Theory of Cosmological Perturbations

Part I

— gauge-invariant formalism —

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PDF files available at
http://www2.yukawa.kyoto-u.ac.jp/~misao/PDF/CPT/

§1. Formulation

- Background spacetime

Friedmann-Lematre-Robertson-Walker metric ($K = \pm 1, 0$):

$$ds^2 = -dt^2 + a^2(t) d\sigma_K^2 \,;$$
$$d\sigma_K^2 = \gamma_{ij} dx^i dx^j = d\chi^2 + \frac{\sinh^2(\sqrt{-K}\,\chi)}{\sqrt{-K}} d\Omega_{(2)}^2$$

Conformal time coordinate:

$$d\eta = \frac{dt}{a} \quad \to \quad ds^2 = a^2(\eta)\left(-d\eta^2 + d\sigma_K^2\right)$$

Energy momentum tensor in the perfect fluid form:

$$T^{\mu\nu} = \rho u^\mu u^\nu + p(g^{\mu\nu} + u^\mu u^\nu)\,; \quad u_\nu dx^\nu = -dt = -a d\eta\,.$$

Friedmann equation: $(G^0{}_0 = R^0{}_0 - \frac{1}{2}R = 8\pi G T^0{}_0)$

$$\left(\frac{\dot a}{a}\right)^2 + \frac{K}{a^2} = \frac{8\pi G}{3}\rho\,, \quad \dot\rho + 3\frac{\dot a}{a}(\rho + p) = 0 \quad \left(\dot a = \frac{da}{dt}\right)$$
$$\Updownarrow$$
$$\left(\frac{a'}{a}\right)^2 + K = \frac{8\pi G}{3}\rho\, a^2\,, \quad \rho' + 3\frac{a'}{a}(\rho + p) = 0 \quad \left(a' = \frac{da}{d\eta}\right).$$

Hereafter, we use the symbols: $H = \frac{\dot a}{a}\,, \quad \mathcal{H} = \frac{a'}{a}$

• Spatial harmonics

· <u>scalar harmonics</u>

$$\left(\Delta + k^2\right) Y_{\boldsymbol{k}} = 0 \qquad (\Delta \equiv \gamma^{ij} D_i D_j).$$

$$k^2 \geq (-K) \qquad \text{for} \quad K \leq 0 \quad (k^2 : \text{ continuous})$$
$$k^2 = n(n+2)K \quad \text{for} \quad K > 0 \quad (n = 0, 1, 2, \cdots)$$

examples

$K = 0$:

$$Y_{\boldsymbol{k}} = e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\,; \quad k^2 = |\boldsymbol{k}^2|\,,$$

or

$$Y_{\boldsymbol{k}} = j_\ell(k\chi) Y_{\ell m}(\Omega)\,; \quad \boldsymbol{k} = (k, \ell, m).$$

$K = -1$:

$$Y_{\boldsymbol{k}} = f_{k\ell}(\chi) Y_{\ell m}(\Omega)\,; \quad \boldsymbol{k} = (k, \ell, m).$$

$$f_{k\ell}(\chi) = \frac{1}{\sinh\chi} P^{-\ell-1/2}_{ip-1/2}(\cosh\chi)\,; \quad p \equiv \sqrt{k^2 - 1}\,.$$

· <u>vector harmonics</u> (index $\boldsymbol{k}$ omitted)

scalar type:

$$Y_i = -k^{-1} D_i Y\,, \quad \big(\Delta + (k^2 - 2K)\big)\, Y_i = 0\,.$$

vector (transverse) type:

$$D_i Y^{(1)i} = 0\,, \quad \big(\Delta + k^2\big)\, Y^{(1)i} = 0\,;$$

$$k^2 = (n(n+2) - 1)K \text{ for } K > 0 \; (n = 1, 2, \cdots)\,, \quad k^2 \geq (-2K) \text{ for } K \leq 0\,.$$

· <u>tensor harmonics</u>

scalar (traceless) type:

$$Y_{ij} = \left(k^{-2} D_i D_j + \frac{1}{3}\gamma_{ij}\right) Y\,, \quad \big(\Delta + k^2 - 6K\big)\, Y_{ij} = 0\,.$$

vector (traceless) type (with index $(\sigma, \boldsymbol{k})$: $\sigma = 1, 2$):

$$Y^{(1)}_{ij} = -\frac{1}{2k}\left(D_i Y^{(1)}_j + D_j Y^{(1)}_i\right)\,, \quad \big(\Delta + k^2 - 4K\big)\, Y^{(1)}_{ij} = 0\,.$$

tensor (transverse-traceless) type (with index $(\sigma, \boldsymbol{k})$: $\sigma = 1, 2$):

$$D^j Y^{(2)}_{ij} = Y^{(2)i}{}_i = 0\,, \quad \big(\Delta + k^2\big)\, Y^{(2)}_{ij} = 0\,;$$

$$k^2 = (n(n+2) - 2)K \text{ for } K > 0 \; (n = 2, 3, \cdots)\,, \quad k^2 \geq (-3K) \text{ for } K \leq 0\,.$$

