The evolution equation for P^τ_A.

$$
\begin{aligned}
\left(\frac{dP^\tau_{1\Theta}}{d\tau}\right)_{\leq 3PN} &= -\epsilon^2\frac{m_1m_2}{r_{12}^2}\left[4(\vec{n}_{12}\cdot\vec{v}_1)-3(\vec{n}_{12}\cdot\vec{v}_2)\right] \\
&+\epsilon^4\frac{m_1m_2}{r_{12}^2}\Big[-\frac{9}{2}(\vec{n}_{12}\cdot\vec{v}_2)^3+\frac{1}{2}v_1^2(\vec{n}_{12}\cdot\vec{v}_2)+6(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)^2 \\
&-2v_1^2(\vec{n}_{12}\cdot\vec{v}_1)+4(\vec{v}_1\cdot\vec{v}_2)(\vec{n}_{12}\cdot\vec{V})+5v_2^2(\vec{n}_{12}\cdot\vec{v}_2)-4v_2^2(\vec{n}_{12}\cdot\vec{v}_1) \\
&+\frac{m_1}{r_{12}}(-4(\vec{n}_{12}\cdot\vec{v}_2)+6(\vec{n}_{12}\cdot\vec{v}_1))+\frac{m_2}{r_{12}}(-10(\vec{n}_{12}\cdot\vec{v}_1)+11(\vec{n}_{12}\cdot\vec{v}_2))\Big] \\
&+\epsilon^6\frac{m_1m_2}{r_{12}}\Big[-\left(\frac{3}{2}v_1^4+2v_1^2v_2^2+4v_2^4\right)(\vec{n}_{12}\cdot\vec{v}_1)+\left(\frac{5}{8}v_1^4+\frac{3}{2}v_1^2v_2^2+7v_2^4\right)(\vec{n}_{12}\cdot\vec{v}_2) \\
&+\left(2v_1^2+4v_2^2\right)(\vec{n}_{12}\cdot\vec{v}_1)(\vec{v}_1\cdot\vec{v}_2)-\left(2v_1^2+8v_2^2\right)(\vec{n}_{12}\cdot\vec{v}_2)(\vec{v}_1\cdot\vec{v}_2) \\
&+\left(3v_1^2+12v_2^2\right)(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)^2-\left(\frac{3}{4}v_1^2+12v_2^2\right)(\vec{n}_{12}\cdot\vec{v}_2)^3 \\
&+2(\vec{n}_{12}\cdot\vec{v}_2)(\vec{v}_1\cdot\vec{v}_2)^2-6(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)^2(\vec{v}_1\cdot\vec{v}_2)+6(\vec{n}_{12}\cdot\vec{v}_2)^3(\vec{v}_1\cdot\vec{v}_2) \\
&-\frac{15}{2}(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)^4-\frac{45}{8}(\vec{n}_{12}\cdot\vec{v}_2)^5 \\
&+\frac{m_1}{r_{12}}\Big(\left(-42v_1^2-\frac{117}{4}v_2^2\right)(\vec{n}_{12}\cdot\vec{v}_1)+60(\vec{n}_{12}\cdot\vec{v}_1)^3 \\
&+\left(\frac{137}{4}v_1^2+\frac{37}{2}v_2^2\right)(\vec{n}_{12}\cdot\vec{v}_2) \\
&+\frac{297}{4}(\vec{n}_{12}\cdot\vec{v}_1)(\vec{v}.\vec{v}_2)-\frac{219}{4}(\vec{n}_{12}\cdot\vec{v}_2)(\vec{v}.\vec{v}_2)-151(\vec{n}_{12}\cdot\vec{v}_1)^2(\vec{n}_{12}\cdot\vec{v}_2) \\
&+109(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)^2-23(\vec{n}_{12}\cdot\vec{v}_2)^3\Big) \\
&+\frac{m_2}{r_{12}}\Big(-\left(13v_1^2+18v_2^2\right)(\vec{n}_{12}\cdot\vec{v}_1)+\left(17v_1^2+25v_2^2\right)(\vec{n}_{12}\cdot\vec{v}_2) \\
&+26(\vec{n}_{12}\cdot\vec{v}_1)(\vec{v}_1\cdot\vec{v}_2)-28(\vec{n}_{12}\cdot\vec{v}_2)(\vec{v}_1\cdot\vec{v}_2)+2(\vec{n}_{12}\cdot\vec{v}_1)^2(\vec{n}_{12}\cdot\vec{v}_2) \\
&+16(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)^2-20(\vec{n}_{12}\cdot\vec{v}_2)^3\Big) \\
&+\frac{m_1^2}{r_{12}^2}\left(\frac{33}{4}(\vec{n}_{12}\cdot\vec{v}_1)-\frac{13}{2}(\vec{n}_{12}\cdot\vec{v}_2)\right)-\frac{m_1m_2}{r_{12}^2}\left(\frac{35}{4}(\vec{n}_{12}\cdot\vec{v}_1)+\frac{17}{4}(\vec{n}_{12}\cdot\vec{v}_2)\right) \\
&+\frac{m_2^2}{r_{12}^2}\left(-12(\vec{n}_{12}\cdot\vec{v}_1)+\frac{23}{2}(\vec{n}_{12}\cdot\vec{v}_2)\right)\Big], \qquad (1)
\end{aligned}
$$

