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Virial relation and first law of thermodynamics in scalar-tensor theories of gravity

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Virial relations are satisfied for equilibria and quasiequilibria of self-gravitating systems in general relativity and Newtonian gravity. First-law relations are also satisfied along sequences of equilibria and quasiequilibria of self-gravitating systems. These relations are useful for analyzing numerical solutions of equilibria and quasiequilibria. We derive these relations in scalar-tensor theories of gravity in an explicit form. The implication of these relations to compact binary evolution is also discussed.

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I. INTRODUCTION

In general relativity, virial relations are satisfied for equilibria (stationary spacetime) and quasiequilibria (e.g., quasistationary spacetime of binary neutron stars and binary black holes in quasicircular orbits) of self-gravitating systems [1–5]. Also, first-law relations are satisfied for sequences of equilibria and quasiequilibria of self-gravitating systems (e.g., Refs. [4–11]). These are valuable relations for characterizing the properties of equilibria and quasiequilibria, and of their sequences as well. In numerical computation, these relations are also important because they are not trivially satisfied in general for the numerical results, and hence they can be used for checking how accurate the numerical solution is.

The virial relations in general relativity have been derived by several authors in different contexts [1–5]. These relations are useful for confirming the accuracy of the numerical solutions of rotating stars in equilibrium (e.g., Ref. [12]) and of binary neutron stars in quasiequilibrium with circular orbits (e.g., Refs. [13,14]). The virial relation in general relativity for both cases can be written as

$$M_{\rm K} = M_{\rm ADM}, \tag{1}$$

where $M_{\rm K}$ and $M_{\rm ADM}$ are the Komar mass [15] and Arnowitt-Deser-Misner (ADM) mass [16], respectively.

In this paper, we derive the virial relation in a class of scalar-tensor theories of gravity. Although the derived relation in this theory is similar to that in general relativity, the modification is in general necessary for *nonvacuum* spacetimes in which nontrivial profiles of scalar fields are present and a scalar charge, $M_{\rm S}$, is excited. Here, the scalar charge is derived from the coefficient of the monopole part of the scalar field in the asymptotic region [see, e.g., Refs. [17,18] and Eq. (38)]. In the presence of the scalar charge, the virial relation is written as (see Sec. IV)

$$M_{\rm K} = M_{\rm ADM} + 2M_{\rm S}\phi_0^{-1}, \tag{2}$$

where ϕ_0 is the value of the scalar field ϕ at spatial infinity. We will derive this relation for a stationary spacetime, and for binaries in quasiequilibrium with circular orbits in the framework of the so-called conformal flatness formulation for three-geometry [or the Isenberg-Wilson-Mathews (IWM) formulation] [19].

The first law is the relation among the variations of the mass, angular momentum, area, and surface gravity of a black hole, and the angular velocity, which is satisfied along a sequence of black holes, rotating objects, and binary systems. This law was first derived by Bardeen, Carter, and Hawking [6] for stationary axisymmetric black hole spacetimes in general relativity (see also Refs. [7,8]). It was also explored for a variety of theories of gravity [9], for a nonstationary black hole spacetime [10], and for quasiequilibrium binary systems, such as binary black holes and binary neutron stars [4,5,11]. For the binary systems in quasiequilibrium, this law in general relativity is written as [4,11]

$$\delta M_{\rm ADM} = \Omega\delta J + \frac{1}{8\pi}\sum_i \kappa_i \delta A_{\mathcal{H}i}, \tag{3}$$

with the assumption that the baryon rest mass, entropy, and vorticity of the fluid are conserved along the sequence. Here, Ω and J are the orbital angular velocity and angular momentum, and κ_i and $A_{\mathcal{H}i}$ are the surface gravity and area of ith black hole horizon, respectively. This relation is useful for clarifying the properties of binary sequences. It has also been a helpful relation for checking the accuracy of numerical results for the sequences of binary systems in quasiequilibrium with circular orbits. In this paper, we will derive the first-law relation for a class of scalar-tensor theories and show that the relation is modified in the presence of nontrivial scalar fields; e.g., for nonvacuum binary systems (i.e., binary neutron stars and black hole-neutron star binaries) in the Jordan-Brans-Dicke frame [20], the first law is written as

$$\delta M_{\rm T} = \Omega\delta J + \frac{1}{8\pi\phi_0}\kappa\delta(\phi_{\mathcal{H}} A_{\mathcal{H}}), \tag{4}$$

with the assumption that the baryon rest mass, entropy, and vorticity of the fluid are conserved. Here, $\phi_{\mathcal{H}}$ is the value of ϕ on the horizon, which is in general different from ϕ_0 (cf. the last paragraph of Sec. III A). $M_{\rm T}$ is the so-called tensor mass defined by $M_T := M_{ADM} + M_S\phi_0^{-1}$ [17,18] (see also Sec. V).

This paper is organized as follows. In Sec. II, we review a class of scalar-tensor theories of gravity, and derive basic equations in the $3+1$ formulation. We also derive the basic equations in the IWM formalism. After we review the formulas for the Komar mass and ADM mass in Sec. III, the virial relations satisfied for stationary spacetimes and for quasiequilibria in the IWM formalism are derived in Sec. IV. In Sec. V, the first law is also derived for stationary and quasiequilibrium spacetimes, extending previous results in general relativity [4,6]. Sections VI and VII are devoted to discussions and a summary. Throughout this paper, we employ the geometrical units $c = 1 = G$, where c and G are the speed of light and bare gravitational constant, respectively. Subscripts $a, b, c, \ldots$ denote the spacetime components, while i, j, k, and l denote the spatial components, respectively.

II. 3 + 1 FORMULATION FOR SCALAR-TENSOR THEORIES

A. Basic equations in the scalar-tensor theory

We briefly summarize the basic equations of a class of scalar-tensor theories of gravity with a special attention to $3+1$ formulation. Scalar-tensor theories of the simplest form are composed of the spacetime metric g_{ab} and a scalar field ϕ that determines the strength of the coupling between the matter and gravitational fields. The action for this theory is written as (see, e.g., Ref. [18] for a review)

$$I = \frac{1}{16\pi}\int\left(\phi R - \frac{\omega(\phi)}{\phi}(\nabla^a\phi)\nabla_a\phi\right)\sqrt{-g}d^4x + I_{\rm matter}, \tag{5}$$

where R and ∇_a denote the Ricci scalar and covariant derivative associated with g_{ab}, respectively, and $\omega(\phi)$ determines the strength of the coupling between the gravitational and scalar fields: ω is constant for the Brans-Dicke theory [22] and $\omega + 3/2 = (C_{\rm DEF}\ln\phi)^{-1}$ for one particularly interesting case of the theory in Refs. [23,24] with $C_{\rm DEF}$ being a constant. $I_{\rm matter}$ denotes the action associated with the matter term and is independent of ϕ. For the case of the perfect fluid, it may be written as (e.g., Ref. [4])

$$I_{\rm matter} = -\int\rho(1+\varepsilon)\sqrt{-g}d^4x, \tag{6}$$

where ρ and ε denote the rest-mass density and specific internal energy. Assuming that the matter is composed of a perfect fluid, we define the Lagrangian density by

$$\mathcal{L} := \left[\frac{1}{16\pi}\left(\phi R - \frac{\omega(\phi)}{\phi}(\nabla^a\phi)\nabla_a\phi\right) - \rho(1+\varepsilon)\right]\sqrt{-g}. \tag{7}$$