• Perturbation variables

· scalar type perturbations

metric:

$$\tilde{g}_{00} = -a^2(1 + 2A(\eta)Y)\,, \qquad \tilde{g}_{0j} = -a^2 B Y_j \ \left(= a^2 \frac{B}{k} D_j Y\right),$$

$$\tilde{g}_{ij} = a^2\left[(1 + 2H_L Y)\gamma_{ij} + 2H_T Y_{ij}\right]$$

$$= a^2\left[(1 + 2\mathcal{R}Y)\gamma_{ij} + 2H_T \frac{1}{k^2} D_i D_j Y\right]\ ; \quad \mathcal{R} \equiv H_L + \frac{1}{3}H_T\,.$$

matter:

$$\tilde{T}^{\mu}{}_{\nu} = \tilde{\rho}\tilde{u}^{\mu}\tilde{u}_{\nu} + \tilde{\tau}^{\mu}{}_{\nu}\,; \quad \tilde{\tau}^{\mu}{}_{\mu} = \tilde{p}(\delta^{\mu}_{\nu} + \tilde{u}^{\mu}\tilde{u}_{\nu}) + \pi^{\mu}{}_{\nu} \quad (\pi^{\mu}{}_{\mu} = \pi^{\mu}{}_{\nu}u^{\nu}u_{\mu} = 0),$$

$$\tilde{\rho} = \rho(1 + \delta\, Y),$$

$$\tilde{v}^i \equiv \frac{\tilde{u}^i}{u^0} = v\, Y^i \ \left(= -\frac{v}{k}D^i Y\right) \Leftrightarrow \ \tilde{u}_j = a(v - B)Y_j \quad (u_0 = -a)$$

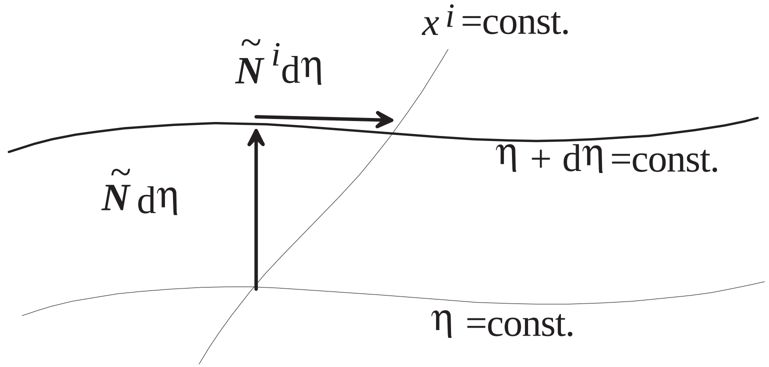
$$\left(\tilde{u}^0 = u^0(1 - AY)\,,\ \tilde{u}_0 = u_0(1 + AY) \quad \text{from} \quad \tilde{u}^{\mu}\tilde{u}_{\mu} = -1\right)$$

$$\tilde{\tau}^i{}_j = p\left[\delta^i_j(1 + \pi_L Y) + \pi_T Y^i{}_j\right]$$

· <u>geometrical meaning</u>

(3 + 1)-decomposition:

$$ds^2 = a^2(\eta)d\hat{s}^2\,; \quad d\hat{s}^2 = -\tilde{N}^2 d\eta^2 + \tilde{\gamma}_{ij}(dx^i + \tilde{N}^i d\eta)(dx^j + \tilde{N}^j d\eta)\,.$$



$$\tilde{n}_\mu dx^\mu = -\tilde{N} d\eta \quad \cdots \text{ hypersurface normal :}$$

$$\tilde{N} = 1 + AY \quad \cdots \text{ lapse function,}$$

$$\tilde{N}^i = -BY^i \quad \cdots \text{ shift vector.}$$

extrinsic curvature:

$$\hat{K}_{ij} = (H_T' - kB)Y_{ij} = k\sigma_g Y_{ij}$$

$$\sigma_g \equiv \frac{1}{k}H_T' - B \quad \cdots \text{ shear of } \eta = \text{const. hypersurface.}$$

"expansion":

$$\tilde{\theta} = 3H(1+\mathcal{K}_g Y) = \frac{3}{a}\mathcal{H}(1+\mathcal{K}_g Y)\,;$$
$$\mathcal{K}_g = -A + \mathcal{H}^{-1}\left(\mathcal{R}' - \frac{k}{3}\sigma_g\right).$$

$$\tilde{\theta} d\tau_{\rm prop} = \tilde{\theta}(1+AY)\, a\, d\eta = \big(3\mathcal{H} + (3\mathcal{R}' - k\sigma_g)Y\big)d\eta\,.$$

3-curvature:

$$\delta^s \hat{R}^i{}_{jmn} = D_m \delta^s\Gamma^i{}_{jn} - D_n \delta^s \Gamma^i{}_{jm}\,;$$
$$\delta^s\Gamma^i{}_{jm} = \frac{1}{2}\gamma^{i\ell}\left(D_m\delta\gamma_{\ell j} + D_j\delta\gamma_{\ell m} - D_\ell \delta\gamma_{jm}\right),$$
$$\delta\gamma_{ij} = 2H_L\gamma_{ij} + 2H_T Y_{ij}\,.$$