where $\vec{V}\equiv\vec{v}_1-\vec{v}_2$.

This equation is sloved functionally to give 3PN mass-energy relation

$$P^\tau_{1\Theta} = m_1\left(1+\epsilon^2{}_2\Gamma_1+\epsilon^4{}_4\Gamma_1+\epsilon^6{}_6\Gamma_1\right)+O(\epsilon^7), \qquad (2)$$

with

$$
\begin{aligned}
{}_2\Gamma_1 &= \frac{1}{2}v_1^2+\frac{3m_2}{r_{12}}, \qquad (3) \\
{}_4\Gamma_1 &= -\frac{3m_2}{2r_{12}}(\vec{n}_{12}\cdot\vec{v}_2)^2+\frac{2m_2}{r_{12}}v_2^2+\frac{7m_2}{2r_{12}}v_1^2-\frac{4m_2}{r_{12}}(\vec{v}_1\cdot\vec{v}_2)+\frac{3}{8}v_1^4+\frac{7m_2^2}{2r_{12}^2}-\frac{5m_1m_2}{2r_{12}^2}. \qquad (4)
\end{aligned}
$$

$$
\begin{aligned}
{}_6\Gamma_1 \;=\;& \frac{m_1^2 m_2}{2r_{12}^3} + \frac{21 m_1 m_2^2}{4r_{12}^3} + \frac{5m_2^3}{2r_{12}^3} + \frac{5}{16}v_1^6 \\
&+ \frac{m_2^2}{r_{12}^2}\left(\frac{45}{4}v_1^2 + \frac{19}{2}v_2^2 + \frac{1}{2}(\vec{n}_{12}\cdot\vec{v}_1)^2 - 19(\vec{v}_1\cdot\vec{v}_2) - (\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2) - 3(\vec{n}_{12}\cdot\vec{v}_2)^2\right) \\
&+ \frac{m_1 m_2}{r_{12}^2}\left(\frac{43}{8}v_1^2 + \frac{53}{8}v_2^2 - \frac{69}{8}(\vec{n}_{12}\cdot\vec{v}_1)^2 - \frac{53}{4}(\vec{v}_1\cdot\vec{v}_2) + \frac{85}{4}(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)\right. \\
&\left. - \frac{69}{8}(\vec{n}_{12}\cdot\vec{v}_2)^2\right) + \frac{m_2}{r_{12}}\left(\frac{33}{8}v_1^4 + \frac{3}{2}v_1^2 v_2^2 + v_2^4 - 6v_1^2(\vec{v}_1\cdot\vec{v}_2) - 4v_2^2(\vec{v}_1\cdot\vec{v}_2)\right. \\
&- \frac{7}{4}v_1^2(\vec{n}_{12}\cdot\vec{v}_2)^2 - \frac{5}{2}v_2^2(\vec{n}_{12}\cdot\vec{v}_2)^2 + 2(\vec{v}_1\cdot\vec{v}_2)^2 \\
&\left. + 2(\vec{n}_{12}\cdot\vec{v}_2)^2(\vec{v}_1\cdot\vec{v}_2) + \frac{9}{8}(\vec{n}_{12}\cdot\vec{v}_2)^4\right). \qquad (5)
\end{aligned}
$$

