Motion of a small mass in curved spacetime: A review of the foundations

Talk at Capra6, based on gr-qc/0306052

The Capra scientific mandate

To formulate the equations of motion of a small body of mass m in a specified background spacetime, beyond the geodesic approximation.

This first step was solved back in 1997. The equations of motion are now known as the MiSaTaQuWa equations.

To concretely describe this motion for situations of astrophysical interest (generic orbits of a Kerr black hole).

Much recent progress on this front.

To properly incorporate this information into a wave-generation formalism.

The holy grail: still elusive.

This talk

The MiSaTaQuWa equations of motion take the form

$$\boxed{a^\mu = -\frac{1}{2}\left(g^{\mu\nu} + u^\mu u^\nu\right)\left(2h^{\text{tail}}_{\nu\lambda\rho} - h^{\text{tail}}_{\lambda\rho\nu}\right)u^\lambda u^\rho}$$

where u^μ and a^μ are respectively the particle's velocity and acceleration vectors in the background spacetime, and

$$\boxed{h^{\text{tail}}_{\mu\nu\lambda} = 4m\int_{-\infty}^{\tau^-} \nabla_\lambda\left(G_{+\mu\nu\mu'\nu'} - \frac{1}{2}g_{\mu\nu}G_{+}{}^{\rho}{}_{\rho\mu'\nu'}\right)u^{\mu'}u^{\nu'}\,d\tau'}$$

is the tail integral, which involves the retarded Green's function for a gravitational perturbation of the background spacetime.

In this talk I will review four derivations of these equations, and discuss the relative strengths and weaknesses of each derivation.

1. **Toy problem: scalar charge**
2. **Coordinate systems**
3. **World-tube conservation**
4. **Averaged field (Quinn & Wald)**
5. **Detweiler-Whiting potentials**
6. **Matched asymptotic expansions**
7. **Conclusion**

1. Toy problem: scalar charge

To simplify the expressions I will refer to a toy problem involving a point particle coupled to a massless scalar field $\Phi(x)$.

The scalar field obeys the wave equation

$$\boxed{\big(g^{\alpha\beta}\nabla_\alpha\nabla_\beta - \xi R\big)\Phi(x) = -4\pi q \int_\gamma \delta_4(x,z)\, d\tau}$$

where ξ is an arbitrary coupling constant and q is the scalar charge of a particle moving on a world line γ described by relations $z^\mu(\tau)$.

The scalar charge moves according to

$$\boxed{m(\tau)a^\mu = q\big(g^{\mu\nu} + u^\mu u^\nu\big)\Phi_\nu}$$

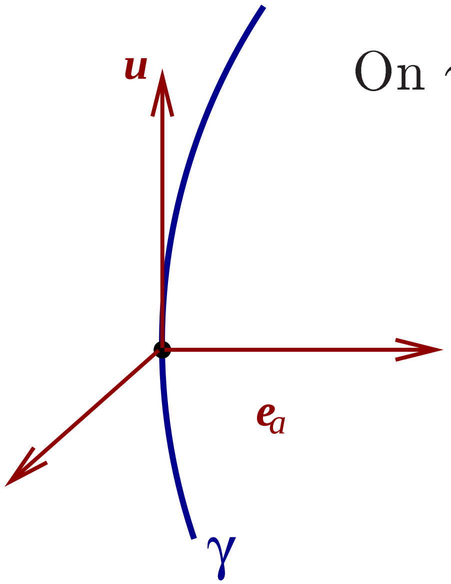
where $\Phi_\nu \equiv \nabla_\nu\Phi$ and $m(\tau)$ is the particle's dynamical mass.

Because the field produced by a point charge is singular on the world line, the equations of motion are meaningless as they stand.

To make sense of them we must carefully analyze the field's singularity structure and regularize its behaviour.

2. Coordinate systems

It is useful to construct two coordinate systems to chart a neighbourhood of the world line γ.



On γ we erect a basis (u^μ, e^μ_a) of orthonormal vectors.

These are transported on γ so as to preserve their orthonormality property.