The basic equation is derived by taking the variation of the action. Employing the so-called Jordan-Brans-Dicke frame, the basic equations are written as (e.g., Ref. [18])

$$G_{ab} = 8\pi\phi^{-1}T_{ab} + \omega(\phi)\phi^{-2}\left[(\nabla_a\phi)\nabla_b\phi - \frac{1}{2}g_{ab}(\nabla_c\phi)\nabla^c\phi\right] + \phi^{-1}(\nabla_a\nabla_b\phi - g_{ab}\Box\phi), \tag{8}$$

$$\Box\phi = \frac{1}{2\omega(\phi)+3}\left[8\pi T - \frac{d\omega}{d\phi}(\nabla_c\phi)\nabla^c\phi\right], \tag{9}$$

$$\nabla_a T^a{}_b = 0, \tag{10}$$

where $\Box := \nabla_a\nabla^a$ and T_{ab} is the stress-energy tensor of the matter field, which for the perfect fluid is written as

$$T^{ab} = \rho h u^a u^b + P g^{ab}, \tag{11}$$

with $T = T_a{}^a$. Here, h is the specific enthalpy defined by $1 + \varepsilon + P/\rho$, with P being the pressure, and u^a is the four-velocity. We note that the matter is coupled only to the gravitational field in the Jordan-Brans-Dicke frame as Eq. (10) shows, and hence the equations for the matter field are the same as those in general relativity in this frame. For this reason, we perform all the calculations in the Jordan-Brans-Dicke frame, because the matter field equations—including the equations for the variation developed in general relativity—can be used without any modification.

In the following, we write Eqs. (8) and (9) in a $3+1$ formulation. The left-hand side of Eq. (9) is written as

$$\Box\phi = \frac{1}{\alpha\sqrt{\gamma}}\partial_a[\alpha\sqrt{\gamma}(\gamma^{ab} - n^a n^b)\partial_b\phi] = D_k D^k\phi + (D_k\ln\alpha)D^k\phi + (\nabla_a n^a)\Pi + n^a\partial_a\Pi, \tag{12}$$

where D_k is the covariant derivative with respect to the spatial metric $\gamma_{ab} := g_{ab} + n_a n_b$, with n^a being the unit timelike normal to the spatial hypersurface. α and β^k are the lapse function and shift vector, respectively. Π is defined by $\Pi := -n^a\nabla_a\phi$. Thus, Eq. (9) is written as the set of equations

$$(\partial_t - \beta^k\partial_k)\phi = -\alpha\Pi, \tag{13}$$

$$(\partial_t - \beta^k\partial_k)\Pi = -\alpha D_i D^i\phi - (D_i\alpha)D^i\phi + \alpha K\Pi + \frac{\alpha}{2\omega+3}\left[8\pi T - \frac{d\omega}{d\phi}(\nabla_c\phi)\nabla^c\phi\right], \tag{14}$$

where K is the trace part of the extrinsic curvature, K_{ab}.

The basic equations in the $3+1$ formulation for the gravitational field equation is simply derived by operating

$n^a n^b$, $n^a \gamma^b{}_i$, and $\gamma^a{}_i \gamma^b{}_j$ to Eq. (8). The left-hand side and the first term on the right-hand side are the same as those in general relativity, and hence we focus only on the terms associated with the scalar field on the right-hand side. Here, only the nontrivial part is $\nabla_a \nabla_b \phi$. For this, we have

$$n^a n^b \nabla_a \nabla_b \phi = \frac{1}{\alpha}[-\partial_t \Pi + \beta^i \partial_i \Pi - (D^i \alpha) D_i \phi] = D_i D^i \phi - K\Pi - \frac{1}{2\omega + 3}\left[8\pi T - \frac{d\omega}{d\phi}(\nabla_c \phi)\nabla^c \phi\right], \quad (15)$$

where we have used Eq. (14) and

$$n^a \gamma^b{}_j \nabla_a \nabla_b \phi = -D_j \Pi + K^i{}_j D_i \phi, \quad (16)$$

$$\gamma^a{}_i \gamma^b{}_j \nabla_a \nabla_b \phi = D_i D_j \phi - K_{ij} \Pi. \quad (17)$$

Equations (14) and (15) yield

$$n^a n^b \nabla_a \nabla_b \phi + \Box \phi = D_i D^i \phi - K\Pi. \quad (18)$$

Thus, the Hamiltonian constraint is written as

$$^{(3)}R + K^2 - K_{ij}K^{ij} = 16\pi \phi^{-1} \rho_{\rm h} + \omega \phi^{-2}[\Pi^2 + (D_i \phi) D^i \phi] + 2\phi^{-1}(-K\Pi + D_i D^i \phi), \quad (19)$$

where $^{(3)}R$ is the Ricci scalar on the spatial hypersurface and we have used

$$(\nabla_a \phi)\nabla^a \phi = (D_i \phi) D^i \phi - \Pi^2 \quad (20)$$

and $\rho_{\rm h} := T_{ab} n^a n^b$.

The momentum constraint is written as

$$D_i K^i{}_j - D_j K = 8\pi \phi^{-1} J_j + \omega \phi^{-2} \Pi D_j \phi + \phi^{-1}(D_j \Pi - K^i{}_j D_i \phi), \quad (21)$$

and the evolution equation of K_{ij} is

$$\partial_t K_{ij} = \alpha {}^{(3)}R_{ij} - 8\pi \alpha \phi^{-1}\left[S_{ij} - \frac{1}{2}\gamma_{ij}(S - \rho_{\rm h})\right] + \alpha(-2K_{ik}K_j{}^k + KK_{ij}) - D_i D_j \alpha + \beta^k D_k K_{ij} + K_{ik} D_j \beta^k + K_{kj} D_i \beta^k - \alpha \omega \phi^{-2}(D_i \phi) D_j \phi - \alpha \phi^{-1}\left[D_i D_j \phi - K_{ij}\Pi + \frac{\gamma_{ij}}{2(2\omega + 3)}\left\{8\pi T + \frac{d\omega}{d\phi}(\Pi^2 - (D_k \phi) D^k \phi)\right\}\right], \quad (22)$$

where $J_i = -T_{ab} n^a \gamma^b{}_i$, $S_{ij} = T_{ab}\gamma^a{}_i \gamma^b{}_j$, $S = S_{ij}\gamma^{ij}$, and $^{(3)}R_{ij}$ is the Ricci tensor with respect to γ_{ij}. Equation (22) together with the Hamiltonian constraint yields the following evolution equation for K:

$$(\partial_t - \beta^k \partial_k) K = 4\pi \alpha \phi^{-1}(S + \rho_{\rm h}) + \alpha K_{ij} K^{ij} - D_i D^i \alpha + \alpha \omega \phi^{-2} \Pi^2 + \alpha \phi^{-1}\left[D_i D^i \phi - K\Pi - \frac{3}{2(2\omega + 3)}\left\{8\pi T + \frac{d\omega}{d\phi}(\Pi^2 - (D_k \phi) D^k \phi)\right\}\right]. \quad (23)$$