REMARK

At least up to the 3 PN order

$$
P^{\tau}_{A\Theta} = m_A[\sqrt{-g}u^{\tau}_A]^{ext}_A. \qquad (6)
$$

In the above equation, $[f]^{ext}_A$ means that we regularize the quantity f at the star A by Hadamard Partie Finie.

The equation of motion up to 3PN order.

$$
\begin{aligned}
m_1\frac{dv_1^i}{d\tau} &= -\frac{m_1m_2}{r_{12}^2}n_{12}^i \\
&+ \epsilon^2\frac{m_1m_2}{r_{12}^2}n_{12}^i\left[-v_1^2-2v_2^2+4(\vec{v}_1\cdot\vec{v}_2)+\frac{3}{2}(\vec{n}_{12}\cdot\vec{v}_2)^2+\frac{5m_1}{r_{12}}+\frac{4m_2}{r_{12}}\right] \\
&+ \epsilon^2\frac{m_1m_2}{r_{12}^2}V^i\left[4(\vec{n}_{12}\cdot\vec{v}_1)-3(\vec{n}_{12}\cdot\vec{v}_2)\right] \\
&+ \epsilon^4\frac{m_1m_2}{r_{12}^2}n_{12}^i\left[-2v_2^4+4v_2^2(\vec{v}_1\cdot\vec{v}_2)-2(\vec{v}_1\cdot\vec{v}_2)^2+\frac{3}{2}v_1^2(\vec{n}_{12}\cdot\vec{v}_2)^2+\frac{9}{2}v_2^2(\vec{n}_{12}\cdot\vec{v}_2)^2\right. \\
&\quad -6(\vec{v}_1\cdot\vec{v}_2)(\vec{n}_{12}\cdot\vec{v}_2)^2-\frac{15}{8}(\vec{n}_{12}\cdot\vec{v}_2)^4-\frac{57}{4}\frac{m_1^2}{r_{12}^2}-9\frac{m_2^2}{r_{12}^2}-\frac{69}{2}\frac{m_1m_2}{r_{12}^2} \\
&\quad +\frac{m_1}{r_{12}}\left(-\frac{15}{4}v_1^2+\frac{5}{4}v_2^2-\frac{5}{2}(\vec{v}_1\cdot\vec{v}_2)+\frac{39}{2}(\vec{n}_{12}\cdot\vec{v}_1)^2-39(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)\right. \\
&\quad \left.+\frac{17}{2}(\vec{n}_{12}\cdot\vec{v}_2)^2\right) \\
&\quad \left.+\frac{m_2}{r_{12}}\left(4v_2^2-8(\vec{v}_1\cdot\vec{v}_2)+2(\vec{n}_{12}\cdot\vec{v}_1)^2-4(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)-6(\vec{n}_{12}\cdot\vec{v}_2)^2\right)\right] \\
&+ \epsilon^4\frac{m_1m_2}{r_{12}^2}V^i\left[\frac{m_1}{r_{12}}\left(-\frac{63}{4}(\vec{n}_{12}\cdot\vec{v}_1)+\frac{55}{4}(\vec{n}_{12}\cdot\vec{v}_2)\right)+\frac{m_2}{r_{12}}\left(-2(\vec{n}_{12}\cdot\vec{v}_1)-2(\vec{n}_{12}\cdot\vec{v}_2)\right)\right. \\
&\quad +v_1^2(\vec{n}_{12}\cdot\vec{v}_2)+4v_2^2(\vec{n}_{12}\cdot\vec{v}_1)-5v_2^2(\vec{n}_{12}\cdot\vec{v}_2)-4(\vec{v}_1\cdot\vec{v}_2)(\vec{n}_{12}\cdot\vec{V}) \\
&\quad \left.-6(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)^2+\frac{9}{2}(\vec{n}_{12}\cdot\vec{v}_2)^3\right] \\
&+ \epsilon^5\frac{4m_1^2m_2}{5r_{12}^3}\left[n_{12}^i(\vec{n}_{12}\cdot\vec{V})\left(-6\frac{m_1}{r_{12}}+\frac{52}{3}\frac{m_2}{r_{12}}+3V^2\right)+V^i\left(2\frac{m_1}{r_{12}}-8\frac{m_2}{r_{12}}-V^2\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
+\ & \epsilon^6\frac{m_1m_2}{r_{12}^2}n_{12}^i\left[\frac{35}{16}(\vec{n}_{12}\cdot\vec{v}_2)^6-\frac{15}{8}(\vec{n}_{12}\cdot\vec{v}_2)^4v_1^2+\frac{15}{2}(\vec{n}_{12}\cdot\vec{v}_2)^4(\vec{v}_1\cdot\vec{v}_2)\right. \\
