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Effects of Tidal Resonances in Coalescing Compact Binary Systems

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Effects of tidal resonances to the evolution of compact star binaries, such as binary neutron stars or neutron star-black hole binaries, are studied. We apply the Press-Teukolsky algorithm to a simple neutron star model whose companion moves with a circular orbit. To see effects of the tidal resonances, we compare the deposited energy in the star with the dissipated energy of a circular motion due to the emission of gravitational waves. It is found that under realistic situations, the g-mode resonance oscillations do not affect the evolution of binary systems so much, while the f-mode may induce a transition from a steady inspiral to an unstable plunge of compact star binaries.

§ 1. Introduction

Laser interferometric gravitational wave detectors, such as LIGO,[1] will be in operation in this decade. These detectors have the ability to detect the signal of gravitational waves from a coalescing compact star binaries, whose frequency is 10–1000 Hz, using a matched filter technique.[2] Here to use the matched filter technique, we need an accurate theoretical template.[3] In particular, if we try to know the various parameters of binaries, such as masses,[4] spins,[5),6] orbital inclinations to the spin axis,[7] cosmological parameters,[8] and so on, from a signal of gravitational waves, we must prepare the theoretical template whose accuracy is less than 0.01%.[3] Hence, we need to investigate the various effects in coalescing compact binaries to prepare an accurate theoretical template. In this paper, we pay attention to effects of tidal resonances of neutron stars.

Tidal effects in coalescing binary neutron stars have been studied by several authors in detail.[9] In particular, to investigate the phase errors ($\Delta\varphi$) due to the tidally deposited energy, Kochanek performed the numerical calculation of evolution of binary neutron stars taking into account the effects of viscosity. He found that for the realistic values of viscosity, the tidal effects are small ($\Delta\varphi \lesssim 2\pi$) for PSR1913-16[10] type binaries. However, because his neutron star models were very simple, they did not consider the effects of the tidal resonances of the g-modes. As for researches for the tidal resonances of the g-modes, recently, Kokkotas and Schäfer[11] calculated the evolutions of the coalescing binary neutron stars including the tidal resonance effects in the post-Newtonian equations of motion. They analyzed the phase errors due to the g-mode resonances at the final stage of coalescing binary neutron stars ($\gtrsim$100 Hz). However their treatments are not appropriate because models of neutron stars they adopted were too crude; they assumed an adiabatic index of the neutron stars as 5/3, while polytropic indices(n) as 1 and 2. As discussed in § 3, in realistic neutron stars in coalescing compact binaries the adiabatic index does not have such a value, and difference between the adiabatic index and the polytropic constant $(1+1/n)$ seems to be very small.

A star has three oscillation modes, f-, p- and g-modes,[12] but in two body encounters, it has been pointed out that the f- and p-modes are important only for the close encounter, and the g-modes become more important than the f-and p-modes for the wider encounter.[13] This means that the g-modes may affect the evolution of binary systems orbiting with the frequency $\gtrsim 10$ Hz, which is a potentially important frequency band for the gravitational wave detectors. The properties of the g-modes, such as the frequency and the oscillating energy, are considerably affected by the polytropic and adiabatic indices, so that crude treatments of these indices lead to an invalid estimate of the physical properties of the g-modes. Thus, we here consider effects of the tidal resonances, in particular, the g-mode oscillations under simple, but physically reasonable assumptions. We will clarify the conditions in which the g-mode resonances become important from an observational point of view.

In § 2, we show the basic equations, which we use in the analysis, and consider physical and chemical states of neutron stars in coalescing binary systems. In § 3, using a simple neutron star model, we analyze the resonance modes and show the numerical results. In § 4, we discuss the implication of the numerical results to the evolution of compact star binaries.

Throughout this paper, G and c denote the gravitational constant and the light velocity, respectively.

§ 2. Formulation

The basic equations to calculate the energy deposited in the oscillation of non-spinning Newtonian stars due to the two body encounter are described by Press and Teukolsky.[14] In their formalism, the parabolic encounter between two stars is considered. Now, we consider the final evolution of coalescing compact binaries whose orbital separation is less than ~ 1000 km. In such a binary, an orbit becomes circular because of the emission of gravitational waves. Therefore we conider the tidal effects of the star 1 of mass M_1 and radius R_1 by the companion star 2 of mass M_2, which move around each other with a circular orbit.

The tidal potential by the companion star 2 at a point (r, θ, φ) inside the star 1 is written as follows:

$$U=\sum_{l=2}^{\infty}\sum_{m=-l}^{l}\frac{4\pi GM_2}{2l+1}\frac{r^l}{R^{l+1}}Y_{lm}\left(\frac{\pi}{2},0\right)Y_{lm}^*(\theta,\varphi)e^{im\Omega t}$$

$$=\sum_{l=2}^{\infty}\sum_{m=-l}^{l}W_{lm}\frac{GM_2r^l}{R^{l+1}}Y_{lm}^*(\theta,\varphi)e^{im\Omega t}\,, \tag{2·1}$$

where

$$W_{lm}=\frac{4\pi}{2l+1}Y_{lm}\left(\frac{\pi}{2},0\right), \tag{2·2}$$

and R, Y_{lm} and Ω are an orbital radius, the spherical harmonics and an angular velocity.

