

$$UV = e^{-2\pi i \frac{k}{N}} VU. \tag{7.16}$$

Then the potential is not exactly equal to the mass of N D0-branes[45] . Such a difference comes from the $\frac{1}{N}$ order correction which can be seen in other non-abelian world-volume analysis [63, 106, 103]. Any way the noncommutative algebra of the torus (7.15) is a good approximation for large N and we have seen the explicit noncommutativity on the world-volume of the D2-brane.

Finally let us turn to a D1-brane in the Melvin background [87, 71]. As shown in [71], this is equivalent to the previous D2-D0 bound state by the T-duality along Y', which interchanges q and β [24, 26]. The boundary condition of a D1-brane in the free field theory can be rewritten in terms of the fields (ρ, φ, Y) and the result is

$$\partial_2(\varphi + qY) = \partial_1 Y = 0. \tag{7.17}$$

This exactly represents a D1-brane wrapping the 'geodesic line for D-branes' $\varphi + qY =$ const., which is defined[46] for the 'D-brane metric' $e^{-2\phi}(ds)^2$.

As pointed out in the calculation of the boundary state [71], for rational values of $qR = \frac{l}{M}$ the mass of the D1-brane is finite and it winds M times along Y and l times along φ, while for irrational values it becomes infinite because the D1-brane should wind infinitely many times. These facts are all consistent with the T-dual equivalence to the previous D2-D0 bound state.

7.2 D-branes in Higher Dimensional Melvin Background

In previous section we considered D2-D0 bound states in the Melvin background. However, in the presence of the flux this background breaks the target space supersymmetry completely, and the D2-D0 bound system is not a BPS state. Thus we have neglected

[45] Of course this relation exactly holds for the infinite number of D0-branes, however in that case the potential value becomes infinite and this configuration may be singular. This consideration may be related to the D-brane picture in the Melvin background with irrational magnetic parameters [71].

[46] Note that the usual geodesic line for the metric $(ds)^2$ is given by $\varphi + (q \pm \beta)Y =$ const..

quantum corrections without any explanation. Since we are interested in the stability of D-branes, it will be more desirable to discuss BPS D-branes.

As we have already seen in section 5, we can extend the exactly solvable NSNS Melvin background $\mathbf{R}^{1,6} \times \mathrm{M}_3$ to more higher dimensional ones $\mathbf{R}^{1,8-2n} \times \mathrm{M}_{2n+1}$ [26, 27]. Then we can construct BPS D-branes in these backgrounds. They are stable and the classical analysis will be reliable.

Here we consider the example of D-branes in the supersymmetric higher dimensional model ($n = 2$) analysed in section 6.5. We assume rational parameters $\beta_i = \frac{k_i}{N}$ $(i = 1, 2)$, where N, k_1 and k_2 are coprime integers. For simplicity we set q_i $(i = 1, 2)$ to zero. This system becomes a BPS state [71] if the background keeps supersymmetry ($k_1 = \pm k_2$). The analysis is very similar to that in the previous section, while there appears one nontrivial constraint which we will see.

First we examine the boundary condition of D2-D0 bound states in this supersymmetric model. In this case we can transform the original coordinates $(Y, \rho, \varphi, r, \theta)$ in (5.6) into the free fields $(Y', \rho, \varphi'', r, \theta'')$ by using T-duality and several field redefinitions. To analyze this system quantitatively we transform the following boundary conditions of D0-branes

$$\partial_2\varphi'' = \partial_2\theta'' = \partial_2 Y' = 0, \tag{7.18}$$

into those which are represented by the original coordinate (φ, θ, Y). The relation between these coordinates are obtained in the same way as (7.6). Then the result becomes

$$\begin{aligned}
\beta_2\partial_2\varphi - \beta_1\partial_2\theta &= 0, \\
i\partial_2\varphi - \beta_1\partial_1 Y &= 0, \\
i\partial_2 Y + \beta_1\rho^2\partial_1\varphi + \beta_2 r^2\partial_1\theta &= 0.
\end{aligned} \tag{7.19}$$

In these equations the first equation indicates that the following condition should be satisfied on the world-volume of the D2-brane

$$\beta_2\varphi = \beta_1\theta + \text{const.} \tag{7.20}$$

This shows that the world-volume of the D2-brane is given by the torus $\{(\varphi, \theta, Y) \mid \beta_2\varphi = \beta_1\theta + \text{const.},\ 0 \le \varphi < 2\pi k_1,\ 0 \le \theta < 2\pi k_2\}$[47] . We can choose the coordinate of the world-volume such that $\xi_1 = \varphi,\ \xi_2 = Y \quad (0 \le \xi_1 \le 2\pi k_1,\ 0 \le \xi_2 \le 2\pi R)$.

[47] This periodicity of φ and θ is effective one which is only available for the world-volume of a D2-brane.

Then by comparing the result (7.19) to the general formula of the boundary condition (7.9) with an additional constraint (7.20), we can see the following flux on the world-volume of D2-branes

$$\beta_1 F_{\xi_1 Y} = 1. \tag{7.21}$$

Then this flux is properly quantized on the world-volume of the D2-brane

$$\frac{1}{4\pi^2\alpha'}\int F = \frac{1}{4\pi^2\alpha'}\int d\xi_1 d\xi_2 \ F_{\xi_1\xi_2} = N, \tag{7.22}$$

Thus we can see that this system represents a bound state of N D0-branes and one D2-brane.

Moreover we can see the stabilization mechanism of the D2-brane by the analysis of the Dirac-Born-Infeld theory in the same way as (7.3). The total mass turns out to be equal to that of N D0-branes. Namely, the D2-brane part again becomes tensionless by the total effect of the NSNS B-field and the magnetic flux F (7.21). The analysis from the world-volume theory of D0-branes is the same as before. We can see the structure of the fuzzy torus with the noncommutativity $\theta = \frac{1}{N}$. Note that if we take the limit $\beta_2 \to 0$, we have k_1 D2-branes with N D0 charges wrapping in the φ direction. Then we can identify the world-volume as a fuzzy torus with $\theta = \frac{k_1}{N}$ and this is consistent with the result in the previous subsection.