B. Conformal flatness approximation

In Sec. IV B, we will derive a virial relation for a stationary and quasiequilibrium spacetime in the conformal flatness (IWM) formulation [19], in which we impose the condition

$$\gamma_{ij} = \psi^4 f_{ij}, \quad (24)$$

where f_{ij} is the flat spatial metric. This formulation has often been employed for obtaining binary systems in quasiequilibrium with circular orbits. In this subsection, we summarize the basic equations in the IWM formalism of scalar-tensor theories.

In this formulation, the basic equations for the tensor field are obtained from the Hamiltonian and momentum constraints, and Eq. (23) with the slicing condition $K = 0$. Except for the modification resulting from the terms associated with the scalar field ϕ, the equations are essentially the same as those in general relativity: the Hamiltonian and momentum constraints are, respectively, written as

$$\overset{(0)}{\Delta} \psi = -2\pi \phi^{-1} \rho_{\rm h} \psi^5 - \frac{1}{8}\tilde{A}_{ij}\tilde{A}^{ij}\psi^5 - \frac{\psi^5}{8}[\omega \phi^{-2}\{\Pi^2 + (D_i \phi) D^i \phi\} + 2\phi^{-1} D_i D^i \phi] \quad (25)$$

and

$$\overset{(0)}{D}_i(\psi^6 \tilde{A}^i{}_j) = \psi^6\left[8\pi \phi^{-1} J_j + \omega \phi^{-2} \Pi \overset{(0)}{D}_j \phi + \phi^{-1}(\overset{(0)}{D}_j \Pi - \tilde{A}^i{}_j \overset{(0)}{D}_i \phi)\right], \quad (26)$$

where $\overset{(0)}{\Delta}$ and $\overset{(0)}{D}_i$ are the Laplacian and covariant derivative with respect to f_{ij}. $\tilde{A}_{ij}$ is the trace-free conformal extrinsic curvature satisfying $K_i{}^j = \tilde{A}_i{}^j$ for $K = 0$ and its equation is derived from the evolution equation for γ_{ij} with Eq. (24) as

$$\tilde{A}_{ij} = \frac{1}{2\alpha}\left(f_{ik}\overset{(0)}{D}_j \beta^k + f_{jk}\overset{(0)}{D}_i \beta^k - \frac{2}{3} f_{ij} \overset{(0)}{D}_k (0) D_k \beta^k\right), \quad (27)$$

where the indices of $\tilde{A}_{ij}$, $\tilde{A}^{ij}$, and $\overset{(0)}{D}_i$ are raised and lowered by f^{ij} and f_{ij}. [The reason that we define $\tilde{A}_{ij}$ (not $\psi^6 \tilde{A}_{ij}$) is that it is often employed in numerical relativity.]

The condition $K = 0$ yields

$$\overset{(0)}{\Delta}\chi = \chi\psi^4\bigg[2\pi\phi^{-1}(2S+\rho_{\rm h}) + \frac{7}{8}\tilde{A}_{ij}\tilde{A}^{ij} + \frac{1}{8}\omega\phi^{-2}\{7\Pi^2 - (D_i\phi)D^i\phi\} + \frac{3}{4\phi}\bigg\{D_iD^i\phi - \frac{2}{(2\omega+3)}\bigg(8\pi T + \frac{d\omega}{d\phi}(\Pi^2 - (D_k\phi)D^k\phi)\bigg)\bigg\}\bigg], \tag{28}$$

where $\chi := \alpha\psi$. Note that the Laplacian term of $D_iD^i\phi$ will be replaced using the equation for ϕ (see below).

In addition to these equations, we have to solve the equation for ϕ. For this, we have to impose a condition for Π and ϕ. For stationary spacetimes, we simply impose the conditions

$$\partial_t\Pi = 0 = \partial_t\phi. \tag{29}$$

For this case, Eq. (14) is written to an elliptic equation for ϕ.

For binaries in quasicircular orbits, a helically symmetric condition is often imposed (e.g., Ref. [4]). This symmetry requires that in the frame corotating with the orbital angular velocity Ω, the binary is assumed to be in a stationary state. The helically symmetric condition for ϕ is written as

$$(\partial_t + \Omega\partial_\varphi)\phi = 0. \tag{30}$$

By this, Eq. (13) results in

$$\Pi = -\alpha^{-1}(\Omega\partial_\varphi + \beta^i\partial_i)\phi. \tag{31}$$

For this case, the equation for ϕ is not elliptic but of the Helmholtz type, and hence $D_i\phi$ and Π of Eq. (14) behave as $\propto r^{-1}$ in the far zone because standing waves for ϕ are present. Then, the spacetime cannot be asymptotically flat because in the Hamiltonian constraint there exist terms on the right-hand side that are proportional to Π^2 and $(D_i\phi)D^i\phi$. This implies that Π and $D_i\phi$ have to be of order r^{-2} in the far zone for the spacetime to be asymptotically flat. To guarantee this condition, the simplest choice is Eq. (29) or $\Pi = 0$. Then, Eq. (14) becomes an elliptic-type equation and $D_i\phi = O(r^{-2})$ in the far zone is guaranteed [$\Pi = O(r^{-3})$ for $\beta^i = O(r^{-1})$]. The boundary condition to be imposed for ϕ is $\phi \to \phi_0 + O(r^{-1})$ for $r \to \infty$, where ϕ_0 is a constant with $0 \le \phi_0 - 1 \ll 1$ and $\phi = \phi_{\mathcal{H}} =$ const on the horizon in the presence of black holes (see also subsequent sections). We note that the resulting elliptic equation for ϕ may be substituted into the right-hand sides of Eqs. (25) and (28).

For the fluid part, the hydrostatic equations are the same as those in general relativity in the Jordan-Brans-Dicke frame. Thus, the first integrals of the hydrodynamics equations is also the same as those in general relativity (see, e.g., Refs. [5,13,14]).

III. ADM MASS AND KOMAR MASS

A. Stationary spacetime

Here, we remind the readers of the definition of the ADM mass and Komar mass, because they are used in the following sections.