In particular,

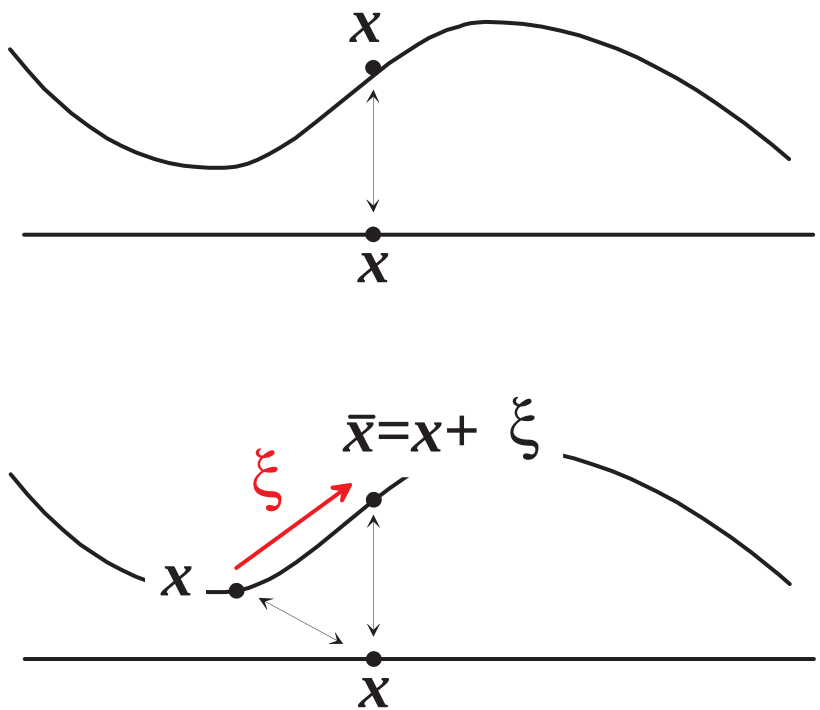
$${}^s\hat{R} = 6K + 4(k^2 - 3K)\left(H_L + \frac{1}{3}H_T\right)Y = 6K + 4(k^2-3K)\mathcal{R}Y\,.$$

$\mathcal{R}$ $\cdots$ intrinsic curvature perturbation (potential).

· Gauge transformation properties

$$x^\mu \to \bar{x}^\mu = x^\mu + \xi^\mu$$

$$\Rightarrow \quad \begin{cases} \bar{\eta} = \eta + TY \,, \\ \bar{x}^i = x^i + LY^i \,. \end{cases}$$



This induces a gauge transformation,

$$\bar{g}_{\mu\nu} = g_{\mu\nu} - (\xi_{\mu;\nu} + \xi_{\nu;\mu})$$

$$\left[\text{in general,} \quad \bar{Q}_{\{A\}} = Q_{\{A\}} - \pounds_\xi Q_{\{A\}} \,. \quad (\{A\} \cdots \text{ spacetime indices}) \right]$$

metric: $\left(\mathcal{H} = \frac{a'}{a}\right)$

$$\bar{A} = A - T' - \mathcal{H}\,T \qquad (\ \cdots \text{ lapse }\ )$$

$$\bar{B} = B + L' + kT$$

$$\bar{H}_L = H_L - \frac{k}{3}L - \mathcal{H}\,T$$

$$\bar{H}_T = H_T + kL$$

$$\bar{\sigma}_g = \sigma_g - kT \qquad \left(\sigma_g = \frac{1}{k}H_T' - B \ \cdots \text{ shear}\right)$$

$$\bar{\mathcal{R}} = \mathcal{R} - \mathcal{H}\,T \qquad \left(\mathcal{R} = H_L + \frac{1}{3}H_T \ \cdots \text{ curvature}\right)$$

matter:

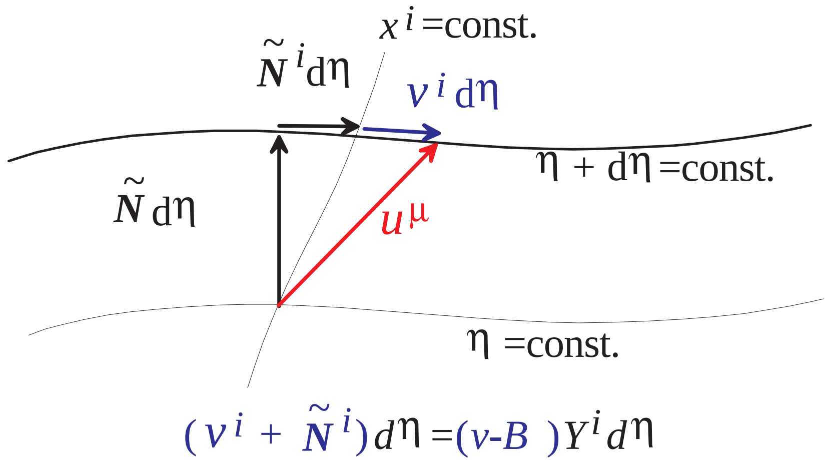
$$\bar{\delta} \ = \ \delta + 3(1+w)\mathcal{H}\,T \qquad \left(w \equiv \frac{p}{\rho}\right)$$

$$\bar{v} \ = \ v + L'$$

$$\Leftrightarrow \qquad (\bar{v} - \bar{B}) = (v - B) - kT$$

$$\bar{\pi}_L \ = \ \pi_L + 3c_w^2 \frac{1+w}{w}\mathcal{H}\,T \qquad \left(c_w^2 \equiv \frac{p'}{\rho'}\right)$$

$$\bar{\pi}_T \ = \ \pi_T$$



⋆ Blue quantities depend only on the choice of time-slicing.

· gauge-invariant variables

(4 metric variables − 2 gauge variables = 2 degrees of freedom)

$$\Psi \equiv A - \frac{1}{k}\left(\sigma_g' + \mathcal{H}\,\sigma_g\right)\,, \qquad \left(\sigma_g = \frac{1}{k}H_T' - B\right)$$

$$\Phi \equiv \mathcal{R} - \frac{\mathcal{H}}{k}\,\sigma_g\,. \qquad \left(\mathcal{R} = H_L + \frac{1}{3}H_T\right)$$

$$\Delta_s \equiv \delta + 3\frac{\mathcal{H}}{k}(1+w)\sigma_g$$

$$V \equiv v - \frac{1}{k}H_T' = (v - B) - \sigma_g \qquad \cdots \text{ shear of 4-velocity } \tilde{u}^\mu$$

$$\Gamma \equiv \pi_L - \frac{c_w^2}{w}\delta \qquad \left(p\Gamma = \delta p - c_w^2 \delta\rho \right) \cdots \text{ entropy perturbation}$$

$$\Pi_T \equiv \pi_T \qquad \cdots \text{ anisotropic stress perturbation.}$$

$\Psi = A$, $\Phi = \mathcal{R}$, $\Delta_s = \delta$, $V = v - B$ on $\sigma_g = 0$ hypersurface.

