Theory of Cosmological Perturbations

Part Ix

— quantum field in curved spacetime—

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§1. Formulation

- Action for real scalar field in curved spacetime

$$S = \int d^4x\sqrt{-g}\left(-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V - \frac{1}{2}\xi R\phi^2\right) \;;\quad \xi = 0 \;:\; \text{minimal coupling}$$

We assume $\xi = 0$ for simplicity. In the case of $\xi \neq 0$, apply a conformal transformation to bring the system into the frame where $\xi = 0$.
We also assume $\phi = 0$ is a miminum and consider free field theory around it. Hence we set $V = \frac{1}{2}m^2\phi^2$.

- $(3+1)$-decomposition (Hamiltonian formulation)

$$ds^2 = -N^2dt^2 + \gamma_{ij}(dx^i + N^i d\eta)(dx^j + N^j d\eta) \quad \Rightarrow$$

$$g_{\mu\nu} = \begin{pmatrix} -N^2 + N^kN_k\,, & N_i \\ N_j\,, & \gamma_{ij} \end{pmatrix}\,, \quad g^{\mu\nu} = \begin{pmatrix} -\dfrac{1}{N^2}\,, & \dfrac{N^i}{N} \\ \dfrac{N^j}{N^2}\,, & \gamma^{ij} - \dfrac{N^iN^j}{N^2} \end{pmatrix}$$

$$S = \int d^3x\, dt \left[\frac{\sqrt{\gamma}}{2N} \left(\dot{\phi} - N^i \phi_{,i} \right)^2 - \frac{N\sqrt{\gamma}}{2} \left(\gamma^{ij} \phi_{,i} \phi_{,j} + m^2 \phi^2 \right) \right]$$

$$\pi \equiv \frac{\delta S}{\delta \dot{\phi}} = \frac{\sqrt{\gamma}}{N} \left(\dot{\phi} - N^i \phi_{,i} \right) = \sqrt{\gamma} N^\mu \phi_{,\mu} \, ; \quad N^\mu = \left(\frac{1}{N} \, , - \frac{N^i}{N} \right)$$

Hamiltonian:

$$H = \int d^3x \mathcal{H} \, ; \quad \mathcal{H} = \pi \dot{\phi} - L = \frac{N}{2} \left[\frac{\pi^2}{\sqrt{\gamma}} + \sqrt{\gamma} \left(\gamma^{ij} \phi_{,i} \phi_{,j} + m^2 \phi^2 \right) \right] + N^j \pi \phi_{,j} \, .$$

• Klein-Gordon inner product

For any (ϕ,ψ) satisfying the field eq.

$$(\phi,\psi)\equiv i\int_{\Sigma(t)} d\Sigma_\mu\big(g^{\mu\nu}\partial_\nu\psi^* - (g^{\mu\nu}\partial_\nu\phi)\psi^*\big)$$
$$= -i\int_t \sqrt{\gamma}d^3x\big(\phi N^\mu\partial_\mu\psi^* - (N^\mu\partial_\mu\phi)\psi^*\big)\,;\quad N^\mu = -Ng^{0\mu}\,.$$

where $\Sigma(t)$ is a Cauchy surface (assume global hyperbolicity for simplicity).
$(\phi,\psi)_{KG}$ is independent of $\Sigma(t)$ because

$$\int d^4V\,\nabla_\mu\big(\phi\nabla^\mu\psi^* - (\nabla^\mu\phi)\psi^*\big) = \int d^4V\,\big(\phi\Box_g\psi^* - (\Box_g\phi)\psi^*\big) = 0$$
$$= \int_{\partial V} d\Sigma_\mu\big(\phi\nabla^\mu\psi^* - (\nabla^\mu\phi)\psi^*\big)$$

In general, any solution ϕ may be expanded in terms of a "complete set" of ortho-normal mode functions u_n:

$$\phi = \sum_n (c_n u_n + c_n^* u_n^*)\,;\quad \left[-\Box_g + m^2\right]u_n = 0\,.$$