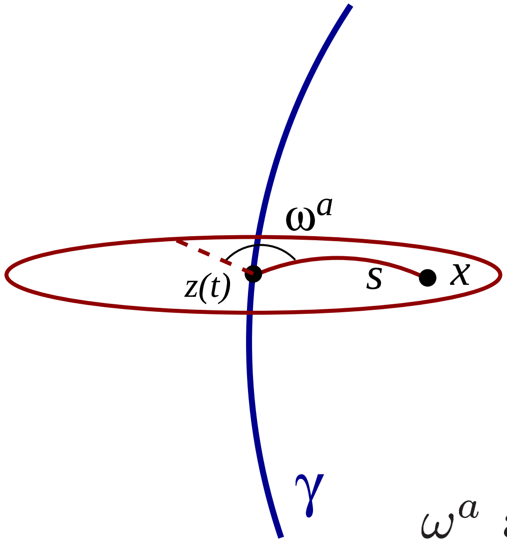
Relevant tensors are decomposed in this basis.

For example,

$$R_{0a0b}(\tau) = R_{\mu\lambda\nu\rho}\Big|_\gamma u^\mu e^\lambda_a u^\nu e^\rho_b$$

are frame components of the Riemann tensor.

Fermi normal coordinates $(\boldsymbol{t}, \boldsymbol{x^a = s\omega^a})$ are constructed as follows:



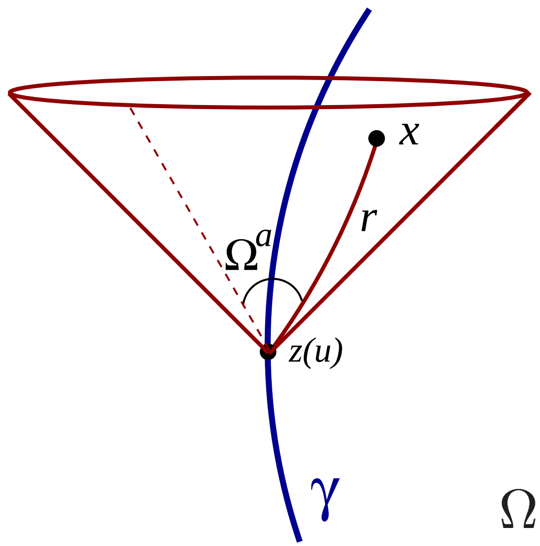
Find the unique spacelike geodesic that passes through x and intersects γ orthogonally; call the intersection point $z(t) \equiv \bar{x}$.

t is proper time at $\bar{x}$.
s is proper distance from $\bar{x}$ to x.
ω^a are direction cosines for the spacelike geodesic.

We parallel transport the basis (u^μ, e^μ_a) at $\bar{x}$ along the spacelike geodesic to obtain another basis $(\bar{e}^\alpha_0, \bar{e}^\alpha_a)$ at x.

Fermi normal coordinates are useful to define the rest frame of the moving particle (needed for field averaging).

Retarded coordinates $(\boldsymbol{u}, \boldsymbol{x}^a = \boldsymbol{r}\boldsymbol{\Omega}^a)$ are constructed as follows:



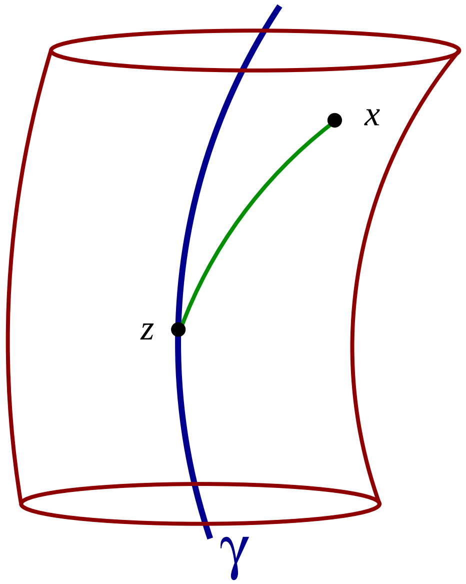
Find the unique null geodesic that passes through x and intersects γ; call the intersection point $z(u) \equiv x'$.

u is proper time at x'.
r is affine-parameter distance from x' to x.
Ω^a are direction cosines for the null geodesic.

We parallel transport the basis (u^μ, e^μ_a) at x' along the null geodesic to obtain another basis (e^α_0, e^α_a) at x.

Retarded coordinates give the simplest description of the scalar field $\Phi_\alpha(x)$; they naturally incorporate the correct causal structure.

