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A Thought on Conformal Frames

observationally

- conformally equivalent metrics are indistinguishable! -

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1. Conformal frames / - why bother? -

In cosmology, we encounter various frames of the metric which are conformally equivalent.

Einstein frame, Jordan frame, string frame, ...

They are mathematically equivalent, so one can work in any frame as long as mathematical manipulations are concerned.

But it is often said that there exists a unique physical frame on which we should consider actual 'physics.'

Is it really so?

➢Consider dimensional reduction of D-dim spacetime

D-dimensions → 4-dimensions

(D-tensor → 4-tensor + 4-vector + 4-scalar)

$$\begin{pmatrix} g^{(D)}_{\mu\nu}(x,y) & g^{(D)}_{\mu B}(x,y) \\ g^{(D)}_{A\nu}(x,y) & g^{(D)}_{AB}(x,y) \end{pmatrix} \to g_{\mu\nu}(x) = \begin{cases} \left\langle g^{(D)}_{\mu\nu} \right\rangle_{D-4} \; ? \\ f(x)\left\langle g^{(D)}_{\mu\nu} \right\rangle_{D-4} \; ? \\ g^{(D)}_{\mu\nu}(x,0) \; ? \end{cases}$$

$x : 4-\mathrm{dim}$

$y : (D-4)-\mathrm{dim}$

or else?

dilatonic scalars will almost always appear.

No natural conformal frame, *a priori*

Is there a unique physical frame?

Two typical frames in scalar-tensor theory

$(\phi + g)$

- Jordan(-Brans-Dicke) frame

"gravitational" part : $F(\phi)R + L(\phi)$

matter part: $L(\psi, A, \ldots)$ ~ minimal coupling with g

(matter assumed to be universally coupled with g
… for baryons, experimentally consistent)

- Einstein frame

"gravitational" part : $R + L(\phi)$ ~ minimal coupling between g and ϕ

matter part: $G(\phi)L(\psi, A, \ldots)$ ψ : fermion, A : vector, ...

(if non-universal coupling:

$$\Rightarrow \sum_A G_A(\phi) L_A(Q_A); \quad Q_A = \psi, A, \cdots.$$
)

2. Conformal transformations

metric and scalar curvature

$$g_{\mu\nu} \to \tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$$

$$R \to \tilde{R} = \Omega^{-2}\left[R - (D-1)\left(2\frac{\Box\Omega}{\Omega} - (D-4)g^{\mu\nu}\frac{\partial_\mu\Omega\partial_\nu\Omega}{\Omega^2} \right) \right]$$

matter fields

$$\phi \to \tilde{\phi} = \Omega^{-(D-2)/2}\phi$$ scalar

$$A_\mu \to \tilde{A}_\mu = \Omega^{-(D-4)/2} A_\mu$$ vector

$$\psi \to \tilde{\psi} = \Omega^{-(D-1)/2}\psi$$ fermion

Standard (baryonic) matter action in 4 dims

'Jordan' frame (= matter minimally coupled to gravity)

$$S = \int d^4x\sqrt{-g}\left[-i\bar{\psi}_X\gamma^\mu\left(\overleftrightarrow{D}_\mu - ie_X A_\mu\right)\psi_X - m_X\bar{\psi}_X\psi_X - \frac{1}{4}g^{\mu\alpha}g^{\nu\beta}F_{\mu\nu}F_{\alpha\beta} + \cdots \right]$$

$$\bar{\psi}\gamma^\mu \overleftrightarrow{D}_\mu \psi = \frac{1}{2}\left[\bar{\psi}\gamma^\mu D_\mu\psi - (D_\mu\bar{\psi})\gamma^\mu\psi\right] ,$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu , \quad D_\mu = \partial_\mu - \frac{1}{4}\omega_{ab\mu}\Sigma^{ab} ,$$

$$\Sigma^{ab} = \frac{1}{2}\left[\gamma^a, \gamma^b\right] , \quad \omega_{ab\mu} = e_{a\nu}\nabla_\mu e^\nu_b .$$

ψ_X : X = electron/proton/...
A : electromagnetic 4-potential

For the moment, ignore/freeze dilatonic degrees of freedom.

