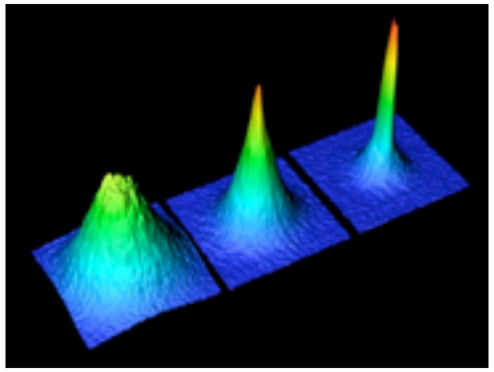
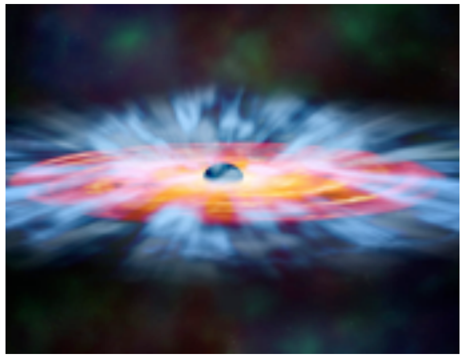
Séminaire "Fluides quantiques" du LPTMS, 16th February, 2011

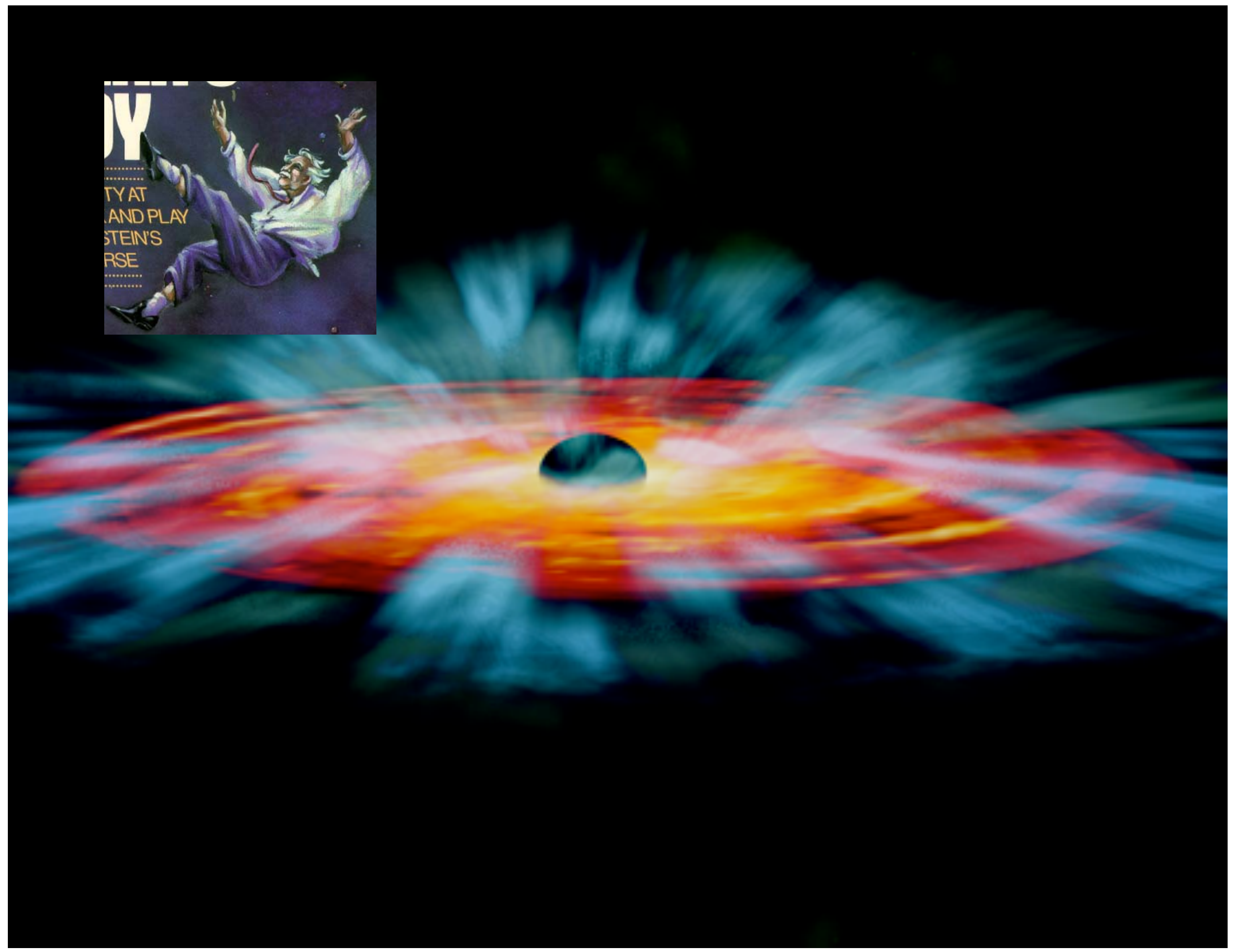
Acoustic Hawking radiation in dynamically expanding Bose-Einstein condensate

Takao Morinari
Yukawa Institute for Theoretical Physics, Kyoto University, JAPAN





What is the Hawking radiation?

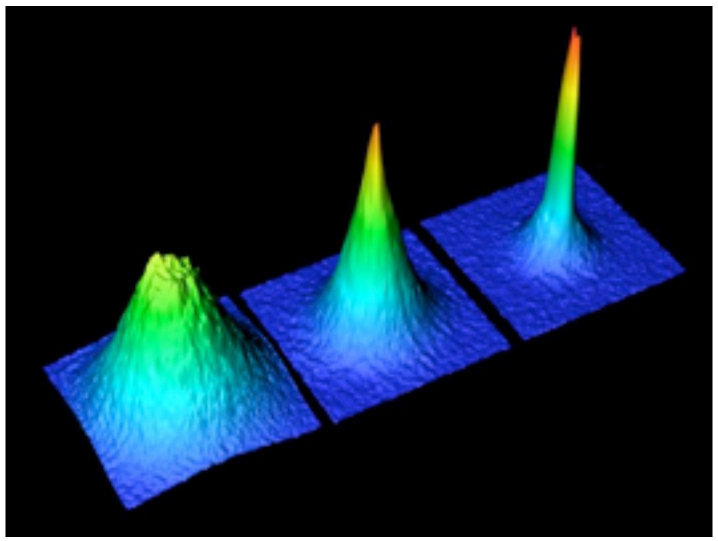


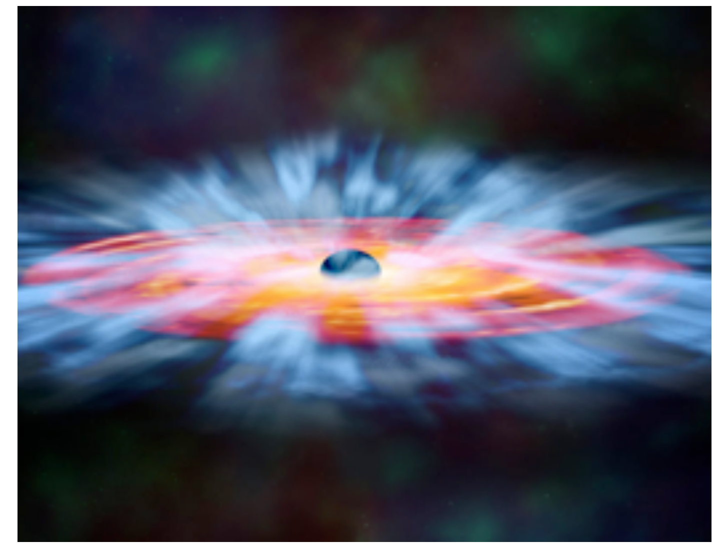
S. Hawking, Nature 248, 30 (1974);
Commun. Math. Phys. 43, 199 (1975)

Black-hole is not really black.
Black-hole evaporation?

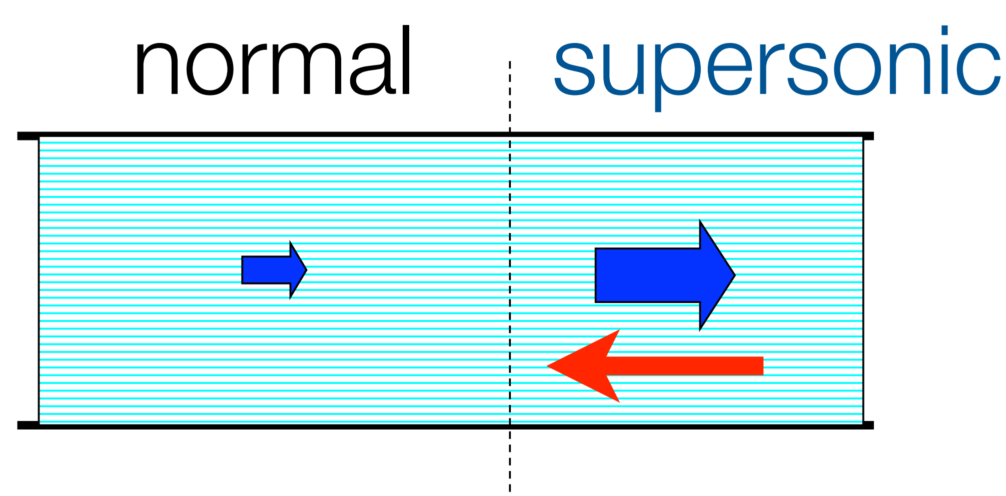
$$n(E) = \frac{1}{\exp\left(\frac{E}{k_B T_H}\right) - 1}$$

Basic idea





Analogy between (quantum) fluid and Black-hole



W.G. Unruh, PRL 46, 1351 (1981)

Outline

1. Bose-Einstein condensation in cold atoms

2. Black hole and Hawking radiation

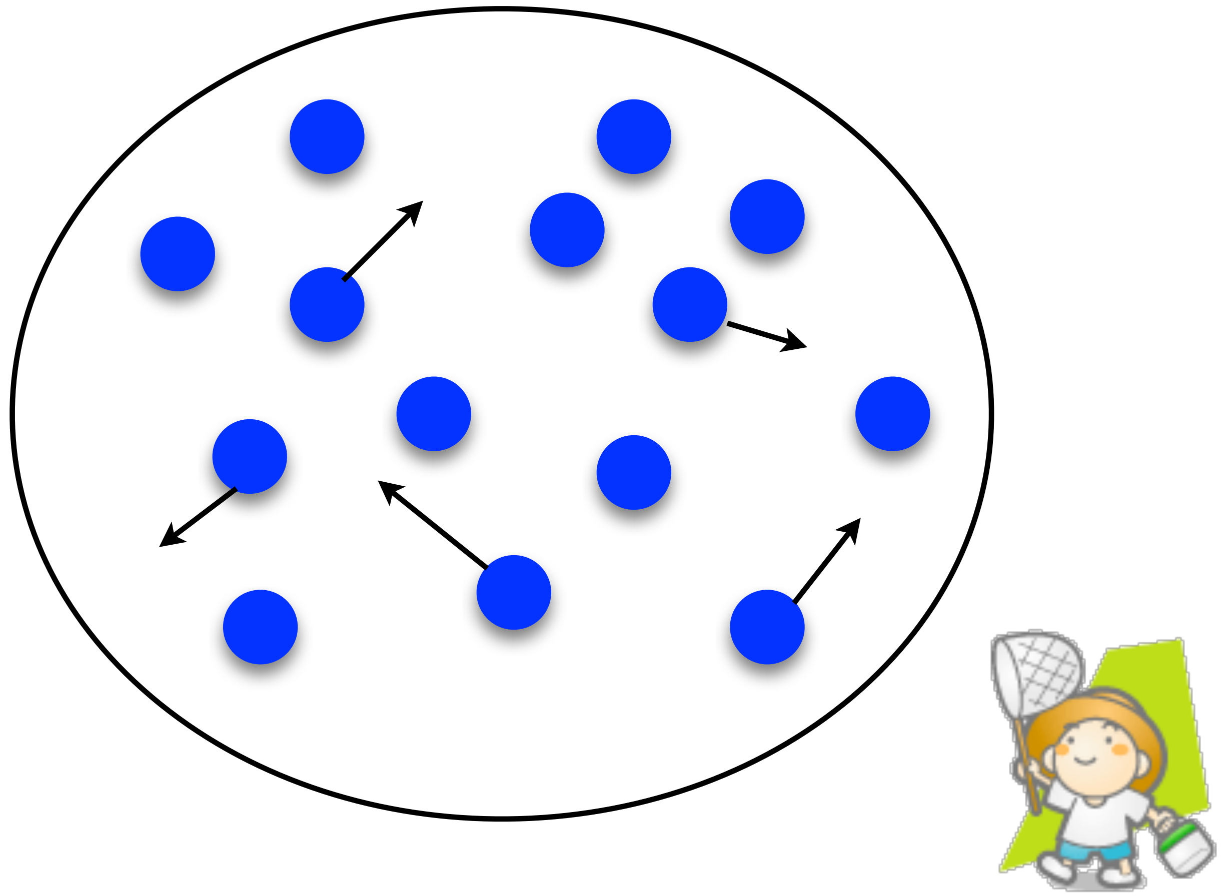
3. “Hawking radiation” in BEC

   - Formulation

   - “Horizon” creation

   - “Hawking radiation”

