STABILITY OF RIGIDLY ROTATING SUPERMASSIVE STARS AGAINST GRAVITATIONAL COLLAPSE

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ABSTRACT

We revisit secular stability against quasi-radial collapse for rigidly rotating supermassive stars (SMSs) in general relativity. We suppose that the SMSs are in a nuclear-burning phase and can be modeled by polytropic equations of state with the polytropic index n_p slightly smaller than 3. The stability is determined in terms of the turning point method. We find a fitting formula of the stability condition for the plausible range of n_p ($2.95 \lesssim n_p \lesssim 3$) for SMSs. This condition reconfirms that while non-rotating SMSs with a mass of $\sim 10^5 M_\odot$–$10^6 M_\odot$ may undergo a general relativistically induced quasi-radial collapse, rigidly rotating SMSs with a ratio of rotational to gravitational potential energy (β) of $\sim 10^{-2}$ are likely to be stable against collapse unless they are able to accrete ~ 5 times more mass during the (relatively brief) hydrogen-burning phase of their evolution. We discuss the implications of our results.

Key words: gravitation – hydrodynamics – instabilities – relativistic processes – stars: massive – stars: rotation

1. INTRODUCTION

A supermassive star (SMS) is a possible progenitor for the formation of a seed of a supermassive black hole (SMBH). Recent star formation calculations in spherical symmetry (Hosokawa et al. 2013; H. Umeda 2016, private communication) suggest that if a high mass accretion rate with $\gtrsim 0.1 M_\odot \, \mathrm{yr}^{-1}$ is preserved in the nuclear-burning phase of $\sim 2 \times 10^6$ years, an SMS with a mass of $\gtrsim 2 \times 10^5 M_\odot$ could be formed. Such a high mass accretion rate requires primordial gas clouds with a virial temperature $\gtrsim 10^4$ K. There are several scenarios proposed to achieve this condition such as Lyman–Werner radiation from nearby local star formation region (Omukai 2001; Dijkstra et al. 2008) or shock heating in the cold accretion flows in the forming first galaxies (Dekel et al. 2009; Inayoshi & Omukai 2012). Subsequently, the SMS may collapse to a seed of an SMBH with a mass $\gtrsim 10^5 M_\odot$. General relativistic radial instability (Iben 1963; Chandrasekhar 1964; Zel'dovich & Novikov 1971) is often referred to as the mechanism leading to the collapse of the SMSs.

The formation process of an SMBH after the collapse of an SMS should be determined by the initial condition at which the instability sets in. In reality it is natural to consider that SMSs are rotating because they likely form in a non-symmetrical environment at a dense core of the galactic center as indicated by recent numerical simulations of the collapse of an atomic cooling halo in the early universe (e.g., Latif et al. 2013; Regan et al. 2014; Becerra et al. 2015). These simulations have suggested that protostellar disks that initially formed in the central gas cloud could be gravitationally unstable and fragment into several clumps, preventing the growth of both the mass and angular momentum of the central protostar. However, the clumps are likely to subsequently migrate inward and eventually fall onto the central protostar, enhancing episodic accretion (Inayoshi & Haiman 2014; Hosokawa et al. 2015). A rotating SMS could be a likely outcome.

This implies that for the realistic exploration of the collapse of SMSs to SMBHs we have to derive the stability condition for *rotating* SMSs. This is particularly the case in this context because the SMSs are very massive and hence they are supported dominantly by the radiation pressure resulting in the adiabatic index, Γ, close to $4/3$.

The condition for the stability of rotating SMSs was first analyzed by Fowler (1966) in his post-Newtonian analysis (see also chapter 14 of Tassoul 1978 for a review). He showed that the rotation plays a significant role for stabilizing the radiation-supported SMSs against gravitational collapse while the general relativistic gravity gives a destabilizing effect. The point to be emphasized is that the energy for these two effects could have the same order of magnitude. In the presence of rotation, the condition for the onset of the general relativistic instability is significantly different from the well-known result for *spherical* stars derived by Chandrasekhar (1964). Indeed, the fully general relativistic study by Baumgarte & Shapiro (1999) showed that rotation would be the important ingredient in their study for the stability of SMSs that were modeled by a simple $\Gamma = 4/3$ polytrope.

The purpose of this paper is to provide a quantitative formula for the stability condition of rotating SMSs which are supported by radiation and gas pressure as well as by rotational centrifugal force. We assume that SMSs are rigidly rotating because their cores in nuclear-burning phases should be in a convective equilibrium (Bond et al. 1984; H. Umeda 2016, private communication; see also the Appendix of Loeb & Rasio 1994) and hence they would be in a turbulent state. We systematically compute a number of equilibrium sequences for rotating SMSs in general relativity employing polytropic equations of state with its polytropic index, n_p, slightly smaller than 3 (i.e., the adiabatic index slightly larger than $4/3$), by which the equations of state for SMSs are well-reproduced (see Section 2).

The paper is organized as follows. In Section 2 we review approximate equations of state for SMSs following Bond et al. (1984). In Section 3 the secular stability of rotating SMSs in general relativity is numerically determined. In Section 4 we predict the final outcomes after the collapse of an SMS assuming that the initial condition is a marginally stable SMS determined in Section 3. Section 5 is devoted to a summary and discussion. Throughout this paper G, c, k_B, and a_r denote the

gravitational constant, speed of light, Boltzmann's constant, and radiation constant, respectively.

2. EQUATIONS OF STATE

We basically suppose that SMSs are composed of hydrogen, helium, electron, and photon. Then the pressure, P, and internal energy density, ϵ, are written as (Bond et al. 1984)

$$P = \frac{a_r T^4}{3} + Y_T n k_B T, \qquad (1)$$

$$\epsilon = a_r T^4 + \frac{3}{2} Y_T n k_B T, \qquad (2)$$

where T is the temperature and n is the baryon number density, respectively. Y_T is defined by

$$Y_T \equiv Y_e + Y_p + Y_\alpha, \qquad (3)$$

where $Y_I = n_I/n$ and n_I for $I = e, p, \alpha$ denotes the number density of electrons, hydrogen, and helium, respectively. For the primordial gas, $Y_p \approx 0.75$, $X_\alpha \equiv 4Y_\alpha \approx 0.25$, and $Y_e \approx 0.88$ yielding $Y_T \approx 1.69$. For pure helium gas, $Y_p = 0$, $X_\alpha = 1$, and $Y_e = 0.50$ yielding $Y_T = 0.75$.

