YITP long-term workshop 2019
Multi-Messenger Astrophysics in the Gravitational Wave Era

Improvements on Initial Data for Spinning Neutron Star Binaries

Hannes Rüter – 03.10.2019
Albert Einstein Institute Potsdam
(work carried out at Friedrich Schiller University Jena)

Fundamental Equations

- Energy-momentum tensor of a perfect fluid:

$$T_{\alpha\beta} = (\epsilon + p)u_\alpha u_\beta + pg_{\alpha\beta}$$
$$= \rho h u_\alpha u_\beta + pg_{\alpha\beta}$$

ϵ – proper energy density

p – fluid pressure

u^α – fluid four-velocity

$g_{\alpha\beta}$ – spacetime metric

h – specific enthalpy

$$h := \frac{\epsilon + p}{\rho}$$

ρ – baryonic mass density

Metric Constraint Equations

- Decomposition of the spacetime metric

$$g_{\mu\nu}dx^{\mu}dx^{\nu} = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$$

$$\alpha - \text{lapse}$$

$$\beta^i - \text{shift}$$

$$\gamma_{ij} - \text{spatial metric}$$

Constraint Equations

Metric Constraint Equations

- Extended conformal thin sandwich (XCTS) equations

$$\bar{D}^j \bar{D}_j \psi = ...$$

$$\bar{D}^j \bar{D}_j \beta^i = -\frac{1}{3} \bar{D}^i \bar{D}_j \beta^j + ...$$

$$\bar{D}^j \bar{D}_j (\alpha\psi) = ...$$

- Conformal decomposition:

$$\gamma_{ij} = \psi^4 \bar{\gamma}_{ij}$$

Fluid Constraint Equations

- Energy-momentum conservation → Euler equations:

$$0 = P^{\alpha}_{\nu}\nabla_{\mu}T^{\mu\nu} = (\epsilon + p)(u^{\mu}\nabla_{\mu}u^{\alpha}) + P^{\alpha}_{\mu}\nabla^{\mu}p$$

- Particle number conservation → Continuity equation:

$$0 = \nabla_{\alpha}(\rho u^{\alpha})$$

$$P^{\alpha}_{\mu} - \text{projection: } P^{\alpha}_{\mu} := \delta^{\alpha}_{\mu} + u^{\alpha}u_{\mu}$$

$$\rho - \text{baryonic mass density}$$

Split of the Enthalpy Current

- Define enthalpy current:

$$\tilde{u}^\mu := h u^\mu$$

- Split off irrotational part:

$$\tilde{u}^\mu = \tilde{u}^\mu_{\mathrm{irr}} + w^\mu$$

$$\tilde{u}^\mu_{\mathrm{irr}} - \text{irrotational part of enthalpy current}$$

$$w^\mu - \text{rotational part / ``spin''}$$

Irrotational Fluids

- Irrotational fluids:

$$\omega_{\alpha\beta} := P^{\mu}_{\alpha} P^{\nu}_{\beta} \frac{1}{2} (\nabla_{\mu} u_{\nu} - \nabla_{\nu} u_{\mu}) = 0$$

- For irrotational barotropic fluids: [Shibata - PRD **58**, 024012 (1998)]

$$\omega_{\alpha\beta} = 0 \iff D_i \, {}^{(3)}\tilde{u}_j - D_j \, {}^{(3)}\tilde{u}_i = 0$$

- $\Longrightarrow {}^{(3)}\tilde{u}_i = D_i \phi$

$$\phi - \text{velocity potential}$$

Split of the Enthalpy Current

- Enthalpy current can be written as:

$$\tilde{u}^\mu = \tilde{u}^\mu_{\mathrm{irr}} + w^\mu = \nabla^\mu \phi + w^\mu \qquad D_i w^i = 0$$

$\tilde{u}^\mu_{\mathrm{irr}}$ – irrotational part of enthalpy current

w^μ – rotational part / "spin"