&+3(\vec{n}_{12}\cdot\vec{v}_2)^2(\vec{v}_1\cdot\vec{v}_2)^2-\frac{15}{2}(\vec{n}_{12}\cdot\vec{v}_2)^4v_2^2+\frac{3}{2}(\vec{n}_{12}\cdot\vec{v}_2)^2v_1^2v_2^2-12(\vec{n}_{12}\cdot\vec{v}_2)^2(\vec{v}_1\cdot\vec{v}_2)v_2^2 \\
&-2(\vec{v}_1\cdot\vec{v}_2)^2v_2^2+\frac{15}{2}(\vec{n}_{12}\cdot\vec{v}_2)^2v_2^4+4(\vec{v}_1\cdot\vec{v}_2)v_2^4-2v_2^6 \\
&+\frac{m_1}{r_{12}}\left(-\frac{171}{8}(\vec{n}_{12}\cdot\vec{v}_1)^4+\frac{171}{2}(\vec{n}_{12}\cdot\vec{v}_1)^3(\vec{n}_{12}\cdot\vec{v}_2)\right. \\
&-\frac{723}{4}(\vec{n}_{12}\cdot\vec{v}_1)^2(\vec{n}_{12}\cdot\vec{v}_2)^2+\frac{383}{2}(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)^3-\frac{455}{8}(\vec{n}_{12}\cdot\vec{v}_2)^4 \\
&+\frac{229}{4}(\vec{n}_{12}\cdot\vec{v}_1)^2v_1^2-\frac{205}{2}(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)v_1^2+\frac{191}{4}(\vec{n}_{12}\cdot\vec{v}_2)^2v_1^2-\frac{91}{8}v_1^4 \\
&-\frac{229}{2}(\vec{n}_{12}\cdot\vec{v}_1)^2(\vec{v}_1\cdot\vec{v}_2)+244(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)(\vec{v}_1\cdot\vec{v}_2)-\frac{225}{2}(\vec{n}_{12}\cdot\vec{v}_2)^2(\vec{v}_1\cdot\vec{v}_2) \\
&+\frac{91}{2}v_1^2(\vec{v}_1\cdot\vec{v}_2)-\frac{177}{4}(\vec{v}_1\cdot\vec{v}_2)^2+\frac{229}{4}(\vec{n}_{12}\cdot\vec{v}_1)^2v_2^2-\frac{283}{2}(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)v_2^2 \\
&\left.+\frac{259}{4}(\vec{n}_{12}\cdot\vec{v}_2)^2v_2^2-\frac{91}{4}v_1^2v_2^2+43(\vec{v}_1\cdot\vec{v}_2)v_2^2-\frac{81}{8}v_2^4\right)
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{m_2}{r_{12}}\Big(-6(\vec{n}_{12}\cdot\vec{v}_1)^2(\vec{n}_{12}\cdot\vec{v}_2)^2 + 12(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)^3 \\
&+ 6(\vec{n}_{12}\cdot\vec{v}_2)^4 + 4(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)(\vec{v}_1\cdot\vec{v}_2) + 12(\vec{n}_{12}\cdot\vec{v}_2)^2(\vec{v}_1\cdot\vec{v}_2) + 4(\vec{v}_1\cdot\vec{v}_2)^2 \\
&- 4(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)v_2^2 - 12(\vec{n}_{12}\cdot\vec{v}_2)^2v_2^2 - 8(\vec{v}_1\cdot\vec{v}_2)v_2^2 + 4v_2^4\Big) \\
&+ \frac{m_2^2}{r_{12}^2}\left(-(\vec{n}_{12}\cdot\vec{v}_1)^2 + 2(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2) + \frac{43}{2}(\vec{n}_{12}\cdot\vec{v}_2)^2 + 18(\vec{v}_1\cdot\vec{v}_2) - 9v_2^2\right) \\
&+ \frac{m_1m_2}{r_{12}^2}\left(\frac{415}{8}(\vec{n}_{12}\cdot\vec{v}_1)^2 - \frac{375}{4}(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2) + \frac{1113}{8}(\vec{n}_{12}\cdot\vec{v}_2)^2\right. \\
&\left.- \frac{615\pi^2}{64}(\vec{n}_{12}\cdot\vec{V})^2 + 18v_1^2 + \frac{123\pi^2}{64}V^2 + 33(\vec{v}_1\cdot\vec{v}_2) - \frac{33}{2}v_2^2\right) \\
&+ \frac{m_1^2}{r_{12}^2}\left(-\frac{2069}{8}(\vec{n}_{12}\cdot\vec{v}_1)^2 + 543(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2) - \frac{939}{4}(\vec{n}_{12}\cdot\vec{v}_2)^2 + \frac{471}{8}v_1^2\right. \\