3. World-tube conservation



In flat spacetime the conservation equations

$$0 = \partial_\beta T^{\alpha\beta}, \qquad T^{\alpha\beta} = T^{\alpha\beta}_{\text{field}} + T^{\alpha\beta}_{\text{particle}}$$

can be converted to a surface integral on a world tube surrounding the world line.

In curved spacetime this method does not work directly.

A trick must be used:

$$\begin{aligned} 0 &= g^{\mu}{}_{\alpha}(z,x)\nabla_\beta T^{\alpha\beta}(x) \\ &= \left(g^{\mu}{}_{\alpha} T^{\alpha\beta}\right)_{;\beta} - g^{\mu}{}_{\alpha;\beta} T^{\alpha\beta} \end{aligned}$$

Integration yields

$$0 = \int_{\text{tube}} g^{\mu}{}_{\alpha} T^{\alpha\beta}\, d\Sigma_\beta + \int_{\text{caps}} g^{\mu}{}_{\alpha} T^{\alpha\beta}\, d\Sigma_\beta - \int_{\text{interior}} g^{\mu}{}_{\alpha;\beta} T^{\alpha\beta}\, dV$$

The tube integration is well defined; it keeps track of the radiation escaping the world tube.

The caps integration diverges as $1/r$ and must be regularized; it represents the particle's bare momentum augmented by the field's contribution.

The volume integration diverges as $\ln(r)$ and must also be regularized; a careful analysis of this term has never been produced.

The world-tube conservation method "works", but regularization must be handled (more) carefully; the method is difficult to make rigourous.

4. Averaged field (Quinn & Wald)

The solution to the wave equation for the scalar potential is

$$\Phi(x) = q \int_\gamma G_+(x, z)\, d\tau$$

where $G_+(x, z)$ is the retarded Green's function.

After decomposition in the tetrad $(\bar{e}^\alpha_0, \bar{e}^\alpha_a)$, the field $\Phi_\alpha \equiv \nabla_\alpha \Phi$ can be expressed in Fermi normal coordinates as an expansion in powers of s.

This expansion involves the tail integral

$$\Phi^{\text{tail}}_\alpha(x) = q \int_{-\infty}^{t^-} \nabla_\alpha G_+(x, z)\, d\tau$$

which is cut short at $\tau = t^- \equiv t - \epsilon$ to avoid the singular behaviour of the Green's function at coincidence.

The spatial components are

$$\begin{aligned}\bar{\Phi}_a \quad = \quad & -\frac{q}{s^2}\omega_a - \frac{q}{2s}\big(a_a - a_b\omega^b\omega_a\big) + \frac{3}{4}qa_b\omega^b a_a - \frac{3}{8}q\big(a_b\omega^b\big)^2\omega_a \\ & + \frac{1}{8}q\dot{a}_0\omega_a + \frac{1}{3}q\dot{a}_a - \frac{1}{3}qR_{a0b0}\omega^b + \frac{1}{6}qR_{b0c0}\omega^b\omega^c\omega_a \\ & + \frac{1}{12}q\big[R_{00} - R_{bc}\omega^b\omega^c - (1-6\xi)R\big]\omega_a \\ & + \frac{1}{6}q\big(R_{a0} + R_{ab}\omega^b\big) + \bar{\Phi}_a^{\text{tail}} + O(s)\end{aligned}$$

The field possesses many singular terms.

How is it supposed to act on the particle?

To regularize we can swell the point particle into a spherical shell and compute the **net force** acting on this shell.

As this should be done in the particle's **rest frame**, this amounts to computing the average of $\Phi_\alpha(x)$ over a two-surface of constant t and s in Fermi normal coordinates.

After averaging,

$$\boxed{\langle \bar{\Phi}_a \rangle = -\frac{q}{3s} a_a + \frac{1}{3} q \dot{a}_a + \frac{1}{6} q R_{a0} + \bar{\Phi}_a^{\rm tail} + O(s)}$$

The remaining singular term in $\langle \Phi_\mu \rangle$ is proportional to a_μ and can be absorbed into a redefinition of the dynamical mass.

After this procedure of **mass renormalization**, the equations of motion become well defined.

