Gravitational Radiation Reaction and Self-force Regularization in Black Hole Perturbation Approach

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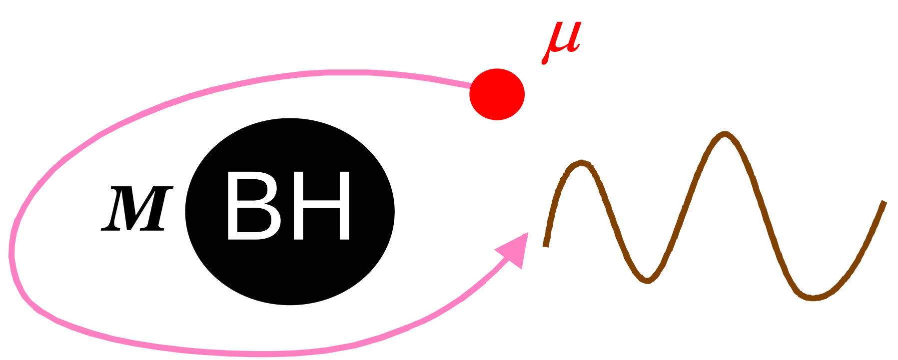
§ 1 Black hole perturbation approach

$$G^{\mu\nu}[g] = 8\pi G T^{\mu\nu}$$

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + h^{(1)}_{\mu\nu} + h^{(2)}_{\mu\nu} + \cdots$$

✧ $M >> \mu$

✧ v/c can be large



Energy-momentum of a point particle

$$\boldsymbol{T}^{\mu\nu}(x) = \mu \int d\tau \; \dot{z}^{\mu} \dot{z}^{\nu} \frac{\delta^4(x - z(\tau))}{\sqrt{-g}} \qquad \left(\dot{z}^{\mu} = \frac{dz^{\mu}}{d\tau} \right)$$

Linear perturbation in μ

$$\delta G^{\mu\nu}\left[\boldsymbol{h}^{(1)}\right] = 8\pi G \boldsymbol{T}^{(1)\mu\nu}$$

$$\boldsymbol{T}^{(1)\mu\nu}(x) = \mu \int d\tau\, \dot{z}^{\mu}\dot{z}^{\nu} \frac{\delta^4(x - z(\tau))}{\sqrt{-g^{(0)}}} \qquad \left(\dot{z}^{\mu} = \frac{dz^{\mu}}{d\tau}\right)$$

geodesic on $g^{(0)}$

background metric

Master variable ζ:

$\zeta = h^{(1)}_{\mu\nu}$ or $\psi^{(1)}_4$ (ψ_4 ~ a component of Weyl tensor)

$$\zeta = \sum_{lm} \phi_{lm}(t, r) Y_{lm}(\Omega)$$

: expanded in spherical (spheroidal) harmonics

$$L[\zeta] = S\left[\boldsymbol{T}^{(1)}\right]$$

Regge-Wheeler / Teukolsky equation

From ζ, we can calculate:

- Waveform at infinity.
- $dE/dt|_{GW}$, $dL_z/dt|_{GW}$, etc. $\sim O\left((G\mu)^2\right)$

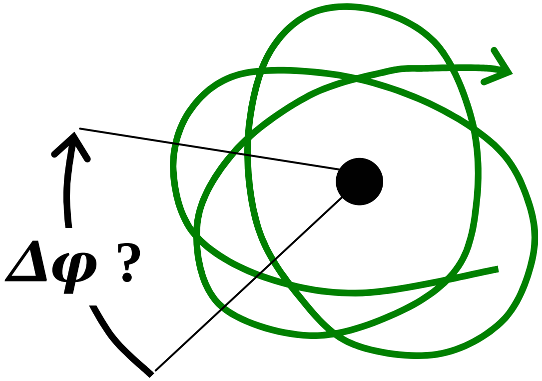
⇨ **the orbit deviates from a geodesic on $g^{(0)}$**

How can we incorporate this deviation?
(focus on the Schwarzschild case)

- Use dE/dt & dL_z/dt to determine the evolution of the orbital parameters (adiabatic approximation).

But, this cannot predict the phase shift in orbit

(**See, however, Y. Mino, PRD67 ('03) 084027**)



Evaluate self-force from $h_{\mu\nu}$ acting on the particle.

§ 2. Regulazitaion of self-force

For point particle,

$$\delta G^{\mu\nu}[\boldsymbol{h}] = 8\pi G \boldsymbol{T}^{\mu\nu} \quad \Longrightarrow \quad \boldsymbol{h}_{\mu\nu} \propto \frac{1}{|\boldsymbol{x} - \boldsymbol{z}(\boldsymbol{\tau})|}$$

$\boldsymbol{h}_{\mu\nu}(x)$ **diverges at** $\boldsymbol{x}^{\alpha} = \boldsymbol{z}^{\alpha}(\boldsymbol{\tau})$

- self-force (back-reaction) in a curved background:

$$\mu \frac{D^2 z^{\alpha}}{d\tau^2} = F^{\alpha}[h] \approx \mu\, \delta\Gamma^{\alpha}_{\mu\nu}[h]\dot{z}^{\mu}\dot{z}^{\nu} = \mu \frac{1}{2}\left(h^{\alpha}{}_{\mu;\nu}(x) + \ldots\ldots\right)\dot{z}^{\mu}\dot{z}^{\nu}$$

~ geodesic eq. on $\boldsymbol{g}^{(0)} + \boldsymbol{h}$

singular !

● Breakdown of perturbation theory ?

Yes! & No!

- Yes, because a point particle is ill-defined in GR.

↔ Mass is non-renormalizable in GR

$$\lim_{r_0 \to 0} \left(m_{\text{bare}} - \frac{G\, m_{\text{bare}}^2}{r_0} \right)$$ has no well-defined limit.

- No, because ∃ regular exact solution (BH) in GR.