Bose-Einstein condensation in cold atoms



BEC in cold atoms: Cooling process

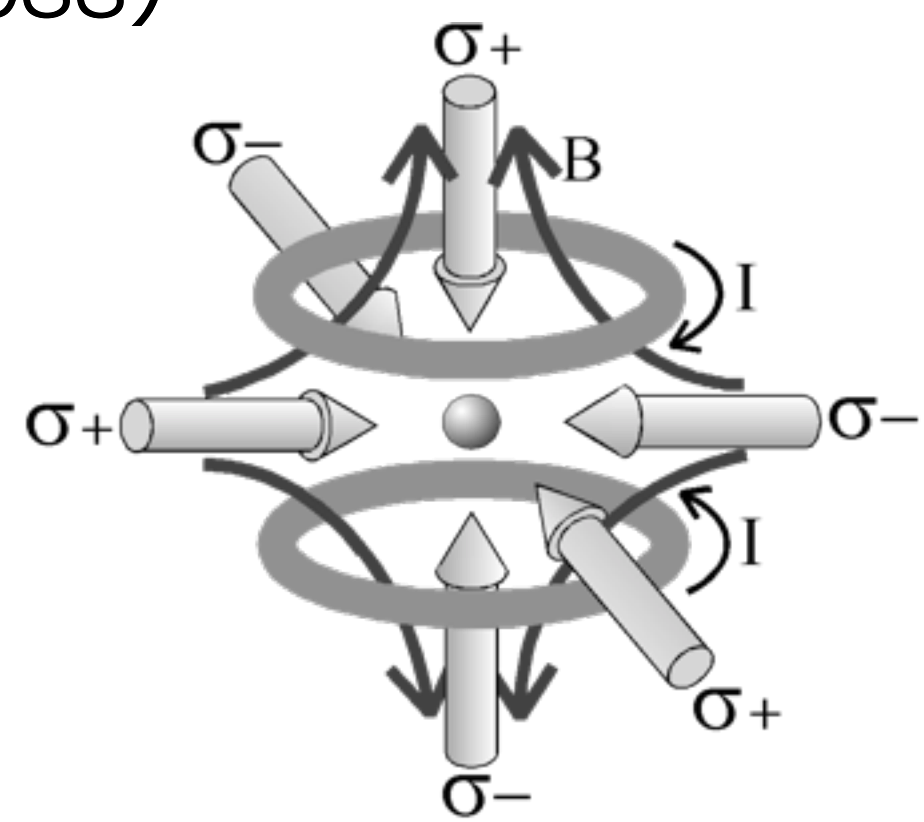
Pethick and Smith, *Bose-Einstein condensation in dilute gases*
(Cambridge University Press, 2002)

- Laser cooling（Doppler process）

  Stop atoms

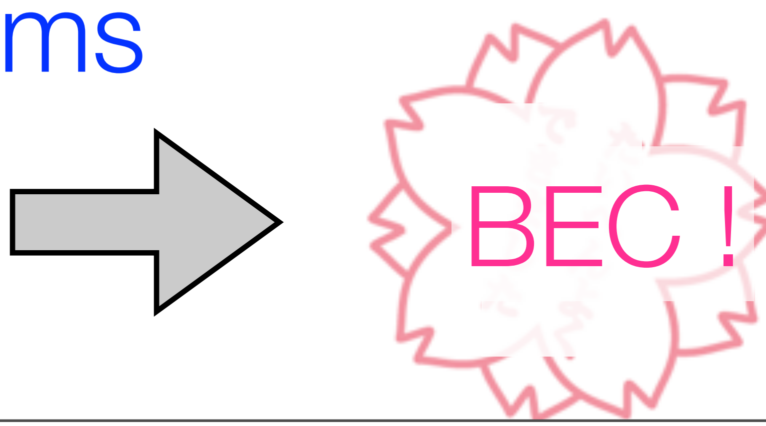
- Magneto-optical trap

  Confine atoms

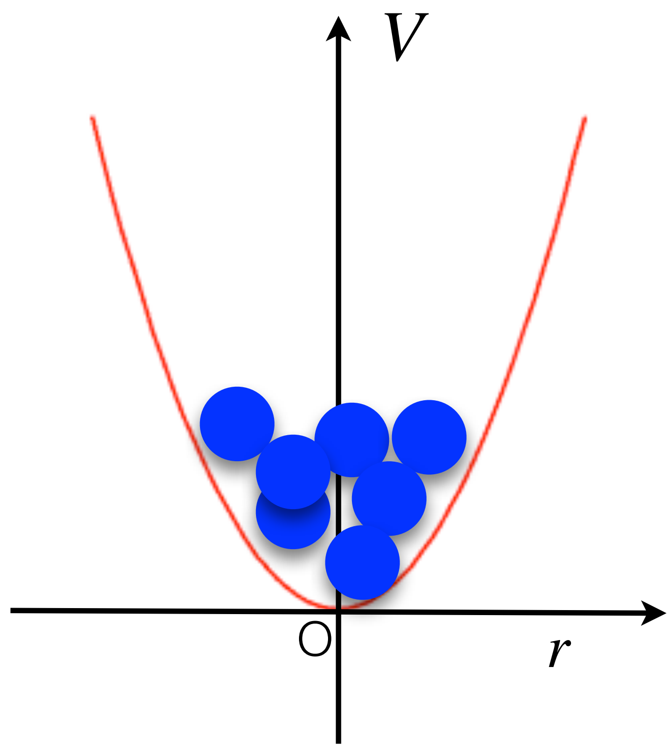


- Evaporative cooling

  Remove high-energy atoms



Model

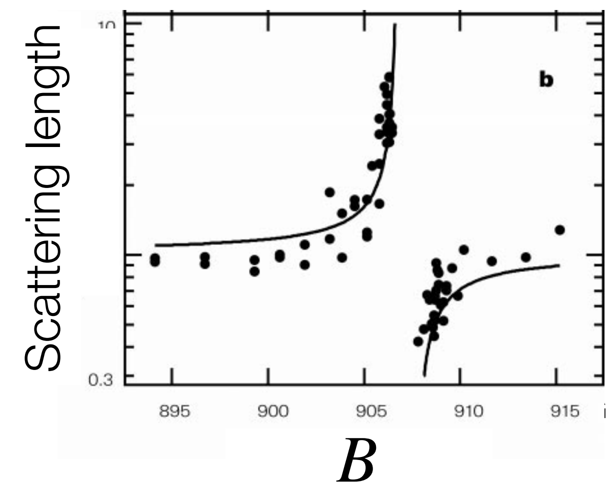


$$V(r) = \frac{1}{2} m \omega_{ho}^2 r^2$$

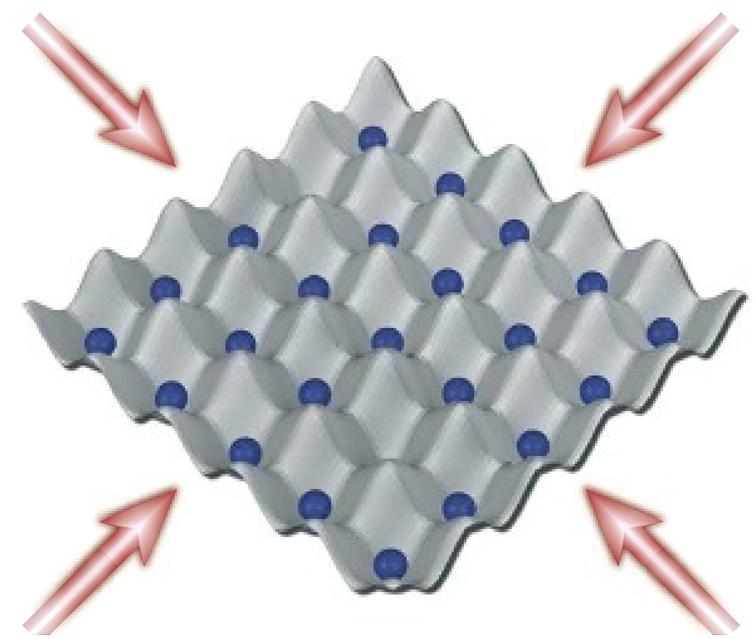
BEC in a harmonic-oscillator potential

Advantages for using cold atoms

- Control system parameters
  - Interaction: Feshbach res.
  - Confining potential
  - Lattice (optical lattice)
  - ...
- Internal degrees of freedom
  Various topological excitations
- boson-fermion mixture



S. Inouye *et al.*, Nature **392**, 152 (1998).



✻ Experimental probes are limited.

Outline

1. Bose-Einstein condensation in cold atoms
2. Black hole and Hawking radiation
3. “Hawking radiation” in BEC
    - Formulation
    - “Horizon” creation
    - “Hawking radiation”

General Relativity

Particle motion under gravity

$$m_I \frac{d^2 r}{dt^2} = \frac{GM}{r^2} m_G = g(r) m_G$$

Accelerated coordinate

$$r' = r - \frac{1}{2} g(r) t^2$$

$$m_I \frac{d^2 r'}{dt^2} = (m_G - m_I) g(r) \longrightarrow 0$$

Equivalence Principle

Einstein equation

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu}$$

“Spacetime tells matter how to move; matter tells spacetime how to curve.”

Black Hole

Spherically symmetric solution: Schwarzschild solution

$$ds^2 = \left(1 - \frac{r_g}{r}\right)c^2dt^2 - \frac{dr^2}{1 - \frac{r_g}{r}} - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2$$

Horizon radius：$r_g = \frac{2GM}{c^2}$

$$r_g^{Sun} = 3\text{km}$$

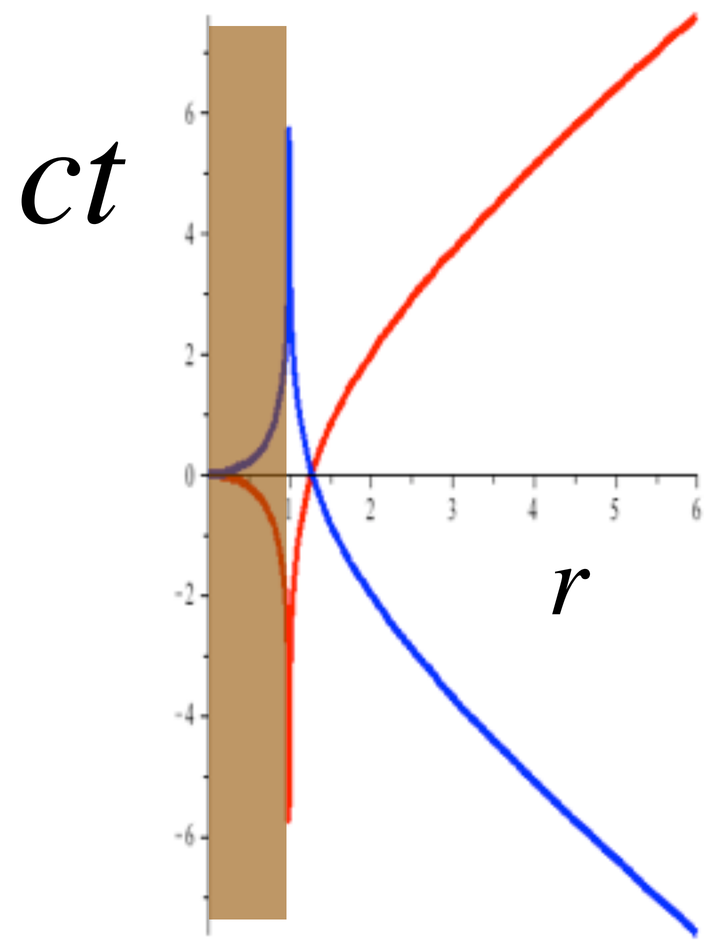
$$r_g^{Earth} = 1\text{cm}$$

$$r_g^{TM} = 1\times 10^{-15}\,\text{Å} = 6\times 10^9\,\ell_P$$

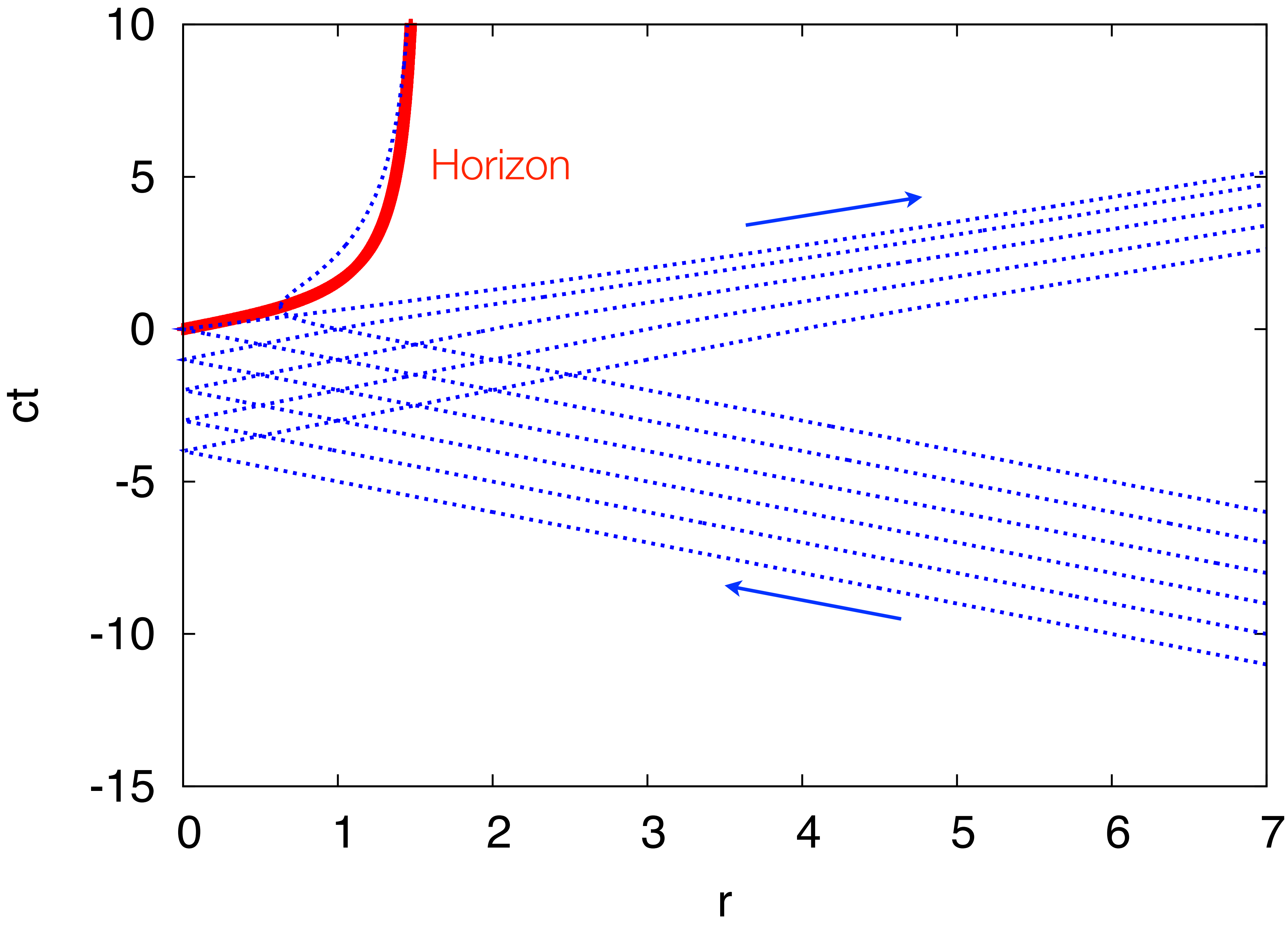
Schwarzschild spacetime

Light path $ds = 0$

$$cdt = \pm \frac{dr}{1 - r_g / r} \quad \Rightarrow \quad ct = \pm \left[r + r_g \ln \left| \frac{r}{r_g} - 1 \right| \right]$$