Inside the SMSs in nuclear-burning phases, in particular for their core region, convection should be highly enhanced and a convective equilibrium is realized (Bond et al. 1984; H. Umeda 2016, private communication). This implies that the SMS core is isentropic, i.e., the specific entropy s is constant and its chemical composition is uniform, i.e., $Y_T =$ constant. For simplicity we assume that these relations are satisfied for the entire SMS or we may say that we focus only on the convective cores ignoring a surrounding low-density envelope.

Then the first law of thermodynamics, $d(\epsilon/n) = -P d(1/n)$, gives the relation between dT and dn (i.e., between dP and dn) from Equations (1) and (2). Using this relation the adiabatic constant is calculated as (Eddington 1918; Chandrasekhar 1939; Bond et al. 1984)

$$\Gamma = \left(\frac{\partial \ln P}{\partial \ln n}\right)_s = \frac{4}{3} + \frac{4\sigma + 1}{3(\sigma + 1)(8\sigma + 1)}, \qquad (4)$$

where σ is the ratio of the radiation pressure to the gas pressure written as

$$\sigma \equiv \frac{a_r T^3}{3 Y_T n k_B} = \frac{s_\gamma}{4 Y_T k_B}. \qquad (5)$$

Here, s_γ denotes the photon entropy per baryon. For SMSs, σ and s_γ/k_B are much larger than unity (see below) and hence Γ can be approximated well by $4/3 + 1/(6\sigma)$.

Because the photon entropy is much larger than the gas entropy and s is assumed to be constant, we may also assume that s_γ and σ are approximately constant. Hence it is reasonable to assume that the equations of state for the SMS core are well approximated by a polytropic form

$$P = K \rho^\Gamma, \quad \Gamma = 1 + \frac{1}{n_p}, \qquad (6)$$

where ρ is the rest-mass density ($\rho = m_B n$ with m_B the mean baryon mass). K and n_p are the adiabatic constant and polytropic index, respectively.

Table 1
Maximum Mass and Related Quantities for Spherical SMSs and Rigidly Rotating SMSs

Γ	$M_{s,\mathrm{max}}$ ($\hat{C}_{n_p}$)	M_{max} ($\hat{C}_{n_p}$)	$\beta_{\mathrm{max}}(\times 10^{-3})$	q_{max}
1.334	4.424 (0.971)	4.461 (0.979)	8.9	0.87
1.335	4.264 (0.936)	4.319 (0.948)	9.0	0.81
1.336	4.123 (0.905)	4.188 (0.919)	9.1	0.77
1.338	3.875 (0.851)	3.950 (0.867)	9.2	0.70
1.340	3.659 (0.803)	3.737 (0.820)	9.4	0.65
1.342	3.466 (0.761)	3.546 (0.778)	9.6	0.61

Note. Γ: adiabatic index. $M_{s,\mathrm{max}}$: the maximum mass for spherical SMSs. M_{max}: the maximum mass for rigidly rotating SMSs. β_{max}: the maximum value of β for rigidly rotating SMSs. q_{max}: the maximum value of q for rigidly rotating SMSs. The units of $M_{s,\mathrm{max}}$ and M_{max} are $M_u = K^{n_p/2}G^{-3/2}c^{3-n_p}$ (i.e., we list C_{n_p}). Here $\hat{C}_{n_p} = C_{n_p}/C_{3,s}$ with $C_{3,s} = 4.555$.

Using Equations (1) and (6) the adiabatic constant is written as

$$K = \left(\frac{Y_T k_B \sigma}{m_B}\right)^{4/3} \left(\frac{3}{a_r}\right)^{1/3} (1 + \sigma^{-1}) \rho^{-1/(6\sigma)}. \qquad (7)$$

Here $\rho^{1/(6\sigma)}$ may be considered to be constant because $\sigma \gg 1$ so that we can consider K to be a constant. From K the quantity of mass dimension is constructed as

$$M_u \equiv K^{n_p/2} G^{-3/2} c^{3-n_p}. \qquad (8)$$

For $\sigma \gg 1$ this quantity is written as

$$M_u = M_{u,3}\left(1 + \frac{3}{2\sigma}\right)\left(\frac{m_B c^2}{Y_T k_B T \sigma}\right)^{3/(4\sigma)}, \qquad (9)$$

where $M_{u,3}$ denotes M_u for $n_p = 3$ ($\Gamma = 4/3$) and is written as

$$M_{u,3} = \left(\frac{Y_T k_B}{m_B}\right)^2 \left(\frac{3}{G^3 a_r}\right)^{1/2} \sigma^2$$

$$\approx 4.01 M_\odot Y_T^2 \sigma^2 \approx 0.251 M_\odot \left(\frac{s_\gamma}{k_B}\right)^2. \qquad (10)$$

To derive Equation (9) we used Equation (5). Note that for $n_p = 3$ polytropic spherical stars in Newtonian gravity the mass is written in the well-known form as (Bond et al. 1984)

$$M = C_3 M_{u,3} \approx 1.14 M_\odot \left(\frac{s_\gamma}{k_B}\right)^2, \qquad (11)$$

where we define $C_{n_p} \equiv M/M_u$ for each value of n_p which will be determined in numerical analysis. In the present context, C_{n_p} should be determined for SMSs at marginally stable states. For the spherical case it decreases slowly with the decrease of n_p from $C_3 = 4.555 \equiv C_{3,s}$ to $C_{2.94} \approx 3.66$ for the change from $n_p = 3$ to $n_p = 2.94$ ($\Gamma \approx 1.340$) (see Table 1).

We note that the correction factor associated with σ^{-1} in Equation (9) is a small number for SMS cores for which $\sigma > 10^2$ although the typical value of $m_B c^2/(Y_T k_B T)$ is of the order of 10^4. For example, for $Y_T = 1.69$, $T = 10^{8.2}$ K, and $\sigma \geqslant 10^2$, $1 \leqslant M_u/M_{u,3} \lesssim 1.06$; hence, $C_{n_p} \approx M/M_{u,3}$. Here, we employ $T = 10^{8.2}$ K as the typical temperature supposing that the SMS is in a hydrogen-burning phase (Bond et al. 1984; H. Umeda 2016, private comminication) and assuming that the hydrogen-burning temperature depends only weakly on its

mass and angular momentum. If the SMS is in a helium-burning phase, T should be slightly higher as $\sim 10^{8.4}$ K.