(scalar gravitational)

Effect of conformal transformation

For $\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$

$$S = \int d^4x \sqrt{-\tilde{g}} \left[i\bar{\tilde{\psi}}\, \tilde{\gamma}^{\mu} \left(\overleftrightarrow{\tilde{D}}_{\mu} - ieA_{\mu} \right) \tilde{\psi} - \tilde{m}\, \bar{\tilde{\psi}} \tilde{\psi} - \frac{1}{4} \tilde{g}^{\mu\alpha} \tilde{g}^{\nu\beta} F_{\mu\nu} F_{\alpha\beta} + \cdots \right]$$

where $\tilde{\gamma}^{\mu} = \Omega^{-1}\gamma^{\mu}, \quad \tilde{\psi} = \Omega^{-3/2}\psi, \quad \tilde{m} = \Omega^{-1} m\,.$

(A_{μ} is invariant in 4 dim)

Conformal transformation from \`Jordan frame' to any other frame results in spacetime-dependent mass.

And this is the only effect, provided
dynamics of dilatons (at short distances) can be neglected.

(dilatons may be dynamical on cosmological scales)

3. Cosmology

Conventional wisdom

$$ds^2 = -dt^2 + a^2(t)d\sigma^2_{(K)} \; ;$$

$d\sigma^2_{(K)}$: homogeneous and isotropic 3-space $\left(K = \pm 1, 0\right)$

$$\left(\frac{\dot{a}}{a}\right)^2 \equiv H^2 = \frac{8\pi G}{3}\rho - \frac{K}{a^2}$$ ··· expanding universe

⇨ cosmological redshift $E_{\text{obs}} = \dfrac{E_{\text{emit}}}{1+z}$

This is how we interpret observational data.

This is regarded as a `proof' of cosmic expansion.

But ....

Conformal transformation:

$$ds^2 \to d\tilde{s}^2 = \Omega^2 ds^2; \quad \Omega = \frac{1}{a}$$

$$\Rightarrow \; d\tilde{s}^2 = -d\eta^2 + d\sigma^2_{(3)}; \quad d\eta = \frac{dt}{a(t)}$$

In this conformal frame, the universe is static.

no Hubble flow.

photons do not redshift...

Is this frame unphysical?

In this static frame,

- electron mass varies in time: $\tilde{m}(\eta) = m\Omega^{-1} = \dfrac{m}{1+z}$

  where "z" is defined by

$$1+z \equiv \Omega = \frac{1}{a(\eta)} \quad \left(a_0 = a(\eta_0) = 1\right)$$

- Bohr radius $\propto m^{-1}$ ⇔ atomic energy levels $\propto m$:

energy level in 'static' frame

$$\tilde{E}_n = \frac{E_n}{1+z}$$

energy level in 'Jordan' frame

Thus frequency of photons emitted from a level transition $n \to n'$ at time when $z = z(\eta)$ is

$$\tilde{E}_{nn'} = \frac{E_{nn'}}{1+z}$$

this is exactly what we observe as Hubble's law!

Gravity in the static frame

Assume canonical Einstein theory with matter minimally coupled to gravity:

Jordan frame = Einstein frame

- Gravity is stronger in the early universe:

$$\frac{1}{G}\sqrt{-g}R = \frac{1}{G\Omega^2}\sqrt{-\tilde{g}}\tilde{R} + \cdots \Rightarrow \tilde{G} = G\Omega^2 = \frac{G}{a^2}$$

- This is what we also observe in the original frame:

$$G\frac{m_1 m_2}{r_{\mathrm{p}}^2} = G\frac{m_1 m_2}{a^2 r^2} = \tilde{G}\frac{m_1 m_2}{r^2}$$

proper distance   comoving distance

(gravity is prop to a^{-2} at a fixed comoving distance)

Interpretation of CMB in this frame

- CMB photons have never redshifted.
- The universe was in thermal equilibrium when the electron mass was small by a factor >10^3, ie, at time z >10^3, at fixed temperature T=2.725K.

Just to check physics...

- Thomson cross section: $\tilde{\sigma}_T \propto \tilde{m}^{-2} \to \tilde{\sigma}_T = \sigma_T (1+z)^2$
  electron density: $\tilde{n}_e = \text{const.} = n_e (1+z)^{-3}$

⇨ rate of scattering/interaction per unit proper time:

$$\tilde{n}_e \tilde{\sigma}_T d\eta = \frac{n_e \sigma_T}{1+z} d\eta = n_e \sigma_T dt$$

local/non-gravitational

Thus physics is the same. It's only the scale that differs.