ϕ – velocity potential

Choice of Time Derivatives

- Quasi-equilibrium: [Tichy - PRD **84**, 024041 (2011)]

$$\mathcal{L}_\xi g_{\mu\nu} = 0 \qquad \mathcal{L}_\xi \left(\rho u^t\right) = 0 \qquad \gamma_i^\nu \mathcal{L}_\xi (\tilde{u}_{\mathrm{irr}\,\nu}) = 0$$

$$\xi^\mu - \text{approximate (helliptical) Killing vector}$$

$$\xi^\mu_{1,2} = (1, \ -\Omega y, \ \Omega(x - x_{1,2}), \ 0)$$

- But for rotational part:

$$\gamma_i^\nu \mathcal{L}_{\tilde{u}_{\mathrm{irr}}/\tilde{u}^t}(w_\nu) = 0$$

Fluid Constraint Equations

- Split off Killing vector: $\tilde{u}^\mu = \tilde{u}^t(\xi^\mu + V^\mu)\,, V^\mu n_\mu = 0$
- Constraint Equations: [Tichy - PRD **84**, 024041 (2011)]

$$0 = \; - D_i C + {}^{(3)}\mathcal{L}_{w/\tilde{u}^t} w_i \approx -D_i C$$

$$0 = D_i \left(\frac{\rho\alpha}{h} (D^i \phi + w^i - \tilde{u}^t(\beta^i + \xi^i)) \right)$$

$$C - \text{constant of integration } C = -\frac{h^2}{\tilde{u}^t} - V^k D_k \phi$$

Fluid Constraint Equations

- Split off Killing vector: $\tilde{u}^\mu = \tilde{u}^t(\xi^\mu + V^\mu)\,, V^\mu n_\mu = 0$
- Constraint Equations: [Tichy - PRD **84**, 024041 (2011)]

$$0 = -D_i C + {}^{(3)}\mathcal{L}_{w/\tilde{u}^t} w_i \approx -D_i C$$

$$0 = D_i \left(\frac{\rho\alpha}{h} (D^i \phi + w^i - \tilde{u}^t(\beta^i + \xi^i)) \right)$$

$$C - \text{constant of integration } C = -\frac{h^2}{\tilde{u}^t} - V^k D_k \phi$$

Exact Integration of the Euler Equations

Exact Integration of the Euler Equations

- Euler equations: $0 = -D_i C + {}^{(3)}\mathcal{L}_{w/\tilde{u}^t} w_i$

- Requires: $$0 \stackrel{!}{=} D_i \, {}^{(3)}\mathcal{L}_{w/\tilde{u}^t} w_j - D_j \, {}^{(3)}\mathcal{L}_{w/\tilde{u}^t} w_i$$

- Satisfied, if w^i is a Beltrami field:

$$0 = w^k (D_k w_i - D_i w_k) \equiv \epsilon_{jki} w^k (\mathrm{curl}\, w)^j$$

Exact Integration of the Euler Equations

- Beltrami fields:

$$0 = w^k (D_k w_i - D_i w_k) \equiv \epsilon_{jki} w^k (\mathrm{curl}\, w)^j$$

$$\Leftrightarrow \epsilon^{jmn} D_m w_n = \lambda w^j$$

- In flat space: $\vec{\nabla} \times \vec{w} = \lambda \vec{w}$
- Planar and axisymmetric flows not allowed!
- Euler equation:

$$0 = -D_i \left(C - \frac{w_j w^j}{\tilde{u}^t} \right)$$

Solution with Helical Rotational Part

- Want: Uniform rotation around a spin-axis
- Example solution:
  rotation around z-axis in flat space:

$$w^i = \omega \left(-y, x, \pm 2\sqrt{c^2 - \frac{x^2+y^2}{2}} \right)$$

$\omega - \text{spin frequency}$

$c - \text{constant}$

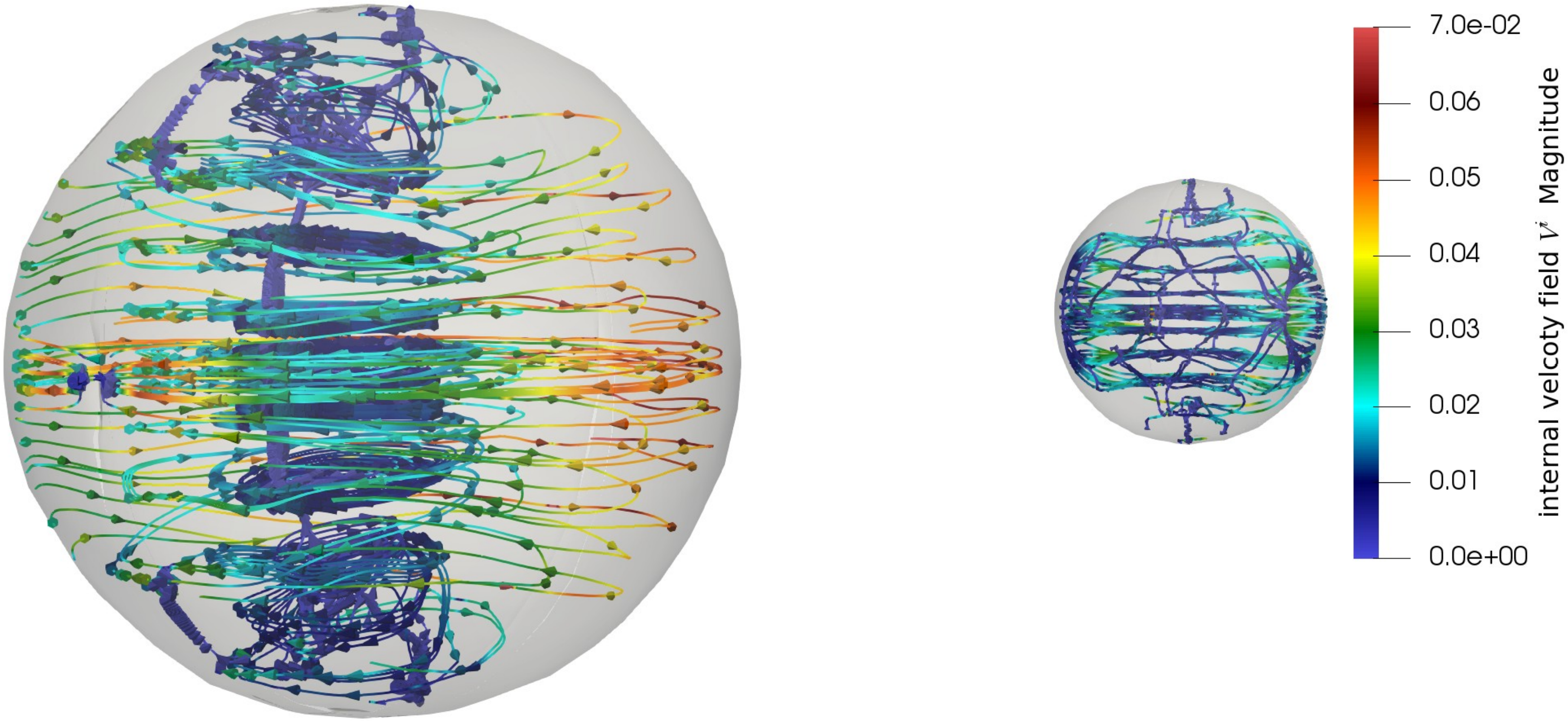
Solution with Planar Rotational Part

- Binary on quasi-circular orbit

- Equal mass
- Anti-aligned spin

- Polytropic EOS

$$\rho = \frac{\kappa(h-1)}{2}$$



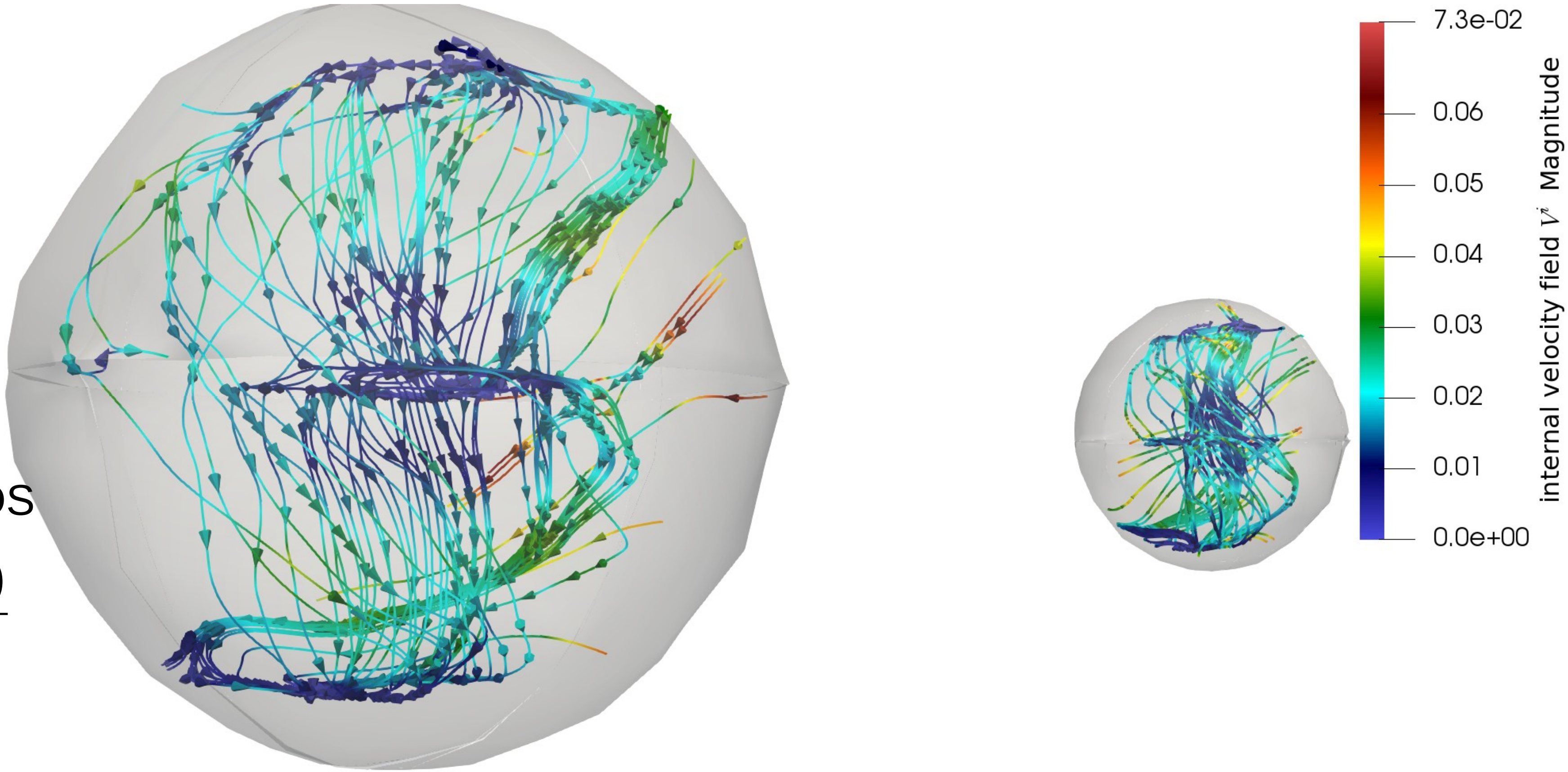
Solution with Helical Rotational Part

- Binary on quasi-circular orbit

- Equal mass
- Anti-aligned spin

- Polytropic EOS

$$\rho = \frac{\kappa(h-1)}{2}$$



New Formalism

Possible Causes for Problems in Previous Formalism

- Split of the enthalpy current: $\tilde{u}^\mu = \tilde{u}^\mu_{\rm irr} + w^\mu$
- $\tilde{u}^\mu_{\rm irr} = \nabla^\mu \phi$ holds only if $w^\mu = 0$
- $D_i{}^{(3)}\tilde{u}_{{\rm irr},j} - D_j{}^{(3)}\tilde{u}_{{\rm irr},i} = 0$ required the relations:
[Shibata - PRD **58**, 024012 (1998)]

normalization: $u^\mu_{\rm irr} u_{\mu\,\rm irr} = -1$

Euler equations: $0 = u^\mu_{\rm irr} \nabla_\mu (h u_{{\rm irr}\nu}) + \nabla_\nu h$