The ADM mass is defined by [16]

$$M_{\rm ADM} := \frac{1}{16\pi}\oint_\infty f^{jk}f^{il}(\partial_k\gamma_{ij} - \partial_i\gamma_{jk})\overset{(0)}{dS_l}, \tag{32}$$

where $\overset{(0)}{dS_l}$ is the surface integral operator in the flat space and $\oint_\infty$ denotes $\oint_{r\to\infty}$. In the conformally flat spatial hypersurface, it may be defined by

$$M_{\rm ADM} = -\frac{1}{2\pi}\oint_\infty Qf^{jk}\partial_k\psi\overset{(0)}{dS_j}, \tag{33}$$

where Q is a function that is unity at $r \to \infty$.

The ADM mass cannot be defined in a covariant way and can be calculated only in a special class of gauge conditions in which the following asymptotic conditions at spatial infinity in the Cartesian coordinates have to be satisfied (e.g., Ref. [25]):

$$g_{ab} - \eta_{ab} = O(r^{-1}), \tag{34}$$

$$\partial_k\gamma_{ij} = O(r^{-2}), \tag{35}$$

$$K_{ij} = O(r^{-2}), \tag{36}$$

$$\partial_k K_{ij} = O(r^{-3}), \tag{37}$$

where η_{ab} is the flat spacetime metric. In the following, we implicitly choose a gauge condition in which the conditions (34)–(37) are satisfied. We also assume the following asymptotic behavior of ϕ at $r \to \infty$:

$$\phi = \phi_0 + \frac{2M_{\rm S}}{r} + O(r^{-2}), \tag{38}$$

where ϕ_0 is a constant close to unity. In addition, we impose the condition that ϕ is constant on the black hole horizon and denote it as $\phi_{\mathcal{H}}$, based on the result of Refs. [21,26].

In the presence of a timelike Killing vector, ξ^a, the Komar mass is defined [15]. In general relativity, it is defined in the covariant way as

$$M_{\rm K} := \frac{1}{4\pi}\oint_{\mathcal{S}} dS_a n_b \nabla^a\xi^b, \tag{39}$$

where the closed surface $\mathcal{S}$ is usually taken in an asymptotically flat region of a spatial hypersurface Σ_t and dS_a denotes the surface integral operator in the general curved space. In the 3 + 1 formulation, the Komar mass is written as

$$M_{\rm K} = \frac{1}{4\pi}\oint_S dS_k(D^k\alpha - \beta^l K_l{}^k), \qquad (40)$$

where have we used $n_a\xi^a = -\alpha$. For gauge conditions in which Eqs. (34)–(37) are satisfied, we have

$$M_{\rm K} = \frac{1}{4\pi}\oint_\infty dS_i D^i\alpha. \qquad (41)$$

In the scalar-tensor theory considered in the present paper, it is more convenient for the following calculation to write the Komar mass by

$$M_{\rm K} := -\frac{1}{4\pi\phi_0}\oint_\infty dS_a n_b \phi\nabla^a\xi^b. \qquad (42)$$

Performing the integral by parts yields

$$\begin{aligned}\phi_0 M_{\rm K} &= -\frac{1}{4\pi}\Big[\oint_{\mathcal{H}} dS_a n_b\phi\nabla^a\xi^b \\ &\quad + \int_{\Sigma_t'} dV n_b\{-\phi R^b{}_a\xi^a + (\nabla_a\phi)\nabla^a\xi^b\}\Big] \\ &= -\frac{1}{4\pi}\oint_{\mathcal{H}} dS_a n_b\phi\nabla^a\xi^b + 2\int_{\Sigma_t'} dV n_b T^b{}_a\xi^a \\ &\quad -\frac{1}{8\pi}\int_{\Sigma_t'} dV\alpha(\phi R - \omega\phi^{-1}(\nabla_c\phi)\nabla^c\phi - 2\Box\phi),\end{aligned} \qquad (43)$$

where $\mathcal{H}$ and Σ_t' denote black hole horizons and the entire spatial hypersurface except for the inside of the horizons, and $dV = \sqrt{\gamma}d^3x$. We used the fact that $\xi^a\nabla_a\phi = 0$ because ξ^a is a timelike Killing vector. Note that the last term of Eq. (43) is written by the action of this theory, and hence the variation of it is calculated in a straightforward manner. This is the reason why the Komar mass is defined in the form of Eq. (42).

In the rest of this subsection, we only consider axisymmetric spacetimes in which the circularity condition is satisfied [7]. In the condition that the value of the scalar field is constant, $\phi = \phi_{\mathcal{H}}$, on the horizon surface [21,26], the first (surface-integral) term of Eq. (43) is written—following the manipulation of Refs. [6–8]—as

$$2\phi_{\mathcal{H}}\Big(\frac{1}{8\pi}\kappa A_{\mathcal{H}} + \Omega_{\mathcal{H}}J_{\mathcal{H}}\Big), \qquad (44)$$

where κ, $A_{\mathcal{H}}$, $\Omega_{\mathcal{H}}$, and $J_{\mathcal{H}}$ are the surface gravity, area, angular velocity, and spin angular momentum of a black hole, respectively (here we restrict our attention to the case where only one black hole is present). Here $J_{\mathcal{H}}$ is defined by

$$J_{\mathcal{H}} := \frac{1}{8\pi}\oint_{\mathcal{H}} dS_a n_b\nabla^a\varphi^b, \qquad (45)$$

where $\varphi^a = (\partial/\partial\varphi)^a$ is the spacelike Killing vector. κ is defined by $\kappa := -\ell^a\chi^b\nabla_b\chi_a$, where $\chi^a := \xi^a + \Omega_{\rm H}\varphi^a$ and ℓ^a is a null vector orthogonal to the horizon surface satisfying $\chi_a\ell^a = -1$ on the horizon [6]. Note that χ^a is a null vector tangent to the horizon on the horizon surface.

$\Box\phi$ is written, in the presence of a timelike Killing vector, as

$$\alpha\Box\phi = D_a(\alpha\nabla^a\phi). \qquad (46)$$

This leads to (see, e.g., Ref. [27] for the surface integral on the horizon in terms of the null vectors)

$$\begin{aligned}\frac{1}{4\pi}\int_{\Sigma_t'} dV\alpha\Box\phi &= \frac{1}{4\pi}\int dV_{\Sigma_t'} D_k(\alpha\nabla^k\phi) \\ &= \frac{1}{4\pi}\oint_\infty \overset{(0)}{dS_k}\sqrt{\gamma}\alpha D^k\phi \\ &\quad - \frac{1}{4\pi}\oint_{\mathcal{H}} dA\chi^a\nabla_a\phi \\ &= -2M_{\rm S}.\end{aligned} \qquad (47)$$

Here dA is the area element. Since χ^a is a Killing vector, $\chi^a\nabla_a\phi = 0$ [21]. We note that in the spacetime we consider here, αn^a agrees with χ^a on the horizon.