Shear-free slicing or Newton slicing

Other popular choices:

$$\mathcal{R}_c \equiv \mathcal{R} - \frac{\mathcal{H}}{k}(v - B) = \Phi - \frac{\mathcal{H}}{k} V \,, \qquad (\sigma_g)_c = V \,,$$

$$A_c \equiv A - \frac{1}{k}\left[(v - B)' + \mathcal{H}(v - B)\right] = \Psi - \frac{1}{k}\left[V' + \mathcal{H}\, V\right]$$

$$\Delta \equiv \delta + 3\frac{\mathcal{H}}{k}(1 + w)(v - B) = \Delta_s + 3\frac{\mathcal{H}}{k}(1 + w)V \,,$$

Velocity-orthogonal or 'comoving' slicing $(v - B = 0)$

$$\zeta \equiv \mathcal{R} + \frac{1}{3(1+w)}\,\delta = \mathcal{R}_c + \frac{1}{3(1+w)}\,\Delta$$

Curvature perturbation on uniform density slices $(\delta = 0)$

$$\Delta_f \equiv \delta + 3(1 + w)\mathcal{R} = \Delta + 3(1 + w)\mathcal{R}_c = 3(1 + w)\zeta$$

Density perturbation on flat slices $(\mathcal{R} = 0)$

· vector type perturbations

$$\tilde{N}^{(1)}_j = -B^{(1)} Y^{(1)}_j \,, \qquad \tilde{v}^{(1)j} = a\tilde{u}^{(1)j} = v^{(1)} Y^{(1)j} \,,$$
$$\tilde{\gamma}^{(1)}_{ij} = \gamma_{ij} + 2H^{(1)}_T Y^{(1)}_{ij} \,. \qquad \tilde{\tau}^{(1)i}{}_j = p\left[\delta^i_j + \pi^{(1)}_T Y^{(1)}_{ij}\right] \,.$$

gauge transformation:

$$\bar{x}^j = x^j + L^{(1)} Y^{(1)j} \quad \Rightarrow \quad \begin{cases} \bar{B}^{(1)} = B^{(1)} + L^{(1)\prime} \,, & \bar{v}^{(1)} = v^{(1)} + L^{(1)\prime} \,, \\ \bar{H}^{(1)}_T = H^{(1)}_T + kL^{(1)} \,. & \bar{\pi}^{(1)}_T = \pi^{(1)}_T \,. \end{cases}$$

gauge-invariant variables:

(2 metric variables − 1 gauge variable = 1 degree of freedom)

$$\sigma^{(1)}_g = \frac{1}{k} H^{(1)\prime}_T - B^{(1)} \,,$$
$$V^{(1)} = v^{(1)} - B^{(1)} \,, \qquad \Pi^{(1)}_T = \pi^{(1)}_T \,.$$

· tensor type perturbations

$$\tilde{\gamma}^{(2)}_{ij} = \gamma_{ij} + 2H^{(2)}_T Y^{(2)}_{ij} \,. \qquad \tilde{\tau}^{(2)i}{}_j = p\left[\delta^i_j + \pi^{(2)}_T Y^{(2)}_{ij}\right] \,.$$

Both $H^{(2)}_T$ and $\pi^{(2)}_T$ are gauge-invariant by themselves.

§2. Einstein equations in terms of gauge-invariant variables

- Scalar type perturbations

Einstein equations:

$$\delta G^{\mu}{}_{\nu} = 8\pi G \delta T^{\mu}{}_{\nu} \quad :$$

$$\delta G^{0}{}_{0} = \frac{2}{a^2}\left[3\mathcal{H}^2 A + \mathcal{H}\, k\sigma_g - 3\mathcal{H}\,\mathcal{R}' - (k^2 - 3K)\mathcal{R}\right] Y$$

$$\delta G^{0}{}_{j} = \frac{2}{a^2}\left[k\mathcal{H}\, A - k\mathcal{R}' + K\sigma_g\right] Y_j$$

$$\delta G^{i}{}_{j} = \frac{2}{a^2}\left[\left(2\mathcal{H}' + \mathcal{H}^2 - \frac{k^2}{3}\right) A + \mathcal{H}\, A' + \frac{k}{3}\left(\sigma_g' + 2\mathcal{H}\,\sigma_g\right) - \mathcal{R}'' - 2\mathcal{H}\mathcal{R}' - \frac{1}{3}(k^2 - 3K)\mathcal{R}\right] \delta^i_j Y + \frac{1}{a^2}\left[-k^2 A + k(\sigma_g' + 2\mathcal{H}\sigma_g) - k^2\mathcal{R}\right] Y^{i}{}_{j}$$

$$\delta T^{0}{}_{0} = -\rho\,\delta\, Y\,,$$

$$\delta T^{0}{}_{j} = (\rho + p)(v - B) Y_j\,,$$

$$\delta T^{i}{}_{j} = (p\Gamma + c_w^2 \rho\,\delta)\delta^i_j Y + p\pi_T Y_{ij}\,. \qquad (\text{N.B.}\ \ p\Gamma + c_w^2\rho\,\delta = \delta p)$$

The above equations depend only on the choice of time slicing.

ie, they are spatially gauge-invariant.