7.3 U-dualization

Finally we would like to investigate how the above discussed D-branes look like in U-dualized theories. Here we concentrate on the ordinary Melvin background (3.1) for simplicity. The higher dimensional generalization will be straightforward.

First let us begin with type IIB NSNS Melvin background. We set the parameter q to zero and assume the fractional value $\beta\alpha'/R = k/N$. Then there exists a bound state of k D3-branes and N D-strings which is expanded due to the NSNS B-field as we have already seen. The world-volume of the bound state is given by $\mathbf{T^2} \times \mathbf{R}$, where the torus $\mathbf{T^2}$ is included in the Melvin geometry M_3 (3.1) . We denote the coordinate of $\mathbf{S}^1$ in M_3 by x_9 and $\mathbf{R}$ by x_8. If we perform S-duality, then we will obtain a bound state of k D3-branes and N F-strings in IIB RR Melvin background. Notice that it is expanded in the radial direction due to the RR-2form flux. The instanton D1-brane charge is produced on the D3-branes due to the electric flux (F-string).

Further we can take T-duality in x^9 direction and then the background becomes equivalent to F7-brane (2.4). There we obtain a bound state k D2-branes and N F-strings. This system is again stabilized by the RR (one form) flux. Another way to see this stabilization in a non-perturbative way is to lift the system to M-theory. Then we can find that the bound state is identified with a M2-brane which wraps k times around the circle in the angular φ direction and N times in x_{11} direction and which also extends in x_8 direction. In other words, the M2-brane wraps along the geodesic line $\varphi + qx_{11}$ =const. in the 'twisted circle' compatification defined by (2.1). This M2-brane is the $9-11$ flip of the spiral D1-brane discussed in section 6.2.3.

8 Conclusions and Discussions

In this thesis we investigated the string theory in Melvin background. This background depends on the radius R in the compactified direction and the two magnetic parameters q and β. The individual effect of the two parameters is roughly as follows. The non-zero value of q twists the plane $\mathbf{R}^2$ spirally along the circle $\mathbf{S}^1$, while the value of β is proportional to the strength of H-flux. Most of our results depend on whether the magnetic parameters qR and $\frac{\beta\alpha'}{R}$ are rational or irrational.

In the first half we discussed its closed string theory. We showed that the theory includes the orbifolds $\mathbf{C}/\mathbf{Z}_N$ in type 0 and type II theory as particular limits for rational values of magnetic parameters. This result will give a new example of type II/type 0 duality relation. For the irrational parameters we encountered a sort of a 'large N limit of the orbifolds $\mathbf{C}/\mathbf{Z}_N$'.

In this background the supersymmetry is completely broken and generically there exist closed string tachyons. We can generalize the model so that it preserves partial supersymmetry by considering a higher dimensional case. Note that this model gives a new example of supersymmetric background with H-flux. We also found that it includes the ALE orbifolds $\mathbf{C}^2/\mathbf{Z}_N$ in the rational case. This result may be related to the T-duality relation between NS5-branes and ALE space [19]. It would be interesting to investigate this further.

In the latter half of this thesis we studied D-branes in the string theory in Melvin background. The boundary states can be constructed in terms of the free field representations which can be obtained by applying T-duality to the original sigma model of the Melvin backgrounds. Then we can impose the ordinary Neumann or Dirichlet boundary conditions on the free fields, though the resulting D-branes are geometrically non-trivial in the Melvin backgrounds.

For the rational parameters the boundary states have the structure similar to orbifold theories and indeed they become the fractional D-branes[48] in the orbifolds $\mathbf{C}^n/\mathbf{Z}_N$ if we take the large radius (or small radius in T-dual picture) limits. The difference between the fractional D0-branes in the orbifolds and the D0-branes in the Melvin background for finite radius is that in the latter case the N kinds of D0-branes which belong to the

[48] More precisely, we have explicitly constructed D-branes in the Melvin background which are reduced to fractional D0-branes (D1-branes) in one of the orbifold limits $R \to \infty$ ($R \to 0$) in section 6. We left the discussion on the other limit as a future problem.

different irreducible representations of $\mathbf{Z}_N$ can be transformed into each other by the monodromy if they go around the compactified circle. Even though a single ('fractional') D-brane cannot move away from the fixed point (or the origin), a system of N different kinds of 'fractional' D-branes can move around as a unit (a 'bulk' D-brane). In the case of irrational parameters one needs the infinite number of the 'fractional' D-branes in order to move away from the fixed point. Note also that if we take the limits which correspond to type 0 orbifolds, we find a combined system of an electric fractional D-brane and a magnetic fractional D-brane. Thus we can connect between type 0 D-branes and type II D-branes. In the same way we can construct the boundary state of a D1-brane, which is transformed into a D0-brane by T-duality.

The above discussion was based on the boundary state constructed by employing the free field representation. We can also view the D-branes in terms of the original sigma of Melvin background. Then we found that a bulk D0-brane at $\rho \neq 0$ in terms of the free fields $(X', \bar{X}', Y')$ is interpreted as a D2-D0 bound state which wraps the two dimensional torus ($\rho =$ fixed) in the original Melvin background (3.1). In this nontrivial interpretation the presence of H-flux plays the most essential role. The important point is why the bound state is stabilized even though it wraps on a topologically trivial cycle. We found the answer to this question by investigating the world-volume theory of the D-brane: it is stabilized by the presence of flux of the gauge field on the world-volume. This mechanism may be similar to the flux stabilization found in the context of D-branes in group manifolds[10, 11, 12].