KG-normalization

$$
\begin{aligned}
(u_n, u_m) &= i\int d\Sigma_\mu \Big(u_n \nabla^\mu u_m^* - (\nabla^\mu n_n) u_m^*\Big) = i\int_t d^3x \Big(u_n \nabla^0 u_m^* - (\nabla^0 n_n) u_m^*\Big) \\
&= -i\int_t d^3x \Big(u_n p_m^* - p_n u_m^*\Big) = \delta_{n,m}
\end{aligned}
$$

$$
p_n \equiv -\sqrt{-g}\nabla^0 u_n = \sqrt{\gamma} N^\nu \partial_\nu u_n = \frac{\sqrt{\gamma}}{N}\left(\dot{u}_n - N^i \partial_i u_n\right)
$$

$$
(u_n, u_m^*) = 0\,, \quad (u_n^*, u_m^*) = -(u_m, u_n) = -\delta_{n,m}\,.
$$

Completeness

$$
\phi(x) = \sum_n \{(\phi, u_n) u_n(x) - (\phi, u_n^*) u_n^*(x)\}
$$

$$
\pi(x) = \sum_n \{(\phi, u_n) p_n(x) - (\phi, u_n^*) p_n^*(x)\}
$$

$$
\Leftrightarrow \quad \text{on } t = const. \text{ surface} \quad
\begin{cases}
\displaystyle\sum_n \Big(u_n(x) p_n^*(y) - u_n^*(x) p_n(y)\Big) = i\delta^3(x-y) \\
\displaystyle\sum_n \Big(u_n(x) u_n^*(y) - u_n^*(x) u_n(y)\Big) = 0 \\
\displaystyle\sum_n \Big(p_n(x) p_n^*(y) - p_n^*(x) p_n(y)\Big) = 0
\end{cases}
$$

- Canonical quantization: $[\phi(\vec{x},t),\pi(\vec{y},t)] = i\hbar\delta^3(\vec{x}-\vec{y})$

Heisenberg eq. $\left(\left[-\Box_g + m^2\right]\phi = 0\right)$:

$$\dot{\phi} = \frac{1}{i\hbar}[\phi, H] - \frac{N}{\sqrt{\gamma}}\pi + N^i\phi_{,i}$$

$$\dot{\pi} = \frac{1}{i\hbar}[\pi, H] = \partial_i(\pi\, N^i + N\sqrt{\gamma}\gamma^{ij}\phi_{,j}) - N\sqrt{\gamma}m^2\phi\,.$$

Mode-by-mode quantization:

$$\phi(x) = \sum_n \left(a_n u_n(x) + a_n^\dagger u_n^*(x)\right)$$

$$\pi(x) = \sum_n \left(a_n p_n(x) + a_n^\dagger p_n^*(x)\right)$$

$$[a_n, a_m^\dagger] = \hbar\delta_{n,m}\,, \quad [a_n, a_m] = [a_n^\dagger, a_m^\dagger] = 0\,.$$

$\Leftrightarrow$ equal time commutation relations:

$$[\phi(x),\pi(y)] = \hbar\sum_n \left(u_n(x)p_n^*(y) - u_n^*(x)p_n(y)\right) = i\hbar\delta^3(x-y)$$

$$[\phi(x),\phi(y)] = \hbar\sum_n \left(u_n(x)u_n^*(y) - u_n^*(x)u_n(y)\right) = 0$$

$$[\pi(x),\pi(y)] = \hbar\sum_n \left(p_n(x)p_n^*(y) - p_n^*(x)p_n(y)\right) = 0$$

§2. QFT in spatially homogeneous & isotropic universe

- harmonic decomposition

Background metric:

$$ds^2 = -N^2 dt^2 + a^2(t) d\sigma_K^2\,;$$

$$d\sigma_K^2 = \gamma_{ij} dx^i dx^j = d\chi^2 + \frac{\sinh^2(\sqrt{-K}\chi)}{(-K)} d\Omega_{(2)}^2\,; \quad K = \pm 1, 0$$

$$\left(\gamma_{ij} \text{ in §1 } \to \ a^2\gamma_{ij}\right)$$

Mode decomposition:

$Y_{\vec{k}}(x) = \text{harmonics on } d\sigma_K^2$

$$\left[\triangle^{(3)} + k^2\right] Y_{\vec{k}} = 0\,, \quad Y_{\vec{k}}^* = Y_{P[\vec{k}]} \quad (P^2 = 1 : \text{ parity trans.})$$

$$\begin{cases} \sqrt{\gamma} \sum_{\vec{k}} Y_{\vec{k}}(x) Y_{\vec{k}}(y) = \delta^3(x-y) \\[2ex] \int d^3x \sqrt{\gamma} Y_{\vec{k}}(x) Y_{\vec{k}'}^*(x) = \delta_{\vec{k},\vec{k}'} \end{cases}$$

mode function $u_{\vec{k}} = \phi_k(t) Y_{\vec{k}}(x)$:

$$\left[-\Box_g + m^2\right] u_{\vec{k}} = 0\,; \quad -\Box_g = \frac{1}{Na^3}\frac{\partial}{\partial t}\left(\frac{a^3}{N}\frac{\partial}{\partial t}\right) - \frac{1}{a^2}\triangle^{(3)}$$

$$\Rightarrow \quad \left[\frac{1}{Na^3}\frac{d}{dt}\left(\frac{a^3}{N}\frac{d}{dt}\right) + \left(\frac{k^2}{a^2} + m^2\right)\right]\phi_k(t) = 0\,.$$

KG-normalization

$$(u_{\vec{k}}, u_{\vec{k}'}) = -i\frac{a^3}{N}\int \sqrt{\gamma}\, d^3x (\phi_k \dot{\phi}^*_{k'} - \dot{\phi}_k \phi^*_{k'}) Y_{\vec{k}}(x) Y^*_{\vec{k}'}(x)$$

$$= -i\frac{a^3}{N}(\phi_k \dot{\phi}^*_{k'} - \dot{\phi}_k \phi^*_{k'})\delta_{\vec{k},\vec{k}'} = \delta_{\vec{k},\vec{k}'}$$

$$\Leftrightarrow \quad \phi_k \dot{\phi}^*_k - \dot{\phi}_k \phi^*_k = i\frac{N}{a^3}$$

$\phi_k(t)$ (or $u_{\vec{k}}(t,x)$) is called a positive frequency (mode) function

mode-by-mode quantization (Fock representation):

$$\phi(t,x) = \sum_{\vec{k}}\left(a_{\vec{k}} u_{\vec{k}}(t,x) + a^\dagger_{\vec{k}} u^*_{\vec{k}}(t,x)\right)$$

$$a_{\vec{k}} = (\phi, u_{\vec{k}})\,, \quad a^\dagger_{\vec{k}} = -(\phi, u^*_{\vec{k}})$$

where $a_{\vec{k}}$ and $a^\dagger_{\vec{k}}$ satisfy

$$[a_{\vec{k}}, a^\dagger_{\vec{k}'}] = \hbar\delta_{\vec{k},\vec{k}'}\,, \quad [a_{\vec{k}}, a_{\vec{k}'}] = [a^\dagger_{\vec{k}}, a^\dagger_{\vec{k}'}] = 0\,.$$

$\star$ checking the equal time commutation relation

$$\pi(t,x) = \frac{a^3}{N}\sqrt{\gamma}\dot{\phi}(t,x) = \frac{a^3}{N}\sqrt{\gamma}\sum_{\vec{k}}\left(a_{\vec{k}}\dot{u}_{\vec{k}} + a^\dagger_{\vec{k}}\dot{u}^*_{\vec{k}}\right)$$

$$\Rightarrow [\phi(t,x), \pi(t,y)] = \hbar\frac{a^3}{N}\sum_{\vec{k}}\Big(u_{\vec{k}}(x)\dot{u}^*_{\vec{k}}(y) - u^*_{\vec{k}}(x)\dot{u}_{\vec{k}}(y)\Big)$$

$$= \hbar\frac{a^3}{N}\sum_{\vec{k}}\Big(\phi_k\dot{\phi}^*_k - \phi^*_k\dot{\phi}_k\Big)\sqrt{\gamma}Y_{\vec{k}}(x)Y^*_{\vec{k}}(y) = i\hbar\delta^3(x-y)$$

- Fock space

“vacuum” state: $a_{\vec{k}}|0\rangle = 0$

in Minkowski space Poincare invariance uniquely determines the vacuum.
$\phi_k(t)$ associated with the vacuum indeed has a positive energy,

$$\phi_k(t) = \frac{1}{\sqrt{2\omega_k}}e^{-i\omega_k t}\,; \quad i\hbar\frac{\partial}{\partial t}u_{\vec{k}}(t) = E_k u_{\vec{k}}(t)\,; \quad E_k = \hbar\omega_k = \hbar\sqrt{k^2+m^2}\,.$$

