Theory of Cosmological Perturbations

Addendum

— a touch on brane-cosmological perturbations —
§ 1. Braneworld scenario

- Our universe is a (timelike) singular surface (brane) in a higher dimensional spacetime.

- Standard matter fields are confined on the brane, while gravity propagates in the whole spacetime (called the bulk).

- One of the most studied models is the Randall-Sundrum model in which the bulk is 5D anti-de Sitter (AdS) space.

- Cosmology on the brane = brane dynamics in the bulk.
• Randall-Sundrum braneworld metric (AdS$_5$ + Minkowski brane):

$$ds^2 = dy^2 + b^2(y) \eta_{\mu\nu} dx^\mu dx^\nu; \quad b(y) = e^{-|y|/\ell}$$

• Generalization to cosmology:

Brane fixed at $y = 0$ (\sim view from the brane)

$$ds^2 = dy^2 + \left( -n^2(y, t) dt^2 + a^2(y, t) d\sigma^2_{(3)} \right)$$

Or, a brane moving in the AdS bulk (\sim view from the bulk)

$$ds^2 = -A(R) dT^2 + \frac{dR^2}{A(R)} + R^2 d\Omega^2_K \quad A(R) = K + \frac{R^2}{\ell^2} \quad (K = \pm 1, 0)$$

with brane trajectory: $(T, R) = (T(\tau), R(\tau))$
§ 2. Large-scale cosmological perturbations on the brane

· General formalism for braneworld perturbations
  Kodama, Ishibashi & Seto ('00), Mukohyama ('00),
  Langlois ('01), Koyama & Soda ('01), · · ·.

· Essentially a 5-dimensional, PDE problem:
  \[ Q(y, x^\mu) = Q_p(y, t)Y_p(x^i) \cdots Q_p(y, t) \text{ is not separable.} \]

· Nevertheless, some simplifications on super-horizon scales.
  Langlois, Maartens, MS & Wands ('01)

• Effective gravitational equations on the brane (in $\text{AdS}_5$ bulk background)
  \[
  G_{\mu\nu} + \Lambda_4 q_{\mu\nu} = 8\pi G_4 T_{\mu\nu} + (8\pi G_5)^2 \Pi_{\mu\nu} - E_{\mu\nu}
  \equiv 8\pi G_4 T_{\mu\nu}^\text{tot} \quad (\Pi_{\mu\nu} \sim \text{quadratic in } T_{\mu\nu})
  \]

  \[
  E_{\mu\nu} \equiv \left(\begin{array}{c}
  \left(\begin{array}{c}
  C_{a_1a_2b_1b_2}n^a n^b ;
  \\
  C_{a_1a_2b_1b_2} \cdots
d  \\
  \left(\begin{array}{c}
  5D \text{ Weyl tensor}
  \\
  \cdots
  \\
  \left(\begin{array}{c}
  \text{unit normal to the brane}
  \\
  \cdots
  \\
  \end{array}\right)
  \end{array}\right)\right)
  \right)
  \]

· $E_{\mu\nu}$ contains all the dynamics of the bulk

· By definition, $E^{\mu}_{\phantom{\mu}\mu} = 0$. In addition, $D^\mu E_{\mu\nu} = 0$ on FLRW background.

  \[ "-E_{\mu\nu}" : \boxed{\text{Weyl fluid}} \text{ (or "dark radiation") } \]
The energy momentum tensor of the brane matter:

\[ T_{\mu\nu} = \rho u_\mu u_\nu + p (u_\mu u_\nu + g_{\mu\nu}) + \pi_{\mu\nu} \]

For FLRW background, with perturbations of \( O(\epsilon) \),

\[ \pi_{\mu\nu} : \text{anisotropic stress} = O(\epsilon) \quad \Rightarrow \quad u^\mu D^\nu E_{\mu\nu} = O(\epsilon^2) \]

On superhorizon scales, only the energy conservation law is important.
\( \because \) momentum cannot be transferred over super-Hubble scales

Weyl fluid decouples from the brane matter on superhorizon scales.

\[ \downarrow \]

Standard 4D theory is applicable with slight modifications due to \( \Pi_{\mu\nu} \) (quadratic in \( T_{\mu\nu} \)).

\[ \rho_{\text{eff}} = \rho \left( 1 + \frac{\rho}{2\sigma} \right) \quad (\sigma = \text{brane tension}) \quad \rho_{\epsilon} = \frac{E^0_0}{8\pi G_4} \quad \text{etc.} \]

What is missing is an equation for anisotropic stress of the Weyl fluid:

\[ \pi_{\epsilon}^{ij} = -\frac{E_T^{ij}}{8\pi G_4}; \quad E_T^{ij} \equiv E^i_j - \frac{1}{3} \delta^i_j E^k_k \]

This must be determined by solving the bulk perturbation equations.

\[ \Rightarrow \quad \text{back to 5D problem} \]
• Some recent ($\gtrsim 2003$) efforts to solve the bulk perturbation equations:
  (in the context of inflation in RS-type brane cosmology)

  · Expanding equations around the brane ($\sim$ gradient expansion).
    Koyama (’03), Battye, Van de Bruck & Mennim (’03), · · ·

  · Approximating the brane by a de Sitter space.
    Minamitsuji, Himemoto & MS (’03), Du, Wang, Abdalla & Su (’04), · · ·

  · Analyzing exactly soluble models.
    Koyama & Takahashi (’03), Yoshiguchi & Koyama (’04), Kobayashi & Tanaka (’04), · · ·

  · Focusing on tensor-type perturbations
    Easther, Langlois, Maartens & Wands (’03), Hiramatsu, Koyama & Taruya (’03), · · ·

  · Elaborating the perturbation theory
    Deffayet (’03), Malik, Rodriguez-Martinez & Langlois (’03), · · ·

But no rigorous treatment of scalar-type perturbations so far