Interestingly, this two dimensional torus can be regarded as a non-commutative torus $\mathcal{A}_\theta$ with the noncommutativity $\theta = \frac{\beta\alpha'}{R}$ if we apply the argument in [2]. The observed dependence of D-branes on the parameters can be now reinterpreted as that on the parameter of the non-commutative torus $\mathcal{A}_\theta$. The corresponding operator algebra K-theory $K(\mathcal{A}_\theta) = \mathbf{Z} + \theta\mathbf{Z}$ [107] shows the analogous difference between the rational and the irrational case, where the value $\mathbf{Z} + \theta\mathbf{Z} (\in \mathbf{R})$ means the dimension of the corresponding projective module. In the irrational case there are only infinitely dimensional representations of $\mathcal{A}_\theta$, while in the rational case (we set $\theta = k/N$) there is a N dimensional representation (fuzzy torus). The two $\mathbf{Z}$ charges in the K-group $K(\mathcal{A}_\theta)$ represent the D2-brane and the D0-brane charges as is clearly explained from the viewpoint of the tachyon condensation in the open string theory [108]. It would also be interesting to examine the noncommutative algebra on D3-branes wrapping the whole manifold by using the free field calculations since this background possesses non-zero H-flux (for a general discussion on the relation between H-flux and noncommutative geometry see [3]).

On the other hand, a D1-brane in terms of the free field representation is interpreted as a D1-brane which wraps the spiral geodesic line (see (3.2)) in the original Melvin background.

It will also be an interesting result that the boundary states of D-branes depend only on either of two parameters. This phenomenon leads to the strange Bose-Fermi degeneracy on specific D-branes even in nonsupersymmetric string backgrounds. We pointed out an interpretation of this as a remnant of the spontaneously broken supersymmetry.

Also we would like to note that we can apply the above results on the original Melvin background (3.1) to its higher dimensional generalizations without any serious modifications as we have seen in section 6. For specific values of parameters the model preserves some supersymmetries and the D-branes there become BPS states. Thus it would be interesting to explore their supersymmetric world-volume theories.

Finally we would like to comment that our analysis of D-branes in Melvin background may be useful for the investigation of closed string tachyon condensation considering a D-brane as a probe in the sense of [33]. We have discussed closed tachyon condensation in Melvin background and found a sign that the system will decay into flat space or another Melvin background with the smaller magnetic parameter. The D-brane probe will give us a good test of this speculation.

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A T-duality in Kaluza-Klein Background

The coordinate of world-sheet is $z = \sigma_1 + i\sigma_2$ and we define its partial derivative as $\partial = \frac{1}{2}(\partial_1 - i\partial_2), \bar{\partial} = \frac{1}{2}(\partial_1 + i\partial_2)$. Let us first consider the following bosonic sigma model

$$S = \frac{1}{\pi\alpha'}\int d^2\sigma\left[(G_{ij}(X) + B_{ij}(X))\partial X^i\bar{\partial}X^j + \frac{1}{4}\alpha' R^{(2)}\phi(X)\right]. \tag{A.1}$$

After substituting the Kaluza-Klein background like (3.1) we obtain

$$\begin{aligned} S &= \frac{1}{\pi\alpha'}\int d^2\sigma\Big[(G_{\mu\nu}(X) + B_{\mu\nu}(X))\bar{\partial}X^\nu\partial X^\mu + e^{2\sigma(X)}(\bar{\partial}Y + A_\mu\bar{\partial}X^\mu)(\partial Y + A_\mu\partial X^\mu) \\ &\quad + B_\mu(\bar{\partial}Y\partial X^\mu - \bar{\partial}X^\mu\partial Y) + \frac{1}{4}\alpha' R^{(2)}\phi(X)\Big]. \end{aligned} \tag{A.2}$$

Then we can perform T-duality along Y direction ($\mathbf{S^1}$ with radius R) since the fields $G_{\mu\nu}(X), B_{\mu\nu}(X), \phi(X)$ do not depend on Y [72]. Introducing the auxiliary vector field $V, \bar{V}$ we can rewrite (A.2) as follows (we show only nontrivial parts)

$$\begin{aligned} S &= \frac{1}{\pi\alpha'}\int d^2\sigma\Big[e^{2\sigma(X)}(\bar{V} + A_\mu\bar{\partial}X^\mu)(V + A_\mu\partial X^\mu) + B_\mu(\bar{V}\partial X^\mu - \bar{\partial}X^\mu V) \\ &\quad + (\bar{V}\partial\tilde{Y} - \bar{\partial}\tilde{Y}V)\Big], \end{aligned} \tag{A.3}$$

where the new field $\tilde{Y}$ is compactified on a circle with the radius $\frac{\alpha'}{R}$. If we first integrate $\tilde{Y}$, then we obtain $\bar{\partial}V - \partial\bar{V} = 0$ and the vector field $V, \bar{V}$ can be written as $V = \partial Y,\ \bar{V} = \bar{\partial}Y$. Indeed one can easily see that this field Y has the periodicity $Y \sim Y + 2\pi R$ as expected[49] .

On the other hand, it is straightforward to integrate out V first and one obtains

$$\begin{aligned} S &= \frac{1}{\pi\alpha'}\int d^2\sigma\Big[\cdots + e^{-2\sigma(X)}(\bar{\partial}\tilde{Y} + B_\mu\bar{\partial}X^\mu)(\partial\tilde{Y} + B_\mu\partial X^\mu) + A_\mu(\bar{\partial}\tilde{Y}\partial X^\mu - \bar{\partial}X^\mu\partial\tilde{Y}) \\ &\quad + \frac{1}{4}\alpha' R^{(2)}(\phi(X) - \sigma(X))\Big], \end{aligned} \tag{A.4}$$

where the shift of the dilaton field can be determined from the condition of the conformal invariance (vanishing beta-function) [72]. Thus we have obtained the following T-duality transformation:

$$\sigma'(X) = -\sigma(X),\quad A'_\mu(X) = B_\mu(X),\quad B'_\mu(X) = A_\mu(X),\ \phi'(X) = \phi(X) - \sigma(X). \tag{A.5}$$

[49] To see this, one should note the relation $d\tilde{Y}$ =(closed form)+(non-trivial cohomology). The non-trivial part is discretized due to the periodicity $\tilde{Y} \sim \tilde{Y} + 2\pi\frac{\alpha'}{R}$. Then the term $\sim \int d\tilde{Y}\wedge V$ in (A.3) leads to the weight $\exp(i\frac{\tilde{w}}{R}\int_C V)$, where $\tilde{w}$ is the winding number of the field $\tilde{Y}$ and C is a one-cycle of the world-sheet. Since one should take a summation over w, the integration is quantized as $\int_C V \in 2\pi R\mathbf{Z}$. This shows that the period of Y is $2\pi R$.