↔ Mass renormalization is unnecessary

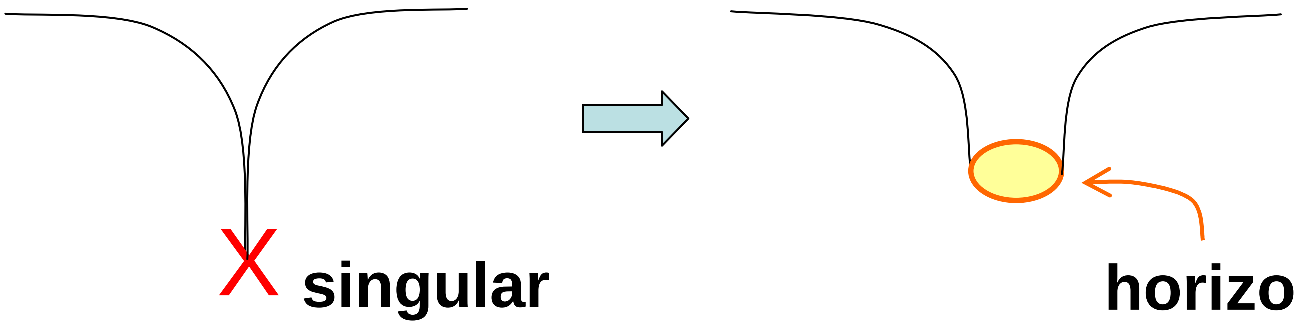
cf. EM theory:

point particle exists ⟺ mass is renormalizable

$$m_{\text{phys}} = \lim_{r_0 \to 0} \left(m_{\text{bare}} + \frac{e^2}{r_0} \right)$$: two parameters to tune the limit

Namely, in GR:

- Identify the point particle with a BH solution of mass μ



- Embed the BH geometry in the linearly perturbed metric $g_{\mu\nu} = g^{(0)}_{\mu\nu} + h_{\mu\nu}$: matching at $|x\text{-}z(\tau)| >> G\mu$

Matched Asymptotic Expansion

Simplest Example

Considr a point particle in the flat background

$$g^{(0)}_{\mu\nu} = \eta_{\mu\nu}$$

$$h_{\mu\nu}(x) = \eta_{\mu\alpha}\eta_{\nu\beta}\frac{2G\mu\,(2\dot{z}^{\alpha}\dot{z}^{\beta}+\eta^{\alpha\beta})}{\dot{z}^{0}\,|\vec{x}-\vec{z}(\tau_{\mathrm{ret}})|}\,; \qquad \ddot{z}^{\alpha}(\tau)=0$$

In the rest frame $\{X^m\}$ of the particle:

$$h_{ab}(X) = \eta_{ac}\eta_{bd}\frac{2G\mu\,(2\dot{Z}^{c}\dot{Z}^{d}+\eta^{cd})}{|\vec{X}|}\,; \quad \dot{Z}^{a} = (1,0,0,0)$$

This is just the Newtonian part of the Schwarzschild metric.

Thus a Schwarzschild black hole of mass μ can be naturally matched to $g_{\mu\nu} = g^{(0)}_{\mu\nu} + h_{\mu\nu}$ **at** $|X| >> G\mu$

⇨ **EOM unchanged. No self-force correction to all orders in $G\mu$**

In General Curved Background:

- Hadamard decomposition of Retarded Green function in harmonic (Lorenz) gauge

$$G^{\mu\nu}_{(\mathrm{ret})\alpha\beta}(x,z) = \theta(x^0 - z^0)\left[u^{\mu\nu}{}_{\alpha\beta}\ \delta(\sigma(x,z)) - v^{\mu\nu}{}_{\alpha\beta}\ \theta(-\sigma(x,z))\right]$$

$\sigma(x,z)$: world interval between x and z $\left(\sim \frac{1}{2}(x\text{-}z)^2\right)$

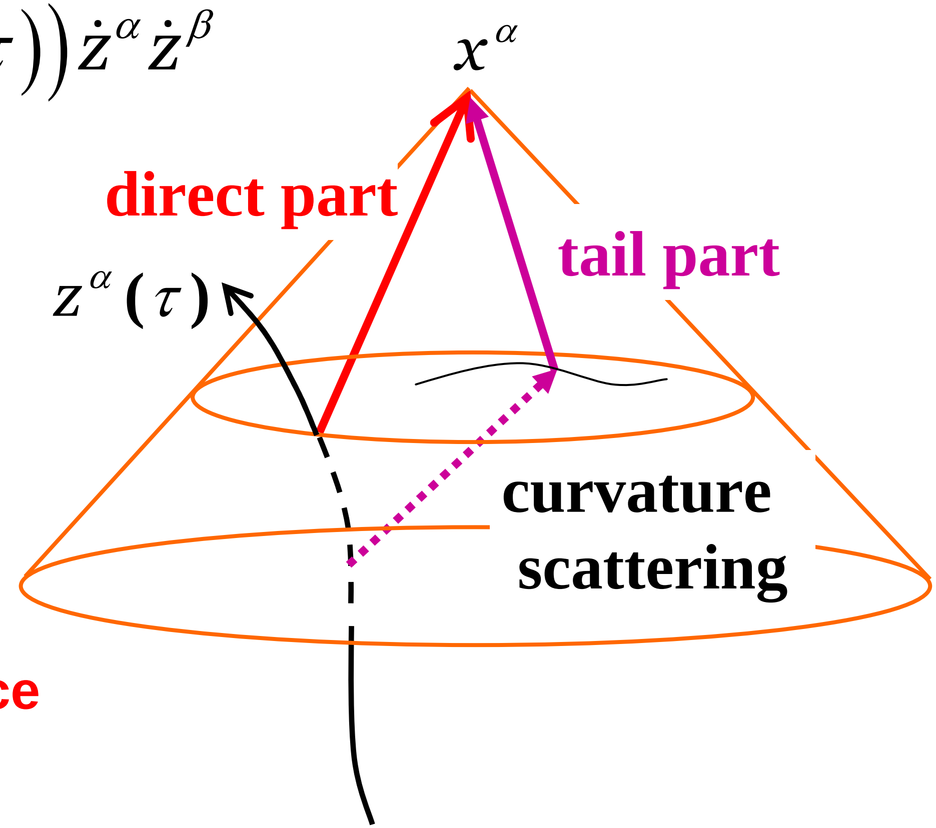
$$h^{\mu\nu}_{(\mathrm{ret})}(x) = \mu \int d\tau\ G^{\mu\nu}_{(\mathrm{ret})\alpha\beta}(x, z(\tau))\,\dot{z}^\alpha \dot{z}^\beta$$