* spatial gauge-invariance $\sim$ $(3+1)$-decomposition

$$
\begin{aligned}
ds^2 = g_{\mu\nu}dx^\mu dx^\nu &= (-n_\mu n_\mu + \gamma_{\mu\nu})dx^\mu dx^\nu \\
&= -(n_\mu dx^\mu)^2 + \gamma_{\mu\nu}dx^\mu dx^\nu \\
&= -N^2 dt^2 + \gamma_{ij}(dx^i + N^i dt)(dx^j + N^j dt)\,;
\end{aligned}
$$

$$
\gamma_{\mu\nu} \equiv g_{\mu\nu} + n_\mu n_\nu\,, \quad n_\mu dx^\mu = -N dt\,.
$$

$$
\begin{aligned}
\delta(G^{\mu\nu}n_\mu n_\nu) &= \delta(-NG^0{}_\nu n^\nu) \\
&= \delta(-G^0{}_0 - NG^0{}_i n^i) = -\delta G^0{}_0 + \text{higher order}\,,
\end{aligned}
$$

$$
\begin{aligned}
\delta(n_\mu G^{\mu\nu}\gamma_{\nu j}) &= \delta(n_\mu G^{\mu\nu} g_{\nu j}) \\
&= \delta(-NG^0{}_j) = -N\delta G^0{}_j + \text{higher order}\,.
\end{aligned}
$$

Energy-momentum conservation:

$$\delta({T_0}^{\nu}{}_{;\nu}) = 0 \ :$$
$$\delta' + 3\mathcal{H}\left[(c_w^2 - w)\delta + w\Gamma\right] + k(1+w)(v-B) + (1+w)(3\mathcal{R}' - k\sigma_g) = 0\,.$$

$$\delta({T_j}^{\nu}{}_{;\nu}) = 0 \ :$$
$$(v-B)' + \mathcal{H}(1-3c_w^2)(v-B) = kA + k\frac{c_w^2\delta + w\Gamma}{1+w} - \frac{2k}{3}\frac{w}{1+w}\left(1 - \frac{3K}{k^2}\right)\pi_T\,.$$

⋆ These are also spatially gauge-invariant.

⋆ To obtain gauge-invariant equations, it is simplest to set a gauge (time-slicing) condition that fixes the time slices completely.

gauge-invariance ⇔ complete gauge-fixing

Gauge-invariant quantities are NOT necessarily directly related to observables

⋆ Advantage of gauge-invariant formalism is because one does not have to choose a particular gauge while dealing only with physical degrees of freedom, not because the values of the gauge-invariant variables are physical by themselves.

Example: choose $\sigma_g = 0$ (Newton slicing), so that $A = \Psi$, $\mathcal{R} = \Phi$, $\delta = \Delta_s$, $v - B = V$.

$$\delta G^0{}_0 = 8\pi G \delta T^0{}_0 : \quad \frac{1}{a^2}\left[3\mathcal{H}(\mathcal{H}\Psi - \Phi') - (k^2 - 3K)\Phi\right] = -4\pi G \rho \Delta_s \,, \tag{1}$$

$$\delta G^0{}_j = 8\pi G \delta T^0{}_j : \quad \frac{k}{a^2}\left[\mathcal{H}\Psi - \Phi'\right] = 4\pi G(\rho + p)V \,, \tag{2}$$

$$(\delta G^i{}_j = 8\pi G \delta T^i{}_j)_{\text{traceless}} : \quad -\frac{k^2}{a^2}\left[\Psi + \Phi\right] = 8\pi p \Pi_T \,. \tag{3}$$

$$(\delta G^i{}_j = 8\pi G \delta T^i{}_j)_{\text{trace}} :$$
$$\frac{1}{a^2}\left[\left(2\mathcal{H}' + \mathcal{H}^2 - \frac{k^2}{3}\right)\Psi + \mathcal{H}\,\Psi' - \Phi'' - 2\mathcal{H}\Phi' - \frac{1}{3}(k^2 - 3K)\Phi\right] = 4\pi G p \left[\Gamma + c_w^2 \rho\, \Delta_s\right] \,. \tag{4}$$

Combining Eqs. (1) and (2) [Hamiltonian and momentum constraints] gives

$$\frac{k^2 - 3K}{a^2}\Phi = 4\pi G \rho \left[\Delta_s + 3\frac{\mathcal{H}}{k}(1 + w)V\right] = 4\pi G \rho \Delta \,. \tag{5}$$

- ⋆ Eqs. (3) and (5) algebraically determine Φ and Ψ in terms of Δ and Π_T.
- ⋆ Combining Eqs. (1), (3) and (4), one can derive 2nd order differential equation for Φ.
- ⋆ Alternatively, one may appeal to the energy-momentum conservation laws.
  ($\because$ contracted Bianchi identities)

$$\delta(T_j{}^\nu{}_{;\nu}) = 0 \text{ on shear-free (Newton) slices :}$$

$$V' + \mathcal{H}(1-3c_w^2)V = k\Psi + k\frac{c_w^2\Delta_s + w\Gamma}{1+w} - \frac{2k}{3}\frac{w}{1+w}\left(1-\frac{3K}{k^2}\right)\pi_T$$

$$\left(\Delta_s = \Delta - 3\frac{\mathcal{H}}{k}(1+w)V\right)$$

$$\Leftrightarrow \quad V' + \mathcal{H}\,V = k\Psi + k\frac{c_w^2\Delta + w\Gamma}{1+w} - \frac{2k}{3}\frac{w}{1+w}\left(1-\frac{3K}{k^2}\right)\pi_T \tag{6}$$