&\left.- \frac{357}{4}(\vec{v}_1\cdot\vec{v}_2) + \frac{357}{8}v_2^2\right) \\
&+ \frac{16m_2^3}{r_{12}^3} + \frac{m_1^2m_2}{r_{12}^3}\left(\frac{547}{3} - \frac{41\pi^2}{16}\right) - \frac{13m_1^3}{12r_{12}^3} + \frac{m_1m_2^2}{r_{12}^3}\left(\frac{545}{3} - \frac{41\pi^2}{16}\right)\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
+ \quad & \epsilon^6\frac{m_1m_2}{r_{12}^2}V^i\left[\frac{15}{2}(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)^4 - \frac{45}{8}(\vec{n}_{12}\cdot\vec{v}_2)^5 - \frac{3}{2}(\vec{n}_{12}\cdot\vec{v}_2)^3v_1^2\right. \\
&+ 6(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)^2(\vec{v}_1\cdot\vec{v}_2) - 6(\vec{n}_{12}\cdot\vec{v}_2)^3(\vec{v}_1\cdot\vec{v}_2) - 2(\vec{n}_{12}\cdot\vec{v}_2)(\vec{v}_1\cdot\vec{v}_2)^2 \\
&- 12(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)^2v_2^2 + 12(\vec{n}_{12}\cdot\vec{v}_2)^3v_2^2 + (\vec{n}_{12}\cdot\vec{v}_2)v_1^2v_2^2 - 4(\vec{n}_{12}\cdot\vec{v}_1)(\vec{v}_1\cdot\vec{v}_2)v_2^2 \\
&+ 8(\vec{n}_{12}\cdot\vec{v}_2)(\vec{v}_1\cdot\vec{v}_2)v_2^2 + 4(\vec{n}_{12}\cdot\vec{v}_1)v_2^4 - 7(\vec{n}_{12}\cdot\vec{v}_2)v_2^4 + \frac{m_2}{r_{12}}\Big(-2(\vec{n}_{12}\cdot\vec{v}_1)^2(\vec{n}_{12}\cdot\vec{v}_2) \\
&+ 8(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)^2 + 2(\vec{n}_{12}\cdot\vec{v}_2)^3 + 2(\vec{n}_{12}\cdot\vec{v}_1)(\vec{v}_1\cdot\vec{v}_2) + 4(\vec{n}_{12}\cdot\vec{v}_2)(\vec{v}_1\cdot\vec{v}_2) \\
&- 2(\vec{n}_{12}\cdot\vec{v}_1)v_2^2 - 4(\vec{n}_{12}\cdot\vec{v}_2)v_2^2\Big) + \frac{m_1}{r_{12}}\left(-\frac{273}{4}(\vec{n}_{12}\cdot\vec{v}_1)^3 + \frac{565}{4}(\vec{n}_{12}\cdot\vec{v}_1)^2(\vec{n}_{12}\cdot\vec{v}_2)\right. \\
&- \frac{269}{4}(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)^2 - \frac{95}{12}(\vec{n}_{12}\cdot\vec{v}_2)^3 + \frac{207}{8}(\vec{n}_{12}\cdot\vec{v}_1)v_1^2 - \frac{137}{8}(\vec{n}_{12}\cdot\vec{v}_2)v_1^2 \\
&\left.- 36(\vec{n}_{12}\cdot\vec{v}_1)(\vec{v}_1\cdot\vec{v}_2) + \frac{27}{4}(\vec{n}_{12}\cdot\vec{v}_2)(\vec{v}_1\cdot\vec{v}_2) + \frac{81}{8}(\vec{n}_{12}\cdot\vec{v}_1)v_2^2 + \frac{83}{8}(\vec{n}_{12}\cdot\vec{v}_2)v_2^2\right) \\
&+ \frac{m_2^2}{r_{12}^2}\left(4(\vec{n}_{12}\cdot\vec{v}_1) + 5(\vec{n}_{12}\cdot\vec{v}_2)\right) \\
&+ \frac{m_1m_2}{r_{12}^2}\left(-\frac{307}{8}(\vec{n}_{12}\cdot\vec{v}_1) + \frac{479}{8}(\vec{n}_{12}\cdot\vec{v}_2) + \frac{123\pi^2}{32}(\vec{n}_{12}\cdot\vec{V})\right) \\
&\left.+ \frac{m_1^2}{r_{12}^2}\left(\frac{311}{4}(\vec{n}_{12}\cdot\vec{v}_1) - \frac{357}{4}(\vec{n}_{12}\cdot\vec{v}_2)\right)\right] \\
&+O(\epsilon^7), \qquad (7)
\end{aligned}
$$