The resulting form of Eq. (43) can be written in several ways. Using the relation [4]

$$T_{ab}n^a\xi^b = \alpha\rho(1+\varepsilon) - \rho h(u^a n_a)u_j v^j, \qquad (48)$$

where $v^j = u^j/u^t$, we have

$$\begin{aligned}\phi_0 M_{\rm K} &= 2\phi_{\mathcal{H}}\Big(\frac{1}{8\pi}\kappa A_{\mathcal{H}} + \Omega_{\mathcal{H}}J_{\mathcal{H}}\Big) - 2M_{\rm S} \\ &\quad - 2\int dV\Big[\frac{\alpha}{\sqrt{-g}}\mathcal{L} + \rho h(u^a n_a)u_j v^j\Big].\end{aligned} \qquad (49)$$

This will be used when the first law is derived. Using the relation

$$T_{ab}n^a\xi^b = \alpha\rho_{\rm h} - \beta^k J_k, \qquad (50)$$

and

$$\phi R = -8\pi T + \frac{\omega}{\phi}(\nabla_c\phi)\nabla^c\phi + 3\Box\phi, \qquad (51)$$

we may also have

$$\begin{aligned}\phi_0 M_{\rm K} &= 2\phi_{\mathcal{H}}\Big(\frac{1}{8\pi}\kappa A_{\mathcal{H}} + \Omega_{\mathcal{H}}J_{\mathcal{H}}\Big) + M_{\rm S} \\ &\quad + \int dV(\alpha\rho_{\rm h} - 2\beta^k J_k + \alpha S).\end{aligned} \qquad (52)$$

It is worthy to note that in the presence of scalar fields, the scalar charge, $M_{\rm S}$, and the scalar fields on the horizon and at spatial infinity, $\phi_{\mathcal{H}}$ and ϕ_0, which are not always unity, modify the expression of $M_{\rm K}$. Such nontrivial scalar fields are induced unless $T = 0$, and hence in the presence of matter fields this modification in general occurs. We note, on the other hand, that in the absence of the matter field (or for $T = 0$) the scalar field should be uniform [21]; $M_{\rm S} = 0$ and $\phi_{\mathcal{H}} = \phi_0$. For this case, the result in general relativity is recovered (after the rescaling of G by G/ϕ_0).

B. Helically symmetric spacetime

In the helically symmetric spacetime, the presence of the following Killing vector field is often assumed [4]:

$$k^a = t^a + \Omega \varphi^a = \left(\frac{\partial}{\partial t}\right)^a + \Omega\left(\frac{\partial}{\partial \varphi}\right)^a, \tag{53}$$

where Ω is an angular velocity which is equal to the orbital one for binary systems in quasicircular orbits. t^a and φ^a are not Killing vectors in the present subsection, although t^a is assumed to be asymptotically a timelike Killing vector at $r \to \infty$. k^a is assumed to be a null vector on the black hole horizon, $k^a k_a = 0$.

In the scalar-tensor theory considered in the present paper, we will consider the following relation (a generalized Smarr formula [28]) in the subsequent sections:

$$M_{\rm K} - 2\Omega J = -\frac{1}{4\pi\phi_0}\oint_\infty dS_a n_b \phi \nabla^a k^b, \tag{54}$$

where J is the angular momentum defined—assuming the asymptotic behavior of Eqs. (34)–(37)—by (e.g., Ref. [25])

$$J := \frac{1}{8\pi}\oint_\infty dS_a n_b \nabla^a \varphi^b = \frac{1}{8\pi}\oint_\infty dS_a K^a{}_b \varphi^b. \tag{55}$$

We note that $M_{\rm K}$ is not always "the mass" in this case. However, it is a useful quantity for the manipulation used when the first law is derived.

Using the same manipulation as that used in Sec. III A, we have [compare with Eq. (49)]

$$\phi_0(M_{\rm K} - 2\Omega J) = \frac{1}{4\pi}\phi_{\mathcal{H}}\kappa A_{\mathcal{H}} - 2M_{\rm S} - 2\int dV\left[\frac{\alpha}{\sqrt{-g}}\mathcal{L} + \rho h(u^a n_a)u_j v^j\right]. \tag{56}$$

Here, u^a is decomposed as $u^a = u^t(k^a + v^a)$ [4] and $\kappa := -\ell^a k^b \nabla_b k_a$, with ℓ^a being a null vector orthogonal to the horizon surface satisfying $k_a \ell^a = -1$ on the horizon.

IV. VIRIAL RELATIONS

A. Virial relation for general stationary spacetimes

As mentioned in Sec. I, the Komar mass and ADM mass agree with each other for stationary spacetimes in general relativity: the relation $M_{\rm ADM} = M_{\rm K}$ is a necessary condition for a stationary spacetime. In this section, we derive the relation (2) that is satisfied in the scalar-tensor theory described in Sec. II, following the prescription developed by Beig [1], who derived a necessary condition for the stationarity using the evolution equation for the extrinsic curvature.

From the evolution equation of K_{ij}, Eq. (22), the following equation is derived:

$$\begin{aligned}\partial_t K_{ij} - \frac{1}{2}\gamma_{ij}\partial_t K = {}& \alpha {}^{(3)}G_{ij} - \left(D_i D_j \alpha - \frac{1}{2}\gamma_{ij} D_k D^k \alpha\right) + \alpha\left(K K_{ij} - 2K_{ik}K^k{}_j - \frac{1}{2}\gamma_{ij}K^2\right) + \beta^k D_k\left(K_{ij} - \frac{1}{2}\gamma_{ij}K\right) \\ &+ K_{ik}D_j\beta^k + K_{jk}D_i\beta^k - 8\pi\alpha\phi^{-1}\left(S_{ij} - \frac{1}{4}\gamma_{ij}(S+\rho_{\rm h})\right) - \alpha\omega\phi^{-2}\left(D_i\phi D_j\phi - \frac{1}{2}\gamma_{ij}(D_k\phi)D^k\phi\right) \\ &- \alpha\phi^{-1}\left[D_i D_j\phi - \frac{1}{2}\gamma_{ij}D_k D^k\phi - \Pi\left(K_{ij} - \frac{1}{2}\gamma_{ij}K\right) - \frac{1}{4}\gamma_{ij}\Box\phi\right],\end{aligned} \tag{57}$$

where ${}^{(3)}G_{ij}$ denotes the Einstein tensor on the spatial hypersurface. Using the prescription of Landau and Lifshitz [29], ${}^{(3)}G^{ij}$ is written as

$$2{}^{(3)}G^{ij} = -16\pi {}^{(3)}t^{ij}_{\rm LL} + \gamma^{-1}\partial_k\partial_l {}^{(3)}H^{ikjl}, \tag{58}$$

where ${}^{(3)}H^{ikjl}$ is a superpotential on the spatial hypersurface defined by

$${}^{(3)}H^{ikjl} := \gamma(\gamma^{ij}\gamma^{kl} - \gamma^{il}\gamma^{jk}), \tag{59}$$

and ${}^{(3)}t^{ij}_{\rm LL}$ is the three-dimensional Landau-Lifshitz pseudotensor. We then define $h^{ikj} = \partial_l {}^{(3)}H^{ikjl}$ for the subsequent calculations, where $h^{ikj} = -h^{kij}$.