$$\left(\Psi = -\Phi - \frac{8\pi G\rho\, a^2 w}{k^2}\Pi_T\,, \quad \Phi = \frac{4\pi G\rho\, a^2}{k^2-3K}\Delta\right)$$

$$\delta(T_0{}^\nu{}_{;\nu}) = 0 \text{ on comoving slices :}$$

$$\Delta' + 3\mathcal{H}\left[(c_w^2 - w)\Delta + w\Gamma\right] + (1+w)\big(3\mathcal{R}_c' + kV\big) = 0\,; \quad \mathcal{R}_c = \Phi - \frac{\mathcal{H}}{k}V$$

$$\left(\frac{k}{a^2}\left[\mathcal{H}\Psi - \Phi'\right] = 4\pi G(\rho+p)V\,, \quad V' = \cdots\,,\right)$$

$$\Leftrightarrow \quad \Delta' - 3w\mathcal{H}\Delta = -\left(1-\frac{3K}{k^2}\right)\left[(1+w)kV + 2\mathcal{H}\,w\Pi_T\right] \tag{7}$$

Combining Eqs. (6) and (7) gives a 2nd order differential equation for Δ.

$$\Rightarrow \qquad \Delta'' + \left(1 + 3(c_w^2 - 2w)\right) \mathcal{H}\, \Delta' + \Bigg[c_s^2 (k^2 - 3K)$$

$$-4\pi G \rho a^2 \left((1+3w)(1-w) + 6(w - c_w^2) \right) + 3(4w - 3c_w^2) K \Bigg] \Delta = \mathcal{S}_c$$

$$\mathcal{S}_c = -(k^2 - 3K) \frac{\left[(\delta p)_c - c_s^2 (\delta \rho)_c \right]}{\rho} - 2 \left(1 - \frac{3K}{k^2} \right) \mathcal{H}\, w \Pi_T'$$

$$+2 \left[(3w^2 - 2w + 3c_w^2) \mathcal{H}^2 + 4(w - c_w^2) K + \frac{c_w^2 k^2}{3} \right] \left(1 - \frac{3K}{k^2} \right) \Pi_T$$

Master equation for Δ ($\sim$ equation for sound waves)

In the Newtonian, non-relativistic limit ($w \ll 1$, $c_s^2 \ll 1$),

$$\Delta'' + \mathcal{H}\Delta' + \left(c_s^2 k^2 - 4\pi G\rho\, a^2\right)\Delta \approx \mathcal{S}_c\,,$$
$$\Leftrightarrow \quad \ddot{\Delta} + 2H\dot{\Delta} + \left(\frac{c_s^2 k^2}{a^2} - 4\pi G\rho\right)\Delta \approx \frac{\mathcal{S}_c}{a^2}$$

· dispersion relation:

$$\omega_{\rm eff}^2 = \frac{c_s^2 k^2}{a^2} - 4\pi G\rho$$

· Jeans instability: (for $c_w^2 \equiv p'/\rho' = \partial p/\partial\rho \equiv c_s^2$)

gravitationally unstable for $\quad \dfrac{k}{a} < \left(\dfrac{k}{a}\right)_J \equiv \sqrt{\dfrac{4\pi G\rho}{c_s^2}} \quad$ or $\quad \lambda > \lambda_J \equiv \sqrt{\dfrac{\pi c_s^2}{G\rho}}$

The above picture is valid only on small scales: $\dfrac{2\pi}{H} \gg \lambda$

- Vector type perturbations

$$\delta G^0{}_j = 8\pi G \delta T^0{}_j \ : \qquad -\frac{k^2 - 2K}{2a^2}\sigma_g^{(1)} = 8\pi G(1+w)\rho\, V^{(1)}\,.$$
$$\delta G^i{}_j = 8\pi G \delta T^i{}_j \ : \qquad \sigma_g^{(1)\prime} + 2\mathcal{H}\sigma_g^{(1)} = 8\pi \rho a^2 \frac{w}{k}\Pi_T^{(1)}\,.$$

Either combining these two or directly from the momentum conservation,

$$\delta(T_i{}^{\mu}{}_{;\mu}) = 0 : \qquad V^{(1)\prime} + (1 - 3c_w^2)\mathcal{H}\, V^{(1)} = -\frac{k^2 - 2K}{2k}\frac{w}{1+w}\Pi_T^{(1)}\,.$$

For $\Pi_T^{(1)} = 0$ (adiabatic),

$$\sigma_g^{(1)} \propto \frac{1}{a^2}\,, \quad V^{(1)} \propto \frac{1}{\rho(1+w)a^4}\left(\propto \frac{1}{a^{1-3w}} \propto \frac{1}{\rho^{w/(1+w)}a} \quad \text{for } w = \text{const.}\right)\,.$$

This is just the vorticity conservation law.

Vorticity Ω:

$$\Omega_{\mu\nu} \equiv P_\mu{}^\alpha u_{[\alpha;\beta]} P^\beta{}_\nu\,; \quad P^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu\,.$$

$$\Rightarrow \ \Omega_{ij} = a\,k\,V^{(1)} Z_{ij} \quad \left(Z_{ij} = \frac{1}{k} D_{[j} Y^{(1)}{}_{i]} \right), \quad \Omega^{ij}\Omega_{ij} = \frac{k^2}{a^2} |V^{(1)}|^2 Z_{ij} Z^{ij}$$

$$\rho^{w/(1+w)} S|\Omega| = \text{const.} \ (S\text{: area} \propto a^2)$$

Shear σ:

$$\sigma_{\mu\nu} \equiv P_\mu{}^\alpha u_{(\alpha;\beta)} P^\beta{}_\nu = -ak(V^{(1)} - \sigma_g^{(1)}) Y_{ij}^{(1)}$$