Our 3 PN equation of motion coincide with a geodesic equation for a test particle in the Schwartzshild metric in the harmonic coordinate (up to the 3 PN order) and is Lorentz invariant (in the post-Newtonian perturbative sense), and admits the conserved energy of the binary orbital motion.

The energy of the binary E associated with the equation of motion.

$$\begin{aligned}
E &= \frac{1}{2}m_1v_1^2 - \frac{m_1m_2}{2r_{12}} \\
&+ \epsilon^2\left[\frac{3}{8}m_1v_1^4 + \frac{m_1^2m_2}{2r_{12}^2} + \frac{m_1m_2}{2r_{12}}\left(3v_1^2 - \frac{7}{2}(\vec{v}_1\cdot\vec{v}_2) - \frac{1}{2}(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)\right)\right]
\end{aligned}$$

$$\begin{aligned}
+ \;& \epsilon^4\left[\frac{5}{16}m_1v_1^6 - \frac{m_1^3m_2}{2r_{12}^3} - \frac{19m_1^2m_2^2}{8r_{12}^3}\right. \\
&+ \frac{m_1^2m_2}{2r_{12}^2}\left(-3v_1^2 + \frac{7}{2}v_2^2 + \frac{29}{2}(\vec{n}_{12}\cdot\vec{v}_1)^2 - \frac{13}{2}(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2) + (\vec{n}_{12}\cdot\vec{v}_2)^2\right) \\
&+ \frac{m_1m_2}{4r_{12}}\left(\frac{3}{2}(\vec{n}_{12}\cdot\vec{v}_1)^3(\vec{n}_{12}\cdot\vec{v}_2) + \frac{3}{4}(\vec{n}_{12}\cdot\vec{v}_1)^2(\vec{n}_{12}\cdot\vec{v}_2)^2 - \frac{9}{2}(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)v_1^2\right. \\
&- \frac{13}{2}(\vec{n}_{12}\cdot\vec{v}_2)^2v_1^2 + \frac{21}{2}v_1^4 + \frac{13}{2}(\vec{n}_{12}\cdot\vec{v}_1)^2(\vec{v}_1\cdot\vec{v}_2) + 3(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)(\vec{v}_1\cdot\vec{v}_2) \\
&\left.\left.- \frac{55}{2}v_1^2(\vec{v}_1\cdot\vec{v}_2) + \frac{17}{2}(\vec{v}_1\cdot\vec{v}_2)^2 + \frac{31}{4}v_1^2v_2^2\right)\right] \\
+ \;& \epsilon^6\left[\frac{35}{128}m_1v_1^8 + \frac{3m_1^4m_2}{8r_{12}^4} + \frac{9707m_1^3m_2^2}{420r_{12}^4}\right. \\
&+ \frac{m_1^2m_2^2}{2r_{12}^3}\left(\frac{547}{6}(\vec{n}_{12}\cdot\vec{v}_1)^2 - \frac{3115}{24}(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2) - \frac{123\pi^2}{32}(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{V})\right. \\
&\left.- \frac{575}{9}v_1^2 + \frac{41\pi^2}{32}(\vec{V}\cdot\vec{v}_2) + \frac{4429}{72}(\vec{v}_1\cdot\vec{v}_2)\right) \\