$u^{\mu\nu}{}_{\alpha\beta}$: direct part

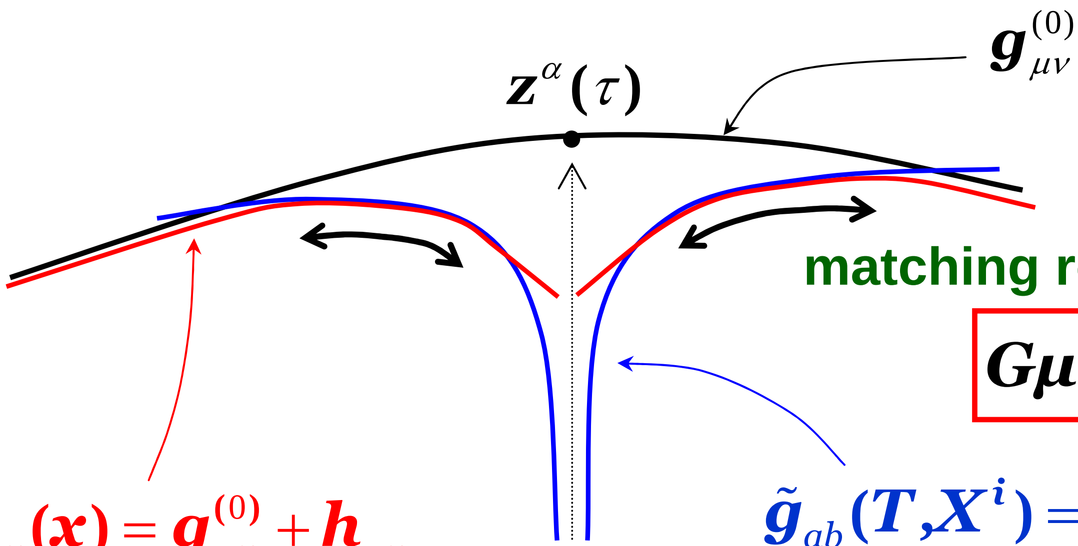
$v^{\mu\nu}{}_{\alpha\beta}$: tail part

$$h^{\mu\nu}_{(\mathrm{ret})}(x) = h^{\mu\nu}_{(\mathrm{direct})} + h^{\mu\nu}_{(\mathrm{tail})}$$

$h^{\mu\nu}_{(\mathrm{direct})}$ contains divergence



Matched asymptotic expansion



$$\left(R^\alpha{}_{\beta\gamma\delta}[g^{(0)}] \sim \frac{1}{L^2} \right)$$

matching region

$$G\mu << X \sim \sqrt{G\mu\, L} << L$$

$$g_{\mu\nu}(x) = g^{(0)}_{\mu\nu} + h_{\mu\nu}$$

(external: valid at $|X| >> G\mu$)

$$\tilde{g}_{ab}(T, X^i) = H^{(\mathrm{BH})}_{ab} + \delta H_{ab}$$

(internal: valid at $|X| << L$)

- **coordinate transformation:** $g_{ab}(X) = \dfrac{\partial x^\mu}{\partial X^a} \dfrac{\partial x^\nu}{\partial X^b} g_{\mu\nu}(x)$

$$\sigma^{;\mu}(x, z(\tau)) \left(\approx -(x^\mu - z^\mu) \right) = -\left(f^\mu{}_i(T) X^i + f^\mu{}_{ij}(T) X^i X^j + \ldots \right)$$

$$\sigma^{;\mu}(x, z(\tau))\, \bar{g}_{\mu\alpha}(x, z) \dot{z}^\alpha = 0 \; ;$$

$\bar{g}_{\mu\alpha}$: parallel transport bi-tensor

- **identify** g_{ab} **and** $\tilde{g}_{ab}$ **in the matching region**

external scheme

$$g_{ab} = g^{(0)}_{ab} + h_{ab}$$

- background Riemann ~ $1/L^2$
- perturbation in $G\mu$

$$g^{(0)}_{ab} = \eta_{ab} + \frac{1}{L}\,{}^{(1)}_{(0)}h_{ab} + \frac{1}{L^2}\,{}^{(2)}_{(0)}h_{ab} + \dots$$

$$h_{ab} = G\mu\left({}^{(0)}_{(1)}h_{ab} + \frac{1}{L}\,{}^{(1)}_{(1)}h_{ab} + \frac{1}{L^2}\,{}^{(2)}_{(1)}h_{ab} + \dots\right)$$

$$+ (G\mu)^2\left({}^{(0)}_{(2)}h_{ab} + \frac{1}{L}\,{}^{(1)}_{(2)}h_{ab} + \frac{1}{L^2}\,{}^{(2)}_{(2)}h_{ab} + \dots\right)$$

internal scheme

$$\tilde{g}_{ab} = H^{(\mathrm{BH})}_{ab} + \delta H_{ab}$$

- background Riemann ~ $G\mu/|X|^3$
- perturbation in $1/L$

$$H^{(\mathrm{BH})}_{ab} = \eta_{ab} + G\mu\,{}^{(0)}_{(1)}H_{ab} + (G\mu)^2\,{}^{(0)}_{(2)}H_{ab} + \dots$$

$$\delta H_{ab} = \frac{1}{L}\left({}^{(1)}_{(0)}H_{ab} + G\mu\,{}^{(1)}_{(1)}H_{ab} + (G\mu)^2\,{}^{(1)}_{(2)}H_{ab} + \dots\right)$$

$$+ \frac{1}{L^2}\left({}^{(2)}_{(0)}H_{ab} + G\mu\,{}^{(2)}_{(1)}H_{ab} + (G\mu)^2\,{}^{(2)}_{(2)}H_{ab} + \dots\right)$$

matching condition:

$${}^{(\mathrm{n})}_{(\mathrm{m})}H_{ab} = {}^{(\mathrm{n})}_{(\mathrm{m})}h_{ab} + O\left((G\mu)^{(\mathrm{m}+1)}/L^{(\mathrm{n}+1)}\right)$$

$${}^{(\mathrm{n})}_{(\mathrm{m})}H_{ab} \sim \frac{(G\mu)^{\mathrm{m}}}{L^{\mathrm{n}}}X^{(\mathrm{n}-\mathrm{m})}$$

