One-Armed Spiral Instability in Differentially Rotating Stars

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1. Introduction

Rotating Configuration

$\beta \equiv \frac{T}{W}$

- **Maclaurin Spheroid**
  Uniformly rotating incompressible stars

- **Jacobi Ellipsoid**
  Ellipsoid with unequal axis, incompressible stars

- **Bar-like Configurations**

$T$: Rotational Kinetic Energy
$W$: Gravitational Binding Energy

Instability $\rightarrow$ Candidate of GW sources

Secular Instability

$\beta_{\text{sec}} \approx 0.14 \quad \tau_{\text{sec}} \sim \tau_{\text{vis}} \text{ or } \tau_{\text{GW}}$

Mass-shedding Limit

Dynamical Instability

$\beta_{\text{dyn}} \approx 0.27 \quad \tau_{\text{dyn}} \sim (G\bar{\rho})^{-1/2} \ll \tau_{\text{sec}}$

Differential rotation release the limit

$\frac{mR\Omega_{\text{eq}}^2}{\beta} \lesssim \frac{Mm}{R^2}$

Centrifugal force $\leq$ Gravity

No. 2

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23rd June 2003 @YITP, Japan
**Effects which enhance the onset of dynamical instability**

- **High degree of differential rotation**
  Tohline, Hachisu (1990); Pickett, Durisen, Davis (1996); Shibata, Karino, Eriguchi (2002)
  \[
  \frac{\Omega_c}{\Omega_{eq}} \gtrsim 10 \quad \Omega_c : \text{Central angular velocity}
  \]
  \[
  \frac{T}{|W|} \lesssim 0.20 \quad \Omega_{eq} : \text{Equatorial surface angular velocity}
  \]

- **Strong relativistic gravitation**
  Shibata, Baumgarte, Shapiro (2000); MS, Shibata, Baumgarte, Shapiro (2001)
  \[
  \frac{\Omega_c}{\Omega_{eq}} \sim 3
  \]
  \[
  \frac{T}{|W|} \sim 0.24 - 0.26
  \]

**m=1 instability in protostar system**

- **Woodward, Tohline, Hachisu (1994)**
  Nonlinear stability analysis of accretion system
  (Point mass + thick self-gravitating disk)

As increasing $T/W$, disk first becomes $m=1$ unstable resembles the “eccentric instability” and further increasing $T/W$ resembles the Papaloizou–Pringle instability
Picket, Durisen, Davis (1996)
Discovered the m=1 instability in n=3/2 polytropic star with strong concentration of the angular momentum in the envelope region of the star

Dynamical One–Armed Spiral Instability (m=1)

Centrella et al. (2001)

- Discovered the m=1 instability in toroidal stars
- Requires soft equation of state (n=3.33) and high degree of differential rotation \( \Omega_c/\Omega_{eq} = 26 \)

→ Dense torus causes the excitation of m=1 instability

Purpose

- Identify necessary conditions for triggering the m=1 dynamical instability in stellar system.
- Discover the nature of the m=1 dynamical instability by comparing with that of the m=2 bar mode instability.
- Find an effect of the m=1 dynamical instability on gravitational waves.
2. Newtonian Hydrodynamics

Features of our Newtonian hydrodynamics code

- Newtonian hydrodynamics with artificial viscosity
- Equatorial plane symmetry
- Adiabatic evolution

Continuity Equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho v^i)}{\partial x^i} = 0$$

Equation of State

$$P = (\Gamma - 1) \rho \varepsilon$$

Energy Equation

$$\frac{\partial e}{\partial t} + \frac{\partial (ev^j)}{\partial x^j} = -\frac{1}{\Gamma} (\rho \varepsilon)^{-1+1/\Gamma} P_{\text{vis}} \frac{\partial v^i}{\partial x^i}$$

Euler Equation

$$\frac{\partial (\rho v_i)}{\partial t} + \frac{\partial (\rho v_i v^j)}{\partial x^j} = -\frac{\partial}{\partial x^i} (P + P_{\text{vis}}) - \rho \frac{\partial \Phi}{\partial x^i}$$
1D Newtonian Wall Shock Problem

Check the validity to treat shock  
(e.g. Hawley, Smarr, Wilson 1984)

Picture

\[ t = 0 \]

\[ \rho \]

\[ v = v_0 \]

\[ P_0 \]

\[ x \]

\[ t = t_1 \]

Shock!

\[ \rho \]

\[ v = v_0 \]

\[ P_0 \]

\[ \rho \]

\[ v = -v_0 \]

\[ P_0 \]

\[ x \]

2 phases of the fluid collide at supersonic speed

Comparison

\[ t = 1.0 \]

Self similar solution
(Analytic solution exists!)

