Massive-Field Approach to the Scalar Self Force in Curved Spacetime

Eran Rosenthal
Goals of this talk

- Proposing an alternative regularization method for the scalar self force.

- This method is based on the difference between two retarded scalar fields: a massless scalar field, and a massive scalar field.

- Show that this method, can be used to calculate the mode-sum regularization parameters.
Scalar self force - introduction

Consider a point like object with a scalar charge \( q \), and a given world line \( z(\tau) \)

and a fixed background spacetime metric \( g_{\mu \nu}(x) \)

The object induces a scalar field (perturbation) \( \phi(x) \)

and a scalar force \( F_\mu(x) \equiv q \phi_{,\mu}(x) \)

\[ \int_\mu^{self}(z_0) \]
Coleman showed that electromagnetic self force in flat spacetime (ALD term)

\[ f_{\mu}^{self} (z_0) = \frac{2}{3} q^2 (\dot{a}_\mu - a^2 u_\mu) \]

can be obtained by replacing the original Green function \( G \) with a new Green function, which depends on a parameter, such that:

\[ \lim_{\Lambda \to \infty} G_\Lambda = G \]

This Green function \( G_\Lambda \) can be obtained from a fictitious massive electromagnetic Green function.

The scalar self force in curved spacetime (after regularization):

\[ f_{self} (z_0) = \text{(local term)} + \lim_{\bar{\varepsilon} \to 0} \int_{-\infty}^{\tau_0 - \bar{\varepsilon}} \partial^\mu G(z_0 | z(\tau)) d\tau \]

T. C. Quinn (2000)
Massive-Field Approach

\[ \Box \phi = -4\pi \rho \]

\[ (\Box - m^2)\phi_m = -4\pi \rho \quad \text{Auxiliary field} \]

\[ \Box \phi \equiv \phi_{\nu} \]

\[ \rho(x) = q \int_{-\infty}^{\infty} \frac{1}{\sqrt{-g}} \delta^4(x - z(\tau)) d\tau \quad \text{The same charge density for both fields} \]
\[ \Delta \phi \equiv \phi - \phi_m \quad u^\mu n_\mu = 0 \quad n^\mu n_\mu = 1 \]

Result

\[
\int_{\mu}^{self} (z_0) = q \lim_{m \to \infty} \left\{ \lim_{\delta \to 0} \Delta \phi_{\mu} (x) + \frac{1}{2} q \left[ m^2 n_\mu (z_0) + m a_\mu (z_0) \right] \right\}
\]
\[
\Delta F_\mu \equiv q \Delta \phi_{\mu}(x)
\]

Retarded solutions for a charge density of a point particle

\[
\phi_m(x) = q \int_{-\infty}^{\infty} G_m(x \mid z(\tau)) d\tau
\]

\[
(\Box - m^2)G_m(x \mid x') = -\frac{4\pi}{\sqrt{-g}} \delta^4(x - x')
\]

\[
\phi(x) = q \int_{-\infty}^{\infty} G(x \mid z(\tau)) d\tau
\]

\[
\Box G(x \mid x') = -\frac{4\pi}{\sqrt{-g}} \delta^4(x - x')
\]

Assumption: these integrals converge off the world line.
Scalar field

Locally: when $x$ is in a local neighborhood of $z$

$$G(x \mid z) = \Theta(\Sigma(x), z)[U(x \mid z)\delta(\sigma) - V(x \mid z)\theta(-\sigma)]$$

$$\phi(x) = q\int_{\tau_1}^{\infty} G(x \mid z) d\tau + q\int_{-\infty}^{\tau_1} G(x \mid z) d\tau$$

$$\phi(x) = \phi^{\text{dir}}(x) + \phi^{\text{tail}}(x)$$

$$\phi^{\text{dir}}(x) = qU(x \mid z(\tau^-))\frac{d\tau}{d\sigma}(\tau^-) \quad \text{diverges as} \quad x \to z(\tau)$$

$$\phi^{\text{tail}}(x) = -q\int_{\tau_1}^{\tau^-} V(x \mid z) d\tau + q\int_{-\infty}^{\tau_1} G(x \mid z) d\tau \quad \text{regular as} \quad x \to z(\tau)$$
Massive scalar field