Asymptotic matching to $O(G\mu)$

flat part
→background EOM

regular (tail) part
→ EOM with self-force

m \ n	1	$1/L$	$1/L^2$
1	η_{ab}	0	$\frac{\lvert X\rvert^2}{L^2}$
$G\mu$	$\frac{G\mu}{\lvert X\rvert}$	0	$\frac{G\mu\lvert X\rvert}{L^2}$

external background

linear perturbation

BH (=point particle)

singular (direct) part
→no effect on EOM

quadrupolar deformation

'Center of mass' gauge condition

Regularized Gravitational Self-force

'MiSaTaQuWa' force: (named by Eric Poisson)

$$F^{\alpha}[h_{(\text{tail})}(x)] \approx \frac{1}{2}(h^{\alpha}{}_{(\text{tail})\,\mu;\nu}(x) + \ldots\ )\,\dot{z}^{\mu}\dot{z}^{\nu}$$

Mino, Sasaki and Tanaka ('97), Quinn and Wald ('99)

Tail part of the metric perturbation

$$h^{\mu\nu}_{(\mathbf{tail})}(x) \approx \int_{-\infty}^{\tau(x)} d\tau'\ v^{\mu\nu}{}_{\alpha\beta}\left(x, z(\tau')\right) T^{\alpha\beta}\left(z(\tau')\right)$$

Regularized self-force is determined by the tail part

E.O.M. with self-force = geodesic on $g^{\mu\nu} + h^{\mu\nu}_{(\text{tail})}$

But $h^{\mu\nu}_{(\text{tail})}(x)$ **is NOT a solution of Einstein equations.**

⇨ meaning of the metric $g^{\mu\nu} + h^{\mu\nu}_{(\text{tail})}$ is unclear

Detweiler - Whiting's S-R decomposition

(improved over "direct-tail" decomposition) PRD 67, 024025 (2003)

$$G^{ret}(x,z) = 2\theta\left(x^0 - z^0\right) G^{sym}(x,z)$$

$$G^{sym}(x,z) = \frac{1}{8\pi}\left[u(x,z)\delta(\sigma) - v(x,z)\theta(-\sigma)\right]$$

$$G^{S}(x,z) = G^{sym}(x,z) + \frac{1}{8\pi}v(x,z) = \frac{1}{8\pi}\left[u(x,z)\delta(\sigma) + \underline{v(x,z)\theta(\sigma)}\right]$$

new term

$$h^{S}(x) = \int d^4x'\sqrt{-g}\; G^{S}(x,x')T(x')$$: satisfies perturbation eqs.

$$G^{R}(x,z) = G^{ret}(x,z) - G^{S}(x,z) = \left(G^{ret}(x,z) - G^{adv}(x,z)\right) - \frac{1}{8\pi}v(x,z)$$

$$h^{R}(x) = h^{ret}(x) - h^{S}(x)$$: satisfies source-free perturbation eqs.

$$h^{R} - h^{tail} = O\left((x-z)^2\right)$$ ⇨ Both give the same force

EOM = geodesic on $g^{(0)}_{\mu\nu} + h^{R}_{\mu\nu}$ → solution of (linearized) vacuum Einstein eqs.

§ 3 Mode-by-mode regularization

Direct evaluation of R-part is difficult. How can we obtain R-part?

⇨ **Subtraction of S-part:**

$$F^{\alpha}[h^{\mathrm{R}}(\tau)] = \lim_{x \to z(\tau)} (F^{\alpha}[h^{\mathrm{full}}(x)] - F^{\alpha}[h^{\mathrm{S}}(x)])$$

Both terms on r.h.s. are divergent ⇒ need regularization

Mode decomposition

$$F^{\alpha}[h^{\mathrm{full}}](x) = \sum_{\ell m} F^{\alpha}_{\ell m}[h^{\mathrm{full}}](x), \quad F^{\alpha}[h^{\mathrm{S}}](x) = \sum_{\ell m} F^{\alpha}_{\ell m}[h^{\mathrm{S}}](x)$$

Each ℓm mode is finite at particle location

e.g.,

$$\frac{1}{|\boldsymbol{r} - \boldsymbol{r}_0|} = \sum_{\ell=0}^{\infty} \frac{1}{r}\left(\frac{r_0}{r}\right)^{\ell} P_{\ell}(\cos\chi) \; ; \quad r > r_0 \, , \quad \cos\chi = \frac{\boldsymbol{r} \cdot \boldsymbol{r}_0}{r \; r_0}$$