Using Equation (10) we have

$$\Gamma - \frac{4}{3} \approx \frac{1}{6\sigma} \approx 3.8 \times 10^{-3} \left(\frac{M}{10^5 M_\odot}\right)^{-1/2} \left(\frac{C_{n_p}}{C_{3,s}}\right)^{1/2} \left(\frac{Y_T}{1.69}\right) \tag{12}$$

This relation will be used in the next section. We note that $(C_{n_p}/C_{3,s})^{1/2}$ is in a narrow range between 0.90 and 1.00 for $2.94 \leqslant n_p \leqslant 3$ (see Table 1).

Because we often refer to it later we also analyze the magnitude of a dimensionless quantity, $P/(\rho c^2)$. This is approximately written as

$$\frac{P}{\rho c^2} \approx \frac{aT^4}{3\rho c^2} = \frac{1}{4}\left(\frac{k_B T}{m_B c^2}\right)\left(\frac{s_\gamma}{k_B}\right)$$
$$\approx 3.7 \times 10^{-6}\left(\frac{T}{10^{8.2} \text{ K}}\right)\left(\frac{s_\gamma}{k_B}\right). \tag{13}$$

Using Equation (10) with $\sigma \gg 1$, we find for spherical SMS cores

$$\frac{P}{\rho c^2} \approx 1.1 \times 10^{-3}\left(\frac{T}{10^{8.2} \text{ K}}\right)$$
$$\times \left(\frac{C_{n_p}}{C_{3,s}}\right)^{-1/2}\left(\frac{M}{10^5 M_\odot}\right)^{1/2}. \tag{14}$$

For $n_p = 3$ spherical polytropes in Newtonian gravity, the central value of P/ρ, P_c/ρ_c, is equal to $(GM/R)/1.1705$, where R is the stellar radius. Thus, the typical compactness of SMS cores defined by $GM/(c^2 R)$ is of the order 10^{-3} for $M \sim 10^5 M_\odot$.

3. NUMERICAL ANALYSIS FOR STABILITY

3.1. Basic Equations

To explore the secular stability of rotating SMSs against general relativistic quasi-radial collapse we systematically compute their equilibrium solutions in general relativity. Assuming that the SMSs are composed of an ideal fluid we write the stress–energy tensor as

$$T^{\mu\nu} = \rho h u^\mu u^\nu + P g^{\mu\nu}, \tag{15}$$

where u^μ is the four velocity, $h \equiv 1 + \varepsilon + P/\rho$ is the specific enthalpy, ε is the specific internal energy (different from ϵ), and $g_{\mu\nu}$ is the spacetime metric. As in Section 2 the polytropic equations of state are employed here. Using the first law of thermodynamics, ε in the polytropic equations of state is written as

$$\varepsilon = \frac{n_p P}{\rho}. \tag{16}$$

As mentioned in Section 2 the SMSs (SMS cores) are likely to be in convective equilibrium. This indicates that a turbulent state would be realized and the angular velocity would be approximately uniform (Baumgarte & Shapiro 1999). Thus, we pay attention only to rigidly rotating stars setting the angular velocity of $\Omega \equiv u^\varphi/u^t$ to be constant.

With the polytropic equation of state, physical units enter the problem only through the polytropic constant K, which can be completely scaled out of the problem. For example, $K^{n_p/2}G^{-3/2}c^{3-n_p}(=M_u)$ has units of mass, $K^{n_p}G^{-2}c^{5-2n_p}$ has units of angular momentum, and $K^{-n_p}c^{2n_p}$ has units of density. Thus, for presenting numerical results we show dimensionless quantities which are rescaled by K. We note that in these units the dimensionless mass M (i.e., M/M_u) is equivalent to C_{n_p}, which is defined in Section 2.

Following Butterworth & Ipser (1976; see also Stergioulas 1998 for a review) the line element is written as

$$ds^2 = -e^{2\nu}dt^2 + B^2 e^{-2\nu}r^2 \sin^2\theta(d\varphi - \omega dt)^2$$
$$+ e^{2\zeta - 2\nu}(dr^2 + r^2 d\theta^2), \tag{17}$$

where ν, B, ω, and ζ are field functions. The first three obey elliptic-type equations in axial symmetry and the last one an ordinary differential equation. These equations are solved using two methods: one is described in Shibata & Sasaki (1998) and the other is a method by Cook et al. (1992). We checked that the results derived by two independent codes agree well with each other for the problems considered in this paper. For instance, the mass and density of the turning points (see below) determined by two methods agree with each other within 0.01% and 1% difference for most of the parameter space (except for the region very close to the mass-shedding limit at which it is not easy to identify the turning points).

The Komar mass (gravitational mass) M, Komar angular momentum J, rotational kinetic energy T_{rot}, and gravitational potential energy W are defined by

$$M = 2\pi \int (-2T_t{}^t + T_\mu{}^\mu) B e^{2\zeta - 2\nu} r^2 dr d(\cos\theta), \tag{18}$$

$$J = 2\pi \int \rho h u^t u_\varphi B e^{2\zeta - 2\nu} r^2 dr d(\cos\theta), \tag{19}$$

$$T_{rot} = \frac{1}{2} J\Omega, \tag{20}$$

$$W = M_p - M + T_{rot} \quad (>0), \tag{21}$$

where M_p is the proper mass defined by

$$M_p = 2\pi \int \rho u^t (1 + \varepsilon) B e^{2\zeta - 2\nu} r^2 dr d(\cos\theta). \tag{22}$$

Note that we define W to be positive. From these quantities the well-known dimensionless parameters are defined as $\beta \equiv T_{rot}/W$ and $q \equiv cJ/(GM^2)$.

In addition to these quantities we often refer to the central density, ρ_c, which is used to specify a rotating star for a given set of β and M, and to the equatorial circumferential radius, R_e, by which a compactness parameter is defined by $GM/(c^2 R_e)$. We also refer to a dimensionless quantity, $P_c/(\rho_c c^2)$, for specifying the compactness of rotating SMSs. As shown in the following, the values of this quantity are close to $GM/(c^2 R_e)$.

3.2. Analysis for Secular Stability

The secular stability for rigidly rotating stars against quasi-radial oscillations can be determined by the turning point method established by Friedman et al. (1988; see, e.g., Cook et al. 1992, 1994; Baumgarte & Shapiro 1999; Shibata 2004 for application). According to the turning point theorem, a change of the sign of $dM/d\rho_c$ along a curve of a constant value of J indicates the change of the secular stability. Thus, in numerical

computation we derive curves of constant values of J in the plane composed of M and ρ_c and then determine the turning points. We always find one turning point of $dM/d\rho_c = 0$ along the $J = $ constant curves if J is smaller than a threshold value. In the present case, the lower-density side is the branch for the stable stars and the other side is the branch for the unstable stars. The stable branch has $(dM/d\rho_c)_J > 0$ while for the unstable branch it is negative.