They take the form

$$\boxed{m a^\mu = q^2 \left(g^{\mu\nu} + u^\mu u^\nu \right) \left(\frac{1}{3} \dot{a}_\nu + \frac{1}{6} R_{\nu\lambda} u^\lambda + \int_{-\infty}^{\tau^-} \nabla_\nu G_+(\tau, \tau')\, d\tau' \right)}$$

which is similar to the MiSaTaQuWa equation.

The averaged-field derivation of the equations of motion is based upon the postulate that the particle moves according to

$$\boxed{m_{\mathrm{bare}}(\tau)a^{\mu} = q(g^{\mu\nu} + u^{\mu}u^{\nu})\langle\Phi_{\nu}\rangle}$$

where the averaging is carried out in the rest frame, on a two-surface of constant proper distance.

The method incorporates a procedure of mass renormalization.

(This can be hidden under an additional subtraction postulate.)

The method is well motivated and sound, but regularization and renormalization are required.

5. Detweiler-Whiting potentials

Detweiler and Whiting have identified a meaningful decomposition of the retarded Green's function into **symmetric-singular** and **regular-radiative** parts.

They successfully generalized the flat-spacetime decomposition first proposed by Dirac.

The singular Green's function is given by

$$G_{\mathrm{S}}(x,x') = \frac{1}{2}\Big[G_+(x,x') + G_-(x,x') - H(x,x')\Big]$$

where $H(x,x')$ is a two-point function that obeys the homogeneous wave equation and agrees locally with the "tail part" of the retarded and advanced Green's functions.

The radiative Green's function is

$$\begin{aligned} G_{\mathrm{R}}(x,x') &= G_+(x,x') - G_{\mathrm{S}}(x,x') \\ &= \frac{1}{2}\Big[G_+(x,x') - G_-(x,x') + H(x,x')\Big] \end{aligned}$$

It also satisfies the homogeneous wave equation.

The singular field $\Phi_{\mathrm{S}}(x)$ is just as singular as the retarded field, but it exerts no force on the particle.

(This can be shown by an averaging procedure.)

The radiative field $\Phi_{\mathrm{R}}(x) = \Phi(x) - \Phi_{\mathrm{S}}(x)$ is smooth on the world line, and is entirely responsible for the self-force.

In retarded coordinates the retarded field is given by

$$\boxed{\begin{aligned}\Phi_a(u,r,\Omega^a) \quad = \quad & -\frac{q}{r^2}\Omega_a - \frac{q}{r}a_b\Omega^b\Omega_a - \frac{1}{3}qR_{b0c0}\Omega^b\Omega^c\Omega_a \\ & - \frac{1}{6}q\big(R_{a0b0}\Omega^b - R_{ab0c}\Omega^b\Omega^c\big) \\ & + \frac{1}{12}q\big[R_{00} - R_{bc}\Omega^b\Omega^c - (1-6\xi)R\big]\Omega_a \\ & + \frac{1}{6}q\big(R_{a0} + R_{ab}\Omega^b\big) + \Phi_a^{\text{tail}} + O(r)\end{aligned}}$$

All singular terms are reproduced by $\nabla_\alpha \Phi_{\text{S}}$.

The radiative field is

$$\boxed{\Phi_a^{\text{R}} = \frac{1}{3}q\dot{a}_a + \frac{1}{6}qR_{a0} + \Phi_a^{\text{tail}} + O(r)}$$

This field's action on the particle gives rise to the same equations of motion as before.

The equations of motion can thus be derived on the basis of the following postulate:

> The motion of the particle is governed by the action of the radiative field only:
>
> $$\boxed{m(\tau)a^\mu = q\bigl(g^{\mu\nu} + u^\mu u^\nu\bigr)\Phi^{\mathrm{R}}_\nu}$$
>
> The role of the singular field is limited to a contribution to the particle's inertia.

This postulate is attractive:

- because the radiative field is smooth, there is no need for averaging, and mass renormalization is bypassed;
- the motion results from a simple interaction between the particle and a free field, and there is no tension with the principle of equivalence.

6. Matched asymptotic expansions

The most compelling derivation of the MiSaTaQuWa equations is produced when the point particle is replaced by a small black hole.

The metric of the black hole perturbed by the tidal gravitational field of the external universe is matched to the metric of the background spacetime perturbed by the moving black hole.

Demanding that this metric be a solution to the vacuum field equation restricts the motion of the black hole.

