Gravitational waves from a particle orbiting around a rotating black hole: Post-Newtonian expansion

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Using the Teukolsky and Sasaki-Nakamura equations for the gravitational perturbation of Kerr spacetime, we calculate the post-Newtonian expansion of the energy and angular momentum luminosities of gravitational waves from a test particle orbiting around a rotating black hole up through post$^{5/2}$-Newtonian (P$^{5/2}$N) order beyond the quadrupole formula. We apply a method recently developed by Sasaki to the case of a rotating black hole. We take into account a small inclination of the orbital plane to the lowest order of the Carter constant. The result to P$^{3/2}$N order is in agreement with a similar calculation by Poisson as well as with the standard post-Newtonian calculation by Kidder, Will, and Wiseman. Using our result, we calculate the integrated phase of gravitational waves from a neutron-star–neutron-star binary and a black-hole–neutron-star binary during their inspiral stage. We find that, in both cases, spin-dependent terms in the P^{2}N and P$^{5/2}$N corrections are important to construct effective template waveforms which will be used for future laser-interferometric gravitational wave detectors.

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I. INTRODUCTION

The last stage of an inspiraling compact binary such as binary neutron stars is one of the promising sources of gravitational waves for the near-future laser-interferometric detectors such as Laser Interferometric Gravitational Wave Observatory (LIGO) [1] and VIRGO [2]. The main reasons why it is most promising are that the event rate is expected to be ~ 3 events/yr within 200 Mpc from a statistical study [3] and that it is possible to theoretically predict the amplitude of gravitational waves with good accuracy. Such a binary is, moreover, not only a strong source of gravitational waves, but also has a possibility to become a treasury of physics of neutron stars [4], cosmology [5], theories of gravity [6], etc., provided we obtain the data about binaries such as masses, spins, distance to the earth, and so on. However, to obtain those data with sufficient accuracy, it is necessary to construct theoretical template waveforms whose phasing has a fractional accuracy of less than 10^{-4} [4]. Hence considerable efforts have been made to construct such theoretical templates [7–15].

To calculate gravitational waves from an inspiraling compact binary, the standard method employed is the post-Newtonian expansion (PNE) of the Einstein equations [7], in which the equations are expanded in terms of a small parameter $v \sim (M/r)^{1/2}$, where M and r are the total mass and orbital length scale of the system, respectively. Despite much effort, however, calculations have been successful to only a few orders in v beyond the leading (Newtonian) order so far. More fundamentally, the nature of PNE has not been clarified due to its complexity: nobody knows the convergence property of PNE nor the validity of the polynomial expansion in v [16]. Given this situation, it is highly desirable to have a method which is complementary to the standard PNE. The perturbative study of a black hole spacetime is one of such, in which we consider gravitational waves radiated by a particle of mass $\mu \ll M$ orbiting around a black hole. This method, though restricted to the case of $\mu \ll M$, is very powerful because we can calculate fully general relativistic corrections of gravitational waves by means of relatively simple analyses. It is then fairly straightforward to evaluate the post-Newtonian corrections which are to be calculated in the standard PNE. This direction of research was first done analytically by Poisson [8] to post$^{3/2}$-Newtonian (P$^{3/2}$N) order and numerically by Cutler *et al.* [9] to P^{3}N order. Then a highly accurate numerical calculation was done by Tagoshi and Nakamura [14] to P^{4}N order and an analytical calculation to the same order was done by Tagoshi and Sasaki [15] which confirmed the result of Tagoshi and Nakamura. In particular, the appearance of $\ln v$ terms in the energy flux at $O(v^6)$ (P^{3}N order) and at $O(v^8)$ (P^{4}N order) is confirmed and it is clarified that the accuracy of the energy flux to at least P^{3}N order is necessary to construct template waveforms for optimal use of the interferometric data.

In this paper, we extend the analysis of Tagoshi and Sasaki [15] to the case of circular orbits around a rotating black hole to see the effect of spin. Some analytical [10,12] and numerical analyses [11,13] of its effect at its leading order (i.e., P$^{3/2}$N order) have been performed. They found that it gives rise to a large error in the integrated phase of gravitational waves from inspiraling binaries for a typical value of the spin angular momentum if it is ignored in the template. The large effect of the $\mathrm{P}^{3/2}\mathrm{N}$ order spin-orbit term causes us great anxiety about the effects of higher-order terms due to the spin angular momentum since the convergence property of the PNE has been found to be slow [4,14,15]. Hence we study the effect of spin to $\mathrm{P}^{5/2}\mathrm{N}$ order in this paper.

The paper is organized as follows. In Sec. II, we show the basic formulas to perform the PNE in our perturbative approach. First we show the PNE of the Teukolsky radial function [17] using the Sasaki-Nakamura equation [18]. We must also perform the PNE of the angular equation, which is given in Appendix D. Then we describe the PNE of the source terms. We consider circular orbits around a Kerr black hole, that is, those at constant Boyer-Lindquist radial coordinate. These circular orbits are, however, not necessarily on the equatorial plane and the emission rates of gravitational waves are different for different orbital inclination angles (or more precisely, different values of the Carter constant) [13]. To see the leading effect of the orbital inclination, we consider orbits with small inclination angles. In Sec. III, the energy and angular-momentum luminosities to $O(v^5)$ beyond Newtonian are derived. In Sec. IV, using the results given in Sec. III, we estimate errors in the accumulated phase of gravitational waves caused by the nonvanishing spin angular momentum. Section V is devoted to the summary. Throughout this paper we use the units of $c = G = 1$ and $GM_\odot/c^2 = 1.477$ km.

II. FORMULATION

To calculate gravitational radiation from a particle orbiting around a Kerr black hole, we start with the Teukolsky equation [17]. We focus on the radiation going out to infinity described by the fourth Newman-Penrose quantity, ψ_4 [19], which may be expressed as

$$\psi_4 = (r - ia\cos\theta)^{-4}\int d\omega e^{-i\omega t}\sum_{l,m}\frac{e^{im\varphi}}{\sqrt{2\pi}}\,{}_{-2}S^{a\omega}_{lm}(\theta)R_{lm\omega}(r) \;, \tag{2.1}$$

where ${}_{-2}S^{a\omega}_{lm}$ is the spheroidal harmonic function of spin weight $s = -2$, which is normalized as

$$\int_0^\pi |{}_{-2}S^{a\omega}_{lm}|^2 \sin\theta d\theta = 1 \;, \tag{2.2}$$

and λ is the eigenvalue. The radial function $R_{lm\omega}(r)$ obeys the Teukolsky equation with spin weight $s = -2$,

$$\Delta^2\frac{d}{dr}\left(\frac{1}{\Delta}\frac{dR_{lm\omega}}{dr}\right) - V(r)R_{lm\omega} = T_{lm\omega}(r) \;, \tag{2.3}$$

where $T_{lm\omega}(r)$ is the source term whose explicit form will be shown later, and $\Delta = r^2 - 2Mr + a^2$. The potential $V(r)$ is given by

$$V(r) = -\frac{K^2 + 4i(r-M)K}{\Delta} + 8i\omega r + \lambda \;, \tag{2.4}$$

where $K = (r^2 + a^2)\omega - ma$.

The solution of the Teukolsky equation at infinity ($r \to \infty$) is expressed as

$$\begin{aligned} R_{lm\omega}(r) &\to \frac{r^3 e^{i\omega r^*}}{2i\omega B^{\rm in}_{lm\omega}}\int_{r_+}^\infty dr'\frac{T_{lm\omega}(r')R^{\rm in}_{lm\omega}(r')}{\Delta^2(r')} \\ &\equiv \tilde{Z}_{lm\omega} r^3 e^{i\omega r^*} \;, \end{aligned} \tag{2.5}$$

where $r_+ = M + \sqrt{M^2 - a^2}$ denotes the radius of the event horizon and $R^{\rm in}_{lm\omega}$ is the homogeneous solution which satisfies the ingoing-wave boundary condition at the horizon,

$$R^{\rm in}_{lm\omega} \to \begin{cases} D_{lm\omega}\Delta^2 e^{-ikr^*}, & r^* \to -\infty \;, \\ r^3 B^{\rm out}_{lm\omega}e^{i\omega r^*} + r^{-1}B^{\rm in}_{lm\omega}e^{-i\omega r^*}, & r^* \to +\infty \;, \end{cases} \tag{2.6}$$

where $k = \omega - ma/2Mr_+$ and r^* is the tortoise coordinate defined by

$$\frac{dr^*}{dr} = \frac{r^2 + a^2}{\Delta} \;. \tag{2.7}$$

For definiteness, we fix the integration constant such that r^* is given explicitly by

$$\begin{aligned} r^* &= \int \frac{dr^*}{dr}dr \\ &= r + \frac{2Mr_+}{r_+ - r_-}\ln\frac{r - r_+}{2M} - \frac{2Mr_-}{r_+ - r_-}\ln\frac{r - r_-}{2M} \;, \end{aligned} \tag{2.8}$$

where $r_\pm = M \pm \sqrt{M^2 - a^2}$.

Thus in order to calculate gravitational waves emitted to infinity from a particle in circular orbits, we need to know the explicit form of the source term $T_{lm\omega}(r)$, which has support only at $r = r_0$ where r_0 is the orbital radius in the Boyer-Lindquist coordinate, the ingoing-wave Teukolsky function $R^{\rm in}_{lm\omega}(r)$ at $r = r_0$, and its incident amplitude $B^{\rm in}_{lm\omega}$ at infinity. We consider the expansion of these quantities in terms of a small parameter, $v^2 \equiv M/r_0$. Note that v is approximately equal to the orbital velocity, but not strictly equal to it in the case of $a \neq 0$.

In addition to these, we need to expand the spheroidal harmonics and their eigenvalues in powers of $a\omega$. Since $\omega = O(\Omega)$ where Ω is the orbital angular velocity of the particle, we have $a\omega = O(M\omega) = O(v^3)$. Thus the expansion in powers of $a\omega$ is also a part of PNE. Note also that the spin parameter of the black hole a does not have to be small but can be of order M. We defer this expansion of the spheroidal harmonics to Appendix D.

In the following, we consider the PNE of the ingoing-wave Teukolsky function and the source term separately. Since the angular eigenvalues λ come into play in the radial equation, we set

$$\lambda = \lambda_0 + a\omega\lambda_1 + a^2\omega^2\lambda_2 + O(v^9) \;, \tag{2.9}$$

where $\lambda_0 = (l-1)(l+2)$ and $\lambda_1 = -2m(l^2+l+4)/(l^2+l)$ have been obtained previously [20], and λ_2 is derived in Appendix D.

A. Homogeneous solution

The PNE of the Teukolsky equation for a rotating black hole was first performed by Poisson up to $O(v^3)$ beyond the quadrupole formula [12]. In his method, he introduced a parameter $\epsilon \equiv 2M\omega = O(v^3)$ and expanded the equation in terms of it to the first order in ϵ, since only the homogeneous solution to $O(\epsilon)$ is needed to obtain the energy flux up to $O(v^3)$ level. The purpose here is to obtain the higher terms, $O(v^n)$ $(n > 3)$. Specifically in this paper we consider the gravitational wave luminosity to $O(v^5)$. Hence we need the homogeneous solution accurate up to $O(\epsilon^2)$. However, it is very difficult to treat the Teukolsky equation itself to obtain the higher-order corrections because the zeroth-order solutions (i.e., the kernel functions) already have a quite complicated form [12]. Hence instead of the Teukolsky equation, we use the Sasaki-Nakamura (SN) equation [18], which is obtained by a certain transformation of the Teukolsky equation. The reason to use it is that the SN equation is a generalization of the Regge-Wheeler (RW) equation for $a = 0$ to the case of $a \neq 0$ and hence has a more tractable structure. In addition, we can make use of algebraic formulas for the PNE of the RW equation developed by Poisson [8] and by Sasaki [21].