3.3. Numerical Results

Figure 1 plots the curves of M as a function of $P_c/(\rho_c c^2)(=K\rho_c^{1/n_p}/c^2)$ for various values of J and for $\Gamma = 1.334$, 1.335, 1.336, 1.338, 1.340, and 1.342. Here the units of M and J are M_u and $K^{n_p}G^{-2}c^{5-2n_p}$, respectively. For all the panels the solid and dotted curves show the relations of M as a function of $P_c/(\rho_c c^2)$ for a given value of J and $J = 0$, respectively, and the dotted–dotted curve denotes the mass-shedding limit (i.e., the sequence of maximally and rigidly rotating SMSs for a given equation of state). Above the dotted–dotted curves no rigidly rotating SMS can be realized.

The maxima of M along $J = $ constant sequences are present for $J/(K^{n_p}G^{-2}c^{5-2n_p}) \lesssim 17.3$, 15.2, 13.4, 10.9, 9.0, and 7.6 for $\Gamma = 1.334$, 1.335, 1.336, 1.338, 1.340, and 1.342, respectively. If the value of $P_c/(\rho_c c^2)$ is smaller than that at this turning point, the SMS is stable against general relativistic quasi-radial collapse. On the other hand, if $P_c/(\rho_c c^2)$ is larger than that at the turning point the SMS is unstable. Rotating SMSs will be unstable if they reach this turning point after some evolution process of increasing the value of $P_c/(\rho_c c^2)$ (i.e., increasing the compactness or mass). The values of the dimensionless mass, M/M_u (C_{n_p}), at the maxima depend only weakly on the values of J for a given value of Γ. On the other hand, the critical values of $P_c/(\rho_c c^2)$ depend strongly on the value of J, in particular, for Γ close to $4/3$. This implies that for more rapidly rotating SMSs the quasi-radial instability sets in at a higher value of compactness. That is, to induce the collapse of a rotating SMS an additional evolution process of increasing the compactness (this is equivalent to increasing the mass; see below) is necessary.

Important quantities for marginally stable and maximally rotating SMSs are listed in Table 1. Definitions of the quantities being tabulated are provided in the "NOTE" that accompanies this table. It should be mentioned that the maximum values of β for the stable SMSs are universally $\sim 9 \times 10^{-3}$ depending weakly on Γ with its plausible values for SMSs. By contrast, the maximum values of q depend strongly on the values of Γ, decreasing far below the Kerr limit, $q = 1$, with the increase of Γ. $M_{s,\max}/M_u$ and $M_{\max}/M_u$ (i.e., C_{n_p}) decrease slowly with Γ; they are well-fitted in the form $C_{n_p} = A - B(\Gamma - 4/3)^{\alpha}$, where $(A, B, \alpha) \approx (4.60, 37.0, 0.734)$ is for the spherical stars and $(A, B, \alpha) \approx (4.60, 45.0, 0.789)$ is for the maximally rotating stars, respectively.

Next we derive a fitting formula for the stability condition of rotating SMSs. Figure 2 shows the relation between β and $P_c/(\rho_c c^2)$ for marginally stable SMSs. For each curve of Γ, SMSs in its right-hand side are unstable. It is found that the relation is approximately linear although it is slightly different from the linear relation around the largest values of β (i.e., near the mass-shedding limit).

To construct a fitting formula, we first pay attention to the spherical case, $\beta = 0$. For $\Gamma \to 4/3$, Chandrasekhar (1964)

showed that the turning point would appear at

$$\Gamma - \frac{4}{3} \approx 2.2489 \frac{GM}{c^2 R_e}. \tag{23}$$

This is equivalent to

$$\Gamma - \frac{4}{3} \approx 2.6324 \frac{P_c}{\rho_c c^2} \equiv y. \tag{24}$$

However, for $\Gamma - 4/3 \gtrsim 10^{-3}$, the deviation from this strictly linear relation is noticeable. By a high-precision numerical analysis of spherical SMSs in general relativity, we find the following better relation

$$\Gamma - \frac{4}{3} + \left(\Gamma - \frac{4}{3}\right)^2 \approx y, \tag{25}$$

or

$$\Gamma - \frac{4}{3} \approx y - y^2. \tag{26}$$

Then we consider the case of $0 < \beta \ll 0.01$. Previous studies in Newtonian gravity also have shown that the rotation plays a significant role for the stabilization (Ledoux 1945; see also Tassoul 1978 for a review) and the relation of Equation (23) is modified as

$$\Gamma - \frac{4}{3} \approx y - 2\left(\frac{5}{3} - \Gamma\right)\beta. \tag{27}$$

Taking this condition into account and taking a close look at numerical results, we fit the numerical data of the turning points in terms of

$$\Gamma - \frac{4}{3} = y - y^2 - \left(\frac{10}{3} - 2\Gamma - y - \beta\right)\beta. \tag{28}$$

Here we have determined the coefficients of the terms of $y\beta$ and β^2 in a rather ad hoc manner. These coefficients could be different from -1 in reality. However, this ad hoc choice is acceptable in the present study because it provides a good-fitting formula as shown by the dotted–dotted curves in Figure 2 which tells us that the order of magnitude of these coefficients is unity. Indeed, the error estimated by

$$\frac{1}{y}\left[\Gamma - \frac{4}{3} - y + y^2 + \left(\frac{10}{3} - 2\Gamma - y - \beta\right)\beta\right] \tag{29}$$

is always smaller than 1% for our numerical data; in particular, for the parameter space far from the mass-shedding limit it is much smaller than 1%. Therefore, we conclude that the fitting formula, (28), is well-suited for determining the condition for the onset of general relativistic quasi-radial instability of rigidly rotating SMSs.

In Equation (28) the coefficients of all the nonlinear terms, y^2, β^2, and $y\beta$, are of the order of unity. This implies that these nonlinear terms give only the minor effect on the SMS stability because for (rigidly rotating) SMSs, $y \lesssim 0.01$ and $\beta \lesssim 0.01$ are satisfied: We have therefore demonstrated that Equation (27), in essence the stability relation provided by Tassoul (1978), can be used as an approximate condition.