<u>vector type perturbations induce both shear & vorticity</u>

$\binom{0}{j}$-component: $$-\frac{k^2 - 2K}{6(\mathcal{H}^2 + K)} \sigma_g^{(1)} = (1+w) V^{(1)}\,.$$

$$k^2 \ll H^2 a^2 = \mathcal{H}^2 \text{ (superhorizon)}: \quad \sigma_g^{(1)} \gg V^{(1)} \ \Rightarrow \ \sigma_{ij}^2 \gg \Omega_{ij}^2\,,$$
$$k^2 \gg H^2 a^2 = \mathcal{H}^2 \text{ (subhorizon)}: \quad \sigma_g^{(1)} \ll V^{(1)} \ \Rightarrow \ \sigma_{ij}^2 \approx \Omega_{ij}^2\,.$$

- Tensor type (gravitational wave) perturbations

$$\delta G^i{}_j = 8\pi G \delta T^i{}_j \ :$$

$$H_T^{(2)\prime\prime} + 2\mathcal{H}\, H_T^{(2)\prime} + (k^2 + 2K) H_T^{(2)} = 8\pi G\, p\, a^2 \Pi_T^{(2)}$$

$$\Leftrightarrow \qquad \ddot{H}_T^{(2)} + 3H\, \dot{H}_T^{(2)} + \frac{k^2 + 2K}{a^2} H_T^{(2)} = 8\pi G\, p\, \Pi_T^{(2)}$$

For $K = 0$, $\Pi_T^{(2)} = 0$, this is the same as the field equation for a massless minimal scalar:

$$\ddot{\varphi} + 3H\dot{\varphi} + \frac{k^2}{a^2}\varphi = 0 \,.$$

⋆ On superhorizon scales ($k^2 \ll H^2 a^2 = \mathcal{H}^2$),

$$H_T^{(2)} \propto \begin{cases} \text{const.} & \cdots \ \text{growing mode} \\ \displaystyle\int^\eta \frac{d\eta'}{a^2} & \cdots \ \text{decaying mode} \end{cases}$$

tensor perturbation ∼ a homogeneous, anisotropic universe

⋆ On subhorizon scales ($k^2 \gg H^2 a^2 = \mathcal{H}^2$),

$$H_T^{(2)} \propto \frac{1}{a} e^{ik\eta} \quad \Rightarrow \quad \rho_{GW} \propto |\dot{H}_T^{(2)}|^2 \propto \frac{1}{a^4}$$

§3. Adiabatic perturbations on superhorizon scales

· Spatially flat universe ($K = 0$) is a good approximation in the early universe. Then,

$$(\rho\, a^3\Delta)'' + (1+3c_w^2)\mathcal{H}(\rho\, a^3\Delta)' + \left[c_s^2 k^2 - \frac{3}{2}(1+w)\mathcal{H}^2\right](\rho\, a^3\Delta) = \rho\, a^3\mathcal{S}_c\,.$$

Here, $c_w^2 \equiv p'/\rho'\,,\ c_s^2 \equiv (\partial p/\partial\rho)_{\text{comoving}} \Leftrightarrow (\delta p)_c = c_s^2(\delta\rho)_c + \text{entropy perturbation}$

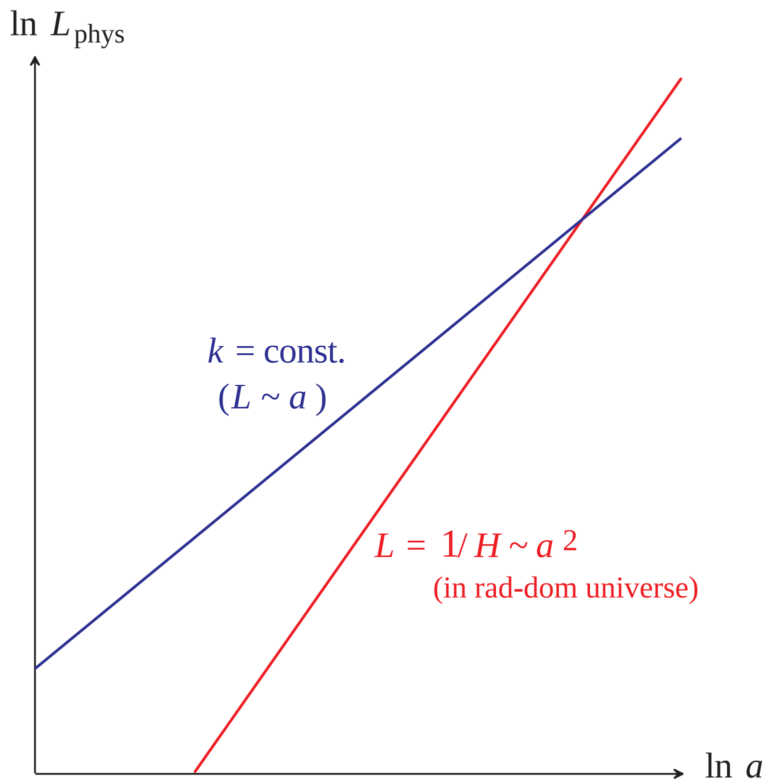
· Also, all cosmologically relevant scales exceed Hubble horizon.

$$\lambda = \frac{2\pi a}{k} \propto a\,,$$

$$H^{-1} \propto \rho^{-1/2}$$

$$\propto \begin{cases} a^{3/2} & \text{for dust } (w=0) \\ a^2 & \text{for radiation } (w=1/3) \end{cases}$$

$$\Rightarrow \quad \frac{\lambda}{H^{-1}} \to \infty \quad \text{for } a \to 0$$