Light passing near Black hole



Plane waves in Schwarzschild spacetime

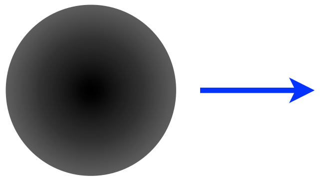
$$\eta^{\mu\nu}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu}\phi = 0 \quad \Rightarrow \quad \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\mu}\left(\sqrt{-g}\,g^{\mu\nu}\frac{\partial\phi}{\partial x^\nu}\right) = 0$$

$$g = \det g_{\mu\nu}$$

$$g_{\mu\nu}g^{\nu\lambda} = \delta_\mu^\lambda$$

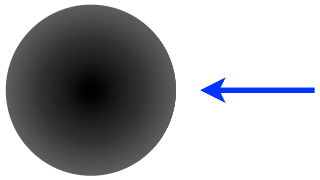
Out-going wave

$$\phi = \frac{1}{r}\exp\left(-i\frac{\omega}{c}\left(ct - r - r_g\ln\left|\frac{r}{r_g}-1\right|\right)\right)$$



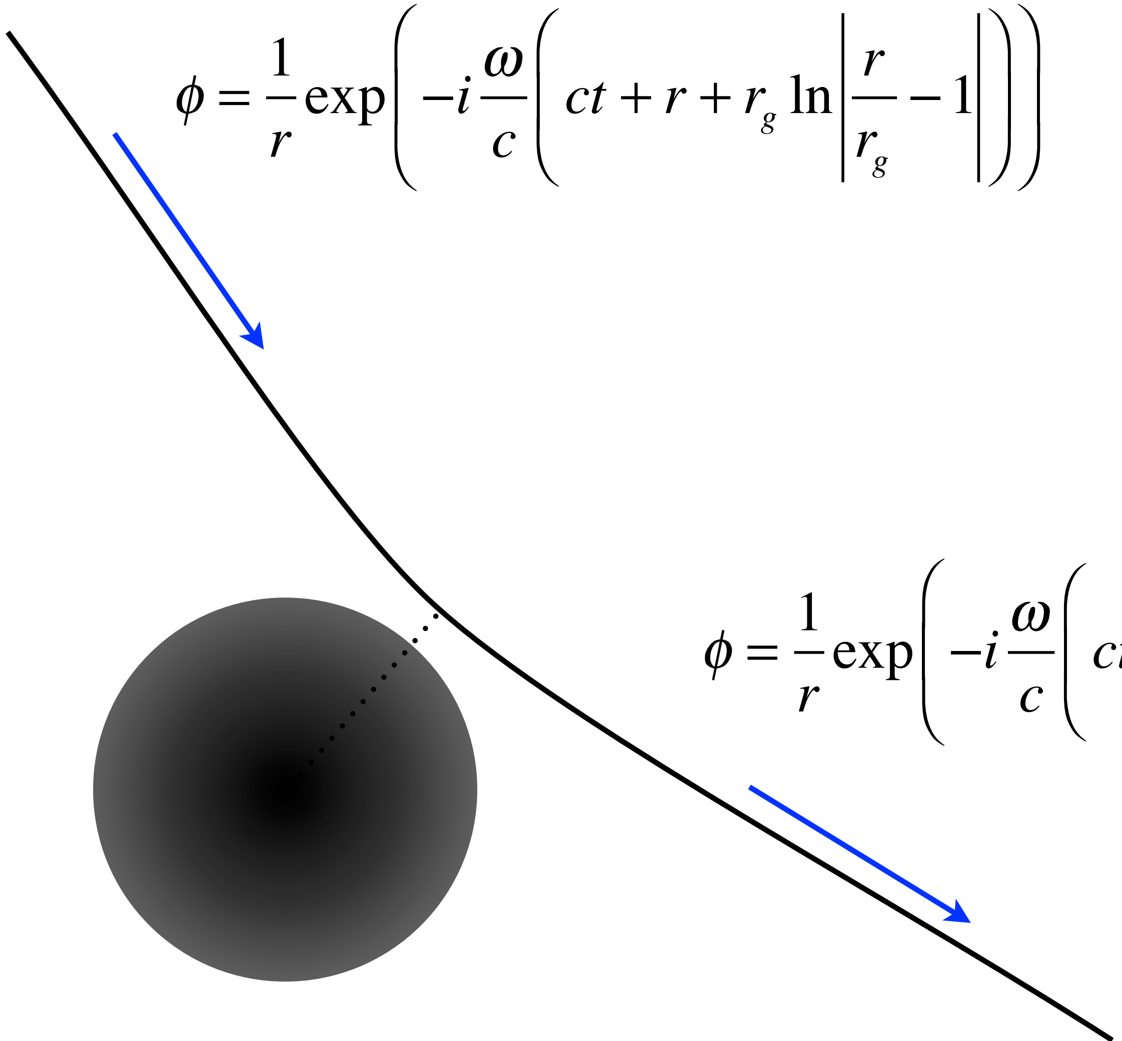
In-coming wave

$$\phi = \frac{1}{r}\exp\left(-i\frac{\omega}{c}\left(ct + r + r_g\ln\left|\frac{r}{r_g}-1\right|\right)\right)$$



Scattering by Black Hole

$$\phi = \frac{1}{r}\exp\left(-i\frac{\omega}{c}\left(ct + r + r_g \ln\left|\frac{r}{r_g} - 1\right|\right)\right)$$



$$\phi = \frac{1}{r}\exp\left(-i\frac{\omega}{c}\left(ct - r - r_g \ln\left|\frac{r}{r_g} - 1\right| + \delta\right)\right)$$

Phase shift

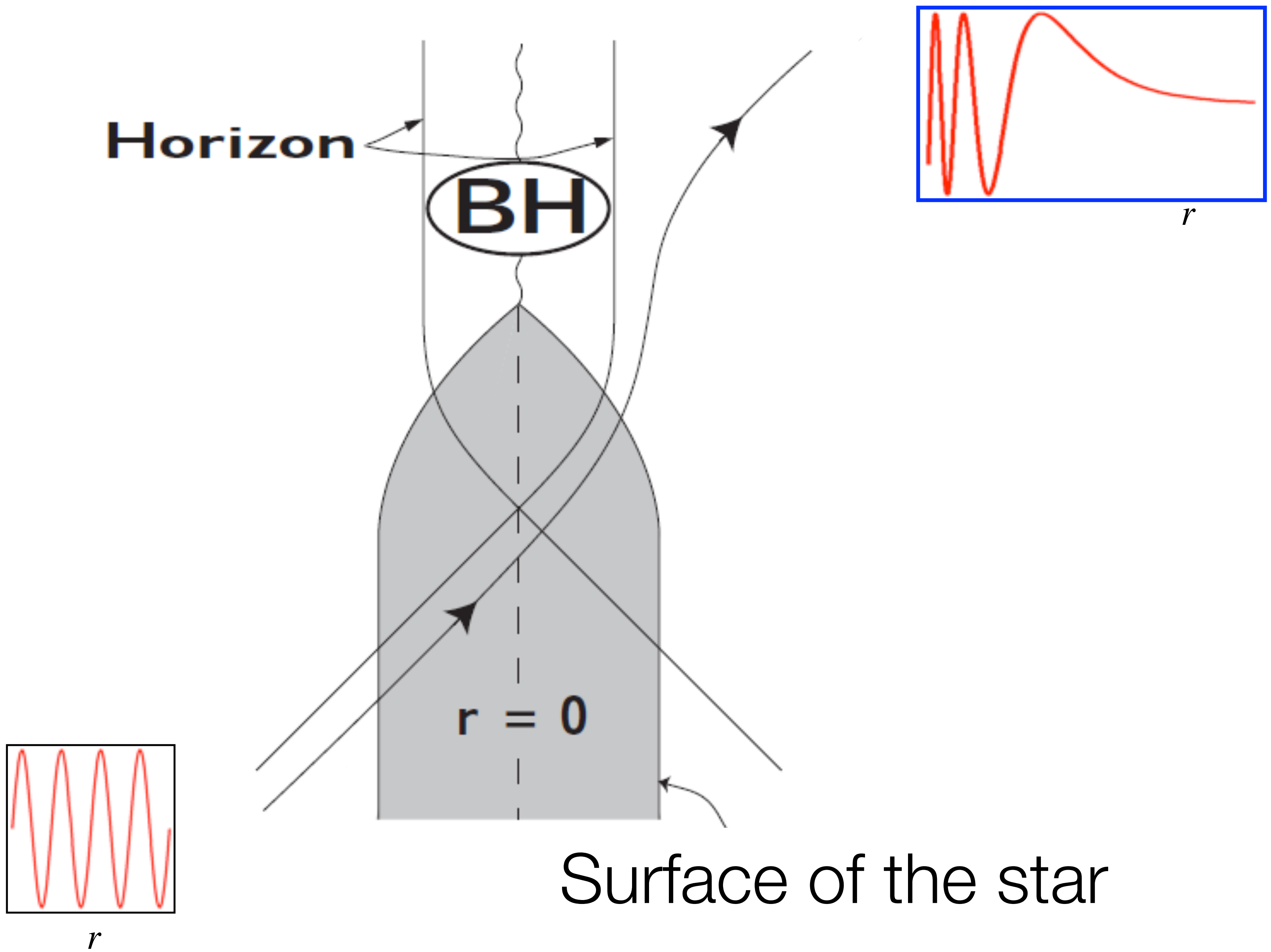
$$\delta = 2r_g + 2r_g \log\frac{r_{\min}}{r_g}$$