Figure 3 plots the relation between $GM/(c^2 R_e)$ and $P_c/(\rho_c c^2)$ for SMSs at the turning points. This illustrates that for $\beta \ll 0.01$ (i.e., for the limits of $GM/(c^2 R_e) \to 0$ and
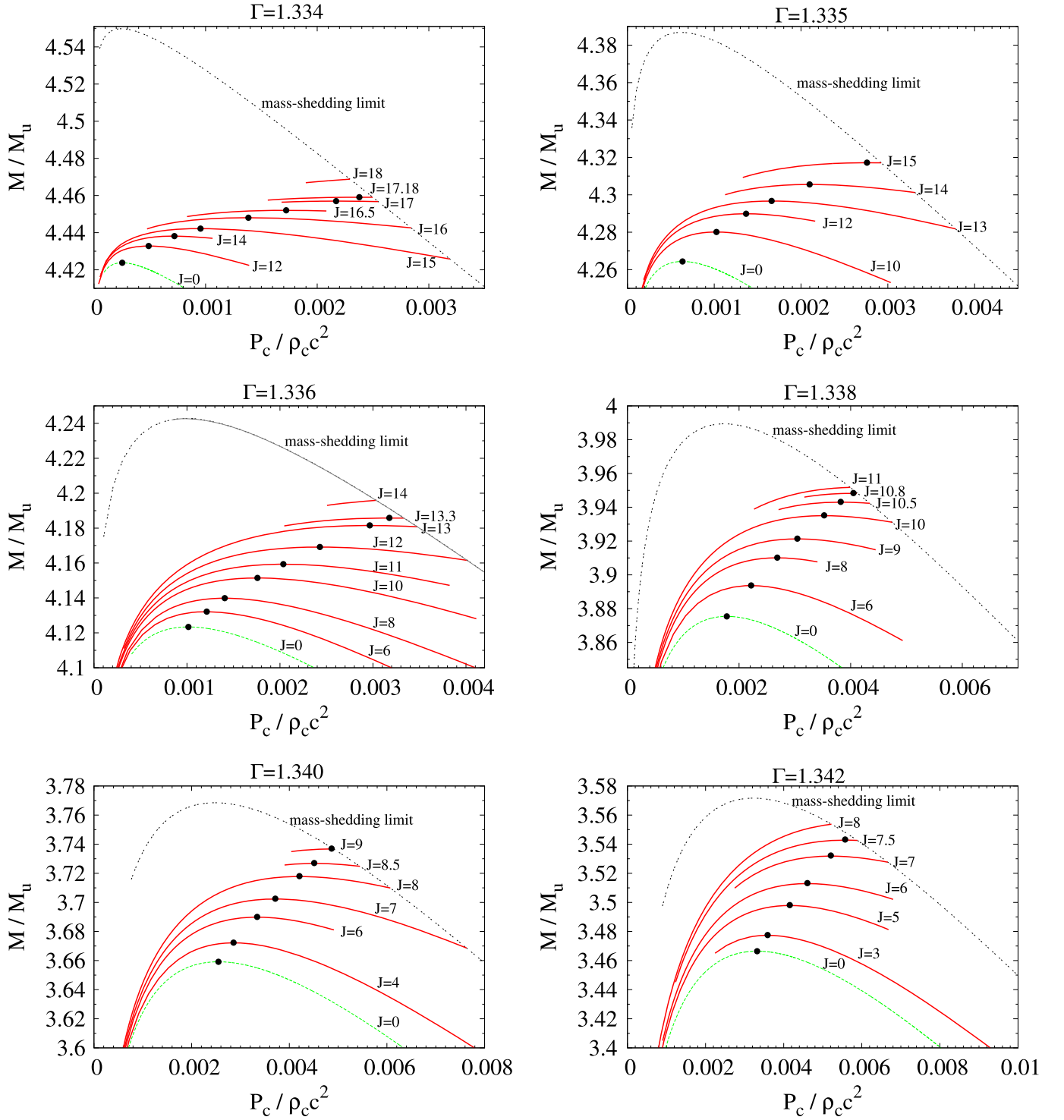
Figure 1. Gravitational mass M as a function of $P_c/(\rho_c c^2)$ for fixed values of J (solid curves; and dotted curve for $J = 0$) and for sequences of rotating stars at the mass-shedding limits (dotted–dotted curves) for $\Gamma = 1.334$, 1.335, 1.336, 1.338, 1.340, and 1.342. The units of the mass and angular momentum are $M_u = K^{n_p/2}G^{-3/2}c^{3-n_p}$ and $K^{n_p}G^{-2}c^{5-2n_p}$, respectively. The filled circles denote the turning points along the several curves of constant angular momentum.

$P_c/(\rho_c c^2) \rightarrow 0$), the relation can be approximated by

$$\frac{GM}{c^2 R_e} \approx 1.1705 \frac{P_c}{\rho_c c^2}. \qquad (30)$$

As already mentioned, the factor 1.1705 is derived from the analysis of $n_p = 3$ spherical polytropes in Newtonian gravity.

For $\beta \gtrsim 0.001$ the linear relation is not satisfied. For the limit that β approaches the maximum value of β_{max} (i.e., for the largest values of $P_c/(\rho_c c^2)$), the ratio of $GM/(c^2 R_e)$ to $P_c/(\rho_c c^2)$ approaches $1.1705 \times 2/3$ (see the dashed line of Figure 3). This stems from the fact that at the mass-shedding limit the ratio of the polar axial length to the equatorial circumferential
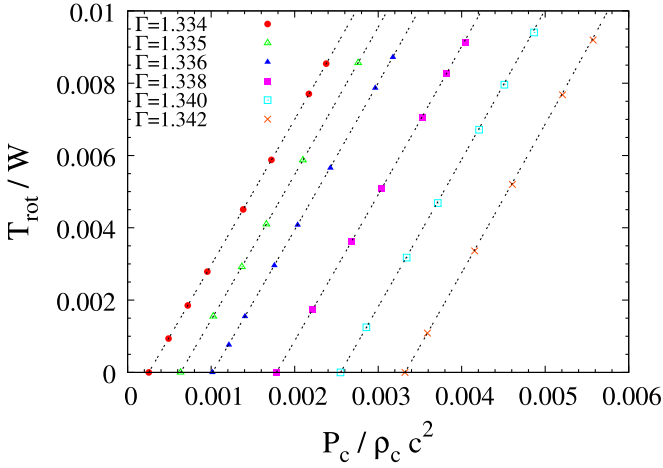
Figure 2. Relation between β ($=T_{\rm rot}/W$) and $P_c/(\rho_c c^2)$ for marginally stable stars with $\Gamma = 1.334$, 1.335, 1.336, 1.338, 1.340, and 1.342. Each dotted–dotted curve denotes the fitting formula of Equation (28). For each value of Γ, SMSs located in the right-hand side of this curve are unstable to general relativistic quasi-radial collapse.
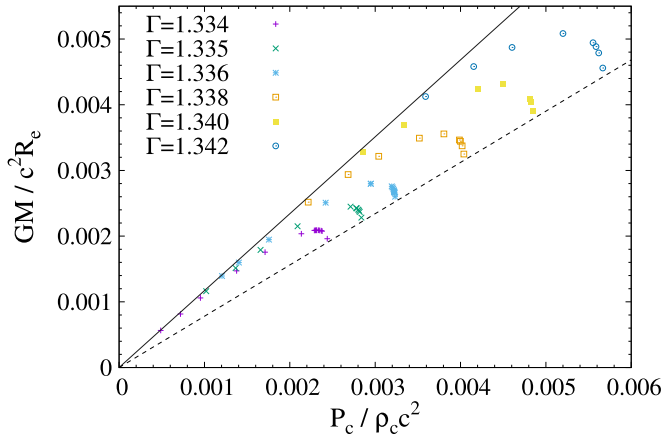