Locally: when $x$ is in a local neighborhood of $z$

\[
G_m(x \mid z) = \Theta(\Sigma(x), z)[U(x \mid z)\delta(\sigma) - V_m(x \mid z)\theta(-\sigma)]
\]

\[
\phi_m(x) = q \int_{\tau_1}^{\infty} G_m(x \mid z) d\tau + q \int_{-\infty}^{\tau_1} G_m(x \mid z) d\tau
\]

\[
\phi_m(x) = \phi_m^{\text{dir}}(x) + \phi_m^{\text{tail}}(x)
\]

\[
\phi_m^{\text{dir}}(x) = qU(x \mid z(\tau^-))\frac{d\tau}{d\sigma}(\tau^-) = \phi_m^{\text{dir}}(x)
\]

\[
\phi_m^{\text{tail}}(x) = -q \int_{\tau_1}^{\infty} V_m(x \mid z) d\tau + q \int_{-\infty}^{\tau_1} G_m(x \mid z) d\tau
\]
\[ \Delta \phi(x) = \phi - \phi_m = \phi_{\text{tail}}(x) - \phi_{\text{m}}^{\text{tail}}(x) \]

\[ \Delta V \equiv V - V_m \quad \Delta G \equiv G - G_m \]

\[ \Delta \phi(x) = -q \int_{\tau_1}^{\tau^-} \Delta V(x \mid z)d\tau + q \int_{-\infty}^{\tau_1} \Delta G(x \mid z)d\tau \]
\[ \Delta F_\mu (x) \equiv q \Delta \phi_{,\mu} (x) \]

\[ \Delta F_\mu (x) = -q^2 (\tau^-)_{,\mu} [\Delta V]_{\tau^-} - q^2 \int_{\tau_1}^{\tau^-} \Delta V_{,\mu} d\tau + q^2 \int_{-\infty}^{\tau_1} \Delta G_{,\mu} d\tau \]

\[ \Delta F_\mu (x) = F^{tail}_\mu (x) - q^2 (\tau^-)_{,\mu} [\Delta V]_{\tau^-} + q^2 \int_{\tau_1}^{\tau^-} V_{m,\mu} d\tau - q^2 \int_{-\infty}^{\tau_1} G_{m,\mu} d\tau \]

\[ F^{tail}_\mu (x) \equiv -q^2 \int_{\tau_1}^{\tau^-} V_{,\mu} d\tau + q^2 \int_{-\infty}^{\tau_1} G_{,\mu} d\tau \]
\[ \Delta F_\mu (x) = F^\text{tail}_\mu (x) - q^2 (\tau^-)_{,\mu} [\Delta V]_{\tau^-} + q^2 \int_{\tau_1}^{\tau} V_{m,\mu} d\tau - q^2 \int_{-\infty}^{\tau_1} G_{m,\mu} d\tau \]

Smooth splitting \( h_1(\tau) + h_2(\tau) = 1 \)

\( h_1(\tau) = 1, \text{ for } \tau \geq \tau_1 + \varepsilon \)

\( h_2(\tau) = 1, \text{ for } \tau \leq \tau_1 \)

\( h_1(\tau_1) = 0 \quad h_2(\tau_1 + \varepsilon) = 0 \)

I. Calculating the limit \( \delta \to 0 \)

II. Then calculating the asymptotic form as \( m \to \infty \)
(1)

\[ F_{\mu}^{\text{tail}}(x) \equiv -q^2 \int_{\tau_1}^{\tau^-} V_{,\mu} d\tau + q^2 \int_{-\infty}^{\tau_1} G_{,\mu} d\tau \]

\[ \lim_{\delta \to 0} F_{\mu}^{\text{tail}} = \lim_{\tilde{\varepsilon} \to 0} q^2 \int_{-\infty}^{\tau_0 - \tilde{\varepsilon}} \partial_{\mu} G(z_0 \mid z(\tau)) d\tau \]
\[ \Delta F_\mu (x) = F_{\mu}^{\text{tail}} (x) - q^2 (\tau^-)_{,\mu} [\Delta V]_{\tau^-} + q^2 \int_{\tau_1}^{\tau^-} h_1 V_{m,\mu} d\tau - q^2 \int_{-\infty}^{\tau_1+\varepsilon} h_2 G_{m,\mu} d\tau \]

(2)+(3)
Hadamard expansion

\[ V(x \mid z) = \sum_{n=0}^{\infty} \frac{\sigma^n}{n!} v_n(x \mid z) \quad V_m(x \mid z) = \sum_{n=0}^{\infty} \frac{\sigma^n}{n!} \tilde{v}_n(x \mid z) \]

Recurrence differential equations

for \( v_n, \tilde{v}_n \)

\[ \tilde{v}_n = \sum_{k=-1}^{n} \left( \frac{m^2}{2} \right)^{n-k} \frac{v_k}{(n-k)!} \]

\[ V_m = mU \frac{J_1(ms)}{s} + \sum_{n=0}^{\infty} v_nJ_n(ms) \left( \frac{-s}{m} \right)^n \]

\[ s(x \mid z) \equiv \sqrt{-2\sigma(x \mid z)} \]