The SN equation has the form

$$\left[\frac{d^2}{dr^{*2}} - F(r)\frac{d}{dr^*} - U(r)\right] X_{lm\omega} = 0 , \tag{2.10}$$

where the explicit forms of F and U are given in Appendix A. It is obtained by the transformation from $R_{lm\omega}$ to $X_{lm\omega}$ as

$$R_{lm\omega} = \frac{1}{\eta}\left\{\left(\alpha + \frac{\beta_{,r}}{\Delta}\right)\chi_{lm\omega} - \frac{\beta}{\Delta}\chi_{lm\omega,r}\right\} , \tag{2.11}$$

where $\chi_{lm\omega} = X_{lm\omega}\Delta/(r^2+a^2)^{1/2}$, and the functions α, β, and η are shown in Appendix A. Conversely, we can express $X_{lm\omega}$ in terms of $R_{lm\omega}$ as

$$X_{lm\omega} = (r^2+a^2)^{1/2} r^2 J_- J_- \left[\frac{1}{r^2} R_{lm\omega}\right] , \tag{2.12}$$

where $J_- = (d/dr) - i(K/A)$. Then the ingoing-wave solution $X^{\rm in}_{lm\omega}$ which corresponds to $R^{\rm in}_{lm\omega}$ has the asymptotic behavior

$$X^{\rm in}_{lm\omega} \to \begin{cases} A^{\rm out}_{lm\omega} e^{i\omega r^*} + A^{\rm in}_{lm\omega} e^{-i\omega r^*}, & r^* \to \infty , \\ C_{lm\omega} e^{-ikr^*}, & r^* \to -\infty , \end{cases} \tag{2.13}$$

where $A^{\rm in}_{lm\omega}$, $A^{\rm out}_{lm\omega}$, and $C_{lm\omega}$ are, respectively, related to $B^{\rm in}_{lm\omega}$, $B^{\rm out}_{lm\omega}$, and $D_{lm\omega}$ defined in Eq. (2.6) as

$$\begin{aligned} B^{\rm in}_{lm\omega} &= -\frac{1}{4\omega^2} A^{\rm in}_{lm\omega} , \\ B^{\rm out}_{lm\omega} &= -\frac{4\omega^2}{c_0} A^{\rm out}_{lm\omega} , \\ D_{lm\omega} &= \frac{1}{d_{lm\omega}} C_{lm\omega} , \end{aligned} \tag{2.14}$$

where c_0 is given in Eq. (A3) of Appendix A and

$$\begin{aligned} d_{lm\omega} = \sqrt{2Mr_+}[&(8 - 24iM\omega - 16M^2\omega^2)r_+^2 \\ &+(12iam - 16M + 16amM\omega + 24iM^2\omega)r_+ \\ &-4a^2m^2 - 12iamM + 8M^2] . \end{aligned} \tag{2.15}$$

Now let us consider the PNE of $X^{\rm in}_{lm\omega}$. It is characterized by the ingoing-wave boundary condition at the horizon, $X^{\rm in}_{lm\omega} \sim e^{-ikr^*}$ as $r^* \to -\infty$. However, since r^* cannot be expanded in terms of ϵ at $r^* \to -\infty$, a naive expansion of the SN equation in terms of ϵ would obscure the boundary condition at the horizon. One prescription to circumvent this difficulty has been suggested by Sasaki [21] in the case of the Schwarzschild black hole, namely, to separate out the factor $e^{-\omega(r^*-r)}$ from $X^{\rm in}_{lm\omega}$ from the beginning. Here we generalize this method to the case of the Kerr black hole.

First we introduce the variable $z = \omega r$ and

$$\begin{aligned} z^* &= z + \epsilon\left[\frac{z_+}{z_+ - z_-}\ln(z - z_+) \right. \\ &\qquad \left. - \frac{z_-}{z_+ - z_-}\ln(z - z_-)\right] \\ &= \omega r^* + \epsilon \ln\epsilon , \end{aligned} \tag{2.16}$$

where $z_\pm = \omega r_\pm$. For later convenience, we also introduce a nondimensional parameter $q = a/M$. Hence, for example, $a\omega = \frac{1}{2}q\epsilon$, $z_\pm = \frac{1}{2}\epsilon(1 \pm \sqrt{1-q^2})$. As mentioned before, q is not necessarily very small but can be of order unity.

Next we define a function $\phi(z)$ as

$$\begin{aligned} \phi(z) &= \int dr \left(\frac{K}{\Delta} - \omega\right) \\ &= z^* - z - \frac{\epsilon}{2} mq \frac{1}{z_+ - z_-} \ln\frac{z - z_+}{z - z_-} , \end{aligned} \tag{2.17}$$

which generalizes the phase function $\omega(r^* - r)$ of the Schwarzschild case. Note that $e^{-i\phi(z)} \sim e^{-ikr^*}$ as $r^* \to -\infty$ and $\sim e^{-i\omega(r^*-r)}$ as $r^* \to +\infty$. Then we set

$$X^{\rm in}_{lm\omega} = \sqrt{z^2 + a^2\omega^2}\,\xi_{lm}(z)\exp[-i\phi(z)] . \tag{2.18}$$

By this prescription, it is easy to implement the ingoing-wave boundary condition on $X^{\rm in}_{lm\omega}$.

Inserting Eq. (2.18) into Eq. (2.10) and expanding it in terms of $\epsilon = 2M\omega$, we obtain

$$\begin{aligned} L^{(0)}[\xi_{lm}] = \epsilon L^{(1)}[\xi_{lm}] &+ \epsilon Q^{(1)}[\xi_{lm}] \\ &+\epsilon^2 Q^{(2)}[\xi_{lm}] + O(\epsilon^3), \end{aligned} \tag{2.19}$$

where $L^{(0)}$, $L^{(1)}$, $Q^{(1)}$, and $Q^{(2)}$ are differential operators given by

$$L^{(0)} = \frac{d^2}{dz^2} + \frac{2}{z}\frac{d}{dz} + \left(1 - \frac{l(l+1)}{z^2}\right) , \tag{2.20}$$

$$L^{(1)} = \frac{1}{z}\frac{d^2}{dz^2} + \left(\frac{1}{z^2} + \frac{2i}{z}\right)\frac{d}{dz} - \left(\frac{4}{z^3} - \frac{i}{z^2} + \frac{1}{z}\right) , \tag{2.21}$$

$$Q^{(1)} = \frac{iq\lambda_1}{2z^2}\frac{d}{dz} - \frac{4imq}{l(l+1)z^3} + \frac{4mq}{l(l+1)z^2} + \frac{\lambda_1 q + 2mq}{2z^2} , \tag{2.22}$$

for arbitrary l and

$$Q^{(2)} = -\frac{q^2}{4z^2}\frac{d^2}{dz^2} - \left(\frac{9q^2 - 2m^2q^2 - 6imq}{36z^3} + \frac{i(2m^2q^2 - 9q^2) - 19mq}{108z^2}\right)\frac{d}{dz} + \left(\frac{9q^2 + 5m^2q^2 - 3imq}{9z^4} + \frac{i(2m^2q^2 - 9q^2)}{54z^3} - \frac{-63imq + 81q^2 - 2m^2q^2}{378z^2}\right) , \tag{2.23}$$

for $l = 2$. We shall see that we do not need $Q^{(2)}$ for $l \geq 3$ for the post-Newtonian order we consider in this paper [i.e., up to $O(v^5)$ beyond Newtonian]. Expanding ξ_{lm} in terms of ϵ as

$$\xi_{lm} = \sum_{n=0}^{\infty} \epsilon^n \xi_{lm}^{(n)}(z) , \tag{2.24}$$

we obtain from Eq. (2.19) the iterative equations

$$L^{(0)}[\xi_{lm}^{(0)}] = 0 , \tag{2.25}$$

$$L^{(0)}[\xi_{lm}^{(1)}] = L^{(1)}[\xi_{lm}^{(0)}] + Q^{(1)}[\xi_{lm}^{(0)}] \equiv W_{lm}^{(1)} , \tag{2.26}$$

$$L^{(0)}[\xi_{lm}^{(2)}] = L^{(1)}[\xi_{lm}^{(1)}] + Q^{(1)}[\xi_{lm}^{(1)}] + Q^{(2)}[\xi_{lm}^{(1)}] \equiv W_{lm}^{(2)} . \tag{2.27}$$

The general solution to Eq. (2.25) is immediately obtained as

$$\xi_{lm}^{(0)} = \alpha_l^{(0)} j_l + \beta_l^{(0)} n_l . \tag{2.28}$$

Now we consider the boundary condition. The argument is parallel to that given in Ref. [21] for the Schwarzschild case. The condition that $X_{lm\omega}^{\rm in} \sim e^{-ikr^*}$ as $r^* \to -\infty$ implies $\sqrt{z^2 + \epsilon^2 q^2/4}\,\xi_{lm}(z)$ should be regular at $z = \omega r_+ \sim \epsilon$. Since ϵ can be made arbitrarily small, $\sqrt{z^2 + \epsilon^2 q^2/4}\,\xi_{lm}^{(n)}(z)$ should be no more singular than $O(z^{-n})$ at $z = 0$. Thus in particular we have $\xi_{lm}^{(0)} = \alpha_l^{(0)} j_l$. For convenience, we set $\alpha_l^{(0)} = 1$. Taking into account the behavior of the lowest-order solution, we then infer that $\sqrt{z^2 + \epsilon^2 q^2/4}\,\xi_{lm}^{(n)}(z)$ must be no more singular than z^{l+1-n} at $z = 0$. Hence for $n \leq 2$, the boundary condition is that $\xi_{lm}^{(n)}$ is regular at $z = 0$.

As noted previously, in the case of a circular orbit of radius r_0, the source term $T_{lm\omega}$ has support only at $r = r_0$ and $\omega r_0 = O(\Omega r_0) = O(v)$. Hence we only need $X_{lm\omega}^{\rm in}$ at $z = O(v) \ll 1$ to evaluate the source integral, apart from the value of the incident amplitude $A_{lm\omega}^{\rm in}$. Hence the PNE of $X_{lm\omega}^{\rm in}$ corresponds to the expansion not only in terms of $\epsilon = O(v^3)$ but also z by assuming $\epsilon \ll z \ll 1$. In order to evaluate the gravitational wave luminosity to $O(v^5)$ beyond the leading order, we must calculate the series expansion of $\xi_{lm}^{(n)}$ in powers of z for $n = 0$ to $l = 4$, for $n = 1$ to $l = 3$, and for $n = 2$ to $l = 2$ (see Appendix C). On the other hand, the accuracy of $A_{lm\omega}^{\rm in}$ we need for this purpose is $O(\epsilon)$ or $n \leq 1$. Thus we need the asymptotic behavior of $\xi_{lm}^{(n)}$ at infinity only for $n = 1$ (that for $n = 0$ is trivially obtained).

To calculate $\xi_{lm\omega}$ to the accuracy discussed above, we rewrite Eqs. (2.26) and (2.27) in the indefinite integral form by using the spherical Bessel functions j_l and n_l:

$$\xi_{lm}^{(n)} = n_l \int^z dz z^2 j_l W_{lm}^{(n)} - j_l \int^z dz z^2 n_l W_{lm}^{(n)} \qquad (n = 1, 2) . \tag{2.29}$$

The series expansion formulas for $\xi_{lm}^{(n)}$ around $z = 0$ are easily obtained if we know those for $W_{lm}^{(n)}$. Then it is straightforward to impose the boundary condition at $z = 0$. Further, if the integrations can be done exactly in closed analytic form, it is easy to extract out the values of the incident amplitude $A_{lm\omega}^{\rm in}$ by examining the asymptotic behavior at infinity of the thus obtained functions. As we shall immediately see, this is easily done for $n = 1$, which is sufficient for our purpose.

For $n = 1$, if we set $q = 0$, Eq. (2.26) becomes the same equation as that discussed in Ref. [21]. Hence the only correction is to include the contribution from the $Q^{(1)}[\xi_{lm}^{(0)}]$ term. Using the formulas developed in that paper, this is easily done to yield

$$\xi_{lm}^{(1)} = \alpha_l^{(1)} j_l + \frac{(l-1)(l+3)}{2(l+1)(2l+1)} j_{l+1} - \frac{l^2 - 4}{2l(2l+1)} j_{l-1} + z^2(n_l j_0 - j_l n_0) j_0 + \sum_{k=1}^{l-1}\left(\frac{1}{k} + \frac{1}{k+1}\right) z^2 (n_l j_k - j_l n_k) j_k + n_l(\mathrm{Ci}2z - \gamma - \ln 2z) - j_l \mathrm{Si}2z + i j_l \ln z + \frac{imq}{2}\left(\frac{l^2+4}{l^2(2l+1)}\right) j_{l-1} + \frac{imq}{2}\left(\frac{(l+1)^2+4}{(l+1)^2(2l+1)}\right) j_{l+1} , \tag{2.30}$$

where $\mathrm{Ci}(x) = -\int_x^\infty dt\, \cos t/t$ and $\mathrm{Si}(x) = \int_0^x dt\, \sin t/t$ are cosine and sine integral functions, γ is the Euler constant, and $\alpha_L^{(1)}$ is an integration constant which represents the arbitrariness of the normalization of $X^{\rm in}_{lm\omega}$. We set $\alpha_L^{(1)} = 0$ for simplicity.