The rate at which the energy is deposited in the star 1 is

$$\frac{dE}{dt}=\int d^3x\rho\partial_t\xi^i\nabla_i U\,, \tag{2·3}$$

where ρ is a density of the non-perturbed star and ξ^i is a Lagrangian displacement vector. Defining the Fourier transforms of U and ξ_i as

$$U=\int d\omega e^{i\omega t}\tilde{U}\,,$$

$$\xi_i=\int d\omega e^{i\omega t}\tilde{\xi}_i\,, \tag{2·4}$$

then the total energy deposited becomes

$$\varDelta E=2\pi\int d\omega(-i\omega)\int d^3x\rho\tilde{\xi}^i(\omega)\nabla_i\tilde{U}(\omega)\,. \tag{2·5}$$

Here, $\tilde{\xi}_i$ can be expanded by the normal mode ξ_i^n, and each normal mode can also be expanded by the spherical harmonics as

$$\xi_i^n\equiv(\xi_{1l}^n(r)Y_{lm},\ r\xi_{2l}^n(r)Y_{lm,\theta},\ r\xi_{2l}^n(r)Y_{lm,\varphi})\,,\quad i=(r,\theta,\varphi)\,. \tag{2·6}$$

The Fourier transform of U is written as

$$\tilde{U}=\sum_{l=2}^{\infty}\sum_{m=-l}^{l}W_{lm}\frac{GM_2r^l}{R^{l+1}}Y_{lm}^*(\theta,\varphi)\lim_{T\to\infty}\frac{1}{\pi(m\Omega-\omega)}\sin\{(m\Omega-\omega)T/2\}\,, \tag{2·7}$$

where we make use of the relation,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}dte^{i(m\Omega-\omega)t}=\lim_{T\to\infty}\frac{1}{\pi(m\Omega-\omega)}\sin\{(m\Omega-\omega)T/2\}\,. \tag{2·8}$$

If we use the above formula in calculating $\varDelta E$, it diverges at $\omega=m\Omega$, because Eq. (2·8) becomes $\delta(\omega-m\Omega)$ for $T\to\infty$. Since an orbital radius of binary becomes small radiating gravitational waves in reality, T is finite. Defining $\varDelta T$ as a finite duration time in which a binary keeps an orbital radius (later we re-define it), $\tilde{U}$ becomes

$$\tilde{U}=\sum_{l=2}^{\infty}\sum_{m=-l}^{l}W_{lm}\frac{GM_2r^l}{R^{l+1}}Y_{lm}^*(\theta,\varphi)\frac{1}{\pi(m\Omega-\omega)}\sin\{(m\Omega-\omega)\varDelta T/2\}\,. \tag{2·9}$$

Inserting $\tilde{U}$ into Eq. (2·5) and using the method of Press and Teukolsky (see Eqs. (9) ~(18) of their paper),[14] $\varDelta E$ becomes

$$\varDelta E=\frac{1}{2}\sum_{nlm}|W_{lm}|^2\frac{G^2M_2^2}{R^{2l+2}}Q_{nl}^2\frac{\sin^2x_{nlm}}{x_{nlm}^2}\varDelta T^2\,, \tag{2·10}$$

where

$$Q_{nl}=l\int_0^{R_1}r^2drr^{l-1}\rho(\xi_{1nl}+(l+1)\xi_{2nl}) \tag{2·11}$$

and

$$x_{nlm}=\frac{(m\Omega-\omega_{nlm})\Delta T}{2}. \tag{2·12}$$

We define the non-dimensional variables,

$$\hat{\rho}=\rho/\rho_0\,,\quad \hat{r}=r/R_1\,,\quad (\hat{\xi}^n_{1l},\hat{\xi}^n_{2l})=\rho_0^{1/2}R_1^{3/2}(\xi^n_{1l},\xi^n_{2l})\,, \tag{2·13}$$

where ρ_0 is a density at the center of the star 1. $\hat{\xi}_i^n$ is normalized for each mode as

$$\int_0^1 \hat{r}^2 d\hat{r}\hat{\rho}[(\hat{\xi}^n_{1l})^2+l(l+1)(\hat{\xi}^n_{2l})^2]=1\,. \tag{2·14}$$

Using them, ΔE is rewritten as

$$\Delta E=\frac{G\rho_0}{2}\frac{GM_1^2}{R_1}\left(\frac{M_2}{M_1}\right)^2\left(\frac{R_1}{R}\right)^6\sum_{nlm}|W_{lm}|^2\left(\frac{R_1}{R}\right)^{2(l-2)}\hat{Q}^2_{nl}\frac{\sin^2 x_{nlm}}{x^2_{nlm}}\Delta T^2\,, \tag{2·15}$$

where

$$\hat{Q}_{nl}=l\int_0^1 \hat{r}^2 d\hat{r}\,\hat{r}^{\,l-1}\hat{\rho}(\hat{\xi}^n_{1l}+(l+1)\hat{\xi}^n_{2l})\,. \tag{2·16}$$