Spectrum

$$n(\omega) = \frac{1}{\exp\left(\dfrac{\hbar\omega}{k_B T_H}\right) - 1}$$

$$k_B T_H = \frac{c^3\hbar}{8\pi G M}$$

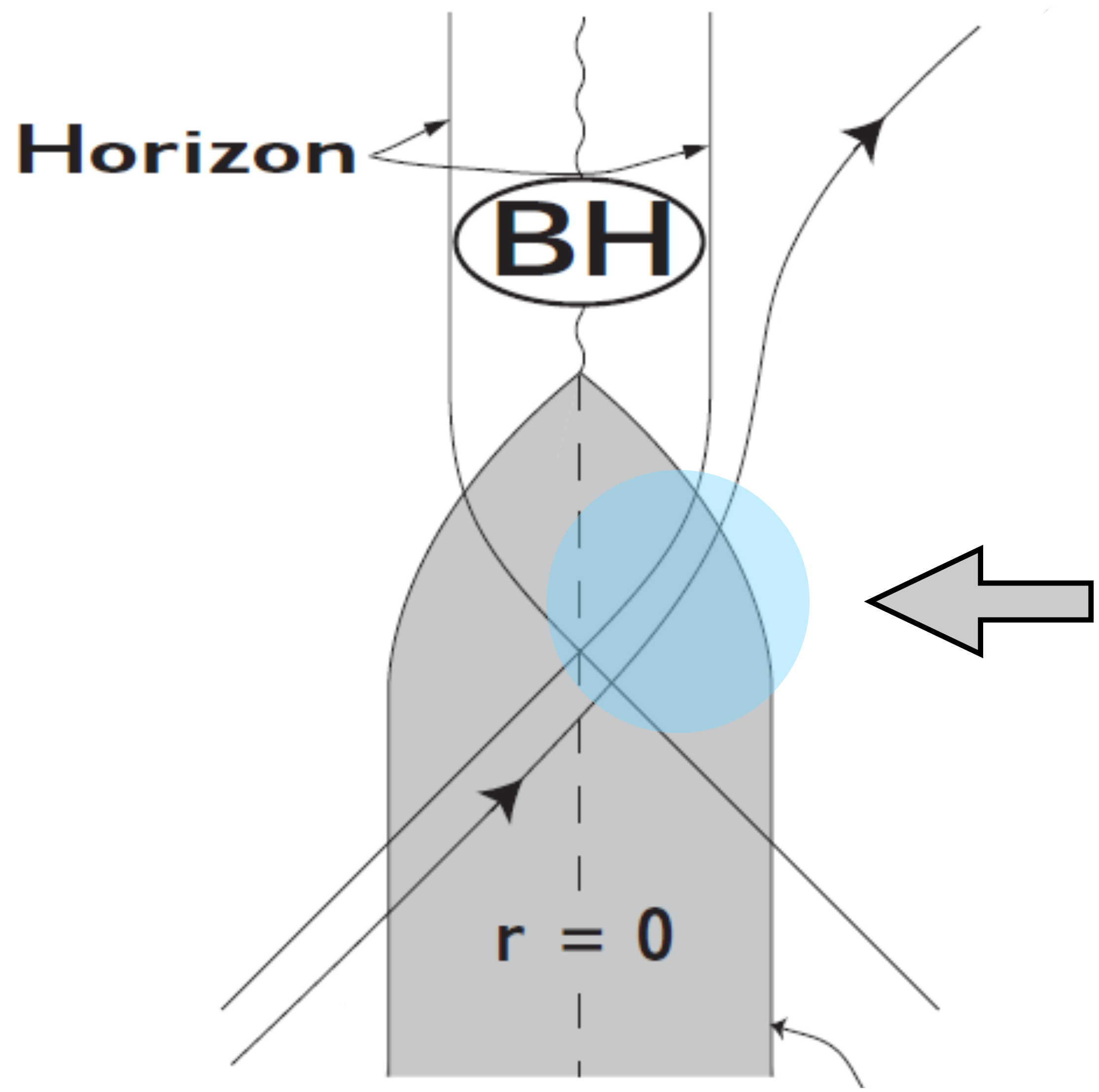
Scattering by Black Hole

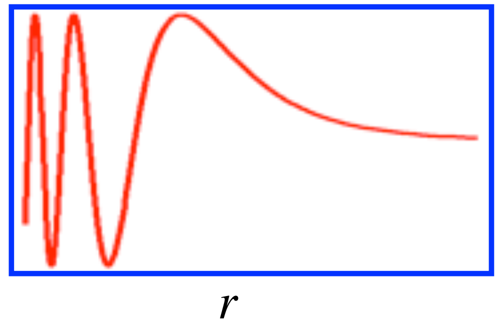


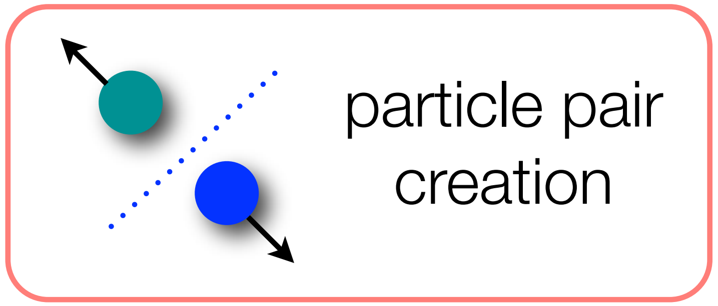
阪上雅昭、大橋　憲、物性研究　76, 328 (2001)

Sakagami and Ohashi, BUSSEI KENKYU 76, 328 (2001) (Japanese)

Hawking radiation







Surface of the star

阪上雅昭、大橋　憲、物性研究　76, 328 (2001)

Sakagami and Ohashi, BUSSEI KENKYU 76, 328 (2001) (Japanese)

Schwarzschild spacetime for a free falling observer

$$ds^2 = \left(1 - \frac{r_g}{r}\right) c^2 dt^2 - \frac{dr^2}{1 - \frac{r_g}{r}} - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2$$

$$\Downarrow$$

$$c d\tilde{t} = c dt - \frac{v / c}{1 - v^2 / c^2} dr, \quad v = -c\sqrt{\frac{r_g}{r}}$$

$$ds^2 = c^2 d\tilde{t}^2 - \left(dr - v d\tilde{t}\right)^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2$$

“Hawking radiation” in BEC

1) Formulation

2) “Horizon” creation

3) “Hawking radiation”

Formulation

F. Dalfovo et al., Rev. Mod. Phys. 71, 463 (1999)

- BEC
  Gross-Pitaevskii equation

- Fluctuations
  Bogoliubov-de Gennes equation

BEC: Gross-Pitaevskii equation

Action

$$S = \int dt \int d^3\mathbf{r} \left[i\hbar\bar{\phi}\partial_t\phi - \bar{\phi}\left(-\frac{\hbar^2}{2m}\nabla^2 + \frac{1}{2}m\omega_{ho}^2 r^2 \right)\phi - \frac{1}{2}U\bar{\phi}\bar{\phi}\phi\phi \right]$$

⇓ Saddle point approx.

$$S = \int dt \int d^3\mathbf{r} \left[i\hbar\Phi^*\partial_t\Phi - \Phi^*\left(-\frac{\hbar^2}{2m}\nabla^2 + \frac{1}{2}m\omega_{ho}^2 r^2 \right)\Phi - \frac{1}{2}U\Phi^*\Phi^*\Phi\Phi \right]$$

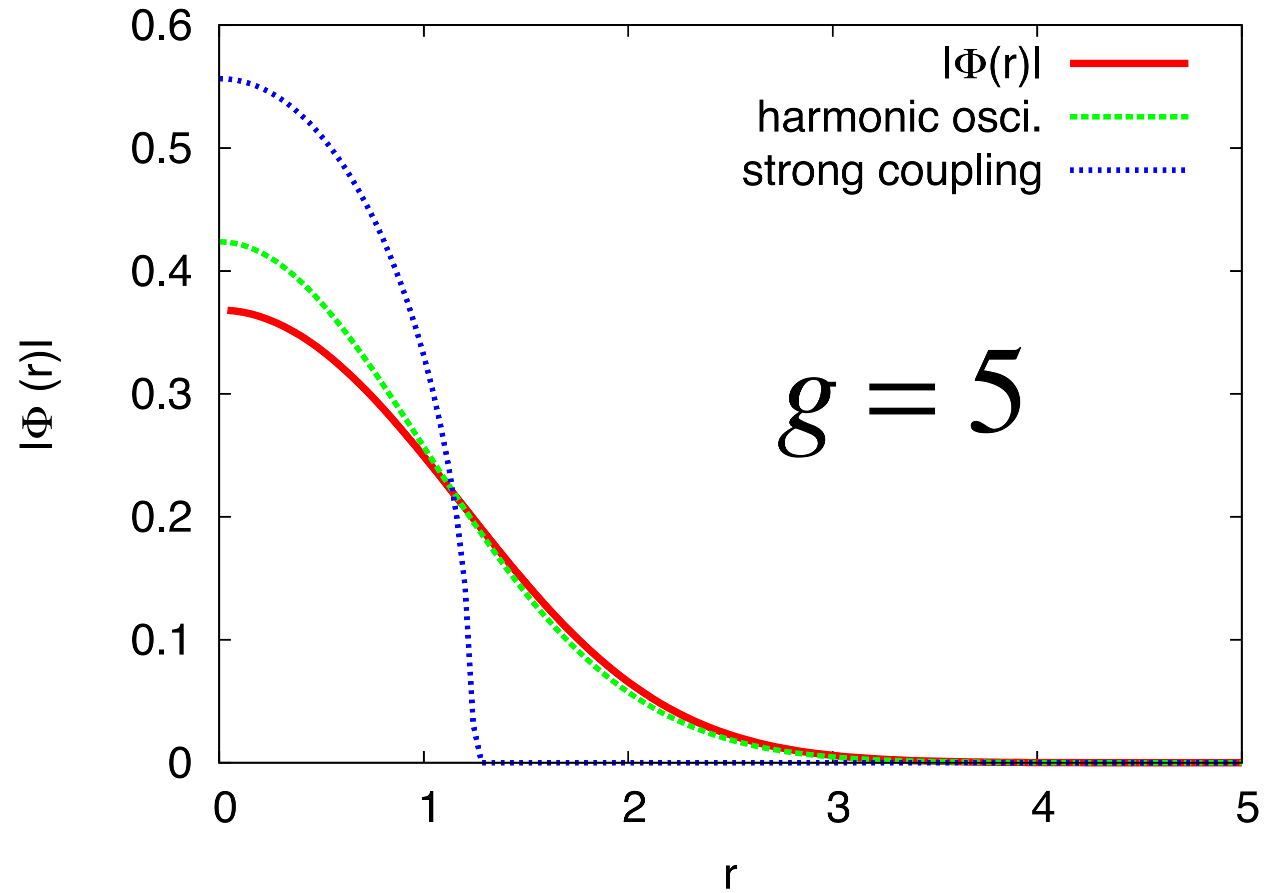
⇓ Take variation

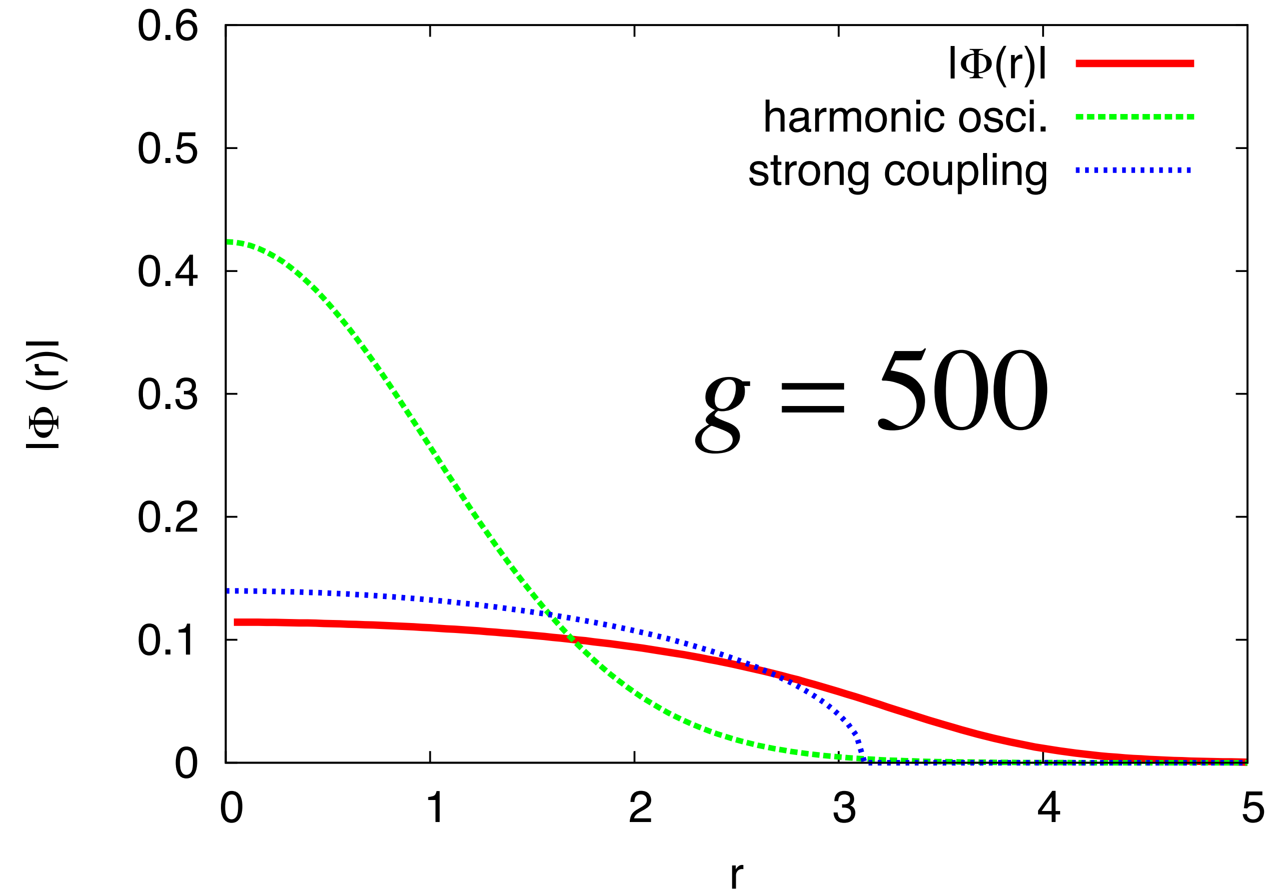
$$i\hbar\partial_t\Phi = \left(-\frac{\hbar^2}{2m}\nabla^2 + \frac{1}{2}m\omega_{ho}^2 r^2 \right)\Phi + U\Phi^*\Phi\Phi$$

Solution of the Gross-Pitaevskii

$$E_0\Phi = \left(-\frac{\hbar^2}{2m}\nabla^2 + \frac{1}{2}m\omega_{ho}^2 r^2 \right)\Phi + U\Phi^*\Phi\Phi$$

$$g = NU\ell^3 \qquad \ell = \sqrt{\frac{\hbar}{m\omega_{ho}}}$$





Fluctuations: Bogoliubov - de Gennes equation

boson field operator

$$\phi = \sum_{\alpha} \phi_\alpha b_\alpha = \phi_0 b_0 + \sum_{\alpha(\neq 0)} \phi_\alpha b_\alpha$$

$$b_0 |g.s.\rangle = \sqrt{N_0}\,|g.s.\rangle$$

$$b_0^\dagger |g.s.\rangle = \sqrt{N_0+1}\,|g.s.\rangle \simeq \sqrt{N_0}\,|g.s.\rangle$$

$$\phi \to \Phi + \phi$$

$$i\hbar\partial_t \phi = \left(-\frac{\hbar^2}{2m}\nabla^2 + \frac{1}{2} m\omega_{ho}^2 r^2 + 2U\Phi^*\Phi \right)\phi + U\Phi^2\phi^\dagger$$

Fluctuations: Bogoliubov - de Gennes equation

$$i\hbar\partial_t\phi = \left(-\frac{\hbar^2}{2m}\nabla^2 + \frac{1}{2}m\omega_0^2 r^2 + 2U\Phi^*\Phi \right)\phi + U\Phi^2\phi^\dagger$$