In the actual calculation of the gravitational radiation to infinity, we need to know $X^{\rm in}_{lm\omega}(z)$ at $\epsilon \ll z \ll 1$. Using Eq. (2.17), we obtain the expansion of Eq. (2.18) as

$$\begin{aligned} X^{\rm in}_{lm\omega} &= \sqrt{z^2 + a^2\omega^2}\xi_{lm}\exp(-i\varphi) \\ &= z\xi^{(0)}_{lm} + \epsilon\left(z\xi^{(1)}_{lm} - \frac{imq}{2}\xi^{(0)}_{lm} - iz\xi^{(0)}_{lm}\ln z\right) \\ &\quad +\epsilon^2\Bigg[z\left(\xi^{(2)}_{lm} - i\xi^{(1)}_{lm}\ln z - \frac{1}{2}\xi^{(0)}_{lm}(\ln z)^2\right) \\ &\quad +i\xi^{(0)}_{lm} - \frac{i}{2}\xi^{(1)}_{lm}mq + \frac{1}{z}\xi^{(0)}_{lm}\left(-\frac{i}{4}mq + \frac{q^2}{8} - \frac{m^2q^2}{8}\right) - \frac{1}{2}\xi^{(0)}_{lm}mq\ln z\Bigg] . \end{aligned} \tag{2.31}$$

For $n = 2$, expanding Eq. (2.30) in terms of z and inserting it into Eq. (2.29), we have the series expansion of $\xi^{(2)}_{lm}$. Inserting it into Eq. (2.31), we obtain

$$\begin{aligned} X^{\rm in}_{2m\omega} &= \frac{z^3}{15} - \frac{z^5}{210} + \frac{z^7}{7560} + O(z^9) + \epsilon\left(\frac{imqz^2}{30} - \frac{13z^4}{630} - \frac{11imqz^4}{3780} + O(z^6)\right) \\ &\quad +\epsilon^2\left(\frac{q^2 + 2imq - m^2q^2}{120}z - \frac{mz}{30}z^2 + O(z^3)\right) , \end{aligned} \tag{2.32}$$

$$X^{\rm in}_{3m\omega} = \frac{z^4}{105} - \frac{z^6}{1890} + O(z^8) + \epsilon\left(-\frac{z^3}{126} + \frac{2imq}{945}z^3 + O(z^5)\right) , \tag{2.33}$$

$$X^{\rm in}_{4m\omega} = \frac{z^5}{945} + O(z^7) . \tag{2.34}$$

Once we have the homogeneous solutions of the SN equation, we have only to perform the transformation (2.11) to obtain the corresponding solutions of the Teukolsky equation. The results are

$$\begin{aligned} \omega R^{\rm in}_{2m\omega} &= \frac{z^4}{30} + \frac{i}{45}z^5 - \frac{11z^6}{1260} - \frac{i}{420}z^7 + \frac{23z^8}{45\,360} + \frac{i}{11\,340}z^9 \\ &\quad +\epsilon\left(\frac{-z^3}{15} - \frac{i}{60}mqz^3 - \frac{i}{60}z^4 + \frac{mqz^4}{45} - \frac{41z^5}{3780} + \frac{277i}{22\,680}mqz^5 - \frac{31i}{3780}z^6 - \frac{7mqz^6}{1620}\right) \\ &\quad +\epsilon^2\left(\frac{z^2}{30} + \frac{i}{40}mqz^2 + \frac{q^2z^2}{60} - \frac{m^2q^2z^2}{240} - \frac{i}{60}z^3 - \frac{mqz^3}{30} + \frac{i}{90}q^2z^3 - \frac{i}{120}m^2q^2z^3\right) , \end{aligned} \tag{2.35}$$

$$\omega R^{\rm in}_{3m\omega} = \frac{z^5}{630} + \frac{i}{1260}z^6 - \frac{z^7}{3780} - \frac{i}{16\,200}z^8 + \epsilon\left(\frac{-z^4}{252} - \frac{i}{1890}mqz^4 - \frac{i}{756}z^5 + \frac{11mqz^5}{22\,680}\right) , \tag{2.36}$$

$$\omega R^{\rm in}_{4m\omega} = \frac{z^6}{11\,340} + \frac{iz^7}{28\,350} . \tag{2.37}$$

Here, it is worth noting that the terms linear in $q(= a/M)$ at $O(v^2)$ beyond the leading term in each $R^{\rm in}_{lm\omega}$ are pure imaginary. This implies there is no linear term in q at the P^{1}N order of the luminosity. Such terms will appear at the P$^{3/2}$N order, which is in fact what was found by Poisson [12]. Further, it is expected that terms linear in q will not appear at the P^{2}N order, but at the P$^{5/2}$N order. By the same argument, terms quadratic in q are expected to appear at the P^{2}N order, but not at the P$^{5/2}$N order.

Next, we consider $A^{\rm in}_{lm\omega}$ to $O(\epsilon) = O(v^3)$. The procedure is the same as that in the Schwarzschild case [21]. Using the relations $j_{l+1} \sim -j_{l-1} \sim (-1)^{l+1}n_{2n-l}$, etc., we obtain the asymptotic behavior of $\xi^{(1)}_{lm}$ at $z = \infty$ as

$$\xi^{(1)}_{lm} \sim -\frac{\pi}{2}j_l + \left(1^{(1)}_{lm} - \ln z\right)n_l + ij_l\ln z , \tag{2.38}$$

where

$$\begin{aligned} q^{(1)}_{lm} &= \frac{1}{2}\left[\psi(l) + \psi(l+1) + \frac{(l-1)(l+3)}{l(l+1)}\right] \\ &\quad -\ln 2 - \frac{2imq}{l^2(l+1)^2} , \end{aligned} \tag{2.39}$$

and $\psi(l)$ is the digamma function,

$$\psi(l) = \sum_{k=1}^{l-1}\frac{1}{k} - \gamma . \tag{2.40}$$

To consider the asymptotic form of $X^{\rm in}_{lm\omega}$, we set $\xi^{(1)}_{lm} = f^{(1)}_{lm} = f^{(1)}_{lm} + ij_l \ln z$ and express the asymptotic form of $f^{(1)}_{lm}$ at $z \to \infty$ as

$$\begin{aligned} f^{(1)}_{lm} &\to P^{(1)}_{lm} j_l + Q^{(1)}_{lm} n_l \\ &= \tfrac{1}{2}(P^{(1)}_{lm} - iQ^{(1)}_{lm}) h^{(1)}_l + \tfrac{1}{2}(P^{(1)}_{lm} + iQ^{(1)}_{lm}) h^{(2)}_l , \end{aligned}$$

where $h^{(1)}_l$ and $h^{(2)}_l$ are the spherical Hankel functions of the first and second kinds, respectively, which are given by

$$\begin{aligned} h^{(1)}_l(z) &= j_l(z) + in_l(z) \to (-i)^{l+1}\frac{e^{iz}}{z} , \\ h^{(2)}_l(z) &= j_l(z) - in_l(z) \to i^{l+1}\frac{e^{-iz}}{z} . \end{aligned} \tag{2.41}$$

From Eq. (2.38), we have

$$P^{(1)}_{lm} = -\frac{\pi}{2}, \quad Q^{(1)}_{lm} = q^{(1)}_{lm} - \ln z . \tag{2.42}$$

Then the asymptotic form of $X^{\rm in}_{lm\omega}$ becomes

$$\begin{aligned} X^{\rm in}_{lm\omega} &= \sqrt{z^2 + a^2\omega^2}\xi_{lm} \exp(-i\varphi) \\ &\sim z[j_l + \epsilon(f^{(1)}_{lm} + ij_l \ln z) + \cdots] \exp\left[-i\left(z^* - z - \frac{mq}{2\sqrt{1-q^2}}\ln\frac{z - z_+}{z - z_-}\right)\right] \\ &\sim \tfrac{1}{2}e^{-iz^*}(ze^{iz}h^{(2)}_{lm})\{1 + \epsilon[P^{(1)}_{lm} + i(Q^{(1)}_{lm} + \ln z)] + \cdots\} \\ &\quad + \tfrac{1}{2}e^{iz^*}(ze^{-iz}h^{(1)}_{lm})\{1 + \epsilon[P^{(1)}_{lm} - i(Q^{(1)}_{lm} + \ln z)] + \cdots\} . \end{aligned} \tag{2.43}$$

From this equation, $A^{\rm in}_{lm\omega}$ can be easily extracted out:

$$A^{\rm in}_{lm} = \frac{1}{2} i^{l+1} e^{-i\epsilon \ln\epsilon}\left[1 + \epsilon\left(-\frac{\pi}{2} + iq^{(1)}_{lm}\right) + \cdots\right] , \tag{2.44}$$

where we use the fact that $\omega r^* = z^* - \epsilon\ln\epsilon$ from our definition of z^*. Specifically for $l = 2$, 3, and 4, to the orders respectively required, we have

$$\begin{aligned} A^{\rm in}_{2m\omega} &= -\frac{i}{2}\left\{1 - \epsilon\frac{\pi}{2} + i\epsilon\left(\frac{5}{3} - \gamma - \ln 2\right) + \frac{mq}{18}\epsilon\right\} \\ &\quad + O(\epsilon^2) , \end{aligned} \tag{2.45}$$

$$\begin{aligned} A^{\rm in}_{3m\omega} &= \frac{1}{2}\left\{1 - \epsilon\frac{\pi}{2} + i\epsilon\left(\frac{13}{6} - \gamma - \ln 2\right) + \frac{mq}{72}\epsilon\right\} \\ &\quad + O(\epsilon^2) , \end{aligned} \tag{2.46}$$

$$A^{\rm in}_{4m\omega} = \frac{i}{2} + O(\epsilon) . \tag{2.47}$$

Finally, the corresponding incident amplitudes $B^{\rm in}_{lm\omega}$ for the Teukolsky function are obtained from Eq. (2.14).

B. The source term

A test particle obeys the equations of motion

$$\begin{aligned} \Sigma\frac{d\theta}{d\tau} &= \pm\left[C - \cos^2\theta\left\{a^2(1 - E^2) + \frac{l_z^2}{\sin^2\theta}\right\}\right]^{1/2} \\ &\equiv \Theta(\theta) , \\ SI\frac{d\varphi}{d\tau} &= -\left(aE - \frac{l_z}{\sin^2\theta}\right) + \frac{a}{\Delta}[E(r^2 + a^2) - al_z] , \end{aligned} \tag{2.48}$$

$$\begin{aligned} \Sigma\frac{dt}{d\tau} &= -\left(aE - \frac{l_z}{\sin^2\theta}\right) a \sin^2\theta \\ &\quad + \frac{r^2 + a^2}{\Delta}[E(r^2 + a^2) - al_z] , \\ \Sigma\frac{dr}{d\tau} &= \pm\sqrt{R} , \end{aligned}$$

where E, l_z, and C are the energy, the z component of the angular momentum, and the Carter constant [22] of a test particle, respectively,[1] $\Sigma = r^2 + a^2\cos^2\theta$ and

$$\begin{aligned} R &= [E(r^2 + a^2) - al_z]^2 \\ &\quad - \Delta[(Ea - l_z)^2 + r^2 + C] . \end{aligned} \tag{2.49}$$

We consider the case in which a particle moves along a constant radius $r = r_0$, but precesses around the symmetric axis. The degree of precession is determined by the value of C. If r_0 and C are given, the energy E and the z component of the angular momentum l_z are obtained by the two equations, $R = 0$ and $dR/dr = 0$. Thus the energy-momentum tensor of a test particle is written as

$$\begin{aligned} T^{\mu\nu} &= \frac{\mu}{\sum \sin\theta dt/d\tau}\frac{dz^\mu}{d\tau}\frac{dz^\nu}{d\tau} \\ &\quad \times\delta(r - r_0)\delta(\theta - \theta(t))\delta(\varphi - \varphi(t)) . \end{aligned} \tag{2.50}$$

The source term of the Teukolsky equation is

$$\begin{aligned} T_{lm\omega} &= -4\int d\Omega dt \rho^{-5}\bar{\rho}^{-1}(B_2' + B_2'^*) \\ &\quad \times e^{-im\varphi + i\omega t}\frac{{}_{-2}S^{a\omega}_{lm}}{\sqrt{2\pi}} , \end{aligned} \tag{2.51}$$

where

[1] In this section, these constants of motion are those measured in units of μ. That is, if expressed in the standard units, E, l_z, and C in Eq. (2.48) are to be replaced with E/μ, l_z/μ, and C/μ^2, respectively.