More on cosmology

(a topic provided by Andrei Linde)

Consider a conformally coupled scalar field

$$S = \frac{1}{2}\int d^4x\mathcal{L} = \frac{1}{2}\int d^4x\sqrt{-g}\left(-g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \left(m^2 + \frac{1}{6}R\right)\phi^2\right)$$

Lagrangean $\mathcal{L}$ is (form-)invariant under conformal trfs.

$\left(\text{with } m^2 \to \tilde{m}^2 = \Omega^{-2}m^2 \text{ for } g \to \tilde{g} = \Omega^2 g\right)$

Stability seems to depend on the sign of $m^2 + \frac{1}{6}R$

ie, on the choice of conformal frame...

For definiteness, assume conformally flat g

$\Leftrightarrow$ $\exists$ a frame in which R=0.

if m^2 <0, ϕ is unstable in the flat space R=0.

$\Updownarrow$ contradiction?

ϕ is stable in dS space where m^2+R/6>0.

resolution:

$$\tilde{\boldsymbol{g}} = a^2(\eta)\left[-d\eta^2 + d\vec{x}^2\right], \quad \boldsymbol{g} = -d\eta^2 + d\vec{x}^2$$

dS space: $a = \dfrac{1}{-H\eta}$, where $-\infty < \eta < 0$

In the flat frame g, time is bounded in the future.

no `time' for the instability to develop.

4. Gravity around localized sources

For simplicity, consider Schwarzschild spacetime

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2 + \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 + r^2 d\Omega_{(2)}^2$$

(discussion applies to any asymptotically flat metric)

Consider a conformal transformation:

$$d\tilde{s}^2 = \Omega^2 ds^2 \;\; ; \qquad \Omega = \left(1 - \frac{2GM}{c^2 r}\right)^{-1/2}$$

$$\Rightarrow \quad ds^2 = -c^2 dt^2 + dr^2 + r^2 d\Omega_{(2)}^2 + O(GM / c^2)$$

post-Newton corrections

No Newton potential?

What is happening to gravity?

In Newton/GR gravity, massive bodies move along geodesics:

Equivalence Principle

In the conformal frame $d\tilde{s}^2$, orbits are no longer geodesics:

Violation of Equivalence Principle

Nevertheless, there exists a scalar field Ω that couples universally to the matter, and it controls the motion of massive bodies:

$$S_A = \int \tilde{m}_A \, d\tilde{\tau} = m_A \int \Omega^{-1} d\tilde{\tau}$$

So the universality still holds.

- In fact, orbits are just geodesics on $g = \Omega^{-2}\tilde{g}$ $\quad \left(d\tau = \Omega^{-1} d\tilde{\tau}\right)$

$$\frac{d}{\Omega^{-1} d\tilde{\tau}}\left(\Omega^{-2}\tilde{g}_{\mu\nu}\frac{dz^{\nu}}{\Omega^{-1} d\tilde{\tau}}\right) - \frac{1}{2}\partial_{\mu}\left(\Omega^{-2}\tilde{g}_{\rho\sigma}\right)\frac{dz^{\rho}}{\Omega^{-1} d\tilde{\tau}}\frac{dz^{\sigma}}{\Omega^{-1} d\tilde{\tau}} = 0$$

- Not only Newtonian but also all relativistic effects on the orbital motion remain the same.

- Light propagation is also unaltered since light paths are conformally invariant.

- Only non-trivial change is in the proper time of the orbit:

$$d\tau \rightarrow d\tilde{\tau} = \Omega\, d\tau$$

One might worry that this would lead to a serious problem...

Shapiro time delay, GPS, etc....

- However, remember that the mass is also affected as

$$m \rightarrow \tilde{m} = \Omega^{-1} m$$

frequencies of an atomic clock: $\nu \rightarrow \tilde{\nu} = \Omega^{-1} \nu$

- Num of ticks within a given proper time interval is invariant:

$$\Delta N = \tilde{\nu}\, \Delta\tilde{\tau} = \nu\, \Delta\tau$$

observational results are indistinguishable

5. Summary

- A variety of conformal frames appear in cosmology.
- There is no unique *physical* frame;
  - all frames are observationally equivalent.
  - interpretations may be very different from frames to frames.
- Caveat: what if two metrics are related by a singular conformal transformation?
  - eg, can we solve the initial cosmological singularity problem by a singular conformal transformation?

Probably not, because physics should be the same.
But maybe worth studying more carefully...