↳ finite in the limit $\boldsymbol{r} \to \boldsymbol{r}_0$

S-part

- **S-part is determined by local expansion near the particle.**

can be expanded in terms of

ε : spatial distance between x and z

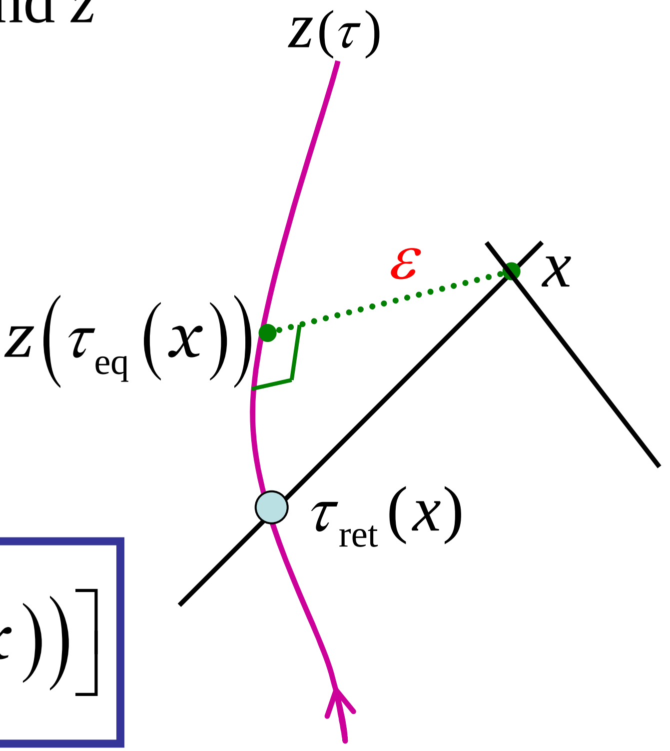
and

$$T = x^0 - z^0$$

$$R = x^r - z^r$$

$$\Theta = x^\theta - z^\theta$$

$$\Phi = x^\varphi - z^\varphi$$



$$S\text{-part} \sim \sum_{n,m,p,q} \frac{R^m \Theta^p \Phi^q}{\varepsilon^n} f^n_{m,p,q}\left[z\left(\tau_{\mathrm{eq}}(x)\right)\right]$$

$f^n_{m,p,q}$ is given in terms of local geometrical quantities.

Spherical extension of S-part

extend $\dfrac{\Theta^n \Phi^m}{\varepsilon^p}$ **over to the whole sphere with accuracy:**

$$h^{\mathrm{S}} = h^{\mathrm{S,approx}} + O\left(\varepsilon^2\right)$$

$$\frac{1}{\varepsilon} \to \sum_{\ell m} \frac{4\pi}{2\ell+1} \frac{1}{r} \left(\frac{r_0}{r}\right)^{\ell} Y^*_{\ell m}(\Omega_0) Y_{\ell m}(\Omega) \quad \text{etc.}$$

$$F^{(\mathrm{S})}_{\alpha,\ell}(r - r_0) = F^{(\mathrm{S,approx})}_{\alpha,\ell}(r - r_0) + O(\varepsilon)$$

･Mode decomposition formula

Barack and Ori ('02), Mino Nakano & Sasaki ('02)

$$F^{(\mathrm{S})}_{\alpha,\ell} = A_\alpha L + B_\alpha + C_\alpha / L + D_{\alpha\ell} \quad \text{where} \quad L = \ell + \tfrac{1}{2}$$

$$\boxed{C_\alpha = \sum_\ell D_{\alpha\ell} = 0}$$

for general geodesic orbit

§ 4 Gauge problem

MiSaTaQuWa Force is defined in harmonic gauge:

$$\lim_{x\to z(\tau)} F^{\alpha}[h^{\mathrm{R,H}}(x)] = \lim_{x\to z(\tau)} (F^{\alpha}[h^{\mathrm{full,H}}(x)] - F^{\alpha}[h^{\mathrm{S,H}}(x)])$$

Difficult to evalutate in harmonic (H) gauge

Defined only in harmonic gauge

$$= \lim_{x\to z(\tau)} (F^{\alpha}[h^{\mathrm{full,G}}(x)] - F^{\alpha}[h^{\mathrm{S,H}}(x)] - F^{\alpha}[\delta h^{\mathrm{full,H\to G}}(x)])$$

Solved in a particular (G) gauge (e.g., RW gauge)

Need to find a gauge transformation

★ $\lim_{x\to z(\tau)} F^{\alpha}[\delta h^{\mathrm{full,H\to G}}(x)]$ **may be divergent**

⇨ **either**

- **introduce a 'hybrid' gauge in which** $h^{\mathrm{S,Hybrid}} = h^{\mathrm{S,H}}$
- **define the self-force (**$\sim h^{\mathrm{R,G}}$**) in G gauge consistently.**

Gauge transformation of the self-force

regularized self-force: $\mu\, u^{\mu}{}_{;\nu} u^{\nu} = F^{\mu}_{(R)} \quad (u^{\mu} = dz^{\mu}/d\tau)$

Gauge transformation of regularized metric

$$\bar{x}^{\mu} = x^{\mu} - \xi^{\mu} \quad \Rightarrow \quad \bar{h}^{\mathbf{R}}_{\mu\nu} = h^{\mathbf{R}}_{\mu\nu} + \delta h_{\mu\nu}$$

$$\delta h_{\mu\nu} = \xi_{\mu;\nu} + \xi_{\nu;\mu} + O\left(\mu^2\right)$$

$$\Rightarrow \delta F^{\mu}_{(R)} = \mu\left[-\left(\delta^{\mu}_{\nu} + u^{\mu}u_{\nu}\right)\ddot{\xi}^{\nu} - R^{\mu}{}_{\mu\rho\sigma} u^{\mu} \xi^{\rho} u^{\sigma}\right]$$

Orbit changes to $\bar{z}^{\mu} = z^{\mu} - \xi^{\mu}$

Gauge-dependence is unimportant for secular orbital evolution, provided that ξ^{μ} stayes small.

$\Rightarrow$ Guaranteed for a 'contact gauge transformation'

contact gauge transformation:

A gauge transformation that is (quasi-)locally and uniquely determined, when harmonic coefficients of $h_{\mu\nu}$ are given.
(like Regge-Wheeler gauge)

Force in hybrid gauge

Gauge transformation to a RW type (G) gauge

$$h^{\mathrm{G}}(x) = h^{\mathrm{H}}(x) + \nabla \boldsymbol{\xi}^{\mathrm{H}\to\mathrm{G}}\left[h^{\mathrm{H}}\right]$$

$$\begin{aligned} h^{\mathrm{R,H}}(x) &= h^{\mathrm{full,H}}(x) - h^{\mathrm{S,H}}(x) \\ &= h^{\mathrm{full,G}}(x) - \nabla \boldsymbol{\xi}^{\mathrm{H}\to\mathrm{G}}\left[h^{\mathrm{full,H}}\right] - h^{\mathrm{S,H}}(x) \\ &= h^{\mathrm{full,G}}(x) - \nabla \boldsymbol{\xi}^{\mathrm{H}\to\mathrm{G}}\left[h^{\mathrm{S,H}}\right] - h^{\mathrm{S,H}}(x) - \underline{\nabla \boldsymbol{\xi}^{\mathrm{H}\to\mathrm{G}}\left[h^{\mathrm{R,H}}\right]} \end{aligned}$$

Last (gauge-dependent) term may be neglected if it does not grow in time (has no secular effect).

$$\boxed{h^{\mathrm{R,Hybrid}}(x) = h^{\mathrm{full,G}}(x) - \nabla \boldsymbol{\xi}^{\mathrm{H}\to\mathrm{G}}\left[h^{\mathrm{S,H}}\right] - h^{\mathrm{S,H}}(x)}$$

$\Rightarrow \; F_{\alpha}^{\mathrm{R}}[h^{\mathrm{R,Hybrid}}]$ **: self-force in hybrid gauge**

Potential problem in this approach is that we have no control of exact gauge condition

Force in RW gauge (equivalent to hybrid gauge?)