($u_{\vec{k}}$ = 1-particle wave function: $u_{\vec{k}} = \langle\phi|\vec{k}\rangle = \langle\phi|a^\dagger_{\vec{k}}|0\rangle$)

· However it is not unique in the expanding background (no Poincare symmetry).
· nor there exists any positive energy eigen function (no time-translation symmetry).
· the two conditions,

$$\left[\frac{1}{Na^3}\frac{d}{dt}\left(\frac{a^3}{N}\frac{d}{dt}\right)+\left(\frac{k^2}{a^2}+m^2\right)\right]\phi_k(t)=0\,,$$

$$\phi_k\dot{\phi}_k^*-\dot{\phi}_k\phi_k^*=i\frac{N}{a^3}\,,$$

are not enough to determine $\phi_k(t)$ uniquely.

⋆ Bogoliubov transformation

$$\phi_k(t)\ \rightarrow\ \tilde{\phi}_k(t)=\alpha_k\phi_k(t)+\beta_k\phi_k^*(t)\,;\quad |\alpha_k|^2-|\beta_k|^2=1$$

$\tilde{\phi}_k$ also satisties the above two conditions.

With the new mode function, $\tilde{u}_{\vec{k}}=\tilde{\phi}_k(t)Y_{\vec{k}}(x)$,

$$\phi(t,x)=\sum_{\vec{k}}\left(\tilde{a}_{\vec{k}}\tilde{u}_{\vec{k}}(t,x)+\tilde{a}^\dagger_{\vec{k}}\tilde{u}^*_{\vec{k}}(t,x)\right)$$

$$\tilde{a}_{\vec{k}}=(\phi,\tilde{u}_{\vec{k}})=\alpha_k^*a_{\vec{k}}-\beta_k^*a^\dagger_{P[\vec{k}]}\,,\quad \tilde{a}^\dagger_{\vec{k}}=-(\phi,\tilde{u}^*_{\vec{k}})=\alpha_k a^\dagger_{\vec{k}}-\beta_k a_{P[\vec{k}]}$$

(spatial homogeneity of the vacuum is assumed: α_k, β_k depend on k not on $\vec{k}$.)

new "vacuum" state: $\tilde{a}_{\vec{k}}|0\rangle = 0$

$$|\tilde{0}\rangle \propto \exp\left[\sum_{\vec{k}} \frac{\beta_k^*}{2\alpha_k} a_{\vec{k}}^\dagger a_{P[\vec{k}]}^\dagger\right]|0\rangle$$

Vacuum expectation values

$$\langle\tilde{0}|a_{\vec{k}}^\dagger a_{\vec{k}}|\tilde{0}\rangle = \langle 0|\tilde{a}_{\vec{k}}^\dagger \tilde{a}_{\vec{k}}|0\rangle = |\beta_k|^2$$

$|\tilde{0}\rangle$ contains $|\beta_k|^2$ particles defined with respect to $|0\rangle$, and vice versa.

Inverse transformation

$$\begin{pmatrix}\tilde{\phi}_k \\ \tilde{\phi}_k^*\end{pmatrix} = \begin{pmatrix}\alpha_k & \beta_k \\ \beta_k^* & \alpha_k^*\end{pmatrix}\begin{pmatrix}\phi_k \\ \phi_k^*\end{pmatrix} \Leftrightarrow \begin{pmatrix}\phi_k \\ \phi_k^*\end{pmatrix} = \begin{pmatrix}\alpha_k^* & -\beta_k \\ -\beta_k^* & \alpha_k\end{pmatrix}\begin{pmatrix}\tilde{\phi}_k \\ \tilde{\phi}_k^*\end{pmatrix}$$

$$\begin{pmatrix}\tilde{a}_{\vec{k}} & \tilde{a}_{P[\vec{k}]}^\dagger\end{pmatrix} = \begin{pmatrix}a_{\vec{k}} & a_{P[\vec{k}]}^\dagger\end{pmatrix}\begin{pmatrix}\alpha_k^* & -\beta_k \\ -\beta_k^* & \alpha_k\end{pmatrix} \Leftrightarrow \begin{pmatrix}a_{\vec{k}} & a_{P[\vec{k}]}^\dagger\end{pmatrix} = \begin{pmatrix}\tilde{a}_{\vec{k}} & \tilde{a}_{P[\vec{k}]}^\dagger\end{pmatrix}\begin{pmatrix}\alpha_k & \beta_k \\ \beta_k^* & \alpha_k^*\end{pmatrix}$$