&+ \frac{m_1^3m_2}{2r_{12}^3}\left(-\frac{44627}{420}(\vec{n}_{12}\cdot\vec{v}_1)^2 + \frac{32027}{420}(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2) + 3(\vec{n}_{12}\cdot\vec{v}_2)^2 + \frac{24187}{1260}v_1^2\right. \\
&\left.- \frac{27967}{1260}(\vec{v}_1\cdot\vec{v}_2) + \frac{5}{2}v_2^2\right) \\
&+ \frac{m_1m_2}{r_{12}}\left(-\frac{5}{16}(\vec{n}_{12}\cdot\vec{v}_1)^5(\vec{n}_{12}\cdot\vec{v}_2) - \frac{5}{16}(\vec{n}_{12}\cdot\vec{v}_1)^4(\vec{n}_{12}\cdot\vec{v}_2)^2 - \frac{5}{32}(\vec{n}_{12}\cdot\vec{v}_1)^3(\vec{n}_{12}\cdot\vec{v}_2)^3\right. \\
&+ \frac{19}{16}(\vec{n}_{12}\cdot\vec{v}_1)^3(\vec{n}_{12}\cdot\vec{v}_2)v_1^2 + \frac{15}{16}(\vec{n}_{12}\cdot\vec{v}_1)^2(\vec{n}_{12}\cdot\vec{v}_2)^2v_1^2 + \frac{3}{4}(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)^3v_1^2 \\
&+ \frac{19}{16}(\vec{n}_{12}\cdot\vec{v}_2)^4v_1^2 - \frac{21}{16}(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)v_1^4 - 2(\vec{n}_{12}\cdot\vec{v}_2)^2v_1^4 + \frac{55}{16}v_1^6 - \frac{19}{16}(\vec{n}_{12}\cdot\vec{v}_1)^4(\vec{v}_1\cdot\vec{v}_2) \\
&- (\vec{n}_{12}\cdot\vec{v}_1)^3(\vec{n}_{12}\cdot\vec{v}_2)(\vec{v}_1\cdot\vec{v}_2) - \frac{15}{32}(\vec{n}_{12}\cdot\vec{v}_1)^2(\vec{n}_{12}\cdot\vec{v}_2)^2(\vec{v}_1\cdot\vec{v}_2) \\
&+ \frac{45}{16}(\vec{n}_{12}\cdot\vec{v}_1)^2v_1^2(\vec{v}_1\cdot\vec{v}_2) + \frac{5}{4}(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)v_1^2(\vec{v}_1\cdot\vec{v}_2) + \frac{11}{4}(\vec{n}_{12}\cdot\vec{v}_2)^2v_1^2(\vec{v}_1\cdot\vec{v}_2) \\
&- \frac{139}{16}v_1^4(\vec{v}_1\cdot\vec{v}_2) - \frac{3}{4}(\vec{n}_{12}\cdot\vec{v}_1)^2(\vec{v}_1\cdot\vec{v}_2)^2 + \frac{5}{16}(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)(\vec{v}_1\cdot\vec{v}_2)^2 + \frac{41}{8}v_1^2(\vec{v}_1\cdot\vec{v}_2)^2 \\
&\left.+ \frac{1}{16}(\vec{v}_1\cdot\vec{v}_2)^3 - \frac{45}{16}(\vec{n}_{12}\cdot\vec{v}_1)^2v_1^2v_2^2 - \frac{23}{32}(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)v_1^2v_2^2 + \frac{79}{16}v_1^4v_2^2 - \frac{161}{32}v_1^2v_2^2(\vec{v}_1\cdot\vec{v}_2)\right) \\
&+ \frac{m_1^2m_2}{r_{12}^2}\left(-\frac{49}{8}(\vec{n}_{12}\cdot\vec{v}_1)^4 + \frac{75}{8}(\vec{n}_{12}\cdot\vec{v}_1)^3(\vec{n}_{12}\cdot\vec{v}_2) - \frac{187}{8}(\vec{n}_{12}\cdot\vec{v}_1)^2(\vec{n}_{12}\cdot\vec{v}_2)^2 + \frac{11}{2}v_1^4\right.
\end{aligned}$$