The method of matched asymptotic expansions relies on the existence of

- an **internal zone** in which $r/\mathcal{R} \ll 1$, where $\mathcal{R}$ is the radius of curvature of the background spacetime;
- an **external zone** in which $m/r \ll 1$;
- a **buffer zone** in which $r/\mathcal{R}$ and m/r are both small.

The metrics are matched in the buffer zone.

The internal-zone metric is expressed in terms of internal retarded coordinates $(\bar{u}, \bar{r}\bar{\Omega}^a)$. For example,

$$\mathrm{g}_{\bar{u}\bar{u}} = -f\big(1 + \bar{r}^2 f \bar{\mathcal{E}}^*\big) + O(\bar{r}^3/\mathcal{R}^3)$$

where $f = 1 - 2m/\bar{r}$ and $\bar{\mathcal{E}}^* = R_{0a0b}(\bar{u})\bar{\Omega}^a\bar{\Omega}^b$ is a component of the tidal gravitational field of the external universe.

The external-zone metric is expressed in terms of external retarded coordinates $(u, r\Omega^a)$. For example,

$$\begin{aligned} \mathrm{g}_{uu} &= -1 - r^2\mathcal{E}^* + O(r^3/\mathcal{R}^3) \\ &\quad + \frac{2m}{r} + h^{\mathrm{tail}}_{00} + r\big(2m\mathcal{E}^* - 2a_a\Omega^a + h^{\mathrm{tail}}_{000} + h^{\mathrm{tail}}_{00a}\Omega^a\big) \\ &\quad + O(mr^2/\mathcal{R}^3) \end{aligned}$$

The coordinate transformation can be written down explicitly. For example,

$$\begin{aligned} \bar{u} &= u - 2m\ln r - \frac{1}{2}\int^u h^{\mathrm{tail}}_{00}\,du - \frac{1}{2}r\Big[h^{\mathrm{tail}}_{00} + 2h^{\mathrm{tail}}_{0a}\Omega^a + h^{\mathrm{tail}}_{ab}\Omega^a\Omega^b\Big] \\ &\quad - \frac{1}{4}r^2\Big[h^{\mathrm{tail}}_{000} + \big(h^{\mathrm{tail}}_{00a} + 2h^{\mathrm{tail}}_{0a0}\big)\Omega^a + \big(h^{\mathrm{tail}}_{ab0} + 2h^{\mathrm{tail}}_{0ab}\big)\Omega^a\Omega^b \\ &\quad + h^{\mathrm{tail}}_{abc}\Omega^a\Omega^b\Omega^c\Big] + O(mr^3/\mathcal{R}^3) \end{aligned}$$

After transformation the $\bar{u}$-$\bar{u}$ component of the external-zone metric becomes

$$
\begin{aligned}
\mathrm{g}_{\bar{u}\bar{u}} &= -1 - \bar{r}^2 \bar{\mathcal{E}}^* + O(\bar{r}^3/\mathcal{R}^3) \\
&\quad + \frac{2m}{\bar{r}} + \bar{r}\left[4m\bar{\mathcal{E}}^* - 2\left(a_a - \frac{1}{2}h^{\mathrm{tail}}_{00a} + h^{\mathrm{tail}}_{0a0}\right)\bar{\Omega}^a\right] \\
&\quad + O(m\bar{r}^2/\mathcal{R}^3)
\end{aligned}
$$

Matching with the internal-zone metric yields

$$
\boxed{a_a = \frac{1}{2}h^{\mathrm{tail}}_{00a} - h^{\mathrm{tail}}_{0a0}}
$$

the MiSaTaQuWa equations decomposed in the tetrad (u^μ, e^μ_a).

This derivation of the equations of motion is free of technical and conceptual pitfalls (no singularities! only retarded fields!).

It validates the other approaches.

7. Conclusion

The MiSaTaQuWa equations of motion can be derived in at least four different ways — their validity is very well established.

But it must be emphasized that

The MiSaTaQuWa equations are not gauge invariant (Barack and Ori) **and they cannot by themselves produce a meaningful answer to a well-posed physical question.**

To obtain such answers it shall always be necessary to combine the equations of motion with the metric perturbation so as to form gauge-invariant quantities that will correspond to observables.

More thought should be put into this.