$$
\begin{aligned}
B_2' &= -\tfrac{1}{2}\rho^8\bar\rho L_{-1}[\rho^{-4}L_0(\rho^{-2}\bar\rho^{-1}T_{nn})] \\
&\quad -\frac{1}{2\sqrt2}\rho^8\bar\rho\Delta^2 L_{-1}[\rho^{-4}\bar\rho^2 J_+(\rho^{-2}\bar\rho^{-2}\Delta^{-1}T_{\bar m n})] , \qquad (2.52)\\
B_2'^{*} &= -\tfrac{1}{4}\rho^8\bar\rho\Delta^2 J_+[\rho^{-4}J_+(\rho^{-2}\bar\rho T_{\bar m\bar m})] \\
&\quad -\frac{1}{2\sqrt2}\rho^8\bar\rho\Delta^2 J_+[\rho^{-4}\bar\rho\Delta^{-1}L_{-1}(\rho^{-2}\bar\rho^{-2}T_{\bar m n})] ,
\end{aligned}
$$

with

$$
\begin{aligned}
\rho &= (r-ia\cos\theta)^{-1} ,\\
L_s &= \partial_\theta + \frac{m}{\sin\theta} - a\omega\sin\theta + s\cot\theta , \qquad (2.53)\\
J_+ &= \partial_r + \frac{iK}{\Delta} ,
\end{aligned}
$$

and $\bar Q$ denoting the complex conjugate of Q. In the present case, the tetrad components of the energy-momentum tensor, T_{nn}, $T_{\bar m n}$, and $T_{\bar m\bar m}$, are in the form

$$
\begin{aligned}
T_{nn} &= \frac{C_{nn}}{\sin\theta}\delta(r-r_0)\delta(\theta-\theta(t))\delta(\varphi-\varphi(t)) ,\\
T_{\bar m n} &= \frac{C_{\bar m n}}{\sin\theta}\delta(r-r_0)\delta(\theta-\theta(t))\delta(\varphi-\varphi(t)) , \qquad (2.54)\\
T_{\bar m\bar m} &= = \frac{C_{\bar m\bar m}}{\sin\theta}\delta(r-r_0)\delta(\theta-\theta(t))\delta(\varphi-\varphi(t)) ,
\end{aligned}
$$

where

$$
\begin{aligned}
C_{nn} &= \frac{\mu}{4\Sigma^3\dot t}[E(r^2+a^2)-al_z]^2 ,\\
C_{\bar m n} &= -\frac{\mu\rho}{2\sqrt2\Sigma^2\dot t}[E(r^2+a^2)-al_z]\left[i\sin\theta\left(aE-\frac{l_z}{\sin^2\theta}\right)+\Theta(\theta)\right] , \qquad (2.55)\\
C_{\bar m\bar m} &= \frac{\mu\rho^2}{2\Sigma\dot t}\left[i\sin\theta\left(aE-\frac{l_z}{\sin^2\theta}\right)+\Theta(\theta)\right]^2 ,
\end{aligned}
$$

and $\dot t = dt/d\tau$.

Substituting Eq. (2.52) into Eq. (2.51) and performing the integration by parts, we obtain

$$
\begin{aligned}
T_{lm\omega} &= -\frac{4}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dt\, e^{i\omega t-im\varphi}\Bigg[-\frac12 L_1^\dagger\{\rho^{-4}L_2^\dagger(\rho^3 S)\}C_{nn}\rho^{-2}\bar\rho^{-1}(r-r_0)\\
&\quad +\frac{\Delta^2\bar\rho^2}{\sqrt2\rho}[L_2^\dagger S + ia(\bar\rho-\rho)\sin\theta S]J_+\{C_{\bar m n}\rho^{-2}\bar\rho^{-2}\Delta^{-1}\delta(r-r_0)\}\\
&\quad +\frac{1}{2\sqrt2}L_2^\dagger\{\rho^3 S(\bar\rho^2\rho^{-4})_{,r}\}C_{\bar m n}\Delta\rho^{-2}\Delta(r-r_0)\\
&\quad -\tfrac14\rho^3\Delta^2 S J_+\{\rho^{-4}J_+[\bar\rho\rho^{-2}C_{\bar m\bar m}\delta(r-r_0)]\}\Bigg]\\
&\equiv \int_{-\infty}^{\infty} dt\, e^{i\omega t-im\varphi}t_{lm\omega} , \qquad (2.56)
\end{aligned}
$$

where

$$
L_s^\dagger = \partial_\theta - \frac{m}{\sin\theta} + a\omega\sin\theta + s\cot\theta , \qquad (2.57)
$$

and S denotes ${}_{-2}S_{lm}^{a\omega}(\theta(t))$ for simplicity.

Equation (2.56) can be further simplified by noting that the orbits of our interest have the properties

$$
\theta(t+\Delta t)=\theta(t),\quad \varphi(t+\Delta t)=\varphi(t)+\Delta\varphi , \qquad (2.58)
$$

where Δt is the orbital period (of the motion in the θ direction) and $\Delta\varphi$ is the phase advancement during Δt. For convenience, we set

$$
\Omega_\theta \equiv \frac{2\pi}{\Delta t},\quad \Omega_\varphi \equiv \frac{\Delta\varphi}{\Delta t} . \qquad (2.59)
$$

Then the source term reduces to the form

$$
\begin{aligned}
T_{lm\omega} &= T_{\rm one}\sum_k e^{ik(\omega\Delta t-m\Delta\varphi)}\\
&= \Omega_\theta T_{\rm one}\sum_n \delta(\omega-\omega_n) , \qquad (2.60)
\end{aligned}
$$

where

$$
\omega_n = n\Omega_\theta + m\Omega_\varphi \quad (n=0,\pm1,\pm2,\dots) , \qquad (2.61)
$$

$$
\begin{aligned}
T_{\rm one} &= \int_{-}^{\Delta t} dt\, e^{i\omega t-im\varphi(t)}t_{lm\omega}\\
&= \Delta^2[(A_{nn0}+A_{\bar m n0}+A_{\bar m\bar m0})\delta(r-r_0)\\
&\quad +\{(A_{\bar m n1}+A_{\bar m\bar m1})\delta(r-r_0)\}_{,r}\\
&\quad +\{A_{\bar m\bar m2}\delta(r-r_0)\}_{,rr}] , \qquad (2.62)
\end{aligned}
$$

and the A's are given in Appendix B. Inserting Eq. (2.60) into Eq. (2.5), we obtain $\tilde Z_{lm\omega}$ as

$$\tilde{Z}_{lm\omega} = \sum_n \delta(\omega - \omega_n) Z_{lm\omega_n} ,$$

$$Z_{lm\omega_n} = \frac{\Omega_\theta}{2i\omega_n B^{\rm in}_{lm\omega_n}} \left[R^{\rm in}_{lm\omega_n} \{A_{nn0} + A_{\bar{m}n0} + A_{\bar{m}\bar{m}0}\} - \frac{dR^{\rm in}_{lm\omega_n}}{dr} \{A_{\bar{m}n1} + A_{\bar{m}\bar{m}1}\} + \frac{d^2 R^{\rm in}_{lm\omega_n}}{dr^2} A_{\bar{m}\bar{m}2} \right]_{r=r_0} . \tag{2.63}$$

Now, let us consider the orbital integral in Eq. (2.62). To perform them, we must know the trajectory of a test particle, i.e., $\theta(t)$ and $\varphi(t)$, but they are not simple analytical functions of t for general trajectories. Thus, here we consider the case in which a particle is in the orbit with small inclination. To be specific, we introduce a dimensionless parameter y defined by

$$y = \frac{C}{Q^2}, \quad Q^2 = l_z^2 + a^2(1 - E^2) , \tag{2.64}$$

and regard it as small. Since $Q^2 \sim l_z^2$ and $C \sim l_x^2 + l_y^2$, [13] this is equivalent to assuming $l_x^2 + l_y^2 \ll l_z^2$. Note also that we do not need the exact expressions for E and l_z in terms of r_0 and C (or y), but only the PNE of them. To the first order of y as well as to the $\mathrm{P}^{5/2}\mathrm{N}$ order, they are given by

$$E = 1 - \frac{M}{2r_0} + \frac{3M^2}{8r_0^2} - \frac{M^{3/2}a}{r_0^{5/2}} \left(1 - \frac{y}{2}\right) - O(v^6) ,$$
$$l_z = (Mr_0)^{1/2} \left[\left(1 - \frac{y}{2}\right) + \frac{3M}{2r_0} \left(1 - \frac{y}{2}\right) - \frac{3M^{1/2}a}{r_0^{3/2}} (1 - y) + \frac{27M^2}{8r_0^2} \left(1 - \frac{y}{2}\right) + \frac{a^2}{r_0^2} (1 - 2y) - \frac{15M^{3/2}a}{2r_0^{5/2}} (1 - y) + O(v^6) \right] . \tag{2.65}$$

To solve the geodesic equations under the assumption $y \ll 1$, we first set $\theta = \pi/2 + y^{1/2}\theta'$ and consider the geodesic equation for θ. It then becomes

$$\left(\frac{d\theta'}{d\tau}\right)^2 = \frac{1}{\Sigma^2} \left[Q^2 - \frac{\sin^2(y^{1/2}\theta')}{y} \left(a^2(1 - E^2) + \frac{l_z^2}{\cos^2(y^{1/2}\theta')} \right) \right] . \tag{2.66}$$

Since the right-hand side (RHS) of Eq. (2.66) contains only even functions of $y^{1/2}\theta'$, we can solve it iteratively by expanding θ' as

$$\theta' = \theta_{(0)} + y\theta_{(1)} + y^2\theta_{(2)} + \cdots . \tag{2.67}$$

This is similar in spirit to the method used by Apostolatos *et al.* [23], who considered gravitational waves from a particle in an elliptical orbit around a nonrotating black hole with small eccentricity $e \ll 1$. However, here we only consider the lowest-order solution $\theta_{(0)}$. This means we take into account the effect of inclination up to $O(y)$, as seen from the structure of the geodesic equations (2.48). The equation for $\theta_{(0)}$ is

$$\left(\frac{d\theta_{(0)}}{d\tau}\right)^2 = \frac{Q^2}{\Sigma^2}(1 - \theta_{(0)}^2) , \tag{2.68}$$

or dividing it by $(dt/d\tau)^2$,

$$\left(\frac{d\theta_{(0)}}{dt}\right)^2 = \frac{Q^2}{\sigma^2}(1 - \theta_{(0)}^2) , \tag{2.69}$$

where

$$\sigma \equiv -a(aE - l_z) + \frac{a^2 + r_0^2}{\Delta(r_0)} \{E(r_0^2 + a^2) - al_z\} . \tag{2.70}$$

Then the solution is easily obtained as

$$\theta_{(0)} = \sin(\Omega_\theta t), \quad \Omega_\theta = \frac{Q}{\sigma} \left(= \frac{2\pi}{\Delta t} \right) , \tag{2.71}$$

where we have chosen $\theta_{(0)} = 0$ at $t = 0$. Thus we have

$$\theta = \frac{\pi}{2} + y^{1/2} \sin(\Omega_\theta t) . \tag{2.72}$$

Note that solution (2.72) implies that the inclination angle θ_t is indeed given by $\theta_i = y^{1/2}$ in the present approximation.

Next, we consider the geodesic equation for φ. Taking account of the terms up to $O(y)$, it becomes

$$\frac{d\varphi}{dt} = \frac{\kappa}{\sigma} \left[1 + \left(\frac{l_z}{\kappa} - \frac{a^2 E}{\sigma} \right) y\theta_{(0)}^2 \right] = \Omega_\varphi - y\frac{\Omega_2}{2} \cos(2\Omega_\theta t) , \tag{2.73}$$

where

$$\kappa \equiv -(aE - l_z) + \frac{a}{\Delta(r_0)} \{E(r_0^2 + a^2) - al_z\}$$

and

$$\Omega_\varphi = \frac{\kappa}{\sigma} + \frac{1}{2} y\Omega_2 \quad \left(= \frac{\Delta\varphi}{\Delta t}\right), \quad \Omega_2 = \frac{\kappa}{\sigma}\left(\frac{l_z}{\kappa} - \frac{a^2 E}{\sigma}\right) . \tag{2.74}$$

The solution to Eq. (2.73) with $\varphi = 0$ at $t = 0$ is

$$\varphi = \Omega_\varphi t - y\frac{\Omega_2}{4\Omega_\theta}\sin(2\Omega_\theta t) . \tag{2.75}$$

Note that $\Omega_\varphi \neq \Omega_\theta$. This means the precession of a test particle orbit around the spin axis of the black hole. Specifically, to the order required for the present purpose (see Sec. III below), we have

$$\Omega_\varphi = \frac{M^{1/2}}{r_0^{3/2}}\left[1 - \frac{M^{1/2}a}{r_0^{3/2}} + \frac{3}{2}y\left(\frac{M^{1/2}a}{r_0^{3/2}} - \frac{a^2}{r_0^2}\right) + O(v^6)\right] , \tag{2.76}$$

$$\Omega_\theta = \frac{M^{1/2}}{r_0^{3/2}}\left[1 - \frac{3M^{1/2}a}{r_0^{3/2}} + \frac{3a^2}{2r_0^2} + O(v^6) + (y)\right] .$$

We see that $\Omega_\varphi - \Omega_\theta \to 2Ma/r_0^3$ for $r_0 \to \infty$ and $y \to 0$, which is just the Lense-Thirring precessional frequency [24].