Note also that the equation of motion of (A.2) is equivalent to that of (A.4) via the rule

$$
\begin{aligned}
\partial\tilde{Y} &= -B_\mu \partial X^\mu - e^{2\sigma}(\partial Y + A_\mu \partial X^\mu), \\
\bar{\partial}\tilde{Y} &= -B_\mu \bar{\partial} X^\mu + e^{2\sigma}(\bar{\partial} Y + A_\mu \bar{\partial} X^\mu).
\end{aligned} \tag{A.6}
$$

In this thesis we discuss superstring models and therefore we need the supersymmetric generalization of the above arguments. The simplest way to do this is to use the $N = 1$ superspace formalism. One has only to replace ∂ and $\bar{\partial}$ with the super covariant derivatives $D_\theta = \partial_\theta + \theta\partial$ and $D_{\bar{\theta}} = \partial_{\bar{\theta}} + \bar{\theta}\bar{\partial}$ and replace the bosonic vector field V with the fermionic vector super field W. The bosonic scalar field $X(z, \bar{z})$ should also be changed into $\mathbf{X}(z, \bar{z}) = X(z, \bar{z}) + i\theta\psi'_L(z) + i\bar{\theta}\psi'_R(\bar{z}) + \cdots$. The calculations are almost the same as before. For example, (A.6) is replaced with

$$
\begin{aligned}
D_\theta\tilde{Y} &= -B_\mu D_\theta X^\mu - e^{2\sigma}(D_\theta Y + A_\mu D_\theta X^\mu), \\
D_{\bar{\theta}}\tilde{Y} &= -B_\mu D_{\bar{\theta}} X^\mu + e^{2\sigma}(D_{\bar{\theta}} Y + A_\mu D_{\bar{\theta}} X^\mu).
\end{aligned} \tag{A.7}
$$

Let us see a useful example: T-duality of the background (3.1). Utilizing the above arguments we can transform q into β by T-duality. This fact is reconfirmed in the mass spectrum (3.23).

B Mode Expansions

As we have seen in section 3, the theory can be represented by the free bosonic fields $X', \bar{X}',\ Y'$ and their superpartners $\psi'_{L,R}, \bar{\psi}'_{L,R},\ \eta'_{L,R}$ with the boundary conditions (3.18), (3.20) and (3.21). Thus we can find the following mode expansions

$$
\begin{aligned}
&X'(z,\bar{z}) = X'_L(z) + X'_R(\bar{z}) = i\sqrt{\alpha'}\sum_m \frac{1}{m-\hat{\gamma}}\alpha_{m-\hat{\gamma}} z^{-m+\hat{\gamma}} + i\sqrt{\alpha'}\sum_m \frac{1}{m+\hat{\gamma}}\tilde{\alpha}_{n+\hat{\gamma}}\bar{z}^{-m-\hat{\gamma}}, \\
&\bar{X}'(z,\bar{z}) = \bar{X}'_L(z) + \bar{X}'_R(\bar{z}) = i\sqrt{\alpha'}\sum_m \frac{1}{m+\hat{\gamma}}\bar{\alpha}_{m+\hat{\gamma}} z^{-m-\hat{\gamma}} + i\sqrt{\alpha'}\sum_m \frac{1}{m-\hat{\gamma}}\bar{\tilde{\alpha}}_{m-\hat{\gamma}}\bar{z}^{-m+\hat{\gamma}}, \\
&\psi'_L(z) = \sqrt{\alpha'}\sum_r \psi_{r-\hat{\gamma}} z^{-r+\hat{\gamma}-\frac{1}{2}}, \quad \psi'_R(\bar{z}) = \sqrt{\alpha'}\sum_r \tilde{\psi}_{r+\hat{\gamma}}\bar{z}^{-r-\hat{\gamma}-\frac{1}{2}}, \\
&\bar{\psi}'_L(z) = \sqrt{\alpha'}\sum_r \bar{\psi}_{r+\hat{\gamma}} z^{-r-\hat{\gamma}-\frac{1}{2}}, \quad \bar{\psi}'_R(\bar{z}) = \sqrt{\alpha'}\sum_r \bar{\tilde{\psi}}_{r-\hat{\gamma}}\bar{z}^{-r+\hat{\gamma}-\frac{1}{2}}, \\
&Y'(z,\bar{z}) = y' - i\frac{\alpha'}{2}P'_L \ln z - i\frac{\alpha'}{2}P'_R \ln\bar{z} + i\sqrt{\frac{\alpha'}{2}}\sum_{m\neq 0}\frac{1}{m}\beta_m z^{-m} + i\sqrt{\frac{\alpha'}{2}}\sum_{m\neq 0}\frac{1}{m}\tilde{\beta}_m \bar{z}^{-m}, \\
&\eta'_L(z) = \sqrt{\alpha'}\sum_r \eta_r z^{-r-\frac{1}{2}}, \quad \eta'_R(\bar{z}) = \sqrt{\alpha'}\sum_r \tilde{\eta}_r \bar{z}^{-r-\frac{1}{2}},
\end{aligned} \tag{B.1}
$$

and (anti)commutation rules

$$\begin{aligned}
&[\alpha_{m-\hat{\gamma}}, \bar{\alpha}_{n+\hat{\gamma}}] = (m-\hat{\gamma})\delta_{m,-n}, \quad [\tilde{\alpha}_{m+\hat{\gamma}}, \bar{\tilde{\alpha}}_{n-\hat{\gamma}}] = (m+\hat{\gamma})\delta_{m,-n}, \\
&\{\psi_{r-\hat{\gamma}}, \bar{\psi}_{s+\hat{\gamma}}\} = \delta_{r,-s}, \quad \{\tilde{\psi}_{r+\hat{\gamma}}, \bar{\tilde{\psi}}_{s-\hat{\gamma}}\} = \delta_{r,-s}, \\
&[\beta_m, \beta_n] = m\delta_{m,-n}, \quad [\tilde{\beta}_m, \tilde{\beta}_n] = m\delta_{m,-n}, \\
&\{\eta_r, \eta_s\} = \delta_{r,-s}, \quad \{\tilde{\eta}_r, \tilde{\eta}_s\} = \delta_{r,-s}.
\end{aligned} \tag{B.2}$$