Bogoliubov transformation

$$\phi(\mathbf{r},t) = \sum_\alpha \left[A_\alpha(\mathbf{r},t)b_\alpha + B_\alpha^*(\mathbf{r},t)b_\alpha^\dagger \right]$$

$$i\hbar\partial_t \begin{pmatrix} A_\alpha(\mathbf{r},t) \\ B_\alpha(\mathbf{r},t) \end{pmatrix} = \begin{pmatrix} K & M \\ -M^* & -K^* \end{pmatrix} \begin{pmatrix} A_\alpha(\mathbf{r},t) \\ B_\alpha(\mathbf{r},t) \end{pmatrix}$$

$$K = -\frac{\hbar^2}{2m}\nabla^2 + \frac{1}{2}m\omega_0^2 r^2 + 2U\Phi^*\Phi, \quad M = U\Phi^2$$

Zero mode

Bogoliubov-de Gennes equation
Zero mode

$$\Rightarrow \begin{pmatrix} \Phi(\mathbf{r},t) \\ -\Phi^*(\mathbf{r},t) \end{pmatrix}$$

$$i\hbar\partial_t \begin{pmatrix} A_\alpha(\mathbf{r},t) \\ B_\alpha(\mathbf{r},t) \end{pmatrix} = \begin{pmatrix} K - E_0 & M \\ -M^* & -K^* + E_0 \end{pmatrix} \begin{pmatrix} A_\alpha(\mathbf{r},t) \\ B_\alpha(\mathbf{r},t) \end{pmatrix}$$

$$K = -\frac{\hbar^2}{2m}\nabla^2 + \frac{1}{2}m\omega_0^2 r^2 + 2U\Phi^*\Phi, \quad M = U\Phi^2$$

Formulation

✓ BEC: Gross-Pitaevskii equation

⇒ $g_{\mu\nu}$

✓ Fluctuations: Bogoliubov-de Gennes equation

⇒ Quantum fields under $g_{\mu\nu}$

Phase and amplitude description: BEC component

$$\Phi = \rho_0^{1/2} \exp\left(i\theta_0\right)$$

Normalized GP equation

$$\partial_t \theta_0 = -\frac{1}{2}\left(\nabla \theta_0\right)^2 - \frac{1}{2} r^2 - g\rho_0 - \frac{1}{8\rho_0^2}\left(\nabla \rho_0\right)^2 + \frac{1}{4\rho_0}\nabla^2 \rho_0$$

$$\partial_t \rho_0 + \nabla \cdot \left(\rho_0 \nabla \theta_0\right) = 0$$

Neglect short-wave length fluctuations

$$\partial_t \theta_0 \simeq -\frac{1}{2}\left(\nabla \theta_0\right)^2 - \frac{1}{2} r^2 - g\rho_0$$

$$\partial_t \rho_0 + \nabla \cdot \left(\rho_0 \nabla \theta_0\right) = 0$$

$$\left(\lambda \gg \xi = \frac{1}{\sqrt{g\rho_0}}\right)$$

Phase and amplitude description: Fluctuations

Fluctuations: $\begin{cases} \theta_0 + \theta \\ \rho_0 + \rho \end{cases}$

$$\partial_t \theta \simeq -(\nabla \theta_0)(\nabla \theta) - g\rho$$

$$\partial_t \rho + \nabla \cdot (\rho \nabla \theta_0) + \nabla \cdot (\rho_0 \nabla \theta) = 0$$

$$(\partial_t + \nabla \cdot \mathbf{v}_0)(\partial_t + \mathbf{v}_0 \cdot \nabla)\theta \simeq \nabla \cdot (c_s^2 \nabla \theta)$$

$$\mathbf{v}_0 = \nabla \theta_0$$

“Metric”

$$\left(\partial_t + \nabla \cdot \mathbf{v}_0\right)\left(\partial_t + \mathbf{v}_0 \cdot \nabla\right)\theta \simeq \nabla \cdot \left(c_s^2 \nabla \theta\right)$$

Field equation in a curved spacetime

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\mu}\left(\sqrt{-g}\, g^{\mu\nu}\frac{\partial}{\partial x^\nu}\psi\right) = 0 \qquad g = \det g_{\mu\nu}$$

$$g_{\mu\nu} = c_s \begin{pmatrix} c_s^2 - v_0^2 & v_0^x & v_0^y & v_0^z \\ v_0^x & -1 & 0 & 0 \\ v_0^y & 0 & -1 & 0 \\ v_0^z & 0 & 0 & -1 \end{pmatrix}_{\mu\nu}$$

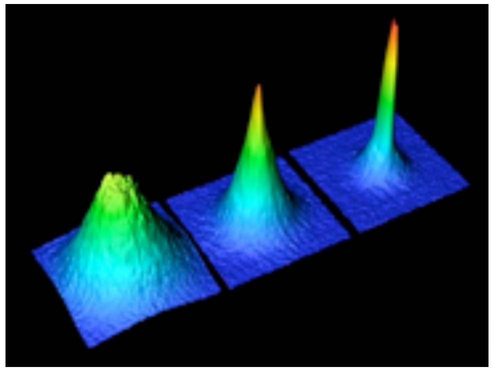
One-to-one correspondence

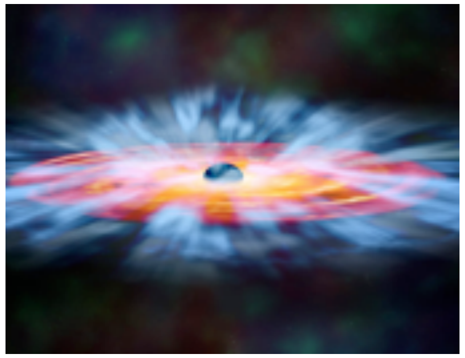
BEC metric

$$ds^2 = c_s^2 dt^2 - \left(v_0^r dt - dr\right)^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2$$

Schwarzschild metric

$$ds^2 = c^2 d\tilde{t}^2 - \left(v d\tilde{t} - dr\right)^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2$$





Quantum fields?

$$\Phi_0 + \phi = (\rho_0 + \rho)^{1/4} \exp(i\theta_0 + i\theta)(\rho_0 + \rho)^{1/4}$$

$$|\theta| \ll |\theta_0|, \quad \rho \ll \rho_0$$

$$\theta = \frac{1}{2i\rho_0}\left(\Phi_0^* \phi - \Phi_0 \phi^\dagger\right)$$

$$\rho = \Phi_0^* \phi + \Phi_0 \phi^\dagger$$

Inner-product

Y. Kurita, M. Kobayashi, T.M., M. Tsubota, and H. Ishihara, Phys. Rev. A 79, 43616 (2009).

Bogoliubov quasiparticles

$$4\pi\int_0^\infty drr^2\left[A_\alpha^*(r)A_\beta(r)-B_\alpha^*(r)B_\beta(r)\right]=\delta_{\alpha\beta}$$

$$\phi(\mathbf{r},t)=\sum_\alpha\left[A_\alpha(\mathbf{r},t)b_\alpha+B_\alpha^*(\mathbf{r},t)b_\alpha^\dagger\right]$$

$$\theta=\sum_\alpha\left(f_\alpha b_\alpha+f_\alpha^* b_\alpha^\dagger\right)$$

$$\rho=\sum_\alpha\left(g_\alpha b_\alpha+g_\alpha^* b_\alpha^\dagger\right)$$

$$\left(f_\alpha,f_\beta\right)=\delta_{\alpha\beta}$$

$$\left(f_\alpha,f_\beta\right)=\frac{4\pi i}{g}\int_0^\infty drr^2\left[f_\alpha^*(r,t)\left(\partial_t+\mathbf{v}_0\cdot\nabla\right)f_\beta(r,t)-\left[\left(\partial_t+\mathbf{v}_0\cdot\nabla\right)f_\alpha^*(r,t)\right]f_\beta(r,t)\right]$$

Summary of the formulation

✓ BEC

$$c_s, v_0 \Longrightarrow g_{\mu\nu}$$

✓ Fluctuations

Phase field of Bogoliubov quasiparticle wave functions

$\Longrightarrow$ Quantum fields under $g_{\mu\nu}$

$$\left(\psi_\alpha, \psi_\beta\right)_{BdG} \Longrightarrow \left(f_\alpha, f_\beta\right)_{g_{\mu\nu}}$$

“Hawking radiation” in BEC

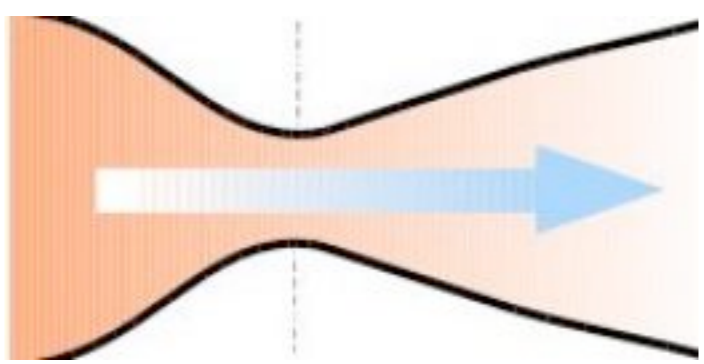
✓ Basic formulation

2) “Horizon” creation $v > c$?

3) “Hawking radiation”

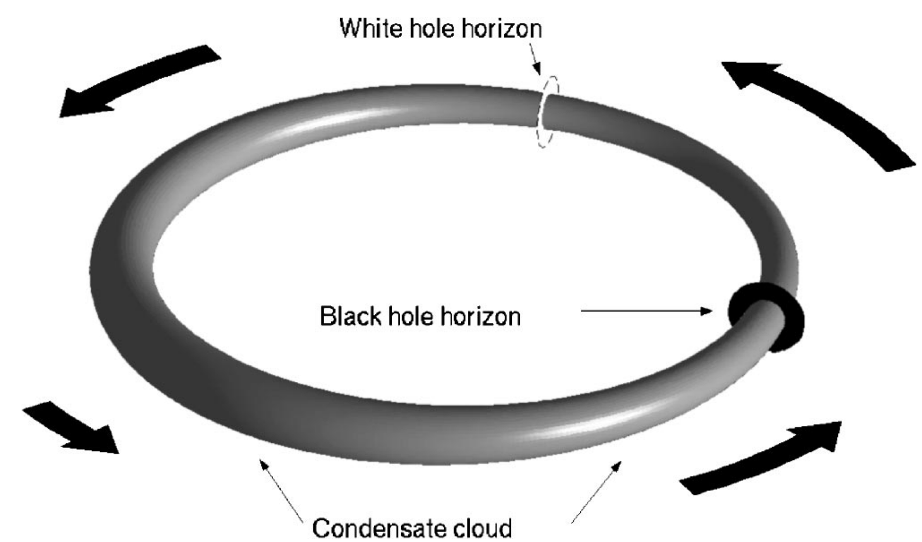
How to create a “horizon”?

De Laval nozzle



M.Sakagami and A. Ohashi, Prog. Theor. Phys. **107**, 1267 (2002).