For $\mathcal{S}_c = 0$ (adiabatic perturbation), one particular solution in the limit $c_s^2 k^2 \to 0$ is

$$\rho\, a^3 \Delta \propto \frac{\mathcal{H}}{a} \quad \Rightarrow \quad \Delta \propto \frac{1}{\mathcal{H}\, a^2} = \frac{1}{H\, a^3} \quad \cdots \text{ decaying mode}$$

Using the Wronskian for 2 independent solutions, the other solution is found as*

$$\rho\, a^3 \Delta \propto \frac{\mathcal{H}}{a} \int_0^\eta (1+w) a^2 d\eta'$$

$$\Rightarrow \quad \Delta \propto \frac{1}{\mathcal{H}\, a^2} \int_0^\eta (1+w) a^2 d\eta' = \frac{1}{H\, a^3} \int_0^t (1+w) a dt' \quad \cdots \text{ growing mode}$$

$$\left(\Delta \propto \eta^2 \quad \text{for } w = \text{const.}\right)$$

*$u,\ v = 2$ independent solutions of $f'' + Af' + Bf = 0$ $(f = \rho a^3 \Delta)$:

$$W \equiv u'v - uv' \,; \quad W' = u''v - uv'' = -A(u'v - uv') = -AW$$

$$\Rightarrow \; W \propto \exp[-\int^\eta A d\eta] \quad \left(A = (1+3c_w^2)\mathcal{H} = -\left[\frac{(\rho+p)'}{\rho+p} + 2\mathcal{H}\right] \right)$$

$$\frac{W}{v^2} = \frac{u'}{v} - \frac{uv'}{v^2} = \left(\frac{u}{v}\right)' \;\Rightarrow\; u = v \int^\eta \frac{W}{v^2} d\eta$$

- Conservation of growing mode amplitude

Let us set

$$\Delta = C_1 \frac{k^2}{\mathcal{H}\,a^2}\int_0^\eta (1+w)a^2 d\eta'$$

This gives, from the Hamiltonian and momentum constraints,

$$\Phi = \frac{3}{2}\frac{\mathcal{H}^2}{k^2}\Delta = \frac{3}{2}C_1\frac{\mathcal{H}}{a^2}\int_0^\eta (1+w)a^2 d\eta'\,, \quad \frac{\mathcal{H}}{k}V = -\frac{2}{3}\frac{1}{(1+w)\mathcal{H}}\left(\Phi' + \mathcal{H}\,\Phi\right)$$

$$\Rightarrow \quad \mathcal{R}_c = \Phi - \frac{\mathcal{H}}{k}V = C_1\,.$$

The growing mode amplitude of $\mathcal{R}_c$ stays constant on superhorizon scales.

The condition for the linear perturbation theory to be valid is $\mathcal{R}_c = C_1 \ll 1$.

linear theory is applicable up to $t = 0$ singularity
if only the growing mode is present.

For the decaying mode,

$$\Delta = C_2 \frac{k^2}{\mathcal{H}\, a^2}\,, \quad \mathcal{R}_c' = -\mathcal{H}\frac{c_s^2\Delta + w\Gamma_c}{1+w}$$

$$\Rightarrow \; \mathcal{R}_c = C_2 \int_\eta^{\eta_f} \frac{c_s^2 k^2}{(1+w)a^2} d\eta' \quad \left(\Gamma_c \equiv \frac{(\delta p)_c - c_s^2(\delta\rho)_c}{\rho} = 0\right)$$

where

$$\eta_f = \begin{cases} \infty & \text{for} \quad w \geq -1/3 \\ 0 \text{ (finite)} & \text{for} \quad w < -1/3 \end{cases}$$

⋆ Decaying mode amplitude for $\mathcal{R}_c$ is not constant.

⋆ The standard lore that $\mathcal{R}_c =$ const. on superhorizon scales is not strictly correct.

⋆ It is correct only if the decaying mode can be ignored.

(This depends on how $\dfrac{c_s^2}{(1+w)a^2}$ behaves in time.)

- Growing mode solution for several other variables (assuming w =const.)

$$\left(a \propto \eta^{2/(3w+1)}, \qquad \mathcal{H} = \frac{2}{(3w+1)\eta}\right)$$

$$\zeta = \mathcal{R}_c + \frac{\Delta}{3(1+w)} \approx C_1 \quad \cdots \quad \zeta \ (= \mathcal{R} \text{ on uniform density slice) is also constant.}$$

This is true not only for GR but also for any metric theory. (Wands et al. 2000)

$$\Delta \approx \frac{(3w+1)^2(1+w)}{2(3w+5)} C_1 (k\eta)^2 \,, \quad A_c = \frac{\mathcal{R}_c'}{\mathcal{H}} = -\frac{c_s^2}{1+w}\Delta \approx -\frac{(3w+1)^2 w}{2(3w+5)} C_1 \,(k\eta)^2 \,,$$

$$(\sigma_g)_c = -V \approx \frac{3w+1}{3w+5} C_1 \, k\eta \,,$$

$$-\Psi = \Phi \approx \frac{3(1+w)}{3w+5} C_1 \,, \qquad \Delta_s \approx -2\Phi \approx -\frac{6(1+w)}{3w+5} C_1$$

⋆ Φ (curvature perturbation on Newton slices) or Ψ (Newton potential) stays constant for w =const., but varies in time when w changes.

⋆ In particular, during inflation when $w \approx -1$, the amplitude of Φ stays very small, but it grows significantly at the end of inflation.

Inflation "hides" (quasi-)nonlinear perturbations ($C_1 \sim 1$) in Newton slicing.