The $\mathcal{N}=1$ super-Virasoro generators $L_m, \tilde{L}_m, G_r, \tilde{G}_r$ are obtained in the usual way because the action is written by free fields

$$\begin{aligned}
L_m &= :\frac{1}{2}\sum_k \beta_{m-k}\beta_k + \sum_k \bar{\alpha}_{m-k+\hat{\gamma}}\alpha_{k-\hat{\gamma}} \\
&\quad + \frac{1}{2}\sum_r (r-\frac{m}{2})\eta_{m-r}\eta_r + \sum_r (r-\hat{\gamma}-\frac{m}{2})\bar{\psi}_{m-r+\hat{\gamma}}\psi_{r-\hat{\gamma}} : \ , \\
G_r &= \sum_k (\beta_k\eta_{r-k} + \alpha_{k-\hat{\gamma}}\bar{\psi}_{r-k+\hat{\gamma}} + \bar{\alpha}_{k+\hat{\gamma}}\psi_{r-k+\hat{\gamma}}),
\end{aligned} \tag{B.3}$$

where :∼: is the normal ordering. For example for $0 < \hat{\gamma} < \frac{1}{2}$, L_0 for NS-sector becomes[50]

$$\begin{aligned}
L_0 &= \frac{\alpha'}{4}P_L'^2 + \sum_{k=1}^{\infty}\beta_{-k}\beta_k + \sum_{k=1}^{\infty}\bar{\alpha}_{-k+\hat{\gamma}}\alpha_{k-\hat{\gamma}} + \sum_{k=0}^{\infty}\alpha_{-k-\hat{\gamma}}\bar{\alpha}_{k+\hat{\gamma}} \\
&\quad + \sum_{r=1/2}^{\infty} r\eta_{-r}\eta_r + \sum_{r=1/2}^{\infty}(r-\hat{\gamma})\bar{\psi}_{-r+\hat{\gamma}}\psi_{r-\hat{\gamma}} + \sum_{r=1/2}^{\infty}(r+\hat{\gamma})\psi_{-r-\hat{\gamma}}\bar{\psi}_{r+\hat{\gamma}} + \frac{\hat{\gamma}}{2},
\end{aligned} \tag{B.4}$$

where P_L' is given by (3.20). Note that here we abbreviate the contributions from other directions than Y', X' and $\bar{X}'$. L_0 for R-sector is almost the same except that r runs integer values and that the zero point energy shifts. The antiholomorphic components $\tilde{L}_m, \tilde{G}_r$ is written in the same form.

For the later use, we define the operators $\hat{N}_L, \hat{N}_R$ (we show the result only in NSNS sector)

$$\begin{aligned}
\hat{N}_L &= \sum_{k=1}^{\infty}\frac{k}{k-\hat{\gamma}}\bar{\alpha}_{-k+\hat{\gamma}}\alpha_{k-\hat{\gamma}} + \sum_{k=1}^{\infty}\frac{k}{k+\hat{\gamma}}\alpha_{-k-\hat{\gamma}}\bar{\alpha}_{k+\hat{\gamma}} + \sum_{k=1}^{\infty}\beta_{-k}\beta_k \\
&\quad + \sum_{r=\frac{1}{2}}^{\infty} r\bar{\psi}_{-r+\hat{\gamma}}\psi_{r-\hat{\gamma}} + \sum_{r=\frac{1}{2}}^{\infty} r\psi_{-r-\hat{\gamma}}\bar{\psi}_{r+\hat{\gamma}} - \frac{1}{2}, \\
\hat{N}_R &= \sum_{k=1}^{\infty}\frac{k}{k-\hat{\gamma}}\tilde{\alpha}_{-k+\hat{\gamma}}\bar{\tilde{\alpha}}_{k-\hat{\gamma}} + \sum_{k=1}^{\infty}\frac{k}{k+\hat{\gamma}}\bar{\tilde{\alpha}}_{-k-\hat{\gamma}}\alpha_{k+\hat{\gamma}} + \sum_{k=1}^{\infty}\tilde{\beta}_{-k}\tilde{\beta}_k
\end{aligned}$$

[50] Note that if $\hat{\gamma}$ is out of this region, the above L_0 is not positive definite, therefore we have to redefine the ground state correctly.

$$+\sum_{r=\frac{1}{2}}^{\infty} r\tilde{\psi}_{-r+\hat{\gamma}}\bar{\tilde{\psi}}_{r-\hat{\gamma}} + \sum_{r=\frac{1}{2}}^{\infty} r\bar{\tilde{\psi}}_{-r-\hat{\gamma}}\tilde{\psi}_{r+\hat{\gamma}} - \frac{1}{2}. \tag{B.5}$$

The last constant term $-1/2$ is replaced by 0 for RR-sector.

The angular momentum operators $\hat{J}_L, \hat{J}_R$, are also defined to be (we show the result for NSNS-sector with $0 < \hat{\gamma} < 1/2$)