In the following, we assume that the following asymptotic conditions are satisfied so that the ADM mass can be defined [cf. Eqs. (34)–(37)]:

$$\alpha = 1 - \frac{M_{\rm K}}{r} + O(r^{-2}), \tag{60}$$

$$\beta^i = O(r^{-1}), \tag{61}$$

$$\gamma_{ij} = f_{ij} + O(r^{-1}), \tag{62}$$

$$\partial_k \gamma_{ij} = O(r^{-2}), \tag{63}$$

$$K_{ij} = O(r^{-2}), \tag{64}$$

$$\partial_k K_{ij} = O(r^{-3}). \tag{65}$$

Note that the leading part of K_{ij} is assumed to be composed only of the ADM linear momentum. We also assume the asymptotic condition of Eq. (38) for ϕ. These assumptions yield the following results: (i) the evolution equation for K_{ij} implies $\partial_t K_{ij} = O(r^{-3})$; (ii) Eq. (60) implies $D_k D^k \alpha = O(r^{-4})$; (iii) ${}^{(3)}t^{ij}_{\rm LL} = O(r^{-4})$; (iv) Eq. (38) implies $D_k D^k\phi = O(r^{-4})$ and $\Box\phi = O(r^{-4})$.

Following Ref. [1], we then evaluate the surface integral at spatial infinity as

$$\begin{aligned} I_1 &= \frac{1}{8\pi}\oint_\infty f_{jk}x^{k(3)}G^{ij}\overset{(0)}{dS_i} \\ &= \frac{1}{16\pi}\oint_\infty f_{jl}x^l\partial_k h^{ikj}\overset{(0)}{dS_i} \\ &= \frac{1}{16\pi}\oint_{\mathcal{H}} f_{jl}x^l\partial_k h^{ikj}\overset{(0)}{dS_i} + \frac{1}{16\pi}\int_{\Sigma_t'}\partial_i(f_{jl}x^l\partial_k h^{ikj})d^3x, \end{aligned} \tag{66}$$

where $\overset{(0)}{dS_l} = dS_l$ for $r\to\infty$. From the identity $\partial_i\partial_k h^{ikj} = 0$, which holds because of the antisymmetry of $h^{ikj} = -h^{kij}$, we have

$$\partial_i(f_{jl}x^l\partial_k h^{ikj}) = f_{ij}\partial_k h^{ikj}. \tag{67}$$

Then,

$$\begin{aligned} I_1 &= \frac{1}{16\pi}\left(\oint_{\mathcal{H}} f_{jl}x^l\partial_k h^{ikj}\overset{(0)}{dS_i} + \int_{\Sigma_t'} f_{jl}\partial_i h^{jil}d^3x\right) \\ &= \frac{1}{16\pi}\left(\oint_{\mathcal{H}} f_{jl}(x^l\partial_k h^{ikj} - h^{jil})\overset{(0)}{dS_i} + \oint_\infty f_{jl}h^{jil}\overset{(0)}{dS_i}\right) \\ &= \frac{1}{16\pi}\left(\oint_{\mathcal{H}} f_{jl}\partial_k(x^l h^{ikj})\overset{(0)}{dS_i} + \oint_\infty f_{jl}h^{jil}\overset{(0)}{dS_i}\right) \\ &= \frac{1}{16\pi}\oint_\infty f^{il}f^{jk}(\partial_i\gamma_{jk} - \partial_k\gamma_{ij})\overset{(0)}{dS_l} =: -M_{\rm ADM}, \end{aligned} \tag{68}$$

where in the last line we have used the definition of the ADM mass. We also used the relation for an antisymmetric tensor $T^{ij}_{\rm ant}$ in flat spaces,

$$\oint \partial_i T^{ij}_{\rm ant}\overset{(0)}{dS_j} = 0, \tag{69}$$

and the asymptotic behavior of $f_{jk}h^{jik}$,

$$f_{jk}h^{jik} \to f^{il}f^{jk}(\partial_l\gamma_{jk} - \partial_k\gamma_{jl}). \tag{70}$$

Note that due to the presence of Eq. (69), the horizon-surface integral term does not appear in Eq. (68).

Using Eq. (57) together with the conditions (60) and (38), I_1 is written as

$$\begin{aligned} I_1 &= \frac{1}{8\pi}\oint_\infty f_{ik}x^k\left(\partial_t K^{ij} - \frac{1}{2}\gamma^{ij}\partial_t K\right)\overset{(0)}{dS_j} \\ &\quad + \frac{1}{8\pi}\oint_\infty x^i(D_iD^j\alpha + \alpha\phi^{-1}D_iD^j\phi)\overset{(0)}{dS_j}, \end{aligned} \tag{71}$$

where we have discarded the terms that are vanishing. Then, the second term of Eq. (71) is calculated to yield

$$\begin{aligned} &\oint_\infty f^{jk}x^i(\partial_i\partial_j\alpha + \phi^{-1}\partial_i\partial_j\phi)\overset{(0)}{dS_k} \\ &\quad = 8\pi(-M_{\rm K} + 2\phi_0^{-1}M_{\rm S}). \end{aligned} \tag{72}$$

Finally, we obtain

$$\begin{aligned} &\frac{1}{8\pi}\oint_\infty f_{ik}x^k\left(\partial_t K^{ij} - \frac{1}{2}\gamma^{ij}\partial_t K\right)\overset{(0)}{dS_j} \\ &\quad = M_{\rm K} - M_{\rm ADM} - 2\phi_0^{-1}M_{\rm S}. \end{aligned} \tag{73}$$

Therefore, for stationary spacetimes for which $\partial_t K^{ij} = 0 = \partial_t K$, the virial relation (2) is a necessary condition [30].

The scalar charge is in general induced in the presence of matter unless $T = 0$, and hence the virial relation is in general different than it is in general relativity for a non-vacuum spacetime. We note that a similar relation is well known for asymptotically flat systems (e.g., Ref. [17]). However, to our knowledge, the virial relation has not yet been explicitly written in the form of Eq. (2).

B. Virial relation for quasiequilibrium in the IWM formalism

Following Appendix A of Ref. [4] in a straightforward manner, we can derive a virial relation for equilibrium and quasiequilibrium states in the IWM formalism. Throughout this section, we will use Cartesian coordinates (t, x^i) for which f_{ij} has components $f_{ij} = \delta_{ij}$. In the IWM formalism, instead of solving the full Einstein's equations, one solves only the Hamiltonian constraint, the momentum constraint, and the equation for the maximal slicing condition. As written in Sec. II B, the extrinsic curvature in this set of equations is replaced by its trace-free part, $\tilde{A}^{ij}$, which is written by the shift vector; see Eq. (27).