The above expression of ΔE means that the oscillation energy is deposited only for $|x_{nlm}|\lesssim 1$. Since $\Delta T\gg\Omega^{-1}$ except for $R\sim 5M$, where $M=M_1+M_2$, $|x_{nlm}|\lesssim 1$ means that $\omega_{nlm}\sim m\Omega$. Hence the oscillation energy is deposited only in the vicinity of a resonant angular frequency, $\omega_{nlm}\sim m\Omega$. Taking into account this property, we define ΔT as the duration time in which the resonance occurs. ΔT is determined by the time scale of the gravitational radiation reaction. Since we here pay attention to the low frequency modes, we use the quadrupole formula to estimate it. According to the quadrupole formula, the energy flux is written by[21]

$$\frac{dE}{dt}=-\frac{32G^4M^3\mu^2}{5c^5R^5}\,, \tag{2·17}$$

where μ is the reduced mass, M_1M_2/M. We should note that in this formula, the higher order terms of GM/Rc^2 are neglected. Although this formula is not rigid, the perturbation studies[16] show that the relative errors of values by the quadrupole formula to the exact one are at most $\sim$10% for $R\gtrsim 10\ GM/c^2$. This means that the errors of estimations for the dissipated energy by gravitational radiation and ΔT are at most $\sim$10%.

Since the total energy of the binary system is

$$E=-\frac{GM\mu}{2R}\,, \tag{2·18}$$

the time in which the orbital radius evolves from R_i to R_f is

$$\begin{aligned}\Delta T&=\frac{5c^5}{256G^3M^2\mu}(R_i^4-R_f^4)\\&=\frac{5c^5}{256G^{5/3}M^{2/3}\mu}(\Omega_i^{-8/3}-\Omega_f^{-8/3})\,.\end{aligned} \tag{2·19}$$

If we write $\Omega_{i,f}=\Omega_0\pm\Delta\Omega$, where $\Omega_0=\omega_{nlm}/m$ and $\Delta\Omega\ll\Omega_0$, then

$$\Delta T \simeq \frac{5c^5}{48G^{5/3}M^{2/3}\mu}\Omega_0^{-8/3}\frac{\Delta\Omega}{\Omega_0}. \tag{2·20}$$

Assuming that the resonance occurs for $|x_{nlm}| \leq x_0$, where x_0 is defined by

$$x_0 \equiv \int_0^\infty dx \frac{\sin^2 x}{x^2} = \frac{\pi}{2}, \tag{2·21}$$

then the condition for ΔT becomes

$$|m|\Delta\Omega\frac{\Delta T}{2} = \frac{\pi}{2}. \tag{2·22}$$

Since the dominant modes of the oscillation are $l=2$, $m=\pm 2$ modes,

$$\Delta T = \frac{\pi}{2\Delta\Omega}. \tag{2·23}$$

From Eqs. (2·20) and (2·23),

$$\Delta T = \left(\frac{5\pi}{96}\right)^{1/2}\left(\frac{Rc^2}{GM}\right)^{3/4}\left(\frac{Rc^2}{G\mu}\right)^{1/2}\Omega_0^{-1}. \tag{2·24}$$

In this case, the total energy dissipated by gravitational radiation and the total deposited energy by the tidal oscillation, respectively, become

$$\Delta E_{\rm GW} = \left(\frac{32\pi}{15}\right)^{1/2}\left(\frac{\mu}{M}\right)^{1/2}\left(\frac{GM}{Rc^2}\right)^{5/4}\frac{GM\mu}{R} \tag{2·25}$$

and

$$\Delta E_{\rm OS} = \frac{5\pi\alpha}{192}\left(\frac{M_2}{\mu}\right)\left(\frac{R}{R_1}\right)^{1/2}\left(\frac{R_1c^2}{GM}\right)^{5/2}\frac{GM\mu}{R}\sum_{nlm}\left(\frac{R_1}{R}\right)^{2(l-2)}|W_{lm}|^2\hat{Q}_{nl}^2, \tag{2·26}$$

where $\alpha \equiv \rho_0 R_1^3/M_1$. Hereafter, we only consider the $l=2$ mode because higher multipole modes, whose contributions are proportional to higher powers of $(R_1/R)^2$, do not contribute for the low frequency modes which we mainly consider in this paper. Then $\Delta E_{\rm OS}$ becomes

$$\Delta E_{\rm OS} = \frac{5\pi\alpha}{96}\left(\frac{M_2}{\mu}\right)\left(\frac{R}{R_1}\right)^{1/2}\left(\frac{R_1c^2}{GM}\right)^{5/2}\frac{GM\mu}{R}\sum_n |W_{22}|^2\hat{Q}_{n2}^2. \tag{2·27}$$