Now that we have the solution of the geodesic equations, we can estimate the A's in Eq. (2.63). Up to $O(y)$, they are integrals of the form (see Appendix B)

$$I_{lm\omega_n} = \int_0^{\Delta t} dt\, e^{i\omega_n t - im\varphi(t)}\Bigg\{Q^0_{lm\omega_n} + y^{1/2}Q^1_{lm\omega_n}\theta_{(0)} + yQ^2_{lm\omega_n}\theta^2_{(0)} + y^{1/2}Q^3_{lm\omega_n}\frac{d\theta_{(0)}}{dt} + yQ^4_{lm\omega_n}\theta_{(0)}\frac{d\theta_{(0)}}{dt} + yQ^5_{lm\omega_n}\left(\frac{d\theta_{(0)}}{dt}\right)^2\Bigg\} + O(y^{3/2}) , \tag{2.77}$$

where $Q^k_{lm\omega_n}$ for $k = 1$–5 are complicated functions of r_0. Substituting Eqs. (2.71), (2.75) into Eq. (2.77), and using the approximation.

$$e^{i\omega_n t - im\varphi(t)} = e^{in\Omega_\theta t}\left(1 + y\frac{m\Omega_2}{8\Omega_\theta}(e^{2i\Omega_\theta t} - e^{-2i\Omega_\theta t}) + O(y^{3/2})\right) , \tag{2.78}$$

we find

$$\begin{aligned} I_{lm\omega_n} = \frac{2\pi}{\Omega_\theta}\Bigg[&\left\{\delta_{n0} + y\frac{m\Omega_2}{8\Omega_\theta}(\delta_{n,-2} - \delta_{n,2})\right\}Q^0_{lm\omega_n} + y^{1/2}\frac{1}{2i}(\delta_{n,-1} - \delta_{n,1})Q^1_{lm\omega_n} \\ &+ y\frac{1}{4}(2\delta_{n,0} - \delta_{n,-2} - \delta_{n,2})Q^2_{lm\omega_n} + y^{1/2}\frac{\Omega_\theta}{2}(\delta_{n,-1} + \delta_{n,1})Q^3_{lm\omega_n} \\ &+ y\frac{\Omega_\theta}{4i}(\delta_{n,-2} - \delta_{n,2})Q^4_{lm\omega_n} + y\frac{\Omega_\theta^2}{4}(2\delta_{n,0} + \delta_{n,-2} + \delta_{n,2})Q^5_{lm\omega_n}\Bigg] + O(y^{3/2}) , \end{aligned} \tag{2.79}$$

where $\delta n, n'$ is Kronecker's delta. Applying these formulas to A's of Eq. (2.63), the amplitude $Z_{lm\omega_n}$ is found to have the form

$$\begin{aligned} Z_{lm\omega_n} = [&(Z^{0,0} + yZ^{0,2})\delta_{n,0} \\ &+ y^{1/2}(Z^{1,1}\delta_{n,1} + Z^{1,-1}\delta_{n,-1}) \\ &+ y(Z^{2,2}\delta_{n,2} + Z^{2,-2}\delta_{n,-2}) + O(y^{3/2})] , \end{aligned} \tag{2.80}$$

where $Z^{i,j}$ are functions of r_0 . In principle, algebraic calculations of Z^{ij} are straightforward, but they are almost impossible by hand in practice. Thus to avoid trivial mistakes as well as to save time, we have made use of the algebraic manipulation program MATHEMATICA to obtain $Z^{i,j}$. We shall not list their explicit forms here since they are too complicated and not much insight can be gained from them.

III. THE ENERGY AND ANGULAR-MOMENTUM FLUXES

In this section, we calculate the energy and angular-momentum fluxes to $O(v^5)$ beyond the quadrupole formula and to $O(y)$ in the orbital inclination. From Eqs. (2.1), ψ_4 at $\to \infty$ to the required order takes the form

$$\psi_4 = \frac{1}{r}\sum_{n=-2}^{2}\sum_{l=2}^{4}\sum_{m=-l}^{l} Z_{lm\omega_n}\frac{{}_{-2}S^{a\omega_n}_{lm}}{\sqrt{2\pi}}e^{i\omega_n(r^* - t) + im\varphi} . \tag{3.1}$$

It is worth noting that there are symmetry relations, ${}_{-2}S^{a\omega_n}_{lm}(\theta) = {}_{-2}S^{a\omega_{-n}}_{l-m}(\pi - \theta)$ and $Z_{l-m\omega_{-n}} = (-1)^n \bar{Z}_{lm\omega_n}$. At infinity, ψ_4 is related to the two independent modes of gravitational waves h_+ and $h_\times$ as

$$\psi_4 = \tfrac{1}{2}(\ddot{h}_+ - i\ddot{h}_\times) . \tag{3.2}$$

From Eqs. (3.1) and (3.2), the energy flux averaged over $t \gg \Delta t$ is given by

$$\left\langle \frac{dE}{dt} \right\rangle = \sum_{l,m,n} \frac{|Z_{lm\omega_n}|^2}{4\pi\omega_n^2} \equiv \sum_{l,m,n} \left(\frac{dE}{dt}\right)_{lmn} . \quad (3.3)$$

In the same way, the angular-momentum flux is given by

$$\begin{aligned} \left\langle \frac{dJ_z}{dt} \right\rangle &= \sum_{l,m,n} \frac{m|Z_{lm\omega_n}|^2}{4\pi\omega_n^3} \\ &\equiv \sum_{l,m,n} \left(\frac{dJ_z}{dt}\right)_{lmn} \\ &= \sum_{l,m,n} \frac{m}{\omega_n} \left(\frac{dE}{dt}\right)_{lmn} . \end{aligned} \quad (3.4)$$

As is clear from Eq. (2.80), since we take the square of $|Z_{lm\omega_n}|$ we only need $n = 0, \pm 1$ modes for the present purpose. Further, since $\omega_n = m\Omega_\varphi + n\Omega_\theta$, we only need Ω_θ at the 0th order of y. The PNE's of Ω_φ and Ω_θ are given in Eq. (2.76). In order to express the post-Newtonian corrections to the luminosity, we define η_{lmn} as

$$\left(\frac{dE}{dt}\right)_{lmn} \equiv \frac{1}{2} \left(\frac{dE}{dt}\right)_N \eta_{lmn} , \quad (3.5)$$

where $(dE/dt)_N$ is the Newtonian quadrupole luminosity:

$$\left(\frac{dE}{dt}\right)_N = \frac{32\mu^2 M^3}{5r_0^5} = \frac{32}{5} \left(\frac{\mu}{M}\right)^2 v^{10} . \quad (3.6)$$

We note that because the mode indices (l, m) here are those associated with the spheroidal harmonics, as is clear from Eq. (3.1), they do not correspond to the usual spherical mode indices. This point should be kept in mind when one attempts to interpret the PN corrections to η_{lmn} in the language of the standard PN approach.

For $l = 2$, the results are as follows. If $|m + n| > 2$ or $m + n = 0$, η_{lmn} becomes $O(v^k)$ $(k > 5)$. The remaining η_{lmn} which contribute to the luminosity to $O(v^5)$ are given by

$$\begin{aligned} \eta_{2\pm2\;0} &= 1 - \frac{107}{21}v^2 + 4\pi v^3 - 6qv^3 + \frac{4784}{1323}v^4 + 2q^2v^4 - \frac{428\pi}{21}v^5 + \frac{4216}{189}qv^5 \\ &\quad + y\left(-1 + \frac{170}{21}v^2 - 4\pi v^3 + 15qv^3 - \frac{4784}{1323}v^4 - 11q^2v^4 + \frac{428\pi}{21}v^5 - \frac{13\,186}{189}qv^5\right) , \\ \eta_{2\pm2\mp1} &= y\left(\frac{1}{36}v^2 - \frac{17}{504}v^4 + \frac{\pi}{18}v^5 + \frac{17}{1134}qv^5\right) , \\ \eta_{2\pm1\;0} &= \frac{1}{36}v^2 - \frac{1}{12}qv^3 - \frac{17}{504}v^4 + \frac{1}{16}q^2v^4 + \frac{\pi}{18}v^5 - \frac{793}{9072}qv^5 \\ &\quad + y\left(-\frac{5}{72}v^2 + \frac{1}{8}qv^3 + \frac{85}{1008}v^4 - \frac{1}{32}q^2v^4 - \frac{5\pi}{36}v^5 + \frac{13\,931}{18\,144}qv^5\right) , \\ \eta_{2\pm1\pm1} &= y\left(1 - \frac{170}{21}v^2 + 4\pi v^3 - 12qv^3 + \frac{4784}{1323}v^4 + \frac{11}{2}q^2v^4 - \frac{428\pi}{21}v^5 + \frac{11\,078}{189}qv^5\right) , \\ \eta_{2\;0\pm1} &= y\left(\frac{1}{24}v^2 - \frac{1}{12}qv^3 - \frac{17}{336}v^4 + \frac{1}{24}q^2v^4 + \frac{\pi}{12}v^5 - \frac{745}{1008}qv^5\right) . \end{aligned} \quad (3.7)$$

Putting together the above results, we obtain $(dE/dt)_l \equiv \sum_{mn}(dE/dt)_{lmn}$ for $l = 2$ as

$$\begin{aligned} \left(\frac{dE}{dt}\right)_2 = \left(\frac{dE}{dt}\right)_N \Bigg\{ & 1 - \frac{1277}{252}v^2 + 4\pi v^3 - \frac{73}{12}qv^3\left(1 - \frac{y}{2}\right) + \frac{37\,915}{10\,584}v^4 \\ & + \frac{33}{16}q^2v^4 - \frac{527}{96}q^2v^4y - \frac{2561\pi}{126}v^5 + \frac{201\,575}{9072}qv^5\left(1 - \frac{y}{2}\right) \Bigg\} . \end{aligned} \quad (3.8)$$

For $l = 3$, the nontrivial η_{lmn} are given by

$$\begin{aligned}
\eta_{3\pm3\ 0} &= \frac{1215}{896}v^2 - \frac{1215}{112}v^4 + \frac{3645\pi}{448}v^5 - \frac{1215}{112}qv^5 \\
&\quad + y\left(-\frac{3645}{1792}v^2 + \frac{3645}{224}v^4 - \frac{10\,935\pi}{896}v^5 + \frac{3645}{112}qv^5\right) , \\
\eta_{3\pm3\mp1} &= \frac{5}{42}v^4 y , \\
\eta_{3\pm2\ 0} &= \frac{5}{63}v^4 - \frac{40}{189}qv^5 + y\left(-\frac{20}{63}v^4 + \frac{100}{189}qv^5\right) , \\
\eta_{3\pm2\pm1} &= y\left(\frac{3645}{1792}v^2 - \frac{3645}{224}v^4 + \frac{10\,935\pi}{896}v^5 - \frac{6075}{224}qv^5\right) , \\
\eta_{3\pm2\mp1} &= y\left(\frac{5}{16\,128}v^2 - \frac{5}{3024}v^4 + \frac{5\pi}{8064}v^5 + \frac{25}{18\,144}qv^5\right) , \\
\eta_{3\pm1\ 0} &= \frac{1}{8064}v^2 - \frac{1}{1512}v^4 + \frac{\pi}{4032}v^5 - \frac{17}{9072}qv^5 \\
&\quad + y\left(-\frac{11}{16\,128}v^2 + \frac{11}{3024}v^4 - \frac{11\pi}{8064}v^5 + \frac{95}{9072}qv^5\right) , \\
\eta_{3\pm1\pm1} &= y\left(\frac{25}{126}v^4 - \frac{80}{189}qv^5\right) , \\
\eta_{3\ 0\pm1} &= y\left(\frac{1}{2688}v^2 - \frac{1}{504}v^4 + \frac{\pi}{1344}v^5 - \frac{11}{1008}qv^5\right) .
\end{aligned} \tag{3.9}$$

The other η_{lmn} are of $O(v^6)$ or higher. Then we obtain

$$\left(\frac{dE}{dt}\right)_3 = \left(\frac{dE}{dt}\right)_N \left\{\frac{1367}{1008}v^2 - \frac{32\,567}{3024}v^4 + \frac{16\,403\pi}{2016}v^5 - \frac{896}{81}qv^5\left(1-\frac{y}{2}\right)\right\} . \tag{3.10}$$