and $\gamma^\mu_i \mathcal{L}_n \tilde{u}_{{\rm irr}\mu} = 0$

Improved Formalism

- New split for u: $u^{\mu} = t(u^{\mu}_{\text{irr}} + u^{\mu}_{\text{rot}})$
- Analogue to Euler eqs.: $0 = u^{\mu}_{\text{irr}} \nabla_{\mu} (X u_{\text{irr}\nu}) + \nabla_{\nu} X$

t – normalization constant

X – some function to be chosen

$$\implies 0 = D_i(X u_{\text{irr}j}) - D_j(X u_{\text{irr}i})$$

$$\implies u_{\text{irr}i} = \frac{1}{X} D_i \sigma$$

σ – new velocity potential

Improved Formalism

- Euler equations → ODE for X

$$0 = \gamma_i^\nu(\delta_\nu^\mu + u^\mu u_\nu)\nabla_\mu \ln\left(\frac{ht}{X}\right)$$

$$+ t u^t_{\mathrm{irr}} \gamma_i^\nu \mathcal{L}_{u_{\mathrm{irr}}/u^t_{\mathrm{irr}}} w_\nu - t u^t_{\mathrm{irr}} D_i \frac{u^\mu_{\mathrm{irr}}}{u^t_{\mathrm{irr}}} w_\mu$$

$$+ \frac{t}{X} w^j (D_j w_i - D_i w_j)$$

here: $w^\mu = X u^\mu_{\mathrm{rot}}$

Summary

- The exact integration of the Euler equations revealed problems in the old formalism for initial data construction for neutron star binaries with spin.
- Mistakes in the old formalism have been identified and are addressed in an improved formalism.

Summary

- The exact integration of the Euler equations revealed problems in the old formalism for initial data construction for neutron star binaries with spin.
- Mistakes in the old formalism have been identified and are addressed in an improved formalism.

Future Objectives

- Implement new formalism in a numerical code
- More accurate initial data for comparison with signals from future high precision gravitational wave detectors

Future Objectives

- Implement new formalism in a numerical code
- More accurate initial data for comparison with signals from future high precision gravitational wave detectors

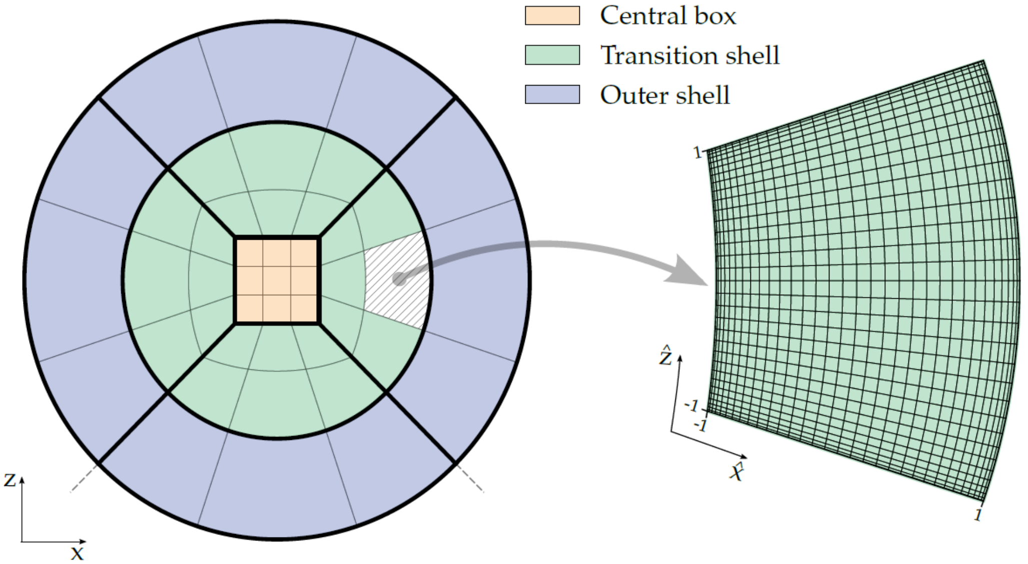
Bonus
New Formalism for Irrotational Fluid

- X = h
- w = 0
- ODE for X follows from

$$\gamma_i^\mu \mathcal{L}_n (X u_{\mathrm{irr}\mu}) = \gamma_i^\mu \mathcal{L}_n \tilde{u}_{\mathrm{irr}\mu} = 0$$