Figure 3. Relation between $GM/(c^2 R_{\rm e})$ and $P_c/(\rho_c c^2)$ for marginally stable SMSs. The solid line is $GM/(c^2 R_{\rm e}) = 1.1705 P_c/(\rho_c c^2)$, which is satisfied for the $n_{\rm p} = 3$ polytrope in Newtonian gravity. The dashed line shows the relation of $GM/(c^2 R_{\rm e}) = (2/3) \times 1.1705 P_c/(\rho_c c^2)$. For each curve of a fixed value of Γ the values of $P_c/(\rho_c c^2)$ increase with the increase of β and the data approach the dashed line (go away from the solid line).

radius is approximately 2/3 depending very weakly on the value of $n_{\rm p}$.

Figure 3 shows that for $\Gamma = 1.334$, the compactness of the SMSs at the turning point increases by a factor of ∼7 from the spherical to the rotating SMSs at the mass-shedding limit. This factor decreases with the increase of Γ. However, even for $\Gamma = 1.335$–1.336 (these could be the typical values for a realistic SMS) this increase factor is ≈2.3–3.3. Therefore, the condition for the onset of the general relativistic quasi-radial instability of rotating SMSs is significantly different from that for spherical SMSs. This fact has to be taken into account for exploring the formation process of SMBHs after the collapse of rotating SMSs.

Finally we approximately determine the condition for the onset of general relativistic quasi-radial instability for realistic SMSs. Specifically, the condition is imposed to the required minimum mass for getting the instability. Using Equation (14)

y is written as

$$y \approx 2.9 \times 10^{-3} \left(\frac{T}{10^{8.2}\,\rm K}\right)\left(\frac{C_{n_{\rm p}}}{C_{3,s}}\right)^{-1/2}\left(\frac{M}{10^5 M_\odot}\right)^{1/2}$$
$$= A \hat{C}_{n_{\rm p}}^{-1/2} M_5^{1/2}, \tag{31}$$

where $M_5 = M/10^5 M_\odot$ and $\hat{C}_{n_{\rm p}} = C_{n_{\rm p}}/C_{3,s}$. Substituting this equation and Equation (12) into Equation (28) and neglecting higher-order terms in y and β, we obtain the equation for M_5 as

$$A\hat{C}_{n_{\rm p}}^{-1} M_5 - \frac{2}{3}\beta\hat{C}_{n_{\rm p}}^{-1/2} M_5^{1/2} - B = 0, \tag{32}$$

where

$$B = 3.8 \times 10^{-3}\left(\frac{Y_T}{1.69}\right). \tag{33}$$

Then, the solution for $M_5^{1/2}$ is

$$M_5^{1/2} = \frac{\hat{C}_{n_{\rm p}}^{1/2}}{3A}\left(\beta + \sqrt{\beta^2 + 9AB}\right). \tag{34}$$

If M_5 is larger than this value SMSs are unstable. Here $\hat{C}_{n_{\rm p}}$ depends only weakly on M and β as already mentioned. Its plausible range is between 0.8 and 1.0; a more specific value can be obtained from Equation (12) and Table 1.

For $\beta = 0$ with the plausible parameters for the SMS core in the hydrogen-burning phase, the threshold is

$$M_5 = \frac{B}{A} \approx 1.3\left(\frac{T}{10^{8.2}\,\rm K}\right)^{-1}\left(\frac{Y_T}{1.69}\right)\hat{C}_{n_{\rm p}}. \tag{35}$$

By contrast for $\beta = 0.009$ (i.e., near the mass-shedding limit), $T = 10^{8.2}$ K, and $Y_T = 1.69$, $M_5 \approx 6.6\hat{C}_{n_{\rm p}}$. Thus, for obtaining unstable and maximally rotating SMSs the mass has to be increased by a factor of ∼5 from that of the spherical SMSs at the marginally stable point. Even for $\beta = 0.005$ we obtain $M_5 \approx 3.4\hat{C}_{n_{\rm p}}$ and hence, significant mass increase is necessary to get an unstable SMS.

As touched on in Section 1, SMSs could be formed in a high mass accretion environment. The often referred to highest mass accretion rate is ∼$0.1 M_\odot$ yr^{-1}. The lifetime of the SMSs, which should shine approximately at the Eddington limit, would be universally ∼2×10^6 years. This suggests that the typically final SMS mass would be at most ∼$2 \times 10^5 M_\odot$. Our present analysis indicates that if they were appreciably rotating, the SMSs would not collapse to a SMBH in their hydrogen-burning phase.

For a helium-burning phase and for $\beta = 0$, $T \approx 10^{8.4}$ K, and $Y_T \approx 0.75$, we obtain $M_5 \approx 0.37\hat{C}_{n_{\rm p}}$ while for $\beta = 0.009$, $T \approx 10^{8.4}$ K, and $Y_T \approx 0.75$ we obtain $M_5 \approx 2.4\hat{C}_{n_{\rm p}}$. Thus, the mass of marginally stable and maximally rotating SMSs has to be by a factor of ∼6 larger than that of marginally stable spherical SMSs. For the non-rotating case, general relativistic quasi-radial collapse can be induced even for $M_5 \sim 0.5$ (Chen et al. 2014). However, for the appreciably rotating case it will not collapse by the general relativistic instability in this phase.

For the oxygen-burning phase the expected values are $T \approx 10^{9.0}$ K and $Y_T \approx 0.56$. Then for $\beta = 0$, $M_5 \approx 0.07\hat{C}_{n_{\rm p}}$, and $\beta = 0.009$, we obtain $M_5 \approx 0.22\hat{C}_{n_{\rm p}}$. Thus, for this case an

SMS core of relatively small core mass can be unstable to general relativistic quasi-radial collapse.