Numerical solution without artificial viscosity

Numerical solution with artificial viscosity

Analytical solution

Can reproduce the analytic solution with the numerical one about 5% of the maximum density.

\[ \rho_0 = 1.0 \times 10^{-3} \]

\[ v_0 = 2v_{\text{sound}} \]

\[ n = 3.33 \]
3. Code Tests

1. Bar formation Test

Confirm the ability of our code to identify bar formation
(Shibata, Karino, Eriguchi 2002)

Initial Condition

<table>
<thead>
<tr>
<th>n</th>
<th>$\Omega_c/\Omega_{eq}$</th>
<th>T/W</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>26.0</td>
<td>0.119</td>
</tr>
</tbody>
</table>

To probe the stability, we put small m=1 and m=2 perturbation in the initial density from equilibrium.

Diagnostics

$\text{m}=2$ Unstable

\[ D = \left\langle e^{im\varphi} \right\rangle_{m=1} \]
\[ Q = \left\langle e^{im\varphi} \right\rangle_{m=2} \]

- Clear evidence for bar structure
- Bar persists without decay for over one surface rotation period.

The amplitude survive without decay until gravitational radiation reaction forces destroy the bar

\[ \sim (R/M)^{5/2}t_{\text{dyn}} \gg t_{\text{dyn}} \]
Final density snapshots in the equatorial plane

Gravitational Waveform using quadrupole formula

Clear evidence for bar structure

Observed from z-axes

Amplitude persists for at least over one surface rotation period
m=1 Dynamical Instability

Confirm the ability of our code to identify the m=1 instability
(Centrella et al. 2001)

Initial Conditions \((n=3.33, \Omega_c/\Omega_{eq}=26)\)

<table>
<thead>
<tr>
<th>Model</th>
<th>T/W</th>
<th>structure</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>0.144</td>
<td>toroidal</td>
</tr>
<tr>
<td>(b)</td>
<td>0.090</td>
<td>spheroidal</td>
</tr>
</tbody>
</table>

To probe stability, we slightly put the m=1 and m=2 perturbation in the initial density from equilibrium.

m=1 and m=2 diagnostics

- D and Q grows up exponentially during the evolution
- Small growth of an m=2 mode

D and Q remains oscillation around zero (absence of exponential growth)
Density snapshots in the equatorial plane

- Clear single spiral arm at the intermediate stage
- Finally the spiral arm is destroyed

Instability rearranges the matter in the star, and, as a consequence, it eliminates the toroidal structure.

Remains equilibrium

Toroidal star triggers $m=1$ instability
4. One–Armed Spiral Instability

Centrally condensed protostellar disk systems are known to experience the m=1 instability

(Picket, Dursen, Davis 1996)

This suggests that in differentially rotating stars

1. Softness of equation of state
2. High degree of differential rotation

might trigger the same m=1 instability

Dependence of m=1 instability on the polytropic index

Initial Conditions (toroidal stars)

<table>
<thead>
<tr>
<th>Model</th>
<th>n</th>
<th>$\Omega_c / \Omega_{eq}$</th>
<th>T/W</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>1</td>
<td>26.0</td>
<td>0.145</td>
</tr>
<tr>
<td>(b)</td>
<td>2</td>
<td>26.0</td>
<td>0.145</td>
</tr>
<tr>
<td>(c)</td>
<td>3</td>
<td>26.0</td>
<td>0.147</td>
</tr>
<tr>
<td>(d)</td>
<td>3.33</td>
<td>26.0</td>
<td>0.144</td>
</tr>
</tbody>
</table>

To probe stability, we put a small m=1 and m=2 perturbation in the initial density from equilibrium.
Diagnostics

- D and Q grows up exponentially during the evolution
- Small growth of an m=2 mode

D and Q remains oscillation around zero
(absence of exponential growth)