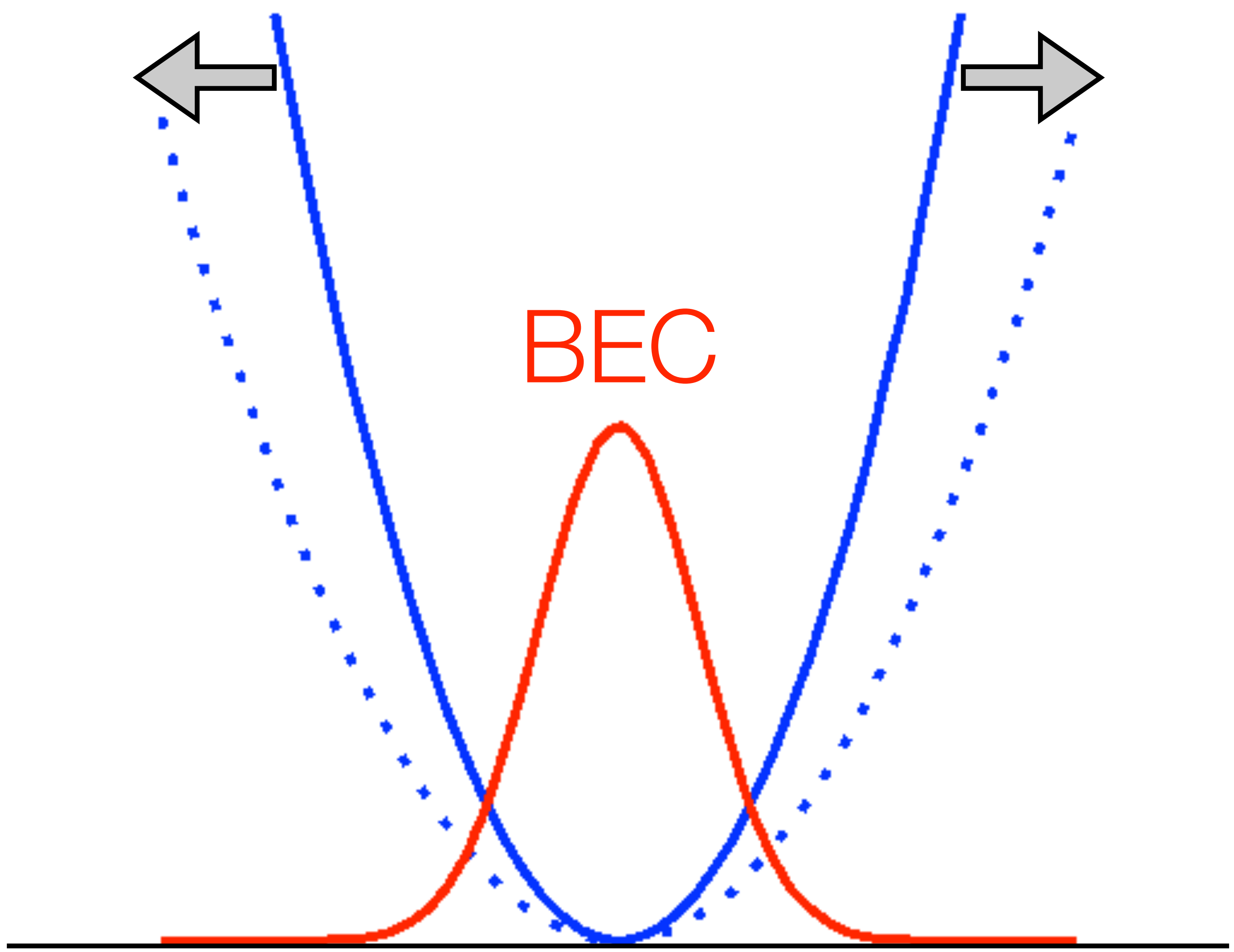
Ring trap



L. J. Garay *et al*., Phys. Rev. Lett. **85**, 4643 (2000).

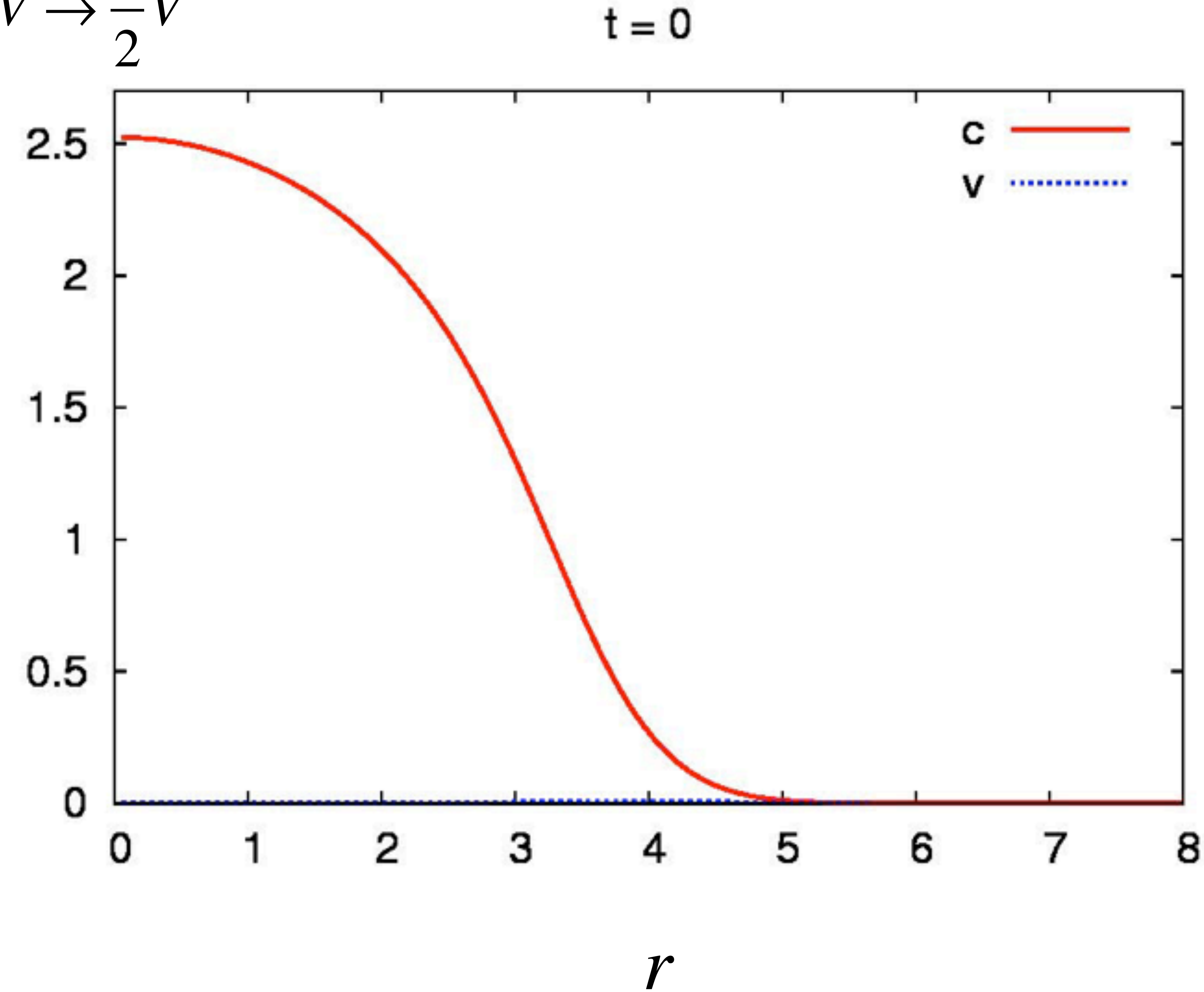
“Horizon” creation?

Y. Kurita, M. Kobayashi, T.M., M. Tsubota, and H. Ishihara,
Phys. Rev. A 79, 43616 (2009).

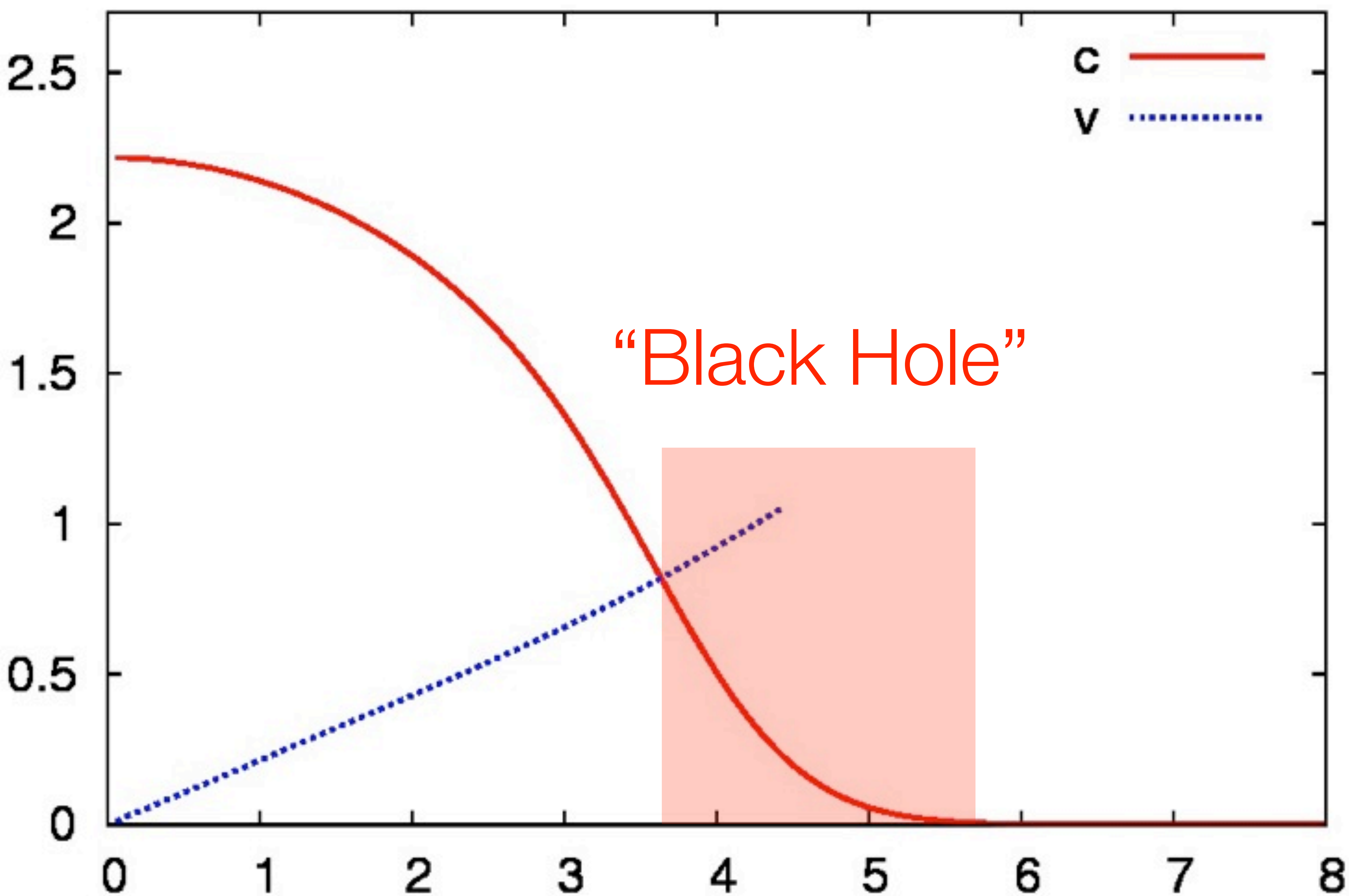


"Horizon" creation

$g = 500 \qquad V \to \frac{1}{2}V$



“Horizon” creation



“Hawking radiation” in BEC

✓ Basic formulation

✓ “Horizon” creation

3) “Hawking radiation”

Simulation steps for "Hawking radiation"

$$\Phi(\mathbf{r})$$ "Spacetime"

$$\{(A_\alpha(\mathbf{r}), B_\alpha(\mathbf{r}))\}$$ "Quantum fields"

$t = 0$ ← Change V

Time-evolution

"Black hole formation"

$$\Phi(\mathbf{r}, t)$$

$$\{(A_\alpha(\mathbf{r}, t), B_\alpha(\mathbf{r}, t))\}$$

Compute particle creation spectrum

Particle creation spectrum calculation

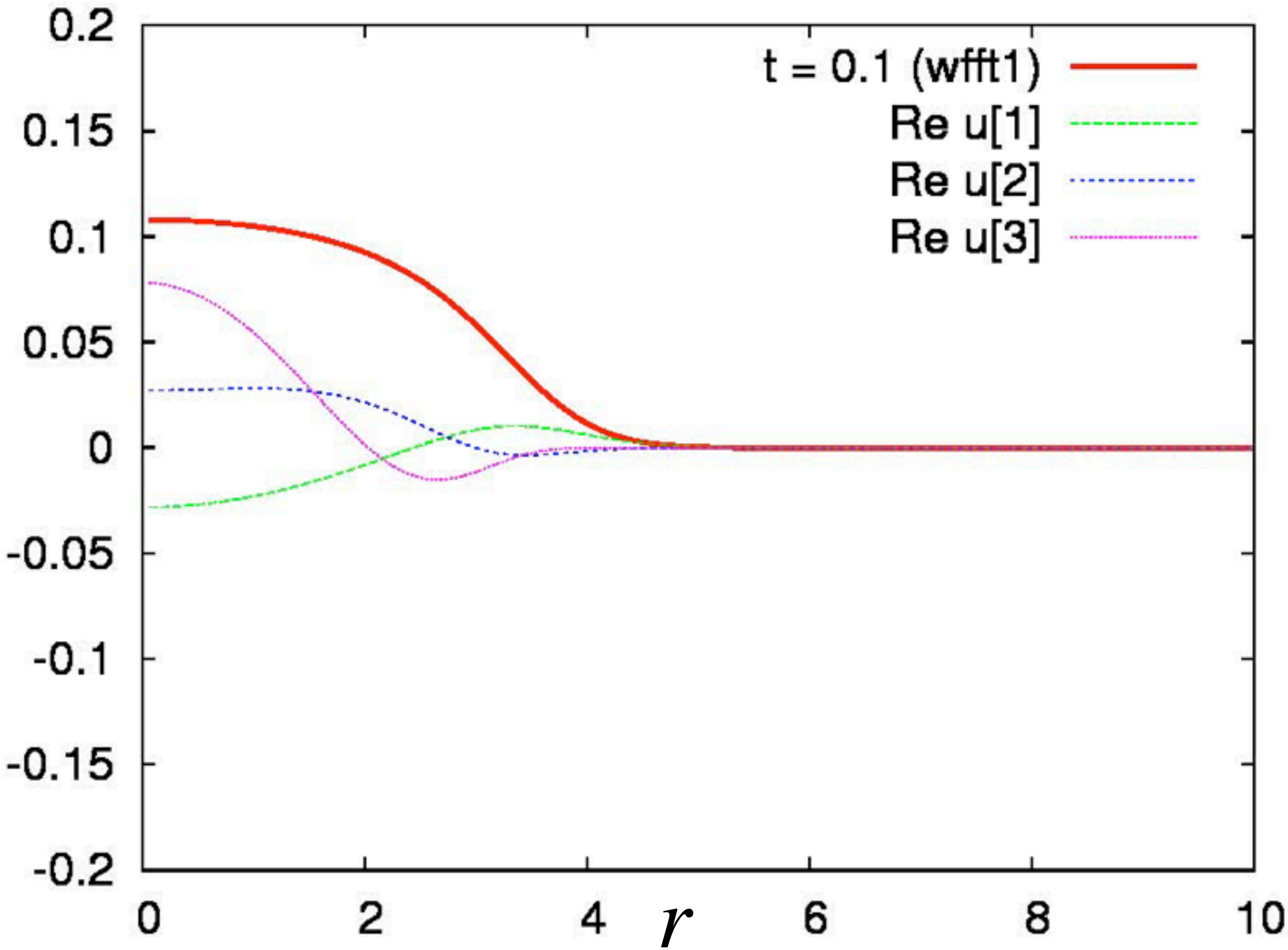
$$b_\alpha^{(1)} = \sum_\beta \left[A_{\alpha\beta} b_\beta^{(2)} + B_{\alpha\beta} b_\beta^{(2)\dagger} \right]$$

$$B_{\beta\alpha}^* = -4\pi \int_0^\infty drr^2 \left[A_\beta^{(2)} B_\alpha^{(1)} - B_\beta^{(2)} A_\alpha^{(1)} \right]$$