4. PREDICTING THE FINAL OUTCOME

The marginally stable SMSs determined in Section 3 are plausible initial conditions for the collapse of SMS cores to a seed of an SMBH. Following Shibata & Shapiro (2002) and Shibata (2004) we predict the remnant of the collapse in the reasonable assumptions that (i) the collapse proceeds in an axisymmetric manner, (ii) the viscous angular-momentum transport during the dynamical collapse is negligible, and (iii) heating effects never halt the collapse. The numerical analysis is carried out in the same manner as that of Shibata (2004). We note that assumption (iii) is not justified for the case of an extremely high CNO abundance in a hydrogen-burning SMS (Montero et al. 2012) or for special SMS mass (Chen et al. 2014). We also note that in the presence of steeply differential rotation the centrifugal force could halt the collapse (Reisswig et al. 2013), although for the rigidly rotating case we would not have to consider this possibility (Shibata & Shapiro 2002; Liu et al. 2007; Montero et al. 2012).

Since viscosity is assumed to be negligible during the collapse, the specific angular momentum j of each fluid element is conserved in axisymmetric systems. Here j is defined by

$$j \equiv h u_\varphi, \qquad (36)$$

and it increases with the cylindrical radius for the SMSs of $\Omega = $ constant.

Next we define rest-mass distribution as a function of j, $m_*(j)$, as

$$m_*(j_0) \equiv 2\pi \int_{j < j_0} \rho u^t B e^{2\zeta - 2\nu} r^2 dr d(\cos\theta). \qquad (37)$$

Here the integration is performed only for the elements with $j < j_0$ for a given value of j_0. In addition we define the total angular momentum with the specific angular momentum less than a given value of j_0:

$$J(j_0) = 2\pi \int_{j < j_0} \rho h u^t u_\varphi B e^{2\zeta - 2\nu} r^2 dr d(\cos\theta). \qquad (38)$$

Then we assume that a seed black hole is formed during the collapse and it dynamically grows with the subsequent infall of ambient matter. For its growth process we consider an innermost stable circular orbit (ISCO) in the equatorial plane around the growing black hole at the center. We then assume that the formed black hole grows sequentially, capturing fluid elements from lower values of j. It is natural to consider that if j of a fluid element is smaller than the value at this ISCO ($j_{\rm ISCO}$) of an instantaneous black hole, the element will fall into the black hole eventually (as long as $j_{\rm ISCO}$ increases with the black hole growth).

To determine $j_{\rm ISCO}$ we assume that at each moment of the black hole growth the instantaneous mass and angular momentum are approximated by $m_*(j)$ and $J(j)$ with its dimensionless spin parameter $q_*(j) \equiv J(j)/m_*(j)^2$. Note that the baryon rest mass of SMSs is nearly equal to the gravitational mass because of their soft equations of state with $n_{\rm p} \approx 3$ and weak general relativistic correction. If we further assume that the spacetime can be approximated instantaneously by a Kerr metric, we can compute $j_{\rm ISCO}$ (Bardeen et al. 1972; chapter 12 of Shapiro & Teukolsky 1983). The value of $j_{\rm ISCO}$ changes as the black hole grows. If $j_{\rm ISCO}$ increases, additional

mass may fall into the black hole. However, if $j_{\rm ISCO}$ decreases, ambient fluid that has $j > j_{\rm ISCO}$ will no longer be captured. This expectation suggests that when $j_{\rm ISCO}$ reaches a maximum value, the dynamical growth of the black hole should have already terminated. Thus, by this consideration we can determine the possible maximum mass of the black hole. In reality the growth of the black hole may be terminated before reaching the maximum of $j_{\rm ISCO}$. Because the mass infall is possible only for the case of $j < j_{\rm ISCO}$ at each instantaneous time, the mass accretion would terminate if the point for $j = j_{\rm ISCO}$ is reached. In the following we consider these two possibilities as in Shibata (2004).

To illustrate the models for the growth of the black hole mass and dimensionless spin, in Figure 4 we plot $j_{\rm ISCO}[m_*, q_*]/M_*$ and q_* (a) for $(\Gamma, J) = (1.334, 17.35)$, $(1.335, 15.20)$, $(1.336, 13.48)$, $(1.338, 10.89)$, $(1.340, 9.02)$, and $(1.342, 7.63)$ and (b) for $\Gamma = 1.335$ and $J = 10, 12, 13$, and 15.2 $(\beta \times 10^3 = 1.54, 2.92, 4.09$, and $8.98)$. Here $M_*(\approx M)$ denotes the total rest mass of the SMSs. We choose the SMSs close to marginally stable states at the mass-shedding limit for (a), while for (b) the degree of the rotation is chosen for a wide range. For the models shown in Figure 4(a) the maximum of $j_{\rm ISCO}$ is reached at $m_*/M_* \approx 0.935, 0.948, 0.957$, $0.970, 0.978$, and 0.983 for $\Gamma = 1.334, 1.335, 1.336, 1.338$, 1.340, and 1.342, respectively (*circles* of Figure 4) while the condition of $j = j_{\rm ISCO}$ is satisfied at $m_*/M_* \approx 0.929, 0.943$, $0.954, 0.968, 0.977$, and 0.982 for $\Gamma = 1.334, 1.335, 1.336$, $1.338, 1.340$, and 1.342 respectively (*crosses* of Figure 4). After the maximum of $j_{\rm ISCO}/M_*$ is reached $j_{\rm ISCO}/M_*$ steeply decreases. Thus, once the black hole reaches this point it will stop entirely growing dynamically. The dynamical growth may be stopped when the point of $j = j_{\rm ISCO}$ is reached as already mentioned. However, m_* for this point is only slightly smaller than that at the maximum of $j_{\rm ISCO}$. This suggests that the dynamical growth will be stopped near the maximum of $j_{\rm ISCO}$. In any case, more than 90% (up to ≈98%) of the SMS matter will fall into a SMBH dynamically (i.e., on a timescale much shorter than dissipation and angular-momentum transport timescales). Lower panels of Figure 4(a) show that $q_* \approx 0.709, 0.681, 0.658, 0.618, 0.586$, and 0.560 at the maximum of $j_{\rm ISCO}$ and $q_* \approx 0.700, 0.675, 0.652, 0.614, 0.584$, and 0.559 at $j = j_{\rm ISCO}$ for $\Gamma = 1.334, 1.335, 1.336, 1.338$, 1.340, and 1.342, respectively. This suggests that the SMBHs formed after the dynamical collapse will not be rapidly rotating.