Here we use the relation, $|W_{22}|=|W_{2-2}|$ and $|W_{21}|=|W_{2-1}|=0$. We can see the importance of the tidal resonant effects by comparing $\Delta E_{\rm OS}$ and $\Delta E_{\rm GW}$. $\Delta E_{\rm GW}$ depends only on their masses and orbital separation, while $\Delta E_{\rm OS}$ strongly depends on the structure of neutron stars because it depends on ω_{nlm} and $\hat{Q}_{nl}$, which depend on the oscillation modes of the neutron stars. To obtain the oscillation modes, several numerical calculations have been performed for both the simple polytropic star and the realistic star, and numerical results of ω_{nlm} and Q_{nl} have been published by several authors.[14),12),13)] The general features of their results are as follows: 1) As for the frequency, the f-mode has the angular frequency comparable to the Kepler angular velocity at the stellar surface $\sim\sqrt{GM_1/R_1^3}$, while the p-modes have the higher frequency than that of the f-mode, and the g-modes have the lower frequency than that of the f-mode. 2) As for the overlap integral $|\hat{Q}_{nl}|$, the f-mode has the maximum value $\sim 0.1-1$, and those of the p-modes and the g-modes are less than that of the f-mode and getting smaller as the number of nodes increases. Nevertheless, the p_1- and g_1-modes have the amplitude $|\hat{Q}_{nl}|\sim 0.1$.

From these properties, it is expected that 1) the p-modes do not contribute to the resonant oscillation so much, 2) the f-mode may induce the resonance at contact of a binary and affect the motion at the moment of the plunging and/or merging of binaries, and 3) the g-modes may contribute to the orbital evolution of binaries, with the orbital frequency $\gtrsim 10$ Hz.

In particular, the g-modes may become important from an observational point of view because a signal of gravitational waves from an inspiral of a binary is accumulated in the low frequency band, 10–100 Hz. If, in realistic neutron stars, the g-modes have the frequency $\gtrsim 10$ Hz and the overlap integral $\gtrsim 0.1$, a serious problem occurs because the energy deposited into the oscillation affects the dissipation of the orbital rotation energy. This means that we are not allowed to treat the binary stars as point particles as usually done,[15),16),6),7)] and must take into account the structure of neutron stars to prepare the theoretical template of gravitational waves. Thus in the next section, we estimate ω_{nlm} and $|\hat{Q}_{nl}|$ of the g-modes.

§ 3. Numerical calculations of the f- and g-modes

3.1. *Basic equation of stellar pulsation*

We consider the adiabatic stellar pulsation of a spherical symmetric neutron star. This treatment is consistent because the time scale of dissipation by the viscosity is much longer than the orbital period in the case of the compact binary.[9)] The basic equations are the perturbed Euler equation,

$$\frac{d^2\xi_i}{dt^2}=-\nabla_i\left(\frac{\delta P}{\rho}+\delta\psi\right)+A\delta_{ir}\frac{\Gamma_1 P}{\rho}\sum_{k=1}^{3}\nabla_k\xi^k\,, \tag{3·1}$$

and the perturbed Poisson equation,

$$\nabla^2\delta\psi=4\pi G\delta\rho\,, \tag{3·2}$$

where δQ denotes the Eulerian perturbation of a quantity Q. The symbol δ_{ir} denotes Kronecker's delta, ψ is the gravitational potential and

$$A=\frac{\nabla_r\rho}{\rho}-\frac{\nabla_r P}{\Gamma_1 P}\,, \tag{3·3}$$

where Γ_1 is an adiabatic index. A is usually related to the g-modes and in the case $A<0$, the stable g-modes exist.[12)]

Performing the Fourier transform with respect to t and the spherical harmonics expansion, Eqs. (3·1) and (3·2) form fourth order ordinary differential equations. These equations are solved under two boundary conditions at the origin and other two boundary conditions at the stellar surface.[12)] Detailed equations and boundary conditions are written in section 17 of Ref. 12), so we do not describe them here, but simply describe the numerical strategy. We use equations of the first form in Ref. 12), and the variables are $u=\xi_1 r^2$, $y=\delta P/\rho$ and $\delta\psi$. Equations for these variables form an eigenvalue equation for ω and four boundary conditions are satisfied only when ω is an eigen angular frequency of the star. Thus we adopt the shooting method as follows: 1) We expect an eigen angular frequency, for the f-mode,

$$\omega \sim \sqrt{GM_1/R_1^3}\,, \tag{3·4}$$

and for the g_1-mode,

$$\omega \sim \sqrt{-AGM_1/R_1^2}\,. \tag{3·5}$$

2) Solve the equations outward from the center and match the solutions at the outer boundary. This procedure is continued until the outer boundary conditions are satisfied with sufficient accuracy.