Nakano, Sago & MS, PRD 68 ('03) 124003

Harmonic to RW gauge by contact gauge transformation

$$h^{\mathrm{R,RW}} = h^{\mathrm{R,H}} + \nabla\xi^{\mathrm{H\to RW}}\left[h^{\mathrm{R,H}}\right]$$

$$= h^{\mathrm{full,H}} - h^{\mathrm{S,H}} + \nabla\xi^{\mathrm{H\to RW}}\left[h^{\mathrm{full,H}}\right] - \nabla\xi^{\mathrm{H\to RW}}\left[h^{\mathrm{S,H}}\right]$$

$$= h^{\mathrm{full,RW}} - h^{\mathrm{S,H}} - \nabla\xi^{\mathrm{H\to RW}}\left[h^{\mathrm{S,H}}\right]$$

$\Rightarrow \quad F_\alpha^{\mathrm{R}}[h^{\mathrm{R,RW}}]$ **:provided $\nabla\xi^{\mathrm{H\to RW}}\left[h^{\mathrm{S,H}}\right]$ can be calculated with sufficient accuracy.**

However, since $h^{\mathrm{S,H}}$ is known only locally, it seems difficult to get rid of ambiguity from the final result.

$\ell = 0,1$ problem ⇨ **Need exact $h^{\mathrm{full,H}}_{\ell=0,1}$**

difficult to solve ℓ=1 even parity (dipole) gauge mode

~ defines "Center of Mass" coordinates

… numerical attempt by Detweiler & Poisson, gr-qc/0312010

▪ Kerr case?

Only known gauge in which h can be obtained is radiation gauge

Chrzanowski, PRD11, 2042 (1975)

Problem: Gauge transformation to radiation gauge is NOT a contact gauge transformation

Gauge condition

$$h_{\mu\nu}^{\text{in rad}} \ell^{\nu} = 0 \qquad \left(\text{or} \quad h_{\mu\nu}^{(\text{out rad})} n^{\nu} = 0 \right)$$

ℓ^{ν} : outgoing principal null vector

n^{ν} : ingoing principal null vector

⬇

$$\left[h_{\mu\nu}^{\text{H}} + \nabla_{(\mu} \xi_{\nu)}^{\text{H}\to\text{rad}} \right] \times (\ell^{\nu} \text{ or } n^{\nu}) = 0$$

Differential equation (in r and t) for gauge parameters

⇨ **no guarantee for $\xi_{\mu}^{\text{H}\to\text{Rad}}$ to remain small**

Some progress made by Barack & Ori, PRL 90 ('03) 111101.

§ 4 Another decomposition of Green function

Hikida, Jhingan, Nakano, Sago, Sasaki & Tanaka, gr-qc/0308068

http://www2.yukawa.kyoto-u.ac.jp/~misao/BHPC

(Black Hole Perturbation Club)

Time / frequency domain problem:

Need *full*-part in time domain to perform subtraction.

	Local expansion (ζ, R, Θ, Φ)	Harmonic expansion
Time domain	S-part $\dfrac{R^b \Theta^c \Phi^d}{\xi^a}$	$F^{S}_{\alpha,\ell} = A_\alpha L + B_\alpha + D_{\alpha\ell}$ $\sum D_{\alpha\ell} = 0$?
Frequency domain		full-part

Regge-Wheeler (or Teukolsky) equation

Green function for a master variable

$$G(x,x') = \sum_{\ell m} \int d\omega \, e^{-i\omega(t-t')} \mathrm{g}^{full}_{\ell m\omega}(r,r') Y_{\ell m}(\Omega) Y^*_{\ell m}(\Omega')$$

Radial part of Green function

$$g^{full}_{\ell m\omega}(r,r') = \frac{1}{W_{\ell m\omega}(R_{\rm in},R_{\rm up})} \left(R_{\rm in}(r) R_{\rm up}(r') \theta(r'-r) + R_{\rm up}(r) R_{\rm in}(r') \theta(r-r') \right)$$

$$W_{\ell m\omega}(R_{\rm in},R_{\rm up}) = r^2 \left(1-\frac{2M}{r}\right) \left(\left(\frac{d}{dr} R_{\rm up}(r)\right) R_{\rm in}(r) - \left(\frac{d}{dr} R_{\rm in}(r)\right) R_{\rm up}(r) \right)$$

$R^{\rm in/up}_{\ell m\omega}(r)$ **: solution of Regge-Wheeler or Teukolsky equation**

Systematic method for solving radial functions

Mano, Suzuki and Takasugi, Prog. Theor. Phys. 95, 1079 (1996)

A solution given in a series of Coulomb wave functions

$$R_C^\nu(r) \approx \sum_{n=-\infty}^{\infty} a_n^\nu p_{n+\nu}(z),$$

$$p_{n+\nu}(z) \approx z^{n+\nu} e^{-iz}\, {}_2F_0(*,*,*;z)$$

$z = \omega r$

ν: eigenvalue

(to be determined later)

Problem reduces to solving 3 terms recursion eqn.

$$\alpha_n^\nu a_{n+1}^\nu + \beta_n^\nu a_n^\nu + \gamma_n^\nu a_{n-1}^\nu = 0$$

$\epsilon = 2M\omega$

$$\alpha_n^\nu = O(\epsilon)$$
$$\beta_n^\nu \approx (n+\nu)(n+\nu+1) - l(l+1)$$
$$\gamma_n^\nu = O(\epsilon)$$

$$\frac{a_n^\nu}{a_{n-1}^\nu} \xrightarrow[n\to+\infty]{} 0 \qquad \frac{a_n^\nu}{a_{n+1}^\nu} \xrightarrow[n\to-\infty]{} 0$$