§3. scalar field on de Sitter space

de Sitter space = time-tike hyperboloid in 5d Minkowski space:

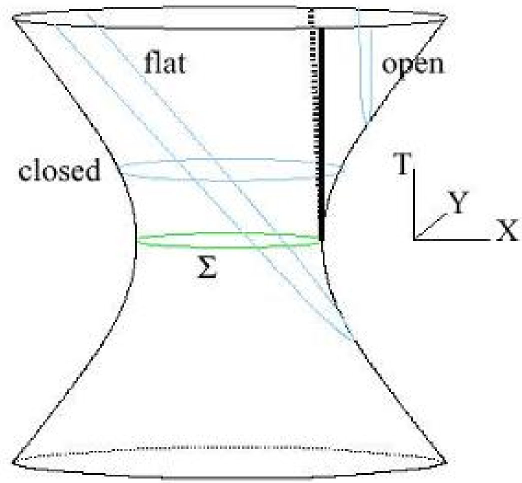
$$\eta_{AB}\xi^A\xi^B = -H^{-2} \quad \text{in} \quad ds^2_{(5)} = -\eta_{AB}d\xi^A d\xi^B \quad (A,B=0,1,2,3,4)$$

$O(4,1)$-symmetry (5d Lorentz symmetry) = de Sitter group

· closed chart (covers whole space)

$$\begin{cases} \xi^0 = H^{-1}\sinh HT\,, \\ \xi^4 = H^{-1}\cos\chi\cosh HT\,, \\ \xi^i = H^{-1}n^i\sin\chi\cosh HT \quad (i=1,2,3) \\ \qquad \sum_{i=1}^3 (n^i)^2 = 1\,, \end{cases}$$

$$\Rightarrow \quad ds^2 = -dT^2 + \frac{1}{H^2}\cosh^2 HT\, d\Omega^2_{(3)}$$



http://www.counterbalance.org/cq-turok/etern-body.html

· flat chart (covers "half" of whole space)

$$\begin{cases} \xi^0 = H^{-1}\left[\sinh Ht + r^2 e^{Ht}/2\right]\,, \\ \xi^4 = H^{-1}\left[\cosh Ht - r^2 e^{Ht}/2\right]\,, \\ \xi^i = H^{-1}n^i r\, e^{Ht} \quad (i=1,2,3)\,. \end{cases}$$

$$\Rightarrow \quad ds^2 = -dt^2 + e^{2Ht}\left[dr^2 + r^2 d\Omega^2_{(2)}\right]\,.$$

flat chart is useful because it is conformally flat:

$$ds^2 = -dt^2 + e^{2Ht}\left[dr^2 + r^2 d\Omega^2_{(2)}\right] = a^2(\eta)\left[-d\eta^2 + \delta_{ij}dx^i dx^j\right]$$
$$= a^2(\eta)\eta_{\mu\nu}dx^\mu dx^\nu\,; \quad a = \frac{1}{-H\eta} \quad (-\infty < \eta < 0)\,,$$

and covers large enough space to render $\Sigma(t)$ $(= \Sigma(\eta))$ a Cauchy surface.

mode expansion

$$\phi = \int \frac{d^3k}{(2\pi)^{3/2}}(a_{\vec{k}}\phi_k(\eta)e^{i\vec{k}\cdot\vec{x}} + a^\dagger_{\vec{k}}\phi^*_k(\eta)e^{-i\vec{k}\cdot\vec{x}})$$
$$\begin{cases} \phi''_k + 2\mathcal{H}\phi'_k + (m^2a^2 + k^2)\phi_k = 0\,, \\ \phi_k\phi^{*\prime}_k - \phi'_k\phi^*_k = \dfrac{i}{a^2}\,. \quad \left(' = \dfrac{d}{d\eta}\right) \end{cases}$$

setting $k\eta = x$,

$$\left[\frac{d^2}{dx^2} - \frac{2}{x}\frac{d}{dx} + 1 + \frac{\mu^2}{x^2}\right]\phi_k = 0\,; \quad \mu^2 = \frac{m^2}{H^2}\,.$$