$$\begin{aligned}
\hat{J}_L &= -\sum_{k=1}^{\infty} \frac{1}{k-\hat{\gamma}}\bar{\alpha}_{-k+\hat{\gamma}}\alpha_{k-\hat{\gamma}} + \sum_{k=0}^{\infty} \frac{1}{k+\hat{\gamma}}\alpha_{-k-\hat{\gamma}}\bar{\alpha}_{k+\hat{\gamma}} - \sum_{r=\frac{1}{2}}^{\infty} \bar{\psi}_{-r+\hat{\gamma}}\psi_{r-\hat{\gamma}} + \sum_{r=\frac{1}{2}}^{\infty} \psi_{-r-\hat{\gamma}}\bar{\psi}_{r+\hat{\gamma}} + \frac{1}{2}, \\
\hat{J}_R &= \sum_{k=1}^{\infty} \frac{1}{k-\hat{\gamma}}\tilde{\alpha}_{-k+\hat{\gamma}}\bar{\tilde{\alpha}}_{k-\hat{\gamma}} - \sum_{k=0}^{\infty} \frac{1}{k+\hat{\gamma}}\bar{\tilde{\alpha}}_{-k-\hat{\gamma}}\alpha_{k+\hat{\gamma}} + \sum_{r=\frac{1}{2}}^{\infty} \tilde{\psi}_{-r+\hat{\gamma}}\bar{\tilde{\psi}}_{r-\hat{\gamma}} - \sum_{r=\frac{1}{2}}^{\infty} \bar{\tilde{\psi}}_{-r-\hat{\gamma}}\tilde{\psi}_{r+\hat{\gamma}} - \frac{1}{2}.
\end{aligned} \tag{B.6}$$

The last constant term for RR-sector is the same as the above.

C Identities of Theta-Functions

Here we summarize the formulae of theta-functions. First define the following theta-functions

$$\begin{aligned}
\eta(\tau) &= q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n), \\
\theta_1(\nu|\tau) &= 2q^{\frac{1}{8}}\sin(\pi\nu)\prod_{n=1}^{\infty}(1-q^n)(1-e^{2i\pi\nu}q^n)(1-e^{-2i\pi\nu}q^n), \\
\theta_2(\nu|\tau) &= 2q^{\frac{1}{8}}\cos(\pi\nu)\prod_{n=1}^{\infty}(1-q^n)(1+e^{2i\pi\nu}q^n)(1+e^{-2i\pi\nu}q^n), \\
\theta_3(\nu|\tau) &= \prod_{n=1}^{\infty}(1-q^n)(1+e^{2i\pi\nu}q^{n-\frac{1}{2}})(1+e^{-2i\pi\nu}q^{n-\frac{1}{2}}), \\
\theta_4(\nu|\tau) &= \prod_{n=1}^{\infty}(1-q^n)(1-e^{2i\pi\nu}q^{n-\frac{1}{2}})(1-e^{-2i\pi\nu}q^{n-\frac{1}{2}}),
\end{aligned} \tag{C.1}$$

where we have defined $q = e^{2i\pi\tau}$.

Next we show the modular properties as follows

$$\begin{aligned}
\eta(\tau) &= (-i\tau)^{-\frac{1}{2}}\eta(-\frac{1}{\tau}), \quad \theta_1(\nu|\tau) = i(-i\tau)^{-\frac{1}{2}}e^{-\pi i\frac{\nu^2}{\tau}}\theta_1(\nu/\tau, -\frac{1}{\tau}), \\
\theta_2(\nu|\tau) &= (-i\tau)^{-\frac{1}{2}}e^{-\pi i\frac{\nu^2}{\tau}}\theta_4(\nu/\tau| -\frac{1}{\tau}), \quad \theta_3(\nu|\tau) = (-i\tau)^{-\frac{1}{2}}e^{-\pi i\frac{\nu^2}{\tau}}\theta_3(\nu/\tau| -\frac{1}{\tau}), \\
\theta_4(\nu|\tau) &= (-i\tau)^{-\frac{1}{2}}e^{-\pi i\frac{\nu^2}{\tau}}\theta_2(\nu/\tau| -\frac{1}{\tau}).
\end{aligned} \tag{C.2}$$

Their quasi periodicity is also given by

$$\begin{aligned}
\theta_1(\nu+\tau|\tau) &= -e^{-2\pi i\nu-\pi i\tau}\theta_1(\nu|\tau), \\
\theta_2(\nu+\tau|\tau) &= e^{-2\pi i\nu-\pi i\tau}\theta_2(\nu|\tau), \\
\theta_3(\nu+\tau|\tau) &= e^{-2\pi i\nu-\pi i\tau}\theta_3(\nu|\tau), \\
\theta_4(\nu+\tau|\tau) &= -e^{-2\pi i\nu-\pi i\tau}\theta_4(\nu|\tau).
\end{aligned} \tag{C.3}$$

It is also useful to note the Jacobi's identity

$$\prod_{a=1}^{4}\theta_3(\nu_a|\tau) - \prod_{a=1}^{4}\theta_2(\nu_a|\tau) - \prod_{a=1}^{4}\theta_4(\nu_a|\tau) + \prod_{a=1}^{4}\theta_1(\nu_a|\tau) = 2\prod_{a=1}^{4}\theta_1(\nu'_a|\tau), \tag{C.4}$$

where we have defined

$$\begin{aligned}
&2\nu'_1 = \nu_1+\nu_2+\nu_3+\nu_4, \quad 2\nu'_2 = \nu_1+\nu_2-\nu_3-\nu_4, \\
&2\nu'_3 = \nu_1-\nu_2+\nu_3-\nu_4, \quad 2\nu'_4 = \nu_1-\nu_2-\nu_3+\nu_4.
\end{aligned} \tag{C.5}$$

D Explicit Metric of the Higher Dimensional Model

Let us define the polar coordinates as $X^1 = \rho e^{i\varphi}, \quad X^2 = re^{i\theta}$. The metric (in string frame) of the higher dimensional Melvin background described by the action (5.1) is given by