For $l = 4$, we have

$$\begin{aligned}
\eta_{4\pm4\ 0} &= \frac{1280}{567}v^4(1-2y) , \\
\eta_{4\pm3\pm1} &= \frac{2560}{567}v^4 y , \\
\eta_{4\pm3\mp1} &= \frac{5}{1134}v^4 y , \\
\eta_{4\pm2\ 0} &= \frac{5}{3969}v^4(1-8y) , \\
\eta_{4\pm1\pm1} &= \frac{5}{882}v^4 y ,
\end{aligned} \tag{3.11}$$

and the others are of $O(v^6)$ or higher. Hence we obtain

$$\left(\frac{dE}{dt}\right)_4 = \left(\frac{dE}{dt}\right)_N \times \frac{8965}{3969}v^4 . \tag{3.12}$$

Finally, gathering all the above results, the total energy flux up to $O(v^5)$ is found to be

$$\begin{aligned}
\left\langle\frac{dE}{dt}\right\rangle = \left(\frac{dE}{dt}\right)_N \Bigg[& 1 - \frac{1247}{336}v^2 + 4\pi v^3 - \frac{73}{12}qv^3\left(1-\frac{y}{2}\right) - \frac{44\,711}{9072}v^4 \\
& + \frac{33}{16}q^2v^4 - \frac{527}{96}q^2v^4y - \frac{8191\pi}{672}v^5 + \frac{3749}{336}qv^5\left(1-\frac{y}{2}\right)\Bigg] .
\end{aligned} \tag{3.13}$$

The terms without q agree with those derived in Ref. [15], and the term $-73qv^3/12$ also agrees with the previous results [10,12].[2]

[2]Note that Poisson [12] defines the PN expansion parameter as $v' \equiv (M\Omega_\varphi)^{1/3}$, which is related to our $v \equiv (M/r_0)^{1/2}$ as $v' = v[1 - qv^3/3 + O(v^6)]$ for $y = 0$. Consequently, his Newtonian quadrupole luminosity differs from ours by a factor $(v'/v)^{10} = [1 - 10qv^3/3 + O(v^4)]$. This explains the apparent difference between his result $-11qv^3/4$ and ours.

From Eq. (3.4), the averaged angular-momentum fluxes for $l = 2$, 3, and 4 are calculated to give

$$\left(\frac{dJ_z}{dt}\right)_2 = \left(\frac{dJ_z}{dt}\right)_N \left[1 - \frac{y}{2} - \frac{1277}{252}v^2\left(1-\frac{y}{2}\right) + 4\pi v^3\left(1-\frac{y}{2}\right) - qv^3\left(\frac{61}{12} - \frac{61}{8}y\right) + \frac{37\,915}{10\,584}v^4\left(1-\frac{y}{2}\right) + q^2v^4\left(\frac{33}{16} - \frac{229}{32}y\right) - \pi v^5\frac{2561}{126}\left(1-\frac{y}{2}\right) + qv^5\left(\frac{22\,229}{1296} - \frac{27\,809}{864}y\right)\right], \tag{3.14}$$

$$\left(\frac{dJ_z}{dt}\right)_3 = \left(\frac{dJ_z}{dt}\right)_N \left[\frac{1367}{1008}v^2\left(1-\frac{y}{2}\right) - \frac{32\,567}{3024}v^4\left(1-\frac{y}{2}\right) + \pi v^5\frac{16\,403}{2016}\left(1-\frac{y}{2}\right) - qv^5\left(\frac{88\,049}{9072} - \frac{9817}{756}y\right)\right], \tag{3.15}$$

$$\left(\frac{dJ_z}{dt}\right)_4 = \left(\frac{dJ_z}{dt}\right)_N \left[\frac{8965}{3969}v^4\left(1-\frac{y}{2}\right)\right], \tag{3.16}$$

where $(dJ_z/dt)_N$ is defined to be

$$\left(\frac{dJ_z}{dt}\right)_N = \frac{32\mu^2 M^{5/2}}{5r_0^{7/2}} = \frac{32}{5}\left(\frac{\mu}{M}\right)^2 Mv^7 . \tag{3.17}$$

Total flux of the angular momentum is then given by

$$\left\langle\frac{dJ_z}{dt}\right\rangle = \left(\frac{dJ_z}{dt}\right)_N \left[\left(1-\frac{y}{2}\right) - \frac{1247}{336}v^2\left(1-\frac{y}{2}\right) + 4\pi v^3\left(1-\frac{y}{2}\right) - \frac{61}{12}qv^3\left(1-\frac{3y}{2}\right) - \frac{44\,711}{9072}v^4\left(1-\frac{y}{2}\right) + q^2v^4\left(\frac{33}{16} - \frac{229}{32}y\right) - \frac{8191}{672}\pi v^5\left(1-\frac{y}{2}\right) + qv^5\left(\frac{417}{56} - \frac{4301}{224}y\right)\right]. \tag{3.18}$$

We note that the result is proportional to $(1 - y/2)$ in the limit $q \to 0$. This is simply because the orbital plane is slightly tilted from the equatorial plane by an angle $\theta_i \sim y^{1/2}$, hence $dJ_z/dt \sim (dJ_{\rm tot}/dt)\cos\theta_i$.

From the above results, we find the following features of the gravitational wave luminosity.

(1) As argued in Sec. II A, the quadratic terms in $q(= a/M)$ appear at the v^4 order, and the linear terms in q appear at v^3 and v^5 orders.

(2) The coefficients of qv^3 and qv^5 in $(dE/dt)_2$ and $(dE/dt)_3$ [and hence in $(dE/dt)_{\rm tot}$] are proportional to $1 - y/2$. Since $1 - y/2 \sim \cos\theta_i$, these terms may be regarded as proportional to the inner product $\mathbf{S}\cdot\mathbf{L}$ of the spin angular momentum $\mathbf{S}$ and the orbital angular momentum $\mathbf{L}$. With this interpretation, our result at the v^3 order is consistent with the PN calculation of the spin-orbit terms by Kidder *et al.* [10] as well as with the numerical result of perturbative calculations by Shibata [13].

(3) Contrary to feature (2), the coefficient of the q^2v^4 term in $(dE/dt)_2$ does not seem to be expressible as a simple function of $\cos\theta_i$. We suspect that a major part of it is attributable to the quadrupolar gravitational field around the Kerr black hole which modifies the particle orbit. In fact, for $y = 0$, the $2q^2v^4$ term of $\eta_{2\pm20}$ in Eq. (3.7) can be explained in terms of the Newtonian quadrupole formula as the contribution from the quadrupole moment of the Kerr black hole.[3] However, the $1/16q^2v^4$ term in $\eta_{2\pm10}$ cannot be explained in this way. An inspection of the expanded form of the $l = 2$ ingoing-wave Teukolsky function given in Eq. (2.35) reveals that the q^2z^2 term at $O(\epsilon^2)$ is proportional to $m^2 - 4$, hence it vanishes for $m = \pm2$ while it remains finite for $m = \pm1$. Since this term will contribute to the q^2v^4 terms in the luminosity, we may interpret the q^2v^4 term in $\eta_{2\pm10}$ as due to the curvature scattering in the near zone field. Incidentally, this suggests that the coincidence of the coefficient 2 of the q^2v^4 term in $\eta_{2\pm20}$ with the Newtonian calculation is rather accidental; naively we would expect the same curvature scattering effect to give rise to some additional contribution to the q^2v^4 term in $\eta_{2\pm20}$. In any case, our result suggests the existence of a new type of spin-dependent terms in the energy flux when a PN analysis beyond the present level is carried out.

[3]This was first pointed out to us by E. Poisson.

IV. IMPLICATIONS TO COALESCING COMPACT BINARIES

In this section, we discuss the effects of PN terms in the luminosity to the orbital evolution of inspiraling binaries composed of neutron stars (NS's) and/or black holes (BH's). Although our results are valid only in the test particle limit, we ignore this fact in the following.

Since we are interested in the orbits off the equatorial plane, we must consider the evolution of C as well as r of the particle as it radiates gravitational waves. Although not proved in any sense, let us assume that the orbit remains quasicircular, i.e., the radius of the orbit is approximately constant for many orbital periods. We then would like to see if the inclination angle $\theta_i(=y^{1/2})$ changes in time as the orbit shrinks. Since the test particle orbit in this case is characterized by two of the four approximate constants of motion, r, C (or y), E, and l_z, we can estimate the change of y with respect to the change of r by equating the energy and angular momentum luminosities of the gravitational waves with those lost by the particle,

$$
\begin{aligned}
-\left\langle\frac{dE}{dt}\right\rangle &= \frac{\partial E(r,y)}{\partial r}\frac{dr}{dt}+\frac{\partial E(r,y)}{\partial y}\frac{dy}{dt},\\
-\left\langle\frac{dJ_z}{dt}\right\rangle &= \frac{\partial l_z(r,y)}{\partial r}\frac{dr}{dt}+\frac{\partial l_z(r,y)}{\partial y}\frac{dy}{dt}.
\end{aligned}
\tag{4.1}
$$

To the leading order in v and y, this gives

$$
\frac{d\ln y}{d\ln r} = -\frac{61}{24}q\left(\frac{M}{r}\right)^{3/2}. \tag{4.2}
$$

Thus y changes only by a small amount during the entire inspiraling stage until $r \lesssim 10M$ even for $q=1$. Hence the approximation $y=$const throughout the evolution of the orbit will be good if the orbit remains quasicircular. Furthermore, by adopting a radiation reaction formula which is valid at least in the Newtonian limit, it has been numerically found by Shibata [13] that the evolution of the inclination angle is small, at least in the low-frequency region, $r/M \gtrsim 30$. This result is consistent with the assumption of quasicircular orbits. Thus we assume $y=$const in the following.

Then the total cycle $N(r_i,r_f)$ of the phase of gravitational waves from an inspiraling compact binary from $r=r_i$ $(t=t_i)$ to $r=r_f$ $(t=t_f)$ is

$$
N \equiv \int_{t_i}^{t_f} f\,dt = \frac{1}{\pi}\int_{r_f}^{r_i} dr\Omega_\varphi \frac{dE/dr}{|dE/dt|}, \tag{4.3}
$$

where f is the frequency of the wave. Expanding Ω_φ, dE/dt, and dE/dr with respect to $v=(M/r)^{1/2}$, N is expressed as

$$
N = \frac{5}{64\pi}\frac{M}{\mu}\int_{r_f}^{r_i}\frac{drr3/2\sum_{k=0}^{\infty}b_k(q)(M/r)^{k/2}\sum_{k=0}^{\infty}c_k(q)(M/r)^{k/2}}{\sum_{k=0}^{\infty}d_k(q)(M/r)^{k/2}}, \tag{4.4}
$$

where the series forms in the denominator and numerator represent the PN corrections to the Ω_φ, dE/dt, and dE/dr, that is,

$$
\begin{aligned}
\sum_{k=0}^{\infty} b_k(q)(M/r)^{k/2} &= \frac{\Omega_\varphi}{(\Omega_\varphi)_N},\\
\sum_{k=0}^{\infty} c_k(q)(M/r)^{k/2} &= \frac{dE/dr}{(dE/dr)_N},\\
\sum_{k=0}^{\infty} d_k(q)(M/r)^{k/2} &= \frac{dE/dt}{(dE/dt)_N},
\end{aligned}
\tag{4.5}
$$

and the argument of $q(=a/M)$ is given to the coefficients b_k, c_k, and d_k to emphasize the q dependence of the PN corrections. To the PN order we consider in this paper, the PN expansions of Ω_φ and dE/dt are given in Eqs. (2.76) and (3.13), respectively. For completeness, here we show the PN expansion of dE/dr:

$$
\frac{dE}{dr} = \frac{\mu M}{2r^2}\left[1-\frac{3M}{2r}+5q\left(1-\frac{y}{2}\right)\left(\frac{M}{r}\right)^{3/2}-\left(\frac{81}{8}+3q^2(1-y)\right)\left(\frac{M}{r}\right)^2-\frac{21}{4}q\left(\frac{M}{r}\right)^{5/2}\right]. \tag{4.6}
$$

Since the effect of PN corrections in the case of $q=0$ has been studied already [15], here we examine only the effect due to nonvanishing q. For this purpose, we introduce the quantity $\Delta N^{(n)}$ defined by

$$
\begin{aligned}
\Delta N^{(n)}(q) = \frac{5}{64\pi}\frac{M}{\mu}\Bigg[&\int_{r_f}^{r_i}\frac{dr\,r^{3/2}}{M^{5/2}}\frac{\sum_{k=0}^{n}b_k(q)(M/r)^{k/2}\sum_{k=0}^{n}c_k(q)(M/r)^{k/2}}{\sum_{k=0}^{n}d_k(q)(M/r)^{k/2}}\\
&-\int_{r_f}^{r_i}dr\,r^{3/2}\left\{\frac{\sum_{k=0}^{n-1}b_k(q)(M/r)^{k/2}+b_n(0)(M/r)^{n/2}}{\sum_{k=0}^{n-1}d_k(q)(M/r)^{k/2}+d_n(0)(M/r)^{n/2}}\right.\\
&\left.\times\left(\sum_{k=0}^{n-1}c_k(q)(M/r)^{k/2}+c_n(0)(M/r)^{n/2}\right)\right\}\Bigg].
\end{aligned}
\tag{4.7}
$$

This describes the effect of q corrections at the $\mathrm{P}^{n/2}\mathrm{N}$ order.