There is a different derivation, see F. G. Friedlander

\[(2)+(3) \quad \Rightarrow \quad \begin{array}{l}
I. \text{ Calculating the limit } \delta \to 0 \\
II. \text{ Then calculating the asymptotic form as } m \to \infty \\
\end{array} \]

\[ q^2 \left[ -\frac{1}{2} (m^2 n_\mu + ma_\mu) + \frac{1}{3} (\dot{a}_\mu - a^2 u_\mu) + \left( \frac{1}{6} R_{\mu \nu}^\eta u_\nu + \frac{1}{6} R_{\eta \nu}^\mu u_\nu + \frac{1}{12} R u_\mu \right) \right] \]
(4) \[ -q^2 \int_{-\infty}^{\tau_1+\epsilon} h_2 G_{m,\mu}(x \mid z) d\tau \]

**Simplified problem:** The world line is within a convex domain (no caustics), spacetime is globally hyperbolic.

Use asymptotic expansion (Friedlander)

\[ G_m(x \mid z) = \Theta(\Sigma(x), z) \left\{ U\delta(\sigma) - [V_m^{\text{asym}} + O(m^{-1/2})] \theta(-\sigma) \right\} \]

\[ V_m^{\text{asym}} = U \sqrt{\frac{2m}{s^3\pi}} \cos(ms - \frac{3\pi}{4}) \]

\[ \lim_{m \to \infty} \lim_{T \to -\infty} (-q^2) \int_{T}^{\tau_1+\epsilon} h_3 h_2 G_{m,\mu}(z_0 \mid z) d\tau \]

\[ 0 \]
(4) \(-q^2 \int_{-\infty}^{\tau_1+\varepsilon} h_2 G_{m,\mu}(x \mid z) d\tau\)  

A more general case: allow caustics

\[ F_{(m)\mu}(z_0) \equiv q \phi_{m,\mu}(z_0) = (\text{depends on } [\tau_1 + \varepsilon, \tau^-]) + q^2 \int_{-\infty}^{\tau_1+\varepsilon} h_2 G_{m,\mu} d\tau \]
Application: calculation of the mode-sum regularization parameters.
Notation change \( m \rightarrow \Lambda \)

Multipole expansion

\[
\phi = \sum_{l=0}^{\infty} \phi^l \quad \phi_\Lambda = \sum_{l=0}^{\infty} \phi_\Lambda^l
\]

\[
n_\mu(z_0) \Rightarrow n_\mu(x) = \sum_{l=0}^{\infty} n^l_\mu(x) \quad a_\mu(z_0) \Rightarrow a_\mu(x) = \sum_{l=0}^{\infty} a^l_\mu(x)
\]

Substitute in \( f^{\text{self}}_\mu \)

\[
f^{\text{self}}_\mu(z_0) = q \lim_{m \rightarrow \infty} \left\{ \lim_{\delta \rightarrow 0} \Delta \phi^l_{\mu}(x) + \frac{1}{2} q \left[ \Lambda^2 n_\mu(z_0) + \Lambda a_\mu(z_0) \right] \right\}
\]

\[
h^l_\mu \equiv A_\mu L - B_\mu - C_\mu / L
\]

\[
f^{\text{self}}_\mu(z_0) = \sum_{l=0}^{\infty} \lim_{\delta \rightarrow 0} (q \phi^l_{\mu} - h^l_\mu) - D_\mu
\]

\[
D_\mu = \lim_{\Lambda \rightarrow \infty} \sum_{l=0}^{\infty} \left\{ \lim_{\delta \rightarrow 0} \left[ q \phi^l_{\Lambda,\mu} - \frac{1}{2} q^2 (\Lambda^2 n^l_\mu + \Lambda a^l_\mu) \right] - h^l_\mu \right\}
\]
Static particle in Schwarzschild spacetime

\[ \phi_{\Lambda, r^*, \varphi}^{lm} - (\mathcal{V}^l + \Lambda^2 f) \phi_{\Lambda}^{lm} = -4\pi \rho_{\Lambda}^{lm} f \quad f \equiv 1 - \frac{2M}{r} \]

For large value of \( l \) - use WKB approximation to solve for \( \phi_{\Lambda}^{lm} \)

\[
A_{r^+} = -q^2 \frac{1}{r_0^2} \frac{1}{\sqrt{f(r_0)}} \\
B_{r^+} = q^2 \frac{M}{2r_0^2} - 1 \frac{1}{f(r_0)} \\
C_{r^+} = 0
\]

For large value of \( \Lambda \) (any value of \( l \)) - use WKB approximation and take \( \Lambda \to \infty \)

\[ D_{r^+} = 0 \]

Conforms with results obtained by Barack and Ori (2000)
References