➡ determines ν and a_n^ν

$\tilde{\mathrm{S}}$-$\tilde{\mathrm{R}}$ decomposition

$$g^{full}_{\ell m\omega}(r,r') = \frac{1}{W_{\ell m\omega}(R_{\mathrm{in}},R_{\mathrm{up}})}\left(R_{\mathrm{in}}(r)R_{\mathrm{up}}(r')\theta(r'-r) + R_{\mathrm{up}}(r)R_{\mathrm{in}}(r')\theta(r-r')\right)$$

$$W_{\ell m\omega}(R_{\mathrm{in}},R_{\mathrm{up}}) = r^2\left(1-\frac{2M}{r}\right)\left(\left(\frac{d}{dr}R_{\mathrm{up}}(r)\right)R_{\mathrm{in}}(r) - \left(\frac{d}{dr}R_{\mathrm{in}}(r)\right)R_{\mathrm{up}}(r)\right)$$

- $R_{\mathrm{in}}(r)$ and $R_{\mathrm{up}}(r)$ in terms of $R_C^\nu(r)$:

$$R_{\mathrm{in}} = R_C^\nu + \beta_\nu R_C^{-\nu-1}, \quad R_{\mathrm{up}} = \gamma_\nu R_C^\nu + R_C^{-\nu-1}$$

We divide the Green function into two parts,

$$g^{full}_{\ell m\omega}(r,r') = g^{(\tilde{S})}_{\ell m\omega}(r,r') + g^{(\tilde{R})}_{\ell m\omega}(r,r')$$

where

$$g^{\tilde{S}}_{\ell m\omega}(r,r') = \frac{1}{W_{\ell m\omega}(R_{\mathrm{C}}^\nu, R_{\mathrm{C}}^{-\nu-1})}\left[\theta(r'-r)R_{\mathrm{C}}^\nu(r)R_{\mathrm{C}}^{-\nu-1}(r') + \theta(r-r')R_{\mathrm{C}}^{-\nu-1}(r)R_{\mathrm{C}}^\nu(r')\right]$$

$$g^{\tilde{R}}_{\ell m\omega}(r,r') = \frac{1}{(1-\beta_\nu\gamma_\nu)W_{\ell m\omega}(R_{\mathrm{C}}^\nu, R_{\mathrm{C}}^{-\nu-1})}\left[\beta_\nu\gamma_\nu\left(R_{\mathrm{C}}^\nu(r)R_{\mathrm{C}}^{-\nu-1}(r') + R_{\mathrm{C}}^{-\nu-1}(r)R_{\mathrm{C}}^\nu(r')\right)\right.$$
$$\left. +\gamma_\nu R_{\mathrm{C}}^\nu(r)R_{\mathrm{C}}^\nu(r') + \beta_\nu R_{\mathrm{C}}^{-\nu-1}(r)R_{\mathrm{C}}^{-\nu-1}(r')\right]$$

$\tilde{\mathrm{S}}$-part:

$$R_c^{\nu}(x) \approx z^{\nu}\left(1 + z^2 + \frac{\epsilon}{z} + \cdots\right)$$

$z = \omega r = O(v)$
$\epsilon = 2M\omega = O(v^3)$

Post-Newtonian expansion

No log ω

- only integer powers of z
- only positive integer powers of ω^2

$\tilde{\mathrm{R}}$-part:

$$g_{\ell m\omega}^{\tilde{R}}(r, r') = O(v^{6\ell}, v^{4\ell-1}, v^{2\ell+1}) \times g_{\ell m\omega}^{\tilde{S}}(r, r')$$

- no step function ⇒ homogeneous solution
- finite ℓ for finite PN order

$\tilde{S}$-part in time domain

$$g^{(\tilde{S})}_{\ell m\omega}(r,r') = \frac{1}{W_{\ell m\omega}(R^{\nu}_{C}, R^{-\nu-1}_{C})}\left[\theta(r'-r)R^{\nu}_{C}(r)R^{-\nu-1}_{C}(r') + \theta(r-r')R^{-\nu-1}_{C}(r)R^{\nu}_{C}(r')\right]$$

$$R^{\nu}_{c}(x) \approx z^{\nu}\left(1 + z^2 + \frac{\epsilon}{z} + \cdots\right) \qquad \begin{array}{l} z = \omega r \\ \epsilon = 2M\omega \end{array}$$

$$= \sum_{k=0}^{\infty} \omega^k X_{\ell m k}(r,r')$$

Since there is no logω, ω–integral is easy

$$\int d\omega\, \omega^n e^{-i\omega(t-t')} = 2\pi(-i)^n \partial^n_{t'} \delta(t-t')$$

In the case of scalar-charge

$$F^{\tilde{S}}_{\alpha,\ell} = q^2 \nabla_\alpha \lim_{x\to z(t)} \sum_{m,k} (i\partial_t)^k \frac{d\tau(t)}{dt} X_{\ell m k}(r, z^r(t)) Y_{\ell m}(\theta,\varphi) Y^*_{\ell m}(z^\theta(t), z^\varphi(t))$$

All singular behavior is in $\tilde{S}$ - part

A_α and B_α are common with S - part :

$$F_{\alpha,\ell}^{\tilde{S}} = A_\alpha L + B_\alpha + \tilde{D}_{\alpha\ell}$$

$$F_{\alpha,\ell}^{S} = A_\alpha L + B_\alpha + D_{\alpha\ell} \quad ; \qquad \sum_{\ell=0}^{\infty} D_{\alpha\ell} = 0$$

Force after subtraction of S - part from $\tilde{S}$ - part :

$$F_\alpha^{\tilde{S}-S} = \sum_\ell (F_{\alpha,\ell}^{\tilde{S}} - F_{\alpha,\ell}^{S}) = \sum_\ell \tilde{D}_{\alpha\ell}$$

Final force is $\quad F_\alpha^R = F_\alpha^{\tilde{R}} + F_\alpha^{\tilde{S}-S}$