In the following, we restrict our attention to the case where black holes are absent. Thus, hereafter $\oint$ without a specification of a surface denotes a surface integral at spatial infinity, $\oint = \oint_\infty$. In the presence of black holes, we have to take care with the boundary condition on the horizons, for which we have to impose physical conditions for χ, ψ, and β^i (e.g., Ref. [31]).

In the IWM formalism, the evolution equation for K_{ij}, Eq. (22), is not satisfied in general. This implies that we cannot use the same method as that in Sec. IV A for deriving the virial relation. Thus, we will attempt to directly perform an integral of the equation of motion for the matter-field equation to derive the virial relation, as is often done in Newtonian gravity.

We here employ the equation of motion in the form

$$\nabla_a\tau^a{}_b = 0, \tag{74}$$

where $\tau^a{}_b$ is composed of the perfect fluid and scalar field as [see Eq. (9)]

$$\tau_{ab} := \phi^{-1}T_{ab} + \frac{1}{8\pi}\Big[\omega(\phi)\phi^{-2}\Big\{(\nabla_a\phi)\nabla_b\phi - \frac{1}{2}g_{ab}(\nabla_c\phi)\nabla^c\phi\Big\} + \phi^{-1}(\nabla_a\nabla_b\phi - g_{ab}\Box\phi)\Big]. \tag{75}$$

We note that in the IWM formalism of the scalar-tensor theory that we consider, Eq. (74) is solved and satisfied. Then, defining

$$\hat\rho_{\rm h} := \tau_{ab}n^a n^b, \qquad \hat J_i := -\tau_{ab}n^a\gamma^b{}_i,$$
$$\hat S_{ij} := \tau_{ab}\gamma^a{}_i\gamma^b{}_j, \qquad \hat S := \tau_{ab}\gamma^{ab},$$

we obtain the equation of motion ($\gamma^a{}_i\nabla_b\tau^b{}_a = 0$) in the form

$$\partial_t(\hat J_i\psi^6) + \partial_j[\psi^6(\alpha\hat S^j{}_i - \hat J_i\beta^j)] + \hat\rho_{\rm h}\psi^5\partial_i\chi - (\hat\rho_{\rm h} + 2\hat S)\chi\psi^4\partial_i\psi - \psi^6\hat J_k\partial_i\beta^k = 0. \tag{76}$$

As in the Newtonian case, we then evaluate

$$\int d^3x x^i\partial_t(\hat J_i\psi^6) = -\int d^3x x^i\Big[\partial_j[\psi^6(\alpha\hat S^j{}_i - \hat J_i\beta^j)] + \hat\rho_{\rm h}\psi^5\partial_i\chi - (\hat\rho_{\rm h} + 2\hat S)\chi\psi^4\partial_i\psi - \psi^6\hat J_k\partial_i\beta^k\Big], \tag{77}$$

and consider the condition to be satisfied for the vanishing right-hand side because—for the stationary spacetime—the left-hand side should be vanishing. Here, we note that by using the momentum constraint the left-hand side is rewritten as

$$\int d^3x x^i\partial_t(\hat J_i\psi^6) = \frac{d}{dt}\int d^3x x^i\partial_j(\psi^6\tilde A^j{}_i) = \frac{d}{dt}\oint \overset{(0)}{dS_j} x^i\psi^6\tilde A^j{}_i, \tag{78}$$

where we have used $\tilde A_i{}^i = 0$. Thus, we are going to essentially integrate the evolution equation for K_{ij} as in Sec. IV A.

Before proceeding, we rewrite the basic equations for the gravitational field as [see Eqs. (25), (26), and (28)]

$$\overset{(0)}{\Delta}\psi = -2\pi\psi^5\hat\rho_{\rm h} - \frac{\psi^5}{8}\tilde A_{ij}\tilde A^{ij} =: -S_\psi, \tag{79}$$

$$\partial_j(\psi^6\tilde A_i{}^j) = 8\pi\hat J_i\psi^6, \tag{80}$$

$$\overset{(0)}{\Delta}\chi = 2\pi\chi\psi^4(\hat\rho_{\rm h} + 2\hat S) + \frac{7}{8}\chi\psi^4\tilde A_{ij}\tilde A^{ij} =: S_\chi. \tag{81}$$

As before, the asymptotic behavior of the geometric and scalar variables is assumed to be

$$\psi = 1 + \frac{M_{\rm ADM}}{2r} + O(r^{-2}), \tag{82}$$

$$\chi = 1 - \frac{M_\chi}{2r} + O(r^{-2}), \tag{83}$$

$$\beta^i = O(r^{-2}), \tag{84}$$

$$\tilde A_i{}^j = O(r^{-3}), \tag{85}$$

$$\phi = \phi_0 + \frac{2M_{\rm S}}{r} + O(r^{-2}), \tag{86}$$

$$\Pi = O(r^{-3}). \tag{87}$$

For this case, we assume for simplicity that the total ADM three-momentum vanishes,

$$P_i := \frac{1}{8\pi}\oint_\infty \tilde A_i{}^j \overset{(0)}{dS_j} = 0. \tag{88}$$

From the momentum constraint (80), we also have

$$0 = \int \hat J_i\psi^6 d^3x. \tag{89}$$

M_χ and $M_{\rm ADM}$ can be defined by the surface integrals

$$M_\chi = \frac{1}{2\pi}\oint_\infty \delta^{ij}\psi\partial_i\chi \overset{(0)}{dS_j}, \qquad M_{\rm ADM} = -\frac{1}{2\pi}\oint_\infty \delta^{ij}\chi\partial_i\psi \overset{(0)}{dS_j}. \tag{90}$$

Using Gauss's law, M_χ and $M_{\rm ADM}$ can be rewritten as

$$M_\chi = \frac{1}{2\pi}\int(\psi S_\chi + \delta^{ij}\partial_i\chi\partial_j\psi)d^3x, \tag{91}$$

$$M_{\rm ADM} = \frac{1}{2\pi}\int(\chi S_\psi - \delta^{ij}\partial_i\psi\partial_j\chi)d^3x. \tag{92}$$

We now derive the virial relation for equilibria and quasiequilibria. As a first step, we derive a relation that will be used several times in the calculations that follow. From $\chi\psi^5\tilde A_i{}^j\tilde A_j{}^i = \psi^6\tilde A_i{}^j\partial_j\beta^i$, we have

$$\int\chi\psi^5\tilde A_i{}^j\tilde A_j{}^i d^3x = \int\psi^6\tilde A_i{}^j\partial_j\beta^i d^3x = -\int\partial_j(\psi^6\tilde A_i{}^j)\beta^i d^3x + \oint\psi^6\tilde A_i{}^j\beta^i\overset{(0)}{dS_j} = -8\pi\int\psi^6\hat J_i\beta^i d^3x, \tag{93}$$

where we have used the asymptotic behaviors at $r\to\infty$ and Eq. (80) to obtain the last line.