First we consider a NS-NS binary of equal mass $M = 1.4M\odot$ $(M/\mu = 4)$ as a typical example. The future laser interferometric gravitational wave detectors such as LIGO [1] have good sensitivity in the frequency band between 10 and 1000 Hz so we choose $r_i = 175M$ and $r_f = 8M$. A NS of mass $1.4M_\odot$ and radius $R \simeq 10$ km has $q \simeq 0.4/P_{\mathrm{ms}}$, where P_{ms} is the period of rotation in units of a ms. In this case, the phase N is accumulated at relatively large radii $r/M \gtrsim 100$. Hence the convergence of the PN series in the numerators and denominators of the integrands in Eq. (4.7) is good enough so that we may expand them further to obtain the approximate formula

$$\begin{aligned}\Delta N^{(n)}(q) \approx \frac{5}{64\pi}\frac{M}{\mu}\int_{r_f}^{r_i}\frac{dr}{M}\left(\frac{M}{r}\right)^{n-3/2} &\times\{[b_n(q)+c_n(q)-d_n(q)] \\ &-[b_n(0)+c_n(0)-d_n(0)]\}\ . \end{aligned} \tag{4.8}$$

This gives

$$\begin{aligned}\Delta N^{(3)} &\sim \left(\frac{70}{P_{\mathrm{ms}}}\right)(1-0.4y)\ , \\ \Delta N^{(4)} &\sim -\left(\frac{2}{P_{\mathrm{ms}}}\right)^2(1-1.5y)\ , \\ \Delta N^{(5)} &\sim \left(\frac{9}{P_{\mathrm{ms}}}\right)(1-0.3y)\ .\end{aligned} \tag{4.9}$$

Although the y corrections in the above are valid only for $y \ll 1$, we expect them to be qualitatively valid even for $y \sim 1$. As mentioned previously, the corrections $\Delta N^{(3)}$ and $\Delta N^{(5)}$ are due to the spin-orbit coupling, hence replacing y with $2(1-\cos\theta_i)$ in the above formula will give a reasonable estimate in the qualitative sense. As for $\Delta N^{(4)}$, although we have not been able to specify the physical meaning of it with certainty, at least we may say that the dominant contribution comes from the quadrupole moment of the gravitational field around the Kerr black hole, as discussed at the end of the previous section. Hence we may also expect the replacement $y \to 2(1-\cos\theta_i)$ to be approximately correct. Thus the inclination angle directly affects the values of these phase corrections whenever they become important.

We know three binary pulsars in our Galaxy which will merge within a Hubble time, PSR1913+16 [25], PSR2127+11C [26], and PSR1534+12 [27]. Hence these may be regarded as a typical target of the gravitational wave detectors. Their rotation periods are $P_{\mathrm{ms}} = 59.0$, 30.5, and 37.9, respectively. If we also assume these values as typical, we have $\Delta N^{(3)} > 1$ as has been discussed previously [11,13] while $\Delta N^{(4)}$ and $\Delta N^{(5)}$ are small. However, we know there are several pulsars with $P_{\mathrm{ms}} \lesssim 2$ in our Galaxy [28], for which both $\Delta N^{(4)}$ and $\Delta N^{(5)}$ exceed unity. Hence it will be safe to construct templates which take account of the $\mathrm{P}^2\mathrm{N}$ and $\mathrm{P}^{5/2}\mathrm{N}$ spin terms.

Note however that a main contribution to the correction $\Delta N^{(4)}$ is due to the quadrupole moment of the gravitational field, the value of which reflects the peculiarity of the Kerr black hole. Concerning this point, Bildsten and Cutler [29] considered the quadrupole moment of the gravitational field induced by the quadrupolar deformation of a NS due to its rotation and evaluated the phase correction for a realistic NS model as

$$\Delta N^{(4)} \sim -\left(\frac{5}{P_{\mathrm{ms}}}\right)^2\ . \tag{4.10}$$

This result is somewhat larger than our estimate. That is, the effect of the quadrupole moment of the gravitational field due to a spinning NS is larger than that due to a spinning BH for the same dimensionless spin parameter q. This suggests that if one body of a compact binary is very rapidly rotating so that we are able to measure $\Delta N^{(4)}$ by the matched filter technique, then together with other terms which carry information of the orbital parameters such as the spin-orbit terms, it will be possible to distinguish a BH from a NS even if the BH has mass $M_{\mathrm{BH}} \sim 1.5M_\odot$.

Next, we consider a BH-NS binary composed of $10M_\odot$ BH and $1.4M_\odot$ NS $(M/\mu = 9.28)$. Our result has more direct applicability to this case. For simplicity, we set $r_i = 68M$ and $r_f = 6M$ irrespective of q and y. In this case, the estimation of $\Delta N^{(n)}$ in terms of the approximate formula (4.8) will not be a good approximation. Instead we must use the original formula for $\Delta N^{(n)}$, Eq. (4.7), as it is. This is because the phase N in the present case is accumulated at smaller r/M than in the case of a NS-NS binary, hence the convergence of the PN expansion becomes slow. As a consequence it is not possible to derive approximate formulas for $\Delta N^{(n)}$ as simple as Eq. (4.9). Here we only quote the critical value of the spin parameter q above which each correction $\Delta N^{(n)}$ $(n = 3,4,5)$ exceeds unity. We find $\Delta N^{(3)} \gtrsim 1$ for $q \gtrsim 0.01$. Hence the correction at this order is important even for a very slowly rotating black hole. As for $\Delta N^{(4)}$ and $\Delta N^{(5)}$, they become larger than unity if $q \gtrsim 0.2$. This indicates that yet higher-order PN corrections will be important if the BH is rapidly rotating $(q \sim 1)$.

Summarizing the above analyses, we obtain the following conclusions.

(1) The spin-orbit coupling term at the $\mathrm{P}^{5/2}\mathrm{N}$ order is important for the evolution of NS-NS binaries with $P_{\mathrm{ms}} \lesssim 2$ and BH-NS binaries with $q \gtrsim 0.2$. Since the rotation of $P_{\mathrm{ms}} \sim 1$ would be the fastest possible period that a NS could have, inclusion of the phase corrections up through the $\mathrm{P}^{5/2}\mathrm{N}$ order seems to be enough for NS-NS binaries. On the other hand, the terms higher than the $\mathrm{P}^{5/2}\mathrm{N}$ order are likely to be important for BH-NS binaries since the rotation of $q \gtrsim 0.2$ for a black hole seems quite possible.

(2) The q^2 terms at the $\mathrm{P}^2\mathrm{N}$ order become important for BH's with $q \gtrsim 0.2$ or NS's with $P_{\mathrm{ms}} \lesssim 2$. However, the latter value is based on our formula which is valid only for a rotating BH. An estimate based on a realistic NS model gives $P_{\mathrm{ms}} \lesssim 5$ [29]. Thus it will be possible to distinguish a small mass BH from a NS if the phase corrections to $\mathrm{P}^2\mathrm{N}$ order can be detected by matched filtering.

(3) At any order of PN corrections, the effect of a finite

inclination angle to the number of the phase cycles must be taken into account whenever the spin terms become important.

V. SUMMARY

In this paper, we have performed a post-Newtonian calculation of the gravitational waves from a particle of mass μ orbiting around a rotating black hole of mass M ($\mu \ll M$). We have considered the orbit of a constant coordinate radius $r = r_0$ but with a small inclination angle $\theta_i \sim y^{1/2}$, where y is a nondimensional parameter proportional to the Carter constant of the orbit.

We have formulated the post-Newtonian expansion of the Teukolsky equation and its source term in terms of a small expansion parameter $v = (M/r_0)^{1/2}$ accurate up through $O(v^5)$ ($\mathrm{P}^{5/2}\mathrm{N}$ order). We have not directly dealt with the Teukolsky equation but first formulated a method to obtain the homogeneous solution for the Sasaki-Nakamura equation, which is a generalization of the Regge-Wheeler equation for the Schwarzschild black hole, by expanding it in powers of $\epsilon = 2M\omega$, where ω is the frequency of a gravitational wave. In particular, to $O(\epsilon)$, we have obtained the ingoing-wave radial functions for arbitrary spherical indices (l, m) in closed analytical form. Then we have obtained all the necessary radial functions to the required accuracy and transformed them to the corresponding Teukolsky radial functions, which have been used to construct the Green's function. Further, we have formulated the post-Newtonian expansion of the source term for circular orbits with small inclination angle. Assuming $y \ll 1$, we have analytically solved the geodesics of a particle accurate to $O(y)$ and obtained the source term to the required accuracy. We have used these results to integrate the Teukolsky equation and derived the formulas for the gravitational energy and angular-momentum fluxes which are correct to $O(v^5)$ and to $O(y)$.

Based on the thus obtained luminosity formula, we have estimated the accumulated phase N of gravitational waves from inspiraling binaries, assuming the orbit remains quasicircular. Specifically we have considered a NS-NS binary of equal mass $1.4M_\odot$ and a BH-NS binary of masses $10M_\odot$ and $1.4M_\odot$, which will be typical targets of the near-future laser interferometric gravitational wave detectors. We have found that if the rotation of a neutron star is moderate, say $P \gtrsim 20$ ms, only the phase correction at its leading $\mathrm{P}^{3/2}\mathrm{N}$ order will be important. However we have also found that if one body of a binary is a rapidly rotating NS ($P \gtrsim 2$ ms) or a rotating BH ($q = J_{\mathrm{BH}}/M^2 \gtrsim 0.2$), the phase correction of $\Delta N > 1$ will be caused by the spin terms at $\mathrm{P}^2\mathrm{N}$ and $\mathrm{P}^{5/2}\mathrm{N}$ orders. Furthermore if one body is a rapidly rotating BH ($q \sim 1$), it is expected that the higher-order corrections such as the $\mathrm{P}^3\mathrm{N}$ ones become important. In all these cases, when the phase correction at a certain PN order becomes significant, that due to a nonvanishing inclination angle at the same PN order becomes equally important.

The above conclusions imply that it is desirable to evaluate yet higher-order PN spin corrections to the gravitational wave luminosity. As for the inclination of the orbit, since we expect the expansion in powers of y to be valid for $y \lesssim 1$ ($\theta_i \lesssim \pi/4$), it will be meaningful and useful to calculate the higher-order corrections in y along with higher-order PN calculations. These problems are left for future work.

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APPENDIX A

In this appendix we show the potential functions F and U of the SN equation (2.10). Details of the derivation are given in Ref. [18].