Figure 4(b) shows that with the decrease of J (and β) the value of m_* at the maximum of $j_{\rm ISCO}$ approaches M_*. However, even for $\beta = O(10^{-3})$, m_* is ≈$0.99M_*$. This property holds irrespective of Γ with its plausible values for SMSs. The dimensionless spin of the formed black hole also decreases with the decrease of J, although the remnant black hole is still likely to be rotating with moderate spin even for the models with $\beta = O(10^{-3})$.

Because a fraction of SMS matter does not fall directly into the SMBH, after the dynamical collapse, a system of a black hole surrounded by disks or tori in a dynamical state likely will be formed. Subsequent evolution of this system will be determined primarily by viscous effects in the accretion disks/tori. The analysis here suggests that disk/torus mass is less than 10% of the initial SMS mass even for maximally rotating SMS models, and for $\beta = O(10^{-3})$, it is ~1%. However, this does not mean that the effect of the disk/torus would be minor, because their mass could be 10^3–$10^5 M_\odot$ due to
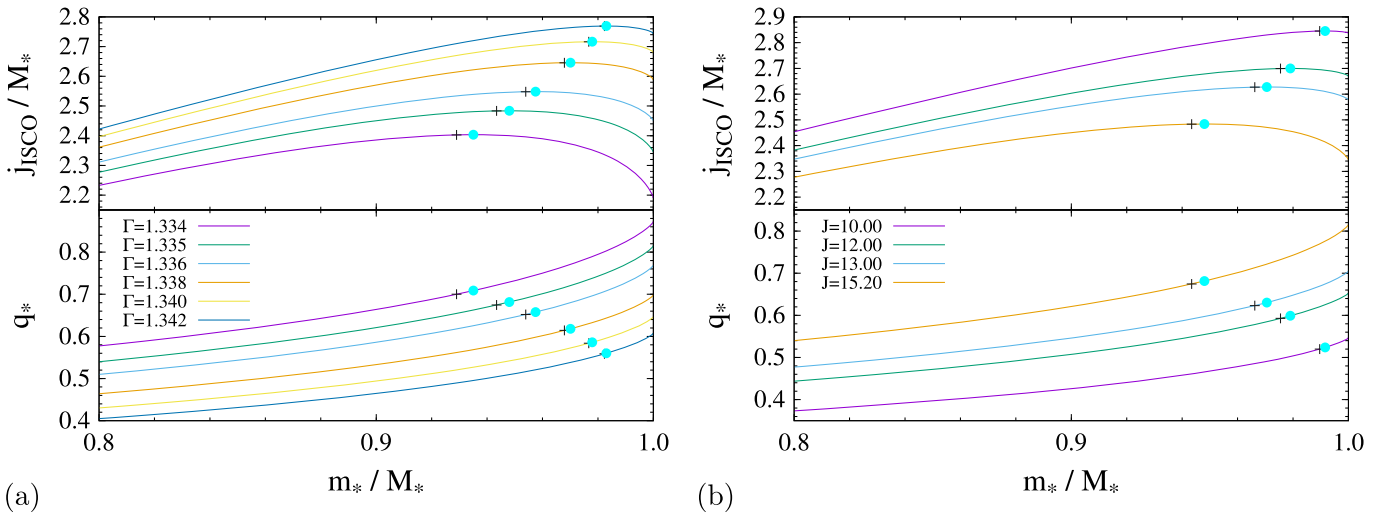
Figure 4. (a) j_{ISCO}/m_* and q_* as functions of m_*/M_* for marginally stable SMSs with $(\Gamma, J) = (1.334, 17.35)$, $(1.335, 15.20)$, $(1.336, 13.48)$, $(1.338, 10.89)$, $(1.340, 9.02)$, and $(1.342, 7.63)$. Here the units of J are $K^{n_p}G^{-2}c^{5-2n_p}$. The filled circles denote the maxima of j_{ISCO}/m_*. The crosses denote the points at which $j = j_{ISCO}$ is satisfied. (b) The same as the upper panel but for $\Gamma = 1.335$ and $J = 10, 12, 13,$ and 15.2 ($\beta \times 10^3 = 1.54, 2.92, 4.09,$ and 8.98).

the fact that the mass of the progenitor SMS core would be quite large as 10^4–$10^6 M_\odot$. Exploring the possible signals from such a high-mass disk/torus by a numerical-relativity simulation is an interesting subject in the future (see Liu et al. 2007 for a previous effort).

5. SUMMARY AND DISCUSSION

We studied the secular stability of rigidly rotating SMSs against quasi-radial collapse in general relativity. We showed that the stability condition of SMSs depends appreciably on the ratio of the rotational kinetic energy to the gravitational potential energy (denoted by β).

Our result implies that for the onset of general relativistic quasi-radial collapse of rapidly rotating SMSs in the hydrogen-burning phase high SMS mass is necessary. In particular, for the possible maximum value of $\beta \sim 0.009$ the mass required for the instability is by a factor of $\gtrsim 5$ larger than that for the spherical SMSs. Since the SMSs shine approximately at the Eddington limit and their lifetimes would be universally $\sim 2 \times 10^6$ years irrespective of their masses, quite a high mass accretion rate would be necessary to produce such high-mass rotating SMSs (e.g., for an SMS of the mass $6 \times 10^5 M_\odot$ the accretion rate has to be at least $0.3 M_\odot$ yr^{-1} and in reality a much higher accretion rate would be necessary because all of the accreted matter does not form the SMS core). This suggests that rapidly rotating SMSs that collapse to SMBHs via general relativistic quasi-radial instability would be rarer than non-rotating or slowly rotating SMSs: Although rapidly rotating SMSs could be formed, few of them would collapse to SMBHs via general relativistic quasi-radial instability in the hydrogen-burning phase. Rather, most of them will evolve as a result of hydrogen and helium burning and eventually the formed oxygen core could collapse to a SMBH via the general relativistic instability or pair instability as in the less massive stars (Bond et al. 1984).

If a large fraction of SMSs are rapidly rotating, the mass of a seed of SMBHs thus formed from an oxygen core could be much smaller than $10^5 M_\odot$, and the seed SMBHs would be initially be surrounded by a huge amount of matter (in the

absence of significant mass loss during the nuclear-burning phases). A fraction of the ambient matter will subsequently accrete onto the central SMBH and surrounding the disk/torus. Exploring this infall and growth phase of the SMBH is an interesting subject for the future. In particular, exploring the resulting signals is an important subject because they could bring information for the SMBH formation process (e.g., Matsumoto et al. 2015).

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