Pattern period of the diagnostics

<table>
<thead>
<tr>
<th>Model</th>
<th>m=1[Pc]</th>
<th>m=2[Pc]</th>
<th>Pattern[Pc]</th>
</tr>
</thead>
<tbody>
<tr>
<td>(b)</td>
<td>2.5</td>
<td>1.2</td>
<td>2.4</td>
</tr>
<tr>
<td>(c)</td>
<td>2.0</td>
<td>1.0</td>
<td>2.0</td>
</tr>
<tr>
<td>(d)</td>
<td>1.6</td>
<td>0.7</td>
<td>1.4</td>
</tr>
</tbody>
</table>

m=2 mode is regarded as a higher mode of m=1 due to the same pattern period.
Density snapshots in the equatorial plane

Instability rearranges the matter in the star, and as a consequence, it finally eliminates the toroidal structure.
Gravitational Waves

- Amplitude saturates due to the spiral arm propagating to the surface
- Oscillation period is related to the central rotation period, which suggest that the instability is generated around the density maximum at $t=0$.

(N.B. We cannot determine the compaction of the star in Newtonian gravity.)
**Dependence of m=1 instability on the degree of differential rotation**

### Initial Conditions

<table>
<thead>
<tr>
<th>Model</th>
<th>n</th>
<th>$\Omega_c/\Omega_{eq}$</th>
<th>T/W</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>1</td>
<td>3.60</td>
<td>0.150</td>
</tr>
<tr>
<td>(b)</td>
<td>2</td>
<td>6.95</td>
<td>0.150</td>
</tr>
<tr>
<td>(c)</td>
<td>3</td>
<td>17.0</td>
<td>0.147</td>
</tr>
<tr>
<td>(d)</td>
<td>3.33</td>
<td>26.0</td>
<td>0.144</td>
</tr>
</tbody>
</table>

To probe stability, we put a small m=1 and m=2 perturbation in the initial density from equilibrium.

- D and Q grows up exponentially during the evolution
- Small growth of an m=2 mode
  - D and Q remains oscillation around zero (absence of exponential growth)

### Diagnostics

**Model m=1 [Pc]**

- (c) 1.6
- (d) 1.6

**Model m=2 [Pc]**

- (c) 0.7
- (d) 0.7

**Pattern [Pc]**

- (c) 1.4
- (d) 1.4

m=2 mode is regarded as a higher mode of m=1 due to the same pattern period.
### Interpretation of m=1 instability

#### Model vs. Stability

<table>
<thead>
<tr>
<th>Model</th>
<th>$t_{ep} / t_{sd}$</th>
<th>Stability</th>
</tr>
</thead>
<tbody>
<tr>
<td>II (a)</td>
<td>3.25</td>
<td>Stable</td>
</tr>
<tr>
<td>II (b)</td>
<td>0.37</td>
<td>Unstable</td>
</tr>
<tr>
<td>II (c)</td>
<td>0.037</td>
<td>Unstable</td>
</tr>
<tr>
<td>II (d)</td>
<td>0.0052</td>
<td>Unstable</td>
</tr>
<tr>
<td>III (a)</td>
<td>233</td>
<td>Stable</td>
</tr>
<tr>
<td>III (b)</td>
<td>1.87</td>
<td>Stable</td>
</tr>
<tr>
<td>III (c)</td>
<td>0.037</td>
<td>Unstable</td>
</tr>
<tr>
<td>III (d)</td>
<td>0.0052</td>
<td>Unstable</td>
</tr>
</tbody>
</table>

#### Typical Timescale

- **Sound crossing time**
  \[
  L = 2\pi \omega_{\text{max}}
  \]
  typical length scale for nonaxisymmetric instability

- **Epicyclic period** (ring instability)
  \[
  t_{\text{epicyclic}} = \frac{2\pi}{\kappa_{\text{epicyclic}}|_{\omega=\omega_{\text{max}}}}
  \]
  \[
  \kappa_{\text{epicyclic}}^2 = 2\frac{\Omega}{\omega} \frac{d}{d\omega} (\omega^2 \Omega)
  \]
  depends on the degree of differential rotation

- $m=1$ instability is excited in the star when $t_{ep} < t_{sd}$.

The mechanism of generating $m=1$ instability could be the same to that of eccentric instability.
5. Conclusions

We investigate the dynamical instability of the one–armed spiral \(m=1\) mode in differential rotating stars by means of hydrodynamical simulations in Newtonian gravitation.

- Both soft equation of state and high degree of differential rotation are necessary for the \(m=1\) instability to be triggered.

- \(m=1\) instability rearranges the matter of the star, and, as a consequence, it eliminates the toroidal structure.

- A quasi–periodic gravitational wave persists for several rotation periods, decaying as the spiral arm instability propagates outward to the surface.