$$
\begin{aligned}
&+\ \frac{247}{24}(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)^3+\frac{49}{8}(\vec{n}_{12}\cdot\vec{v}_1)^2v_1^2+\frac{81}{8}(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)v_1^2-\frac{21}{4}(\vec{n}_{12}\cdot\vec{v}_2)^2v_1^2\\
&-\ \frac{15}{2}(\vec{n}_{12}\cdot\vec{v}_1)^2(\vec{v}_1\cdot\vec{v}_2)-\frac{3}{2}(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)(\vec{v}_1\cdot\vec{v}_2)+\frac{21}{4}(\vec{n}_{12}\cdot\vec{v}_2)^2(\vec{v}_1\cdot\vec{v}_2)-27v_1^2(\vec{v}_1\cdot\vec{v}_2)\\
&+\ \frac{55}{2}(\vec{v}_1\cdot\vec{v}_2)^2+\frac{49}{4}(\vec{n}_{12}\cdot\vec{v}_1)^2v_2^2-\frac{27}{2}(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)v_2^2+\frac{3}{4}(\vec{n}_{12}\cdot\vec{v}_2)^2v_2^2+\frac{55}{4}v_1^2v_2^2\\
&-\ 28(\vec{v}_1\cdot\vec{v}_2)v_2^2+\frac{135}{16}v_2^4\Big)\Big]+(1\leftrightarrow 2)+O(\epsilon^7). \qquad (8)
\end{aligned}
$$