3.2. *Star models and numerical results*

We consider the f- and g-mode oscillations in a neutron star model. For every star, the f-mode always exists, but the g-modes do not always exist: If the star is zero temperature and chemically homogeneous, the frequencies of the g-modes become zero because $A=0$ in Eq. (3·3).[12] However a real neutron star has a finite temperature and chemical inhomogeneities. Such a situation may induce the g-mode oscillations. Thus let us consider physical and chemical states of neutron stars in coalescing compact binary systems.

We know three coalescing binary neutron star systems, PSR1913+16,[10] PSR2127+11C[17] and PSR1534+12,[18] which will merge within the Hubble time. The total lifetimes from their birth to merging are 3×10^8, 3×10^9 and 2×10^8 yr, respectively.[19] According to the cooling property of such old neutron stars,[20),21] the temperature is less than 10^4K.

Frequencies of neutron stars with a finite temperature are investigated by McDermott et al.[22] According to their studies of the pulsation of relativistic neutron stars, the frequencies of the g-mode oscillations are $\lesssim 20$Hz for the temperature $\tau\sim10^8$ K, $\lesssim 3$ Hz for $\tau\sim10^7$ K, and the frequencies become small as the temperature decreases. This suggests that the g-mode frequencies of the neutron stars of temperature $\tau<10^4$ K is much less than 10 Hz. In fact, the thermal pressure of $\tau\sim10^4$ K is only

$$P_T \sim 40\left(\frac{c}{c_s}\right)^3\left(\frac{\tau}{10^4\,\mathrm{K}}\right)^4 \mathrm{dynes\ cm^{-2}}\,, \tag{3·6}$$

where c_s is a sound velocity, $\propto\rho^{(\Gamma_1-1)}$, and $\Gamma_1\sim1.5-2.5$, while the degenerate pressure is[21]

$$P_0 \sim 10^{35}\left(\frac{\rho}{10^{15}\,\mathrm{g\ cm^{-3}}}\right)^{\Gamma_1} \mathrm{dynes\ cm^{-2}}\,. \tag{3·7}$$

Since P_0 is much larger than P_T even for $\rho\sim10^{10}\,\mathrm{g\ cm^{-3}}$, $A(\sim P_T/P_0\Gamma_1)$ is very small. The frequencies of the g-modes are proportional to the square root of A(Eq. (3·5)), so that in old neutron stars, the frequencies of this type of the g-modes are much smaller than 10 Hz, and we do not have to take into account non-zero temperature as an origin of any g-modes with the frequency ≥ 10 Hz.

The second possibility is the chemical inhomogeneity. In real neutron stars, the chemical composition is expected to vary significantly in the surface region where the density is below the nuclear density $\simeq 2.8\times 10^{14}\,\mathrm{g\,cm^{-3}}$.[21] In particular, below the neutron drip density $\simeq 4.3\times 10^{11}\,\mathrm{g\,cm^{-3}}$,[23),21] there are various layers at which phase transitions take place. At these layers, the density discontinuity as well as the chemical one exists. If such a discontinuity exists, another type of the g-modes due to the density discontinuity appears.[24),25] According to the previous authors,[25] such modes have the frequency, 10–100 Hz. In fact, a frequency of the discontinuity mode[25] is approximately written as

$$f\sim\frac{1}{2\pi}\left\{l(l+1)\frac{\Delta\rho}{\rho_d}\frac{R_1-r_0}{R_1}\frac{GM_1}{R_1^3}\right\}^{1/2}\sim 10-100\ \mathrm{Hz}\,, \tag{3·8}$$

where r_0 and ρ_d are the radius at the discontinuity and the density at the outer side of the discontinuity, respectively. This means that this type of the g-modes may affect the evolution of coalescing compact binaries. Thus let us analyze the discontinuity mode.

As a non-perturbed neutron star model, we adopt a simple polytropic star whose equation of state is

$$P=K\rho^2\,. \tag{3·9}$$

To include the chemical inhomogeneity phenomenologically, we regard K as the function of the radius, such as

$$K=K_0+\frac{\Delta K}{2}\left[1+\tanh\left(\frac{r-r_0}{\Delta r}\right)\right],$$

where we put $K=10^{12}\,G$ in cgs units, and we regard ΔK, r_0 and Δr as free parameters. In the case $\Delta r\to 0$, the density becomes discontinuous at $r=r_0$, and our treatment agrees with that of Finn.[25] Because the pressure at r_0 must be continuous, the density discontinuity, $\Delta\rho$, becomes

$$\Delta\rho=\left(\sqrt{1+\frac{\Delta K}{K_0}}-1\right)\rho_d\,. \tag{3·10}$$

From this relation, we choose $\Delta K/K_0=0.01$, 0.1 and 0.4 for examples. These values correspond to about 0.5, 5 and 20% of the density discontinuities. We should note that a typical value of the density discontinuity is $2\sim 6\%$.[23]

In the case $\Delta K=0$, the density profile and the mass, respectively, become

$$\rho=\rho_0\frac{\sin(\pi r/R_p)}{\pi r/R_p}\,,$$

$$M_p=\frac{4\rho_0 R_p^3}{\pi}\,, \tag{3·11}$$

where $R_p=\sqrt{\pi K_0/2G}=12.5$ km. Hereafter, we use M_p and R_p as units of mass and radius. α in Eq. (2·26) is $\pi/4$ and the frequency in these units becomes

Table I. Models of neutron stars considered in this paper are shown. $K_0=10^{12}\,G$, $R_p=12.5$ km, and $M_p=1.26\,M_\odot(\rho_0/10^{15}\,\mathrm{g\,cm^{-3}})$, respectively.