$$\begin{aligned}
(ds)^2 &= d\rho^2 + dr^2 + \frac{1+\beta_2^2 r^2}{F(r,\rho)}\rho^2 d\varphi^2 - 2\frac{\beta_1\beta_2 r^2\rho^2}{F(r,\rho)}d\varphi d\theta + \frac{1+\beta_1^2\rho^2}{F(r,\rho)}r^2 d\theta^2 \\
&\quad + \frac{G(r,\rho)}{F(r,\rho)}dy^2 + 2\frac{q_1\rho^2(1+\beta_2^2 r^2) - \beta_1\beta_2 q_2\rho^2 r^2}{F(r,\rho)}d\varphi dy \\
&\quad + 2\frac{q_2 r^2(1+\beta_1^2\rho^2) - \beta_1\beta_2 q_1\rho^2 r^2}{F(r,\rho)}d\theta dy, \\
&= \frac{G(r,\rho)}{F(r,\rho)}(dy + A_\varphi d\varphi + A_\theta d\theta)^2 - \frac{2r^2\rho^2}{G(r,\rho)F(r,\rho)}\left(\beta_1\beta_2 + q_1 q_2 F(r,\rho)\right)d\varphi d\theta \\
&\quad + d\rho^2 + \frac{\rho^2}{G(r,\rho)F(r,\rho)}\left(1 + \beta_2^2 r^2 + q_2^2 r^2 + q_2^2\beta_2^2 r^4 + \beta_1^2 q_2^2\rho^2 r^2\right)d\varphi^2 \\
&\quad + dr^2 + \frac{r^2}{G(r,\rho)F(r,\rho)}\left(1 + \beta_1^2\rho^2 + q_1^2\rho^2 + q_1^2\beta_1^2\rho^4 + \beta_2^2 q_1^2\rho^2 r^2\right)d\theta^2,
\end{aligned} \tag{D.1}$$

where we have defined

$$\begin{aligned}
F(r,\rho) &\equiv 1 + \beta_1^2\rho^2 + \beta_2^2 r^2, \quad G(r,\rho) \equiv 1 + (\beta_1 q_2 - \beta_2 q_1)^2\rho^2 r^2 + q_1^2\rho^2 + q_2^2 r^2, \\
A_\varphi &= \frac{q_1\rho^2 + \beta_2(q_1\beta_2 - q_2\beta_1)\rho^2 r^2}{G(r,\rho)}, \quad A_\theta = \frac{q_2 r^2 + \beta_1(q_2\beta_1 - q_1\beta_2)\rho^2 r^2}{G(r,\rho)}.
\end{aligned} \tag{D.2}$$

The B-field and the dilaton is

$$B_\varphi = -\frac{\beta_1\rho^2}{F(r,\rho)}, \quad B_\theta = -\frac{\beta_2 r^2}{F(r,\rho)}, \quad e^{2(\phi-\phi_0)} = \frac{1}{F(r,\rho)}. \tag{D.3}$$

One can check that the curvature of the above metric is not singular.

It is not so difficult to show that these satisfy the equations of motion in supergravity

$$\begin{aligned}
&R_{\mu\nu} + \nabla_\mu\nabla_\nu\phi - \frac{1}{4}H_{\mu\alpha\beta}H_\nu^{\alpha\beta} = 0,\\
&-\frac{1}{2}\nabla^2\phi + \nabla_\mu\phi\nabla^\mu\phi - \frac{1}{24}H_{\mu\nu\rho}H^{\mu\nu\rho} = 0,\\
&\nabla_\rho H^\rho_{\mu\nu} - 2(\nabla_\rho\phi)H^\rho_{\mu\nu} = 0.
\end{aligned} \tag{D.4}$$

It is also worth noting that we can relate $B_{\varphi,\theta}$ to $A_{\varphi,\theta}$ by T-duality: $R \leftrightarrow \frac{\alpha'}{R}$ and $q_i \leftrightarrow \beta_i$ if $(\beta_1 q_2 - \beta_2 q_1) = 0$. This includes the supersymmetric cases $q_1 = \pm q_2, \quad \beta_1 = \pm\beta_2$.

E Summary of World-Sheet Fields

Here we summarize various world-sheet fields which appear in this thesis and their relations.

Melvin sigma model

The original Melvin sigma model (3.4) : (ρ, φ, Y) or $(X = \rho e^{i\varphi}, \bar{X}, Y)$.

The second sigma model (3.5) : $(\rho, \tilde{\varphi}, Y)$.

The free field (third) sigma model (3.7) : $(\rho, \varphi', Y' = Y + \beta\tilde{\varphi})$
or $(X' = \rho e^{i\varphi''} = \rho e^{i(\varphi' + qY')}, \bar{X}', Y')$.

The relations (A.6) of derivatives of world-sheet fields when we perform the T-dual transformations are given by

$$\begin{aligned}
&\partial\tilde{\varphi} = -\frac{\rho^2}{1+\beta^2\rho^2}\partial(\varphi + qY) - \frac{\beta\rho^2}{1+\beta^2\rho^2}\partial Y, \quad \partial\varphi = -\frac{1+\beta^2\rho^2}{\rho^2}(\partial\tilde{\varphi} + \frac{\beta\rho^2}{1+\beta^2\rho^2}\partial Y) - q\partial Y,\\
&\bar{\partial}\tilde{\varphi} = \frac{\rho^2}{1+\beta^2\rho^2}\bar{\partial}(\varphi + qY) - \frac{\beta\rho^2}{1+\beta^2\rho^2}\bar{\partial} Y, \quad \bar{\partial}\varphi = \frac{1+\beta^2\rho^2}{\rho^2}(\bar{\partial}\tilde{\varphi} + \frac{\beta\rho^2}{1+\beta^2\rho^2}\bar{\partial} Y) - q\bar{\partial} Y,\\
&\partial\varphi' = -\frac{1}{\rho^2}\partial\tilde{\varphi} - q\partial Y', \quad \partial\tilde{\varphi} = -\rho^2\partial(\varphi' + qY') = -\rho^2\partial\varphi'',\\
&\bar{\partial}\varphi' = \frac{1}{\rho^2}\bar{\partial}\tilde{\varphi} - q\bar{\partial} Y', \quad \bar{\partial}\tilde{\varphi} = \rho^2\bar{\partial}(\varphi' + qY') = \rho^2\bar{\partial}\varphi''.
\end{aligned} \tag{E.1}$$

Higher dimensional Melvin sigma model

The original Melvin sigma model (5.6) : $(\rho, \varphi, r, \theta)$ or $(X^1 = \rho e^{i\varphi}, \bar{X}^1, X^2 = re^{i\theta}, \bar{X}^2, Y)$.
We define $\check{\varphi} = \varphi + q_1 Y,\ \check{\theta} = \theta + q_2 Y$.