General solution

$$\phi_k = (-x)^{3/2}\left(C_1 H_\nu^{(1)}(-x) + C_2 H_\nu^{(2)}(-x)\right) \ ; \quad \nu = \sqrt{\frac{9}{4} - \mu^2}$$

· $\nu > 0$:

$$H_\nu^{(1)}(-x)^* = H_\nu^{(2)}(-x)\,, \quad H_\nu^{(1)}(-x)\frac{d}{dx}H_\nu^{(2)}(-x) - H_\nu^{(2)}(-x)\frac{d}{dx}H_\nu^{(1)}(-x) = \frac{4i}{\pi(-x)}$$

$$\Rightarrow \quad \phi_k\frac{d}{dx}\phi_k^* - \phi_k^*\frac{d}{dx}\phi_k = \frac{4ix^2}{\pi}\left(|C_1|^2 - |C_2|^2\right) = \frac{iH^2}{k^3}x^2$$

$$\Rightarrow \quad |C_1|^2 - |C_2|^2 = \frac{\pi H^2}{4k^3}\,.$$

· $\nu = i\nu'$ (for real ν'):

$$H_{i\nu'}^{(1)}(-x)^* = e^{\nu'\pi}H_{i\nu'}^{(2)}(-x)\,, \quad H_{i\nu'}^{(2)}(-x)^* = e^{-\nu'\pi}H_{i\nu'}^{(1)}(-x)$$

$$\Rightarrow \quad |C_1|^2 e^{\nu'\pi} - |C_2|^2 e^{-\nu'\pi} = \frac{\pi H^2}{4k^3}\,.$$

· From the above, for any ν,

$$\phi_k = \frac{\sqrt{\pi}H}{2k^{3/2}}(-k\eta)^{3/2}\left(\alpha_k e^{i\nu\pi/2}H_\nu^{(1)}(-x) + \beta_k e^{-i\nu\pi/2}H_\nu^{(2)}(-x)\right) \ ; \quad |\alpha_k|^2 - |\beta_k|^2 = 1\,.$$

$\alpha_k = 1$: Bunch-Davis vacuum (de Sitter invariant & UV behavior $\sim$ Minkowski)

$$\phi_k^{BD} = e^{i\nu\pi/2}\frac{\sqrt{\pi}H}{2k^{3/2}}(-k\eta)^{3/2}H_\nu^{(1)}(-k\eta)$$

For $\mu^2 \ll 1$ ($m^2 \ll H^2$), $\nu \approx 3/2 - \mu^2/3$,

$$\phi_k^{BD} \to \frac{H}{\sqrt{2}k^{3/2}}(-k\eta)^{\mu^2/3} \quad \text{at } (-k\eta) \to 0\,.$$

⋆ Random walk on superhorizon scales (Vilenkin '83)

$$\langle\phi^2\rangle_{reg} = \int_0^{Ha}\frac{4\pi k^2 dk}{(2\pi)^3}\frac{H^2}{2k^3}(-k\eta)^{2\mu^2/3} = \left(\frac{H}{2\pi}\right)^2\frac{3H^2}{2m^2} \quad \left(\text{UV cutoff at } k = aH = -\eta^{-1}\right)\,.$$

IR divergence: $\langle\phi^2\rangle_{reg} \to \infty$ for $m^2 \to 0 \Rightarrow$ no deS invariant vacuum for $m^2 = 0$.
Setting $\langle\phi^2\rangle_{reg} = 0$ at $a = a_*$ ($\leftrightarrow -\eta_* = (a_*H)^{-1}$),

$$\langle\phi^2\rangle_{reg} = \int_H^{Ha}\frac{dk}{k}\left(\frac{H}{2\pi}\right)^2 = \left(\frac{H}{2\pi}\right)^2\ln(a/a_*) = \left(\frac{H}{2\pi}\right)^2 H(t-t_*)\,.$$

Radom walk with $\Delta\phi = \pm\dfrac{H}{2\pi}$ at each time step $\Delta t = H^{-1}$.

This gives rise to large 'classical' fluctuations on superhorizon scales
(origin of curvature perturbation)