Model	r_0/R_p	$\Delta r/R_p$	$\Delta K/K_0$	M_1/M_p	R_1/R_p
I	0.9	0.01	0.01	1.0000	1.0006
II	0.9	0.001	0.01	1.0000	1.0006
III	0.9	0.01	0.1	1.0004	1.0055
IV	0.9	0.001	0.1	1.0004	1.0055
V	0.9	0.001	0.4	1.0013	1.0208
VI	0.99	0.001	0.01	1.0000	1.0001
VII	0.99	0.001	0.1	1.0000	1.0005
VIII	0.99	0.001	0.4	1.0000	1.0019

$$f_p \equiv \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{GM_p}{R_p^3}} = 1.47\left(\frac{\rho_0}{10^{15}\,\mathrm{g\,cm^{-3}}}\right)^{1/2}\mathrm{kHz}\,. \qquad (3\cdot 12)$$

From these relations, we choose $\Delta r/R_p=0.01$, 0.001 and $r_0/R_p=0.9$, 0.99 as examples. We summarize the models in Table I.

Density configuration for each model is solved by numerical integration. In Fig. 1, we show a numerical result of the density configuration for model V as an example. We can see the density gap at $r=r_0$, which is the origin of the g-modes.

In Table II, we show the numerical results of ω_{n22} and $\tilde{Q}_{n2}$ for each model. It is found that both ω_{n22} and $\tilde{Q}_{n2}$ of the f-mode are almost the same for all models. This is consistent with the property of the f-mode, which is deter-

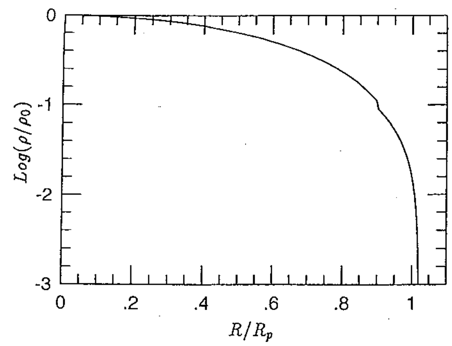

Fig. 1. The density profile for model V.

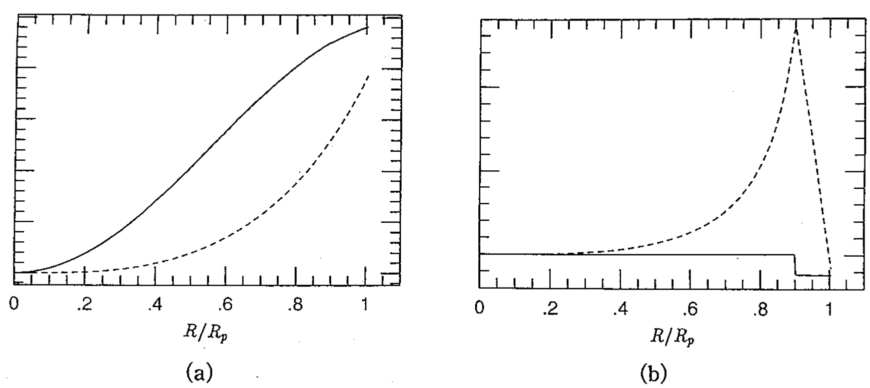

Fig. 2. The profiles of the perturbation variables, $y=\delta P/\rho$ and $u=\xi_1 r^2$ for model IV are shown by the solid and dashed lines, respectively. The scale of the two variables is the same. The f-mode is shown in (a) and the g_1-mode is shown in (b).

mined by the global properties of star, such as the mass and radius which are almost independent of the density discontinuity. In Fig. 2, we show $y=\delta P/\rho$ for the solid line and $u=\xi_1 r^2$ for the dashed line for both the f-mode and g_1-mode for model IV. It is found that those of the f-mode are not affected by the discontinuity at $r=r_0$, while those of the g_1-mode are considerably affected. This also indicates the property of the f- and g-modes; the f-mode is not affected by the local property of the star, but the g-modes are affected.