The function $F(r)$ is given by

$$F(r) = \frac{\eta_{,r}}{\eta} \frac{\Delta}{r^2 + a^2} , \qquad \text{(A1)}$$

where

$$\eta = c_0 + c_1/r + c_2/r^2 + c_3/r^3 + c_4/r^4 , \qquad \text{(A2)}$$

with

$$\begin{aligned} c_0 &= -12i\omega M + \lambda(\lambda + 2) - 12a\omega(a\omega - m) , \\ c_1 &= 8ia[3a\omega - \lambda(a\omega - m)] , \\ c_2 &= -24iaM(a\omega - m) + 12a^2[1 - 2(a\omega - m)^2] , \qquad \text{(A3)} \\ c_3 &= 24ia^3(a\omega - m) - 24Ma^2 , \\ c_4 &= 12a^4 . \end{aligned}$$

The function $U(r)$ is given by

$$U(r) = \frac{\Delta U_1}{(r^2 + a^2)^2} + G^2 + \frac{\Delta G_{,r}}{r^2 + a^2} - FG , \qquad \text{(A4)}$$

where

$$\begin{aligned} G &= -\frac{2(r - M)}{r^2 + a^2} + \frac{r\Delta}{(r^2 + a^2)^2} , \\ U_1 &= V + \frac{\Delta^2}{\beta}\left[\left(2\alpha + \frac{\beta_{,r}}{\Delta}\right)_{,r} - \frac{\eta_{,r}}{\eta}\left(\alpha + \frac{\beta_{,r}}{\Delta}\right)\right] , \\ &\qquad \text{(A5)} \\ \alpha &= -i\frac{K\beta}{\Delta^2} + 3iK_{,r} + \lambda + \frac{6\Delta}{r^2} , \\ \beta &= 2\Delta\left(-iK + r - M - \frac{2\Delta}{r}\right) . \end{aligned}$$

APPENDIX B

In this appendix we show the A's in Eq. (2.60):

$$
\begin{aligned}
A_{nn0} &= \frac{2}{\sqrt{2\pi}\Delta^2}\int_0^{\Delta t} dt\, e^{i\omega t - im\varphi(t)} C_{nn}\rho^{-2}\bar{\rho}^{-1} L_1^+\{\rho^{-4}L_2^+(\rho^3 S)\}\ ,\\
A_{\bar{m}n0} &= -\frac{2}{\sqrt{\pi}\Delta}\int_0^{\Delta t} dt\, e^{i\omega t - im\varphi(t)} C_{\bar{m}n}\rho^{-3}\left[(L_2^+ S)\left(\frac{iK}{\Delta}+\rho+\bar{\rho}\right) - a\sin\theta S\frac{K}{\Delta}(\bar{\rho}-\rho)\right],\\
A_{\bar{m}\bar{m}0} &= \frac{1}{\sqrt{2\pi}}\int_0^{\Delta t} dt\, e^{i\omega t - im\varphi(t)} \rho^{-3}\bar{\rho} C_{\bar{m}\bar{m}} S\left[-i\left(\frac{K}{\Delta}\right)_{,r} - \frac{K^2}{\Delta^2} + 2i\rho\frac{K}{\Delta}\right]\ ,\\
A_{\bar{m}n1} &= -\frac{2}{\sqrt{\pi}\Delta}\int_0^{\Delta t} dt\, e^{i\omega t - im\varphi(t)} \rho^{-3} C_{\bar{m}n}[L_2^+ S + ia\sin\theta(\bar{\rho}-\rho)S]\ ,\\
A_{\bar{m}\bar{m}1} &= \frac{2}{\sqrt{2\pi}}\int_0^{\Delta t} dt\, e^{i\omega t - im\varphi(t)} \rho^{-3}\bar{\rho} C_{\bar{m}\bar{m}} S\left(i\frac{K}{\Delta}+\rho\right)\ ,\\
A_{\bar{m}\bar{m}2} &= \frac{1}{\sqrt{2\pi}}\int_0^{\Delta t} dt\, e^{i\omega t - im\varphi(t)} \rho^{-3}\bar{\rho} C_{\bar{m}\bar{m}} S\ ,
\end{aligned}
\tag{B1}
$$

where S denotes ${}_{-2}S_{lm}^{a\omega}$.

APPENDIX C

In this appendix, we show that the asymptotic form of $X_{lm\omega}^{\rm in}$ at $\epsilon \ll z \ll 1$ has the form

$$
X_{lm\omega}^{\rm in} = z^{l+1}[O(1) + \epsilon O(z^{-1}) + \epsilon^2 O(z^{-2}) + \epsilon^3 O(z^{-3}) + \cdots]\ . \tag{C1}
$$

From Eqs. (2.10) and (2.18), the equation for ξ_{lm} becomes

$$
\left[\frac{\Delta}{r^2}\frac{d^2}{dr^2} + C_1\frac{d}{dr} + C_2\right]\xi_{lm} = 0\ ,
$$

$$
C_1 = \frac{\Delta_{,r} - \Delta F_1 - 2i\Delta\phi_{,r}}{r^2}\ , \tag{C2}
$$

$$
C_2 = \frac{1}{r^2}\left(-U_1 + 2 - \frac{\Delta_{,r}^2}{\Delta} - \Delta\phi_{,r}^2 - \Delta_{,r}F_1 + i\Delta\phi_{,r}F_1 - 2iM\omega\right)\ ,
$$

where $F_1 = \eta_{,r}/\eta$, and ϕ, η, and U_1 are given in Eqs. (2.17), (A2), and (A5), respectively.

Let us now examine the behaviors of the coefficients C_1 and C_2. By dimensional consideration, they must be of the form

$$
C_k = \frac{1}{r^k} f_k(2M/r, \omega r)\ (k = 1, 2)\ , \tag{C3}
$$

where $f_k(y, z)$ are dimensionless functions of their arguments. We then easily see that $f_k(y, z)$ are regular at $y = 0$, since both $C_1 r$ and $C_2 r^2$ have well-defined limits as $M \to 0$. Furthermore, by examining the behaviors of C_1 and C_2 as $\omega \to 0$, we find they are also regular in this limit. Hence $f_k(y, z)$ are regular at $z = 0$ as well. Thus f_k may be expanded as

$$
f_k(y, z) = \sum_{n=0}^{\infty} f_k^{(n)}(z) y^n, \tag{C4}
$$

where $f_k^{(n)}(z)$ are regular at $z = 0$, hence may be further expanded as

$$
f_k^{(n)}(z) = \sum_{m=0}^{\infty} f_k^{(n,m)} z^m\ . \tag{C5}
$$

Note that $f_1^{(0)}(z) = 2$ and $f_2^{(0)}(z) = -l(l+1) + z^2$, which are the coefficients appearing in the lowest-order differential operator $L^{(0)}$ given by Eq. (2.20).

Taking the above consideration into account, scaling r to $z = \omega r$ in Eq. (C2), and noting that $y = 2M/r = \epsilon/z$, we have

$$
\left[L^{(0)} + \left(-\frac{\epsilon}{z} + \frac{q^2}{4}\frac{\epsilon^2}{z^2}\right)\frac{d^2}{dz^2} + \left(\sum_{n=1}^{\infty} f_1^{(n)}(z)\frac{\epsilon^n}{z^n}\right)\frac{1}{z}\frac{d}{dz} + \left(\sum_{n=1}^{\infty} f_2^{(n)}(z)\frac{\epsilon^n}{z^n}\right)\right]\xi_{lm} = 0\ . \tag{C6}
$$

Accordingly, if we expand ξ_{lm} as $\xi_{lm}(\epsilon; z) = \sum_{n=0}^{\infty}\epsilon^n \xi_{lm}^{(n)}(z)$ and set $\xi_{lm}^{(0)}(z) = j_l \sim z^l$, we can easily show recursively that the asymptotic behavior of $\xi_{lm}^{(n)}$ at $z \ll 1$ is

$$
\xi_{lm}^{(n)} = O(z^{-n+l})\ . \tag{C7}
$$

Then the conversion of ξ_{lm} to $X^{\rm in}_{lm\omega}$ by Eq. (2.18) yields the result Eq. (C1). Since $\epsilon = O(v^3)$ and $z = O(v)$, in order to calculate the energy- and angular-momentum fluxes up to $O(v^5)$ beyond Newtonian, we conclude that the necessary power series formulas for $X^{(n)}_{lm\omega}$ around $z = 0$ are those of $n \le 2$ for $l = 2$, $n \le 1$ for $l = 3$, and $n = 0$ for $l = 4$.

APPENDIX D

In this appendix we describe the expansion of the spheroidal harmonics ${}_{-2}S^{a\omega}_{lm}$ and their eigenvalues λ in powers of $a\omega$. Since we are interested in the energy- and angular-momentum fluxes to $O(v^5)$ in this paper, we need the expansions of ${}_{-2}S^{a\omega}_{lm}$ to $O(a\omega)$ for both $l = 2$ and 3, but those of λ to $O((a\omega)^2)$ for $l = 2$ and $O(a\omega)$ for $l = 3$, while we only need the lowest-order formulas for $l = 4$.

The spheroidal harmonics of spin weight $s = -2$ obey the equation

$$\left[\frac{1}{\sin\theta}\frac{d}{d\theta}\left\{\sin\theta\frac{d}{d\theta}\right\} - a^2\omega^2\sin^2\theta - \frac{(m-2\cos\theta)^2}{\sin^2\theta} + 4a\omega\cos\theta - 2 + 2ma\omega + \lambda\right]{}_{-2}S^{a\omega}_{lm} = 0\ . \tag{D1}$$

We assume they are normalized according to Eq. (2.2). We expand ${}_{-2}S^{a\omega}_{lm}$ and λ as

$$\begin{aligned} {}_{-2}S^{a\omega}_{lm} &= {}_{-2}P_{lm} + a\omega S^{(1)}_{lm} + (a\omega)^2 S^{(2)}_{lm} + O((a\omega)^3)\ , \\ \lambda &= \lambda_0 + a\omega\lambda_1 + a^2\omega^2\lambda_2 + O((a\omega)^3)\ , \end{aligned} \tag{D2}$$

where ${}_{-2}P_{lm}$ are the spherical harmonics of spin weight $s = -2$, $\lambda_0 = (l-1)(l+2)$, and $\lambda_1 = -2m(l^2+l+4)/(l^2+l)$ [20].

Inserting Eq. (D2) into Eq. (D1) and collecting the terms of equal orders in $a\omega$ and $(a\omega)^2$, we obtain

$$\mathcal{L}_0 S^{(1)}_{lm} + \lambda_0 S^{(1)}_{lm} = -(4\cos\theta + 2m + \lambda_1)\,{}_{-2}P_{lm}\ , \tag{D3}$$

$$\mathcal{L}_0 S^{(2)}_{lm} + \lambda_0 S^{(2)}_{lm} = -(4\cos\theta + 2m + \lambda_1) S^{(1)}_{lm} - (\lambda_2 - \sin^2\theta)\,{}_{-2}P_{lm}\ , \tag{D4}$$

where $\mathcal{L}_0$ is the operator for the spin-weighted spherical harmonics:

$$\begin{aligned} \mathcal{L}_0[{}_{-2}P_{lm}] &\equiv \left[\frac{1}{\sin\theta}\frac{d}{d\theta}\left\{\sin\theta\frac{d}{d\theta}\right\} - \frac{(m-2\cos\theta)^2}{\sin^2\theta} - 2\right]{}_{-2}P_{lm} \\ &= -\lambda_0\,{}_{-2}P_{lm}\ . \end{aligned} \tag{D5}$$

First we solve Eq. (D3) for $S^{(1)}_{lm}$. Setting

$$S^{(1)}_{lm} = \sum_{l'} c^{l'}_{lm}\,{}_{-2}P_{l'm}\ , \tag{D6}$$

we insert it into Eq. (D3), multiply it by ${}_{-2}P_{l'm}$, and integrate it over θ. Then noting the normalization of the spheroidal harmonics, we have

$$c^{l'}_{lm} = \begin{cases} \dfrac{4}{(l'-1)(l'+2)-(l-1)(l+2)}\displaystyle\int {}_{-2}P_{l'm}\cos\theta\,{}_{-2}P_{lm}\,d\cos\theta, & l' \ne l\ . \\ 0, \quad l' = l\ . \end{cases} \tag{D7}$$

Hence $c^{l'}_{lm}$ is nonzero only for $l' = l \pm 1$, and we obtain

$$\begin{aligned} c^{l+1}_{lm} &= \frac{2}{(l+1)^2}\left[\frac{(l+3)(l-1)(l+m+1)(l-m+1)}{(2l+1)(2l+3)}\right]^{1/2}, \\ c^{l-1}_{lm} &= -\frac{2}{l^2}\left[\frac{(l+2)(l-2)(l+m)(l-m)}{(2l+1)(2l-1)}\right]^{1/2}\ . \end{aligned} \tag{D8}$$

Next we consider λ_2. We set $S^{(2)}_{lm} = \sum_{l'} d^{l'}_{lm}\,{}_{-2}P_{l'm}$ and insert it into Eq. (D4). This time we multiply it by ${}_{-2}P_{lm}$ and integrate it over θ, noting that $d^{l}_{lm} = 0$. The result is

$$\lambda_2 = -4\int {}_{-2}P_{lm}\cos\theta S^{(1)}_{lm} d\cos\theta + \int {}_{-2}P_{lm}\sin^2\theta\, {}_{-2}P_{lm} d\cos\theta$$
$$= -2(l+1)(c^{l+1}_{lm})^2 + 2l(c^{l-1}_{lm})^2 + 1 - \int {}_{-2}P_{lm}\cos^2\theta\, {}_{-2}P_{lm} d\cos\theta \ . \tag{D9}$$

The last integral becomes

$$\int {}_{-2}P_{lm}\cos^2\theta\, {}_{-2}P_{lm} d\cos\theta = \frac{1}{3} + \frac{2}{3}\frac{(l+4(l-3)(l^2+l-3m^2)}{l(l+1)(2l+3)(2l-1)} \ . \tag{D10}$$

We need λ_2 only for $l = 2$. In this case, the final answer becomes

$$\lambda_2(l=2) = \frac{90-10m^2}{189} \ . \tag{D11}$$

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