Summary

	local expansion (ζ, R, Θ, Φ)	Harmonic expansion
Time domain	*S*-part $\dfrac{R^b \Theta^c \Phi^d}{\xi^a}$	$F^{S}_{\alpha,\ell} = A_\alpha L + B_\alpha + D_{\alpha\ell}$ $\sum D_{\alpha\ell} = 0$ $F^{\tilde{S}}_{\alpha,\ell} = A_\alpha L + B_\alpha + \tilde{D}_{\alpha\ell}$
Frequency domain		full-part → $\tilde{S}$-part, $\tilde{R}$-part

Regularized force up to *n* -PN

$$\sum_{\ell=0}^{\infty} F^{R}_{\alpha,\ell} = \sum_{\ell=0}^{\infty} \left(\lim_{x \to z_0} F^{\tilde{S}-S}_{\alpha,\ell} + F^{\tilde{R}}_{\alpha,\ell} \right) = \sum_{\ell=0}^{\infty} \tilde{D}_{\alpha\ell} + \sum_{\ell=0}^{n} F^{\tilde{R}}_{\alpha,\ell}$$

Result for S-$\tilde{S}$ part

For a scalar charge in a geodesic orbit

$$F_t^{\tilde{S}-S} = q^2 \frac{v^r(t)}{4\pi (z^r)^2} \sum_i^4 K_t^{(i)} \delta_{\mathcal{E}}^i \qquad \delta_{\mathcal{E}} = 1 - \frac{1}{\mathcal{E}^2}$$

where the coefficients $K_t^{(i)}$ are

$$\begin{aligned}
K_t^{(0)} = & -\Big[\frac{9}{19} + \frac{10364}{1659}\frac{M}{(z^r)} + \frac{20728}{1659}\frac{M^2}{(z^r)^2} + \left(\frac{5246140232891518}{35013238792623} - \frac{27\pi^2}{2}\right)\frac{M^3}{(z^r)^3} - \Big(\frac{303577852378782158933 9}{121429413456695826 3} \\
& + \frac{2007\pi^2}{16}\Big)\frac{M^4}{(z^r)^4}\Big] + \Big[\frac{38844}{10507} + \frac{12572900}{1775683}\frac{M}{(z^r)} + \left(\frac{1443514854479884}{11671079597541} - \frac{585M^2}{64}\right)\frac{M^2}{(z^r)^2} - \Big(\frac{7173\pi^2}{32} \\
& + \frac{2092360235136608219218 04}{4250029470984353920 5}\Big)\frac{M^3}{(z^r)^3}\Big]\frac{\mathcal{L}^2}{(z^r)^2} - \Big[\frac{3078617}{253669} + \frac{13691383240}{513172387}\frac{M}{(z^r)} - \Big(\frac{433220205623858418591 1}{2023823557611597105} \\
& - \frac{111825\pi^2}{1024}\Big)\frac{M^2}{(z^r)^2}\Big]\frac{\mathcal{L}^4}{(z^r)^4} + \Big[\frac{16973925730}{513172387} + \frac{1332278699099876}{23654681178765}\frac{M}{(z^r)}\Big]\frac{\mathcal{L}^6}{(z^r)^6} - \frac{94008905915838}{1126413389465}\frac{\mathcal{L}^8}{(z^r)^8}, \\
K_t^{(1)} = & -\Big[\frac{145329}{105070} + \frac{224752726}{26635245}\frac{M}{(z^r)} + \left(\frac{3961114172666372}{58355397987705} - \frac{117\pi^2}{32}\right)\frac{M^2}{(z^r)^2} - \Big(\frac{609653268515710331648 9}{607147067283479131 5} \\
& + \frac{1413\pi^2}{16}\Big)\frac{M^3}{(z^r)^3}\Big] + \Big[\frac{81010078}{8878415} + \frac{26029992074}{2565861935}\frac{M}{(z^r)} - \left(\frac{5613396674353868549121 4}{4250029470984353920 5} + \frac{5985}{64}\right)\frac{M^2}{(z^r)^2}\Big]\frac{\mathcal{L}^2}{(z^r)^2} \\
& - \Big[\frac{163684391287}{5131723870} + \frac{98611138120}{13251922229}\frac{M}{(z^r)}\Big]\frac{\mathcal{L}^4}{(z^r)^4} + \frac{21116821648567}{225282677893}\frac{\mathcal{L}^6}{(z^r)^6}, \\
K_t^{(2)} = & -\Big[\frac{230022021}{99438248} + \frac{5642118696757}{538831006350}\frac{M}{(z^r)} + \left(\frac{9361340681317465627159}{2125014735492176960 25} - \frac{2061\pi^2}{128}\right)\frac{M^2}{(z^r)^2}\Big] + \Big[\frac{535002897207}{35922067090} \\
& + \frac{130651280359811}{78848937262550}\frac{M}{(z^r)}\Big]\frac{\mathcal{L}^2}{(z^r)^2} - \frac{3424552998566313}{63079149810040}\frac{\mathcal{L}^4}{(z^r)^4} \qquad \cdots\cdot
\end{aligned}$$

Conclusion

- **Gravitational self-force is given by R-part of metric perturbation (EOM = geodesic on $g^{(0)}+h^{\mathrm{R}}$).**
- **Mode-by-mode regularization seems promising.**
- **Gauge problem not completely solved yet.**

  $\ell = 1$ **problem**

- **Extension to Kerr background still at preliminary stage.**
- **$\tilde{\mathrm{S}}$ - $\tilde{\mathrm{R}}$ decomposition instead of S - R decomposition.**

makes it possible to perform subtraction in time domain for any orbit at the expence of PN expansion.

Regularized force up to n-PN order

$$\sum_{\ell=0}^{\infty} F^{R}_{\alpha,\ell} = \sum_{\ell=0}^{\infty} \left(\lim_{x \to z_0} F^{\tilde{S}-S}_{\alpha,\ell} + F^{\tilde{R}}_{\alpha,\ell} \right) = \sum_{\ell=0}^{\infty} \tilde{D}_{\alpha\ell} + \sum_{\ell=0}^{n} F^{\tilde{R}}_{\alpha,\ell}$$