If we consider binary neutron stars with the same masses and radii, ω_{n22} of the f-mode is about 20% larger than 2Ω, so the f-mode does not seem to contribute to the resonance. However in the present calculations, we use the Newtonian gravity, whereas the general relativity plays an important role in the real neutron stars. For example, if the general relativistic effects are taken into account, the radius may become small for about a few ten percent because of the strong gravity. In this case, the f-mode resonance may occur near $r\sim 5M$ and induce a transition from a steady

Table II. The resonant frequencies and the overlap integrals for every model are shown. Here, the angular frequencies are written in units of $(GM_1/R_1^3)^{1/2}\sim(GM_p/R_p^3)^{1/2}$. In the units, 1, 0.1 and 0.01 represent 1.47 kHz, 147 Hz and $14.7(\rho_0/10^{15}\,\mathrm{gcm}^{-3})^{1/2}$Hz of the frequency f, respectively.

Model	mode	$\omega_{n22}(R_1^3/GM_1)^{1/2}$	$\hat{Q}_{n2}$
I	f	1.226	0.6296
	g_1	4.208×10^{-2}	8.8×10^{-4}
	g_2	9.605×10^{-3}	2.8×10^{-4}
II	f	1.226	0.6296
	g_1	4.404×10^{-2}	8.6×10^{-4}
	g_2	$<6.\times10^{-3}$	—
III	f	1.219	0.6300
	g_1	.1299	8.99×10^{-3}
	g_2	3.168×10^{-2}	2.81×10^{-3}
	g_3	1.791×10^{-2}	2.10×10^{-3}
IV	f	1.219	0.6300
	g_1	.1355	8.80×10^{-3}
	g_2	1.019×10^{-2}	9.7×10^{-4}
V	f	1.200	0.6311
	g_1	.2499	3.65×10^{-2}
	g_2	1.960×10^{-2}	3.7×10^{-3}
	g_3	1.135×10^{-2}	2.8×10^{-3}
VI	f	1.227	0.6296
	g_1	1.185×10^{-2}	9×10^{-5}
VII	f	1.225	0.6296
	g_1	3.640×10^{-2}	9.0×10^{-4}
	g_2	8.007×10^{-3}	3.0×10^{-4}
VIII	f	1.224	0.6296
	g_1	6.659×10^{-2}	3.49×10^{-3}
	g_2	1.510×10^{-2}	1.13×10^{-3}
	g_3	8.999×10^{-3}	8.5×10^{-4}

inspiral to an unstable plunge because $\Delta E_{\rm GW} \simeq \Delta E_{\rm os}$.

From Table II, we also find the following features of the g-modes.

1) The value of ω_{n22} and $\hat{Q}_{n2}$ of the g_1-mode are almost independent of Δr, but for the higher g-modes, the angular frequency becomes small as Δr becomes small. (Compare models I and II, and III and IV.)

2) The larger r_0 corresponds to the smaller ω_{n22} and $\hat{Q}_{n2}$.

3) The larger ΔK corresponds to the larger ω_{n22} and $\hat{Q}_{n2}$.

Property 1) is consistent with the theorem that there is only one g-mode with non-zero frequency for each point r_i where A is infinite and negative.[26] This means that if the thickness of the boundary layer, between which the phase transition occurs, is very thin, the g-modes for >10 Hz do not seem to exist for real neutron stars except for the g_1-mode.

The dependences of ω_{n22} on r_0 and ΔK are consisent with Eq. (3.8). Properties 2) and 3) state that there is a correlation between ω_{n22} and $\hat{Q}_{n2}$; large (small) ω_{n22} corresponds to large (small) $\hat{Q}_{n2}$, and the ratio $\hat{Q}_{n2}/\omega_{n22}$ does not change so much for each model.

From these properties in mind, let us discuss the effects of the g-modes to the evolutions of binary systems. We consider the two typical binaries, one is the binary neutron star with the same masses, 1.4 $M_\odot$, the other is the neutron star-black hole binary, with the masses, 1.4 $M_\odot$ and 10 $M_\odot$, respectively. In our models, 1.4 $M_\odot$ means $\rho_0 \simeq 1.11 \times 10^{15}$ g cm^{-3}. In the former case, $f \geq 10$ Hz corresponds to $R \leq 174\, GM/c^2 = R_m$ and in the latter case $R \leq 68\, GM/c^2 = R_m$. Defining the accumulated cycles[3] for each radius as $N(R) = \Delta T \Omega_0/\pi$, for both cases, $N(R) = 164(R/R_m)^{5/4}$ and $N(R) = 77(R/R_m)^{5/4}$, respectively. If N is affected by the g-mode resonances even for several cycles, it is a problem because we must take into account the structure of neutron stars to make a theoretical template even in the low frequency band.[3]