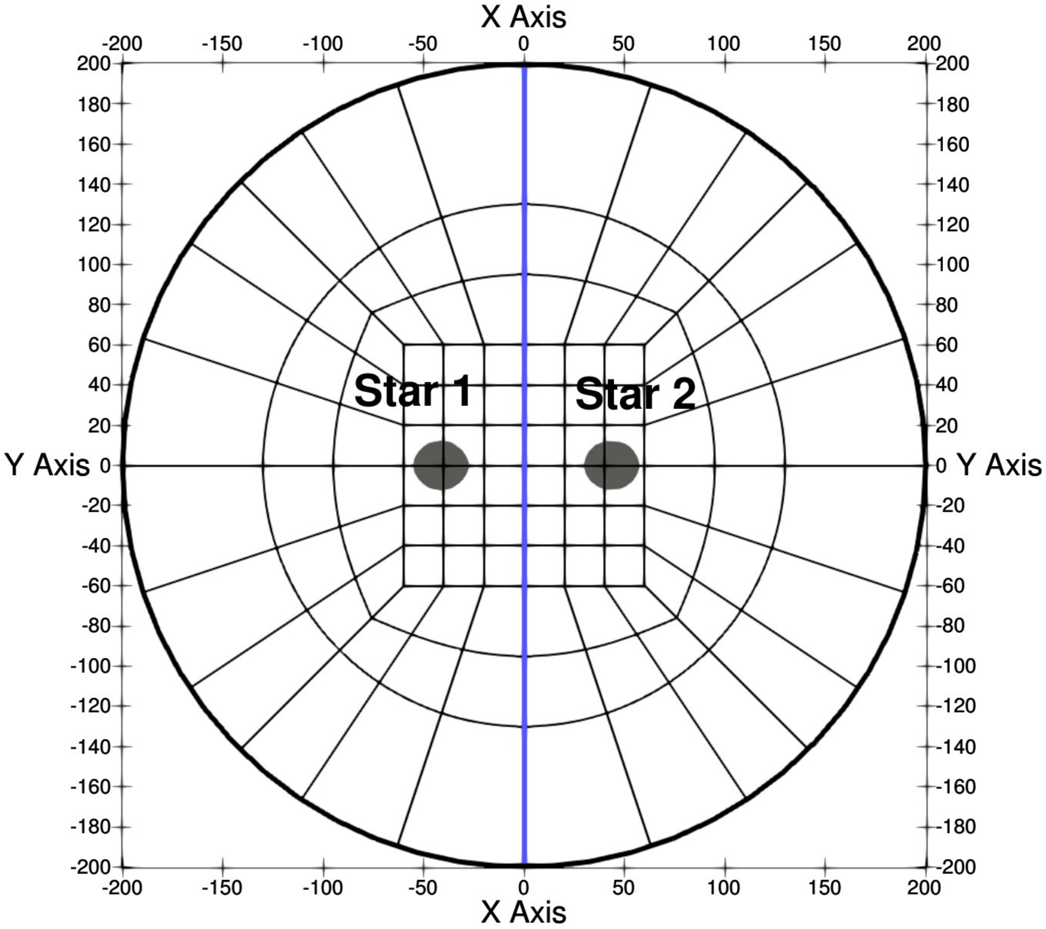
Bonus: Numerical details - BAMPS

- Pseudospectral evolution code for hyperbolic first order PDEs
- 4th order Runge-Kutta time integrator



<u>BAMPS:</u>
D. Hilditch, A. Weyhausen, B. Brügmann - PRD **93**, 063006 (2016)

Bonus: Numerical details



Bonus: Rotational Part as a Beltrami Field for arbitrary spin

- Want: Uniform rotation around a spin-axis
- Flat space solution for arbitrary spin ω_i:

$$w^i = \epsilon^{ijk}\omega_j(x_k - x_{Ck})$$

$$\pm 2\omega^i\sqrt{c^2 - \frac{\delta_{ij}(x^i - x^i_C)(x^j - x^j_C)}{2} + \frac{\delta_{ij}(\omega^j(x^i - x^i_C))^2}{2\delta_{kl}\omega^l\omega^k}}$$