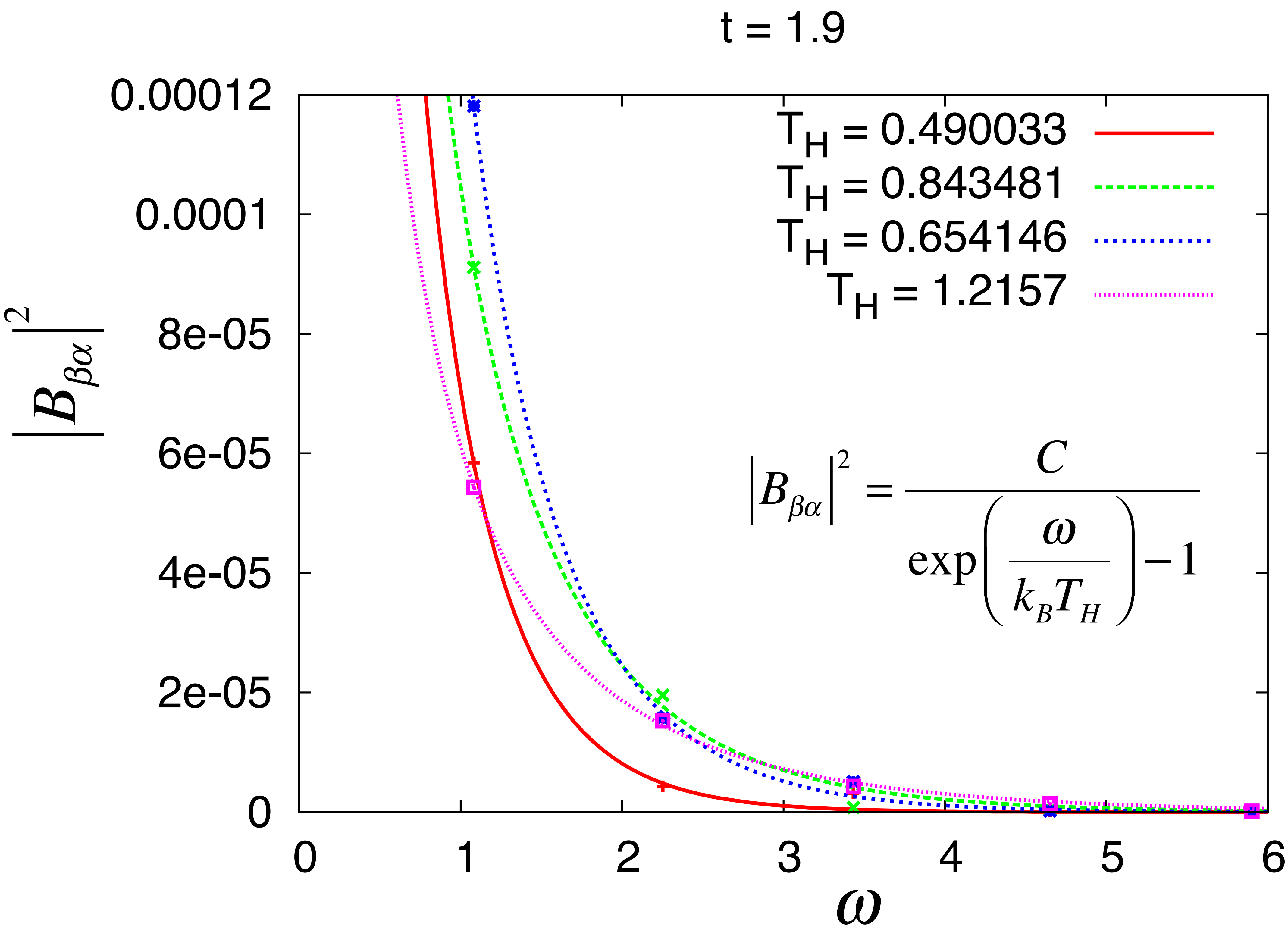
$$B_{\beta\alpha}^* = \frac{i}{g} \int_0^\infty dr \left[f_\beta^{(2)}(r) \left(\partial_t + \mathbf{v}_0 \cdot \nabla \right) f_\alpha^{(1)}(r) - \left[\left(\partial_t + \mathbf{v}_0 \cdot \nabla \right) f_\beta^{(2)}(r) \right] f_\alpha^{(1)}(r) \right]$$

$$\left\langle b_\alpha^{(1)\dagger} b_\alpha^{(1)} \right\rangle = \sum_\beta \left| B_{\alpha\beta} \right|^2 \left\langle b_\beta^{(2)} b_\beta^{(2)\dagger} \right\rangle = \sum_\beta \left| B_{\alpha\beta} \right|^2$$

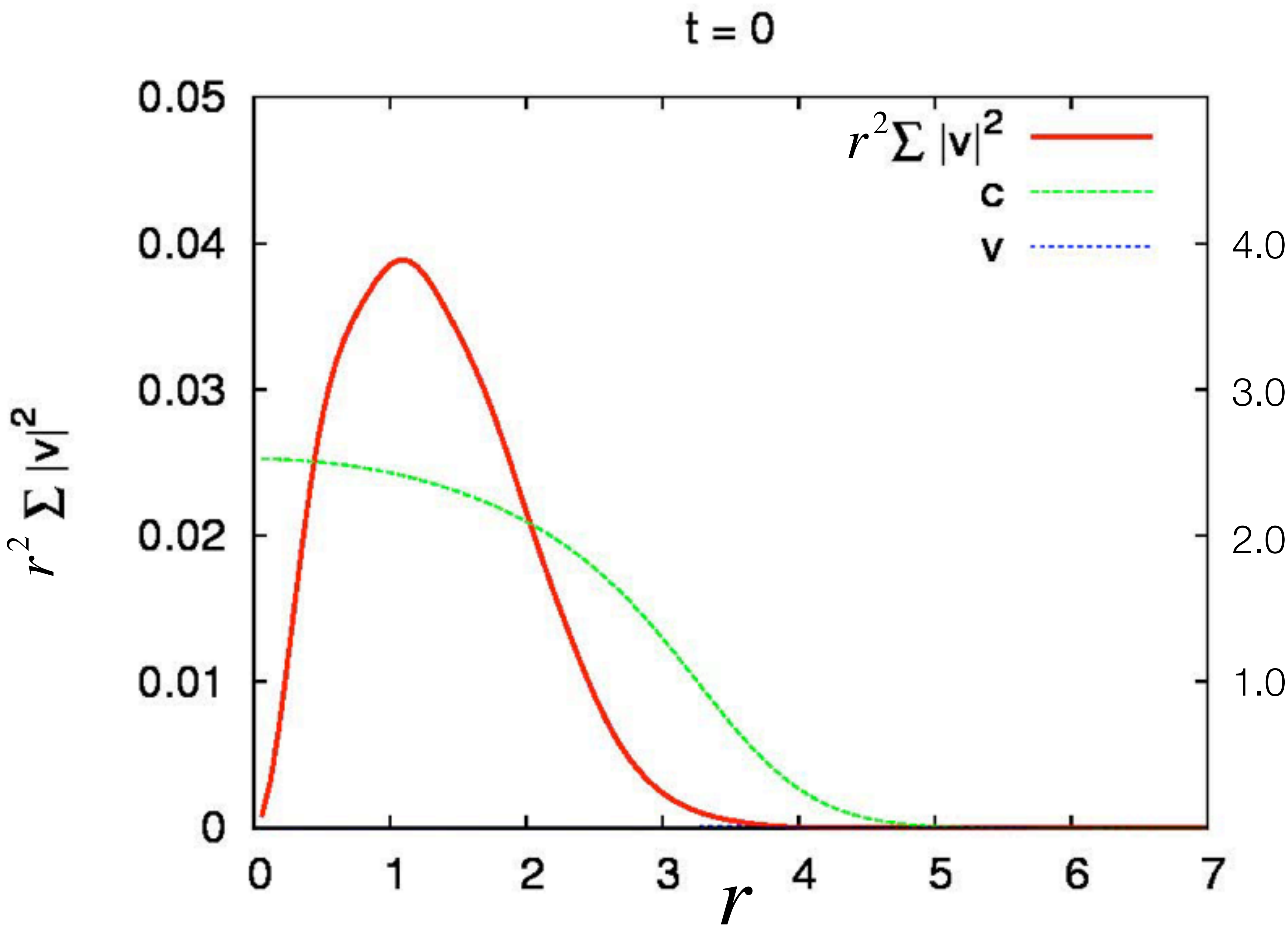
Time-evolution



Particle creation spectrum

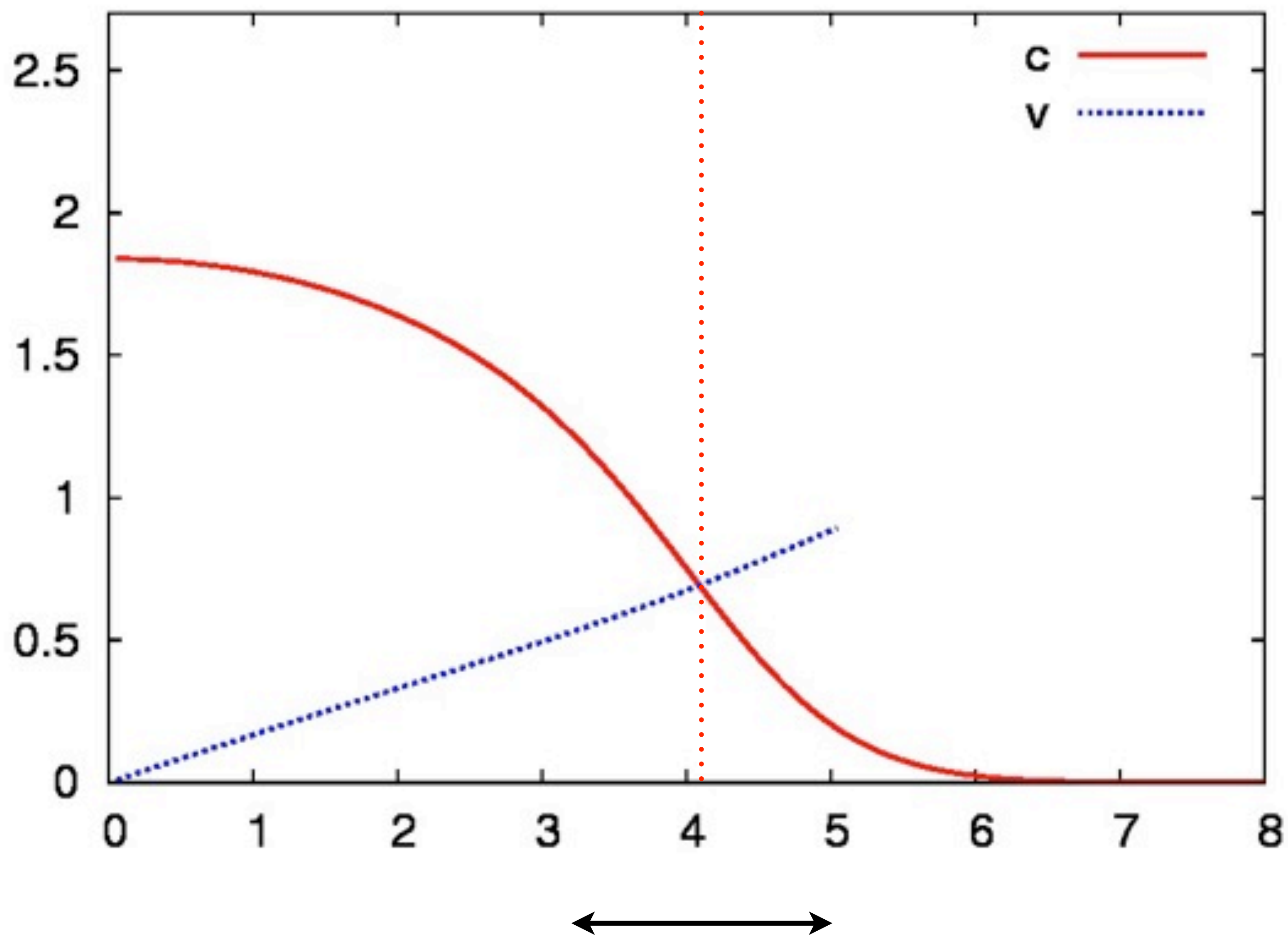


Origin?



t = 0
0.05
0.04
0.03
0.02
0.01
0
0
1
2
3
4
5
6
7
4.0
3.0
2.0
1.0
$r^2\Sigma\,|\mathbf{v}|^2$
c
v
r
$r^2\,\Sigma\,|\mathbf{v}|^2$

How particles are created?



Disentangled!

Conclusion

- Analogy between Bose-Einstein condensate and black hole
  - Bose-Einstein condensate wave function is connected with the Schwarzschild spacetime metric.
  - Bogoliubov quasiparticle wave functions are connected with quantum field in a black hole spacetime.
  - There is correspondence between the inner products.
- Horizon is formed by changing the trap potential.
- Particle creation occurs near the horizon.

Enhancement of quantum depletion near the horizon.