Defining $\eta = \beta \Delta E_{\rm os}/\Delta E_{\rm GW}$, where $\beta = 2$ for the binary neutron star and $\beta = 1$ for the neutron star-black hole binary, the contribution of the resonance oscillations to the accumulated cycles is estimated by $\Delta N = \eta N(R)$. For the binary neutron star model,

$$\eta = 13.5 \left(\frac{Rc^2}{GM} \right)^{7/4} \hat{Q}_{n2}^2 , \tag{3·13}$$

and for the neutron star-black hole model,

$$\eta = 2.5 \left(\frac{Rc^2}{GM} \right)^{7/4} \hat{Q}_{n2}^2 . \tag{3·14}$$

Hence $N(R)$ for the binary neutron star model and neutron star-black hole binary model, respectively, become

$$\Delta N = 18.5 \left(\frac{R}{R_m} \right)^3 \left(\frac{\hat{Q}_{n2}}{10^{-3}} \right)^2 = 18.5 \left(\frac{f}{10\ \rm Hz} \right)^{-2} \left(\frac{\hat{Q}_{n2}}{10^{-3}} \right)^2 , \tag{3·15}$$

$$\Delta N = 0.31 \left(\frac{R}{R_m} \right)^3 \left(\frac{\hat{Q}_{n2}}{10^{-3}} \right)^2 = 0.31 \left(\frac{f}{10\ \rm Hz} \right)^{-2} \left(\frac{\hat{Q}_{n2}}{10^{-3}} \right)^2 . \tag{3·16}$$

It is found that for neutron star-black hole binaries, ΔN is smaller than unity for every g-mode of eight models. These results do not change if the mass of the black hole is larger than $\sim 6\,M_{\odot}$. In the case of the binary neutron star of masses $1.4\,M_{\odot}$, for models I, II, VI and VII, $\Delta N<1$, but for models III, IV and VIII, ΔN becomes $\sim 5-10$, and especially for model V, ΔN becomes $\sim 20-40$ for all of the g-modes. These results mean that 1) if the considerably large discontinuities ($\Delta\rho/\rho\sim 20\%$) exist or 2) if the large discontinuities ($\Delta\rho/\rho\sim 5\%$) exist in the inner crust ($r_0\sim 0.9\,R_1$) of neutron stars, the orbital evolution of the binary is affected by the g-mode resonances from an observational point of view. However, as long as the physically reasonable density discontinuities are concerned, the orbital evolution does not seem to be affected by the g-mode resonances. This is because 1) $\Delta\rho/\rho$ is at most 6% in realistic neutron stars,[23] and 2) the density discontinuities exist only in the outer crust in which the density is below the neutron drip density ($\sim 4.3\times 10^{11}\,\mathrm{g\,cm^{-3}}$).[21] Therefore we do not have to take into account the effects of the g-modes to the orbital evolution of the binary under a physically reasonable situation.

§ 4. Summary

In this paper, we considered the effects of the tidal resonances of neutron stars on the evolution of compact star binaries. In realistic neutron stars, not the thermal pressure, but the density discontinuity generates the g-mode oscillations with $f\gtrsim 10$ Hz, so that we adopted the neutron stars with density discontinuities as the model stars. Using the Press-Teukolsky formalism, we numerically calculate the tidally deposited energy of the f- and g-modes, and comparing the deposited energy with the energy flux of gravitational waves, we consider the effects of the tidal resonances to the orbital evolution of binaries. It is found that if the companion of a neutron star is a black hole of mass $\gtrsim 6\,M_{\odot}$, the g-mode resonance is unimportant. As for binary neutron stars, we obtain the following results: 1) The f-mode resonance may affect the orbital evolution just before the merging. 2) Unless a considerably large density discontinuity ($\Delta\rho/\rho\sim 20\%$) exists or a large discontinuity ($\Delta\rho/\rho\sim 5\%$) exists in the inner crust ($r_0\sim 0.9\,R_1$), the g-mode resonances do not affect the orbital evolution of the binary because the deposited energy by the resonances are very small compared with the dissipated energy by gravitational radiation. 3) In the case that a considerably large density discontinuity exists or a large discontinuity exists in the inner crust, the g-mode resonances affect the orbital evolution of the binary. However such a situation does not seem to realize in the neutron stars, so that the g-modes do not affect the orbital evolution so much, in reality. This means that if we try to prepare a theoretical template of gravitational waves from a coalescing compact binary, we can regard binary stars as the point masses for low frequency region, $10\sim$ a few 100 Hz.

As for the high frequency region, 300—1000 Hz, the f-mode may become important. However in this region, not only such a fluid effect, but also the general relativistic effects are important; 1) the higher order post-Newtonian gravitational potentials must be taken into account to calculate the orbital motion,[15] and 2) the radiation reaction of gravitational waves for higher multipole modes ($l\geq 3$) is needed.[16] This means that in such a region, we must treat the problem including the effects of both the fluid and the general relativity. Therefore the fully general relativistic 3-dimensional simulations[27] are urgent.

We only consider effects of the resonances in the tidal oscillations in this paper. When we treat this problem in a general relativistic manner, not only resonant oscillations but also resonant gravitational waves[28] must be taken into account. As shown in this paper, the oscillating energy of the g-modes is small in the realistic neutron star, so that the emission rate of gravitational waves by the g-mode resonances will not be also so large even in a general relativistic case. On the other hand, the emission rate of gravitational waves by the f-mode resonance will be large.[28] Hence it is necessary to investigate the emission rate of gravitational waves by the effect of the f-mode resonance at the final phase of the coalescing compact binaries.

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