The free field (third) sigma model (5.7) : $(\rho, \varphi'' = \check{\varphi} - \beta_1 \tilde{Y}', r, \theta'' = \check{\theta} - \beta_2 \tilde{Y}')$
or $(X^{1'} = \rho e^{i\varphi''}, \bar{X}^1, X^{2'} = re^{i\theta''}, \bar{X}^2, Y')$.

The relations of derivatives are given by

$$\begin{aligned}
-\partial Y' = \partial \tilde{Y}' &= -\frac{1}{1+\beta_1^2\rho^2+\beta_2 r^2}\partial Y + \frac{\beta_1\rho^2}{1+\beta_1^2\rho^2+\beta_2 r^2}\partial\check{\varphi} + \frac{\beta_2 r^2}{1+\beta_1^2\rho^2+\beta_2 r^2}\partial\check{\theta}, \\
\bar{\partial} Y' = \bar{\partial}\tilde{Y}' &= +\frac{1}{1+\beta_1^2\rho^2+\beta_2 r^2}\bar{\partial} Y + \frac{\beta_1\rho^2}{1+\beta_1^2\rho^2+\beta_2 r^2}\bar{\partial}\check{\varphi} + \frac{\beta_2 r^2}{1+\beta_1^2\rho^2+\beta_2 r^2}\bar{\partial}\check{\theta}.
\end{aligned} \tag{E.2}$$

F Review of Fractional D-branes

Here we would like to briefly review basic facts on fractional D-branes in orbifold theories and their boundary states. These are useful for the investigation of D-branes in Melvin background as we see in section 6.

Let us consider noncompact abelian orbifolds $\mathbf{C}^n/\mathbf{Z}_N$. The generator g of discrete group $\mathbf{Z}_N$ acts on the coordinate $(X_1, X_2, \cdots, X_n)$ of $\mathbf{C}^n$ as follows

$$X_i \to e^{2\pi i \frac{k_i}{N}} X_i. \tag{F.1}$$

If the group $\mathbf{Z}_N$ is the subgroup of $SU(n)$, then we have partial supersymmetries preserved. The results below can be generalized to non-abelian orbifolds $\mathbf{C}^n/\mathbf{\Gamma}$ ($\mathbf{\Gamma}$ is a discrete group) without any difficulty (see e.g. [90, 92, 93, 94]).

The closed string theory in the orbifold consists of an untwisted sector and $N-1$ twisted sectors [6]. The g^m twisted sector $(m = 1, 2, \cdots, N-1)$ is defined by the following boundary condition (we show only the result for bosonic fields)

$$X_i(\tau, \sigma + 2\pi) = e^{2\pi i \frac{mk_i}{N}} X_i(\tau, \sigma). \tag{F.2}$$

The open string spectrum is computed by defining the $g \in \mathbf{\Gamma}$ action on Chan-Paton factors. Following the prescription in [89] the action is regarded as representations of $\mathbf{Z}_N$. There are N different types of irreducible representations, which are all one dimensional and we denote these by ρ_a $(0 \le a \le N-1)$. Any representation ρ can be decomposed into irreducible representations $\rho = \oplus_{a=0}^{N-1} n_a \rho_a$ $(n_a \in \mathbf{Z})$. In other words we can classify D-branes in the orbifold theory in terms of ρ_a. Thus there are N different types of fundamental D-branes corresponding to irreducible representations. These are called fractional D-branes [91] and are labeled by the integers a. The other systems of D-branes in this theory correspond to reducible representations. Then we can see that the Chan-Paton factor Λ of open string between a system of D-branes (corresponding to ρ_1) and another one (ρ_2) is acted by $g \in \mathbf{Z}_N$ as follows

$$\Lambda' = \gamma_{\rho_1}(g) \cdot \Lambda \cdot \gamma_{\rho_2}(g)^{-1}, \tag{F.3}$$

where the matrix γ_ρ denotes the representation matrix in the ρ representation. The explicit form of γ_ρ for irreducible representations is given by $\gamma_{\rho_a}(g^m) = e^{2\pi i \frac{am}{N}}$. The open string is projected by the $\mathbf{Z}_N$ group in this way. The resulting gauge theories on D-branes is known to be represented by the quiver diagram conveniently [89].

Next let us construct the boundary states of fractional D-branes. They should be constructed so that they reproduces the correct cylinder amplitudes equivalent to the open string spectrum given above (Cardy's condition). The boundary states include all of twisted sectors and each of them should satisfy the boundary conditions just like (6.3) or (6.4). To be exact in our case we have n sets of the conditions of (6.3) or (6.4) and we should regard each γ as $k_i m/N$. These boundary conditions can be easily solved like (6.7) and we can obtain the complete boundary states as the appropriate linear combinations of them satisfying the Cardy's condition [67, 92, 93, 94] as follows

$$|B_a\rangle_{NSNS,RR} = \sum_{m=0}^{N-1} e^{2\pi i \frac{am}{N}} |B, g^m\rangle_{NSNS,RR}. \tag{F.4}$$

The boundary state $|B, g^m\rangle_{NSNS,RR}$ denotes the boundary state for g^m twisted sector in NSNS or RR sector.

We can also show that the fractional D-branes possess the fractional $(1/N)$ tension and RR-charges [91]. Another important property of fractional D-branes is that they cannot move away from the origin $X_i = 0$. This is explained from the observation that the boundary states (F.4) include the twisted sectors where there are no zero-modes. Then one may ask whether one can construct movable D-branes. We can obtain such a D-brane as the regular representation $\rho_{reg} = \oplus_{a=0}^{N-1} \rho_a$ (N dimensional). The boundary state is given by

$$
\begin{aligned}
|B_{bulk}\rangle_{NSNS,RR} &= \sum_{a=0}^{N-1} |B_a\rangle_{NSNS,RR} \\
&= N|B_0\rangle_{NSNS,RR}. \qquad \text{(F.5)}
\end{aligned}
$$

We can see that the D-brane have an ordinary tension and RR-charge. The absence of twisted sectors show that it can move from the origin. Thus we have a movable D-brane and call it a bulk D-brane.

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