Appendix

Black hole entropy

Hawking temperature

$$k_B T_H = \frac{\hbar c^3}{8\pi G M}$$

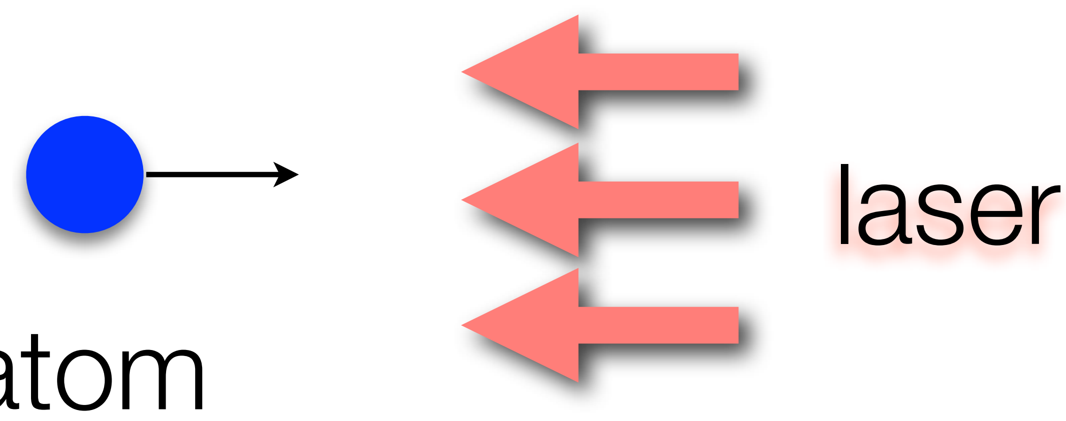
Thermodynamical relation

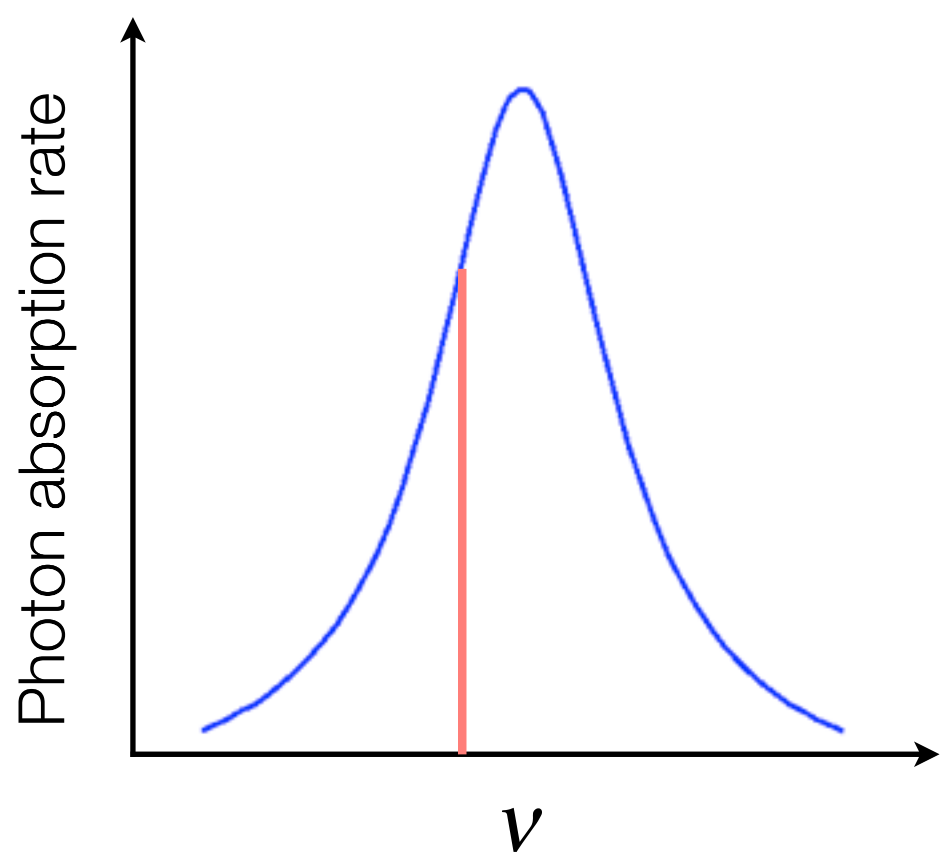
$$dU = c^2 dM = T_H dS$$

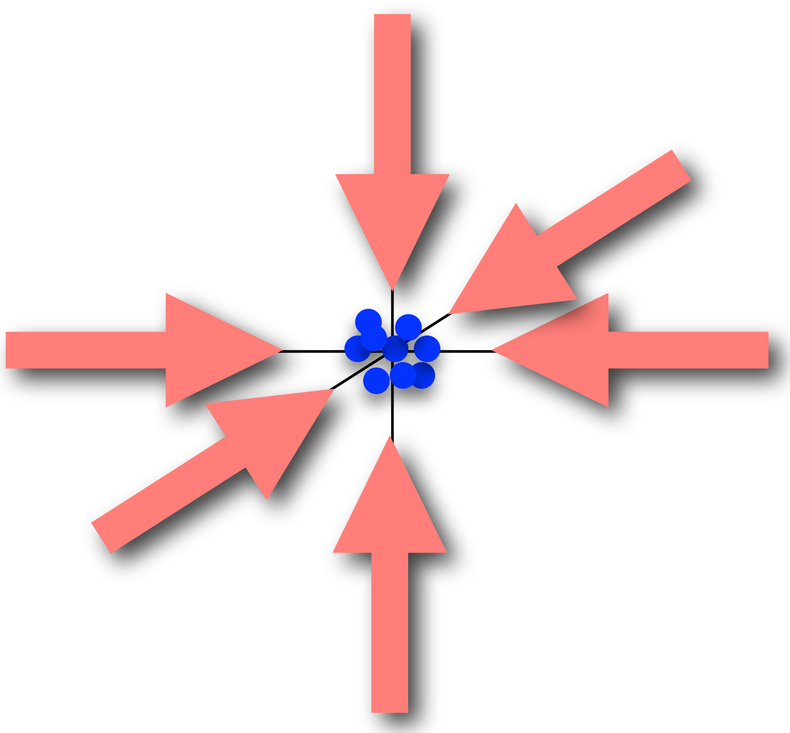
⇩

$$S / k_B = \frac{1}{4} \times \frac{4\pi r_g^2}{\ell_P^2}$$

Laser cooling







^{87}Rb, ^{23}Na, ^{7}Li, ^{40}K, ...

WKB analysis

$$\left(\partial_t + \nabla\cdot\mathbf{v}_0\right)\left(\partial_t + \mathbf{v}_0\cdot\nabla\right)\theta \simeq \nabla\cdot\left(c_s^2\nabla\theta\right)$$

Near the horizon

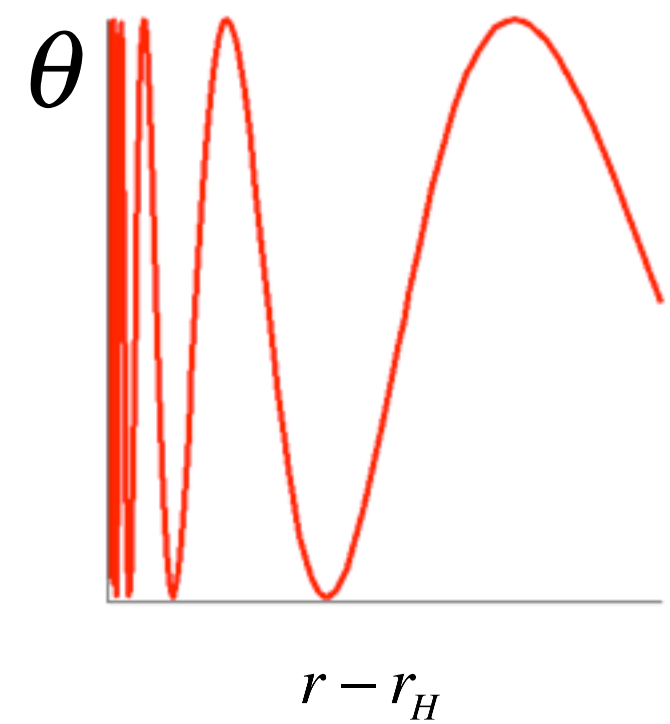
$$v = c + \kappa\left(r - r_H\right) + \ldots$$

$$\theta = \exp\left(-i\omega t + iK\left(r\right)\right)$$

WKB analysis

$$\omega K'' + \kappa (K')^2 = 0$$

$$\theta = \exp\left(-i\omega t + i\frac{\omega}{\kappa}\ln\left|\frac{\kappa}{\omega}(r - r_H)\right| \right)$$



Fourier transform

$$\theta_k = \frac{i}{k}\left(\frac{\kappa}{\omega k}\right)^{i\frac{\omega}{\kappa}} e^{-\frac{\pi\omega}{2\kappa}}\Gamma\left(1 + i\frac{\omega}{\kappa}\right)$$

$$|\theta_k|^2 = \frac{1}{k^2}\frac{2\pi\omega}{\kappa}\frac{1}{\exp\left(\frac{2\pi\omega}{\kappa}\right) - 1}$$

WKB analysis: Hawking Temperature

$$|\theta_k|^2 = \frac{1}{k^2}\frac{2\pi\omega}{\kappa}\frac{1}{\exp\left(\frac{2\pi\omega}{\kappa}\right)-1}$$

Hawking temperature

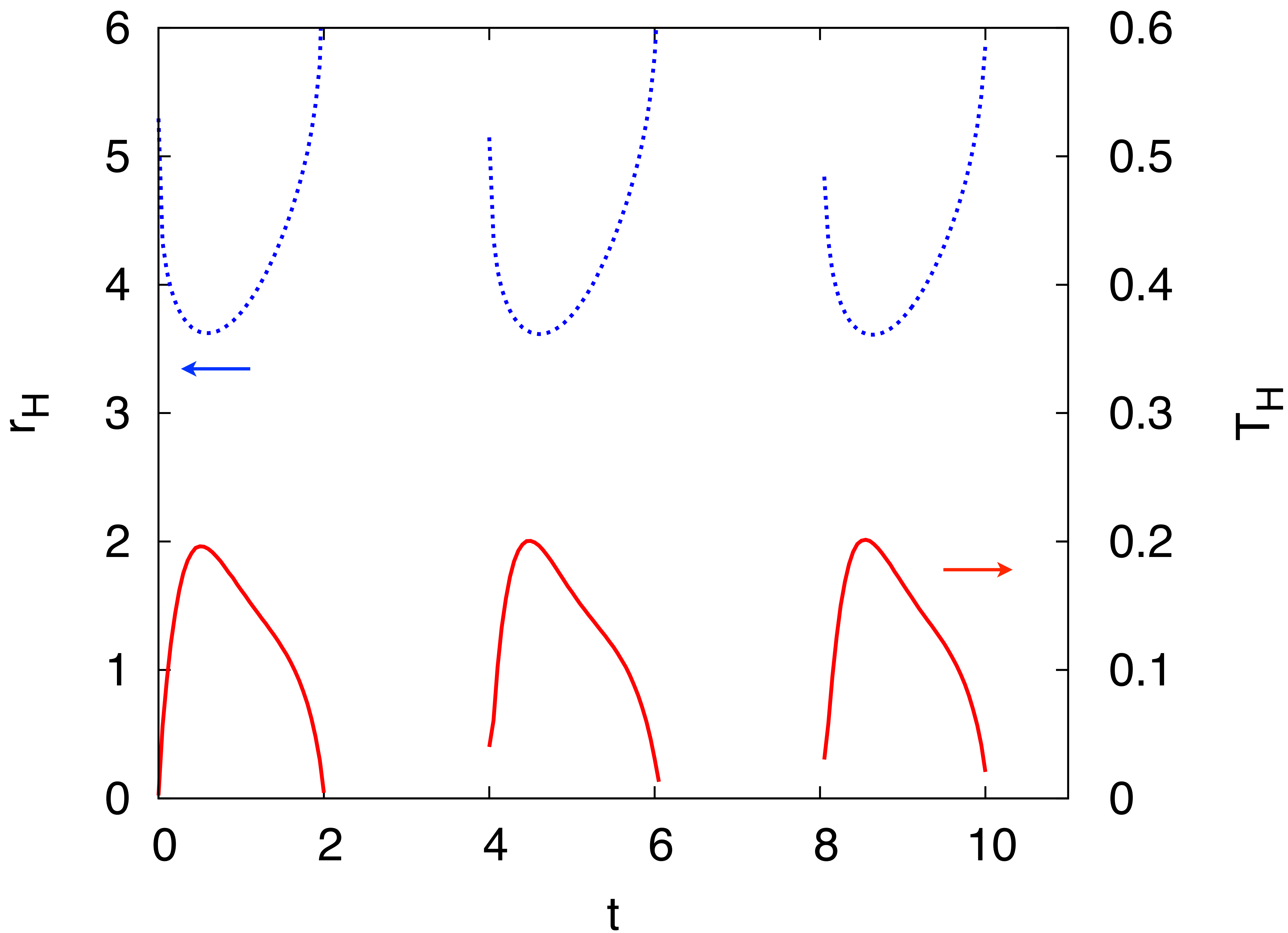
$$k_B T_H = \frac{\hbar}{2\pi}\kappa$$

$$\kappa = \frac{\partial}{\partial r}(v-c)\Big|_{r=r_H}$$

$$v = c + \kappa(r - r_H) + \ldots$$

Semiclassical analysis

Horizon and approximate Hawking temperature



“Hawking radiation”

Dynamical evolution of “spacetime”

Gross-Pitaevskii equation

Spacetime dynamics

$$i\hbar\partial_t\Phi = \left(-\frac{\hbar^2}{2m}\nabla^2 + \frac{1}{2}m\omega_{ho}^2 r^2\right)\Phi + U\Phi^*\Phi\Phi$$

Bogoliubov-de Gennes equation

Quantum field dynamics

$$i\hbar\partial_t\begin{pmatrix} A_\alpha(\mathbf{r},t) \\ B_\alpha(\mathbf{r},t)\end{pmatrix} = \begin{pmatrix} K-E_0 & M \\ -M^* & -K^*-E_0\end{pmatrix}\begin{pmatrix} A_\alpha(\mathbf{r},t) \\ B_\alpha(\mathbf{r},t)\end{pmatrix}$$

$$K = -\frac{\hbar^2}{2m}\nabla^2 + \frac{1}{2}m\omega_0^2 r^2 + 2U\Phi^*\Phi, \quad M = U\Phi^2$$