Using Eqs. (91)–(93), the difference between $M_{\rm ADM}$ and M_χ is written as

$$M_\chi - M_{\rm ADM} = \frac{1}{\pi}\int\!\!\Big[2\pi\chi\psi^5\hat{S} + \frac{3}{8}\chi\psi^5\tilde{A}_i{}^j\tilde{A}_j{}^i + \delta^{ij}\partial_i\psi\partial_j\chi\Big]d^3x = 2\int\!\!\Big[\chi\psi^5\hat{S} - \frac{3}{2}\psi^6\hat{J}_j\beta^j + \frac{1}{2\pi}\delta^{ij}\partial_i\psi\partial_j\chi\Big]d^3x. \quad (94)$$

This will be used later.

In the following, we evaluate the terms on the right-hand side of Eq. (77) separately.

(i) First term: An integration by parts immediately yields

$$-\int x^k\partial_j(\alpha\psi^6\hat{S}^j{}_k)d^3x = -\oint x^k\alpha\psi^6\hat{S}^j{}_k \overset{(0)}{dS_j} + \int\alpha\psi^6\hat{S}d^3x = -2\phi_0^{-1}M_{\rm S} + \int\alpha\psi^6\hat{S}d^3x. \quad (95)$$

The contribution to the surface-integral term comes from the term $\phi^{-1}D^jD_k\phi$ and is equal to $-2\phi_0^{-1}M_{\rm S}$.

(ii) Second term: An integration by parts immediately yields

$$\int x^k\partial_j(\beta^j\hat{J}_k)d^3x = -\int\beta^j\hat{J}_jd^3x. \quad (96)$$

In this case, the contribution of the surface integral at spatial infinity is vanishing.

(iii) Third and fourth terms: Using Eqs. (79) and (81), we can rewrite these terms as

$$-\hat{\rho}_{\rm h}\psi^5\partial_k\chi + (\hat{\rho}_{\rm h} + 2\hat{S})\chi\psi^4\partial_k\psi = \frac{1}{2\pi}\Big[\overset{(0)}{\Delta}\psi\partial_k\chi + \overset{(0)}{\Delta}\chi\partial_k\psi\Big] + \frac{\psi^{12}\tilde{A}_i{}^j\tilde{A}_j{}^i}{16\pi}\partial_k\Big(\frac{\alpha}{\psi^6}\Big). \quad (97)$$

Taking into account the identity

$$\int\!\!\Big[(x^k\partial_k\psi)\overset{(0)}{\Delta}\chi + (x^k\partial_k\chi)\overset{(0)}{\Delta}\psi\Big]d^3x = \int\delta^{ij}\partial_i\chi\partial_j\psi d^3x, \quad (98)$$

we find

$$-\int x^k[\hat{\rho}_{\rm h}\psi^5\partial_k\chi - (\hat{\rho}_{\rm h} + 2\hat{S})\chi\psi^4\partial_k\psi]d^3x = \int\!\!\Big[\frac{1}{2\pi}\delta^{ij}\partial_i\chi\partial_j\psi + \frac{\psi^{12}\tilde{A}_i{}^j\tilde{A}_j{}^i}{16\pi}x^k\partial_k\Big(\frac{\alpha}{\psi^6}\Big)\Big]d^3x. \quad (99)$$

(iv) Fifth term: Using the same calculation of Eq. (A28) of Ref. [4] (see this reference for details),

$$\int\psi^6\hat{J}_ix^k\partial_k\beta^id^3x = \frac{1}{8\pi}\int\partial_j(\psi^6\tilde{A}_i{}^j)x^k\partial_k\beta^id^3x = -\frac{1}{16\pi}\int\!\!\Big[8\pi\psi^6\hat{J}_k\beta^k + \psi^{12}\tilde{A}_i{}^j\tilde{A}_j{}^ix^k\partial_k\Big(\frac{\alpha}{\psi^6}\Big)\Big]d^3x. \quad (100)$$

Because of the asymptotic behavior assumed, the surface terms at $r\to\infty$ that appear for this term [see Eq. (A28) of Ref. [4]] vanish.

Finally, gathering the results of (i)–(iv), the right-hand side of Eq. (77) is written as

$$-2\phi_0^{-1}M_{\rm S} + \int\Big(\alpha\psi^6\hat{S} - \frac{3}{2}\hat{J}_k\beta^k\psi^6 + \frac{1}{2\pi}\delta^{ij}\partial_i\chi\partial_j\psi\Big)d^3x. \quad (101)$$

Using Eq. (94), Eq. (101) is rewritten as

$$-2\phi_0^{-1}M_{\rm S} + \frac{M_\chi - M_{\rm ADM}}{2} = -2\phi_0^{-1}M_{\rm S} + M_{\rm K} - M_{\rm ADM}, \quad (102)$$

implying that Eq. (2) is the necessary condition for the stationarity.

Because the integrand is the scalar, the left-hand side of Eq. (77) is written as

$$\frac{d}{dt}\int d^3x x^i\hat{J}_i\psi^6 = \int d^3x\partial_t(x^i\hat{J}_i\psi^6) = \int d^3x(\partial_t + \Omega\partial_\varphi)x^i\hat{J}_i\psi^6. \quad (103)$$

This implies that for helically symmetric spacetimes, the virial relation should have the same form as Eq. (2).

V. FIRST LAW

To derive the first law for stationary and axisymmetric spacetimes that satisfy the circularity condition, we simply follow the calculation of Ref. [6] (although the method of Ref. [9] could be more elegant): we simply perform the variation of the Komar mass in the straightforward manner.

First, we take the variation of Eq. (49) as

$$\phi_0 \delta M_{\rm K} = 2\delta\Big[\phi_{\mathcal{H}}\Big(\frac{1}{8\pi}\kappa A_{\mathcal{H}} + \Omega_{\mathcal{H}} J_{\mathcal{H}}\Big)\Big] - 2\delta M_{\rm S} - 2\delta \int dV\Big[\frac{\alpha}{16\pi}\{\phi R - \phi^{-1}\omega(\nabla_c\phi)\nabla^c\phi\} - \alpha\rho(1+\varepsilon) + \rho h(u^a n_a)u_j v^j\Big], \tag{104}$$

where we have assumed that ϕ_0 is fixed to be a constant. Following Ref. [6], we take the variation in the gauge $\delta\xi^a = 0 = \delta\varphi^a$ for the entire spacetime and $\delta\chi_a \propto \chi_a$ on the horizon.

The variation for the last line of Eq. (104) is only one nontrivial part. However, it can be taken without difficulty following the manipulations in the previous works (see, e.g., Refs. [4,6]) due to the choice of the Jordan-Brans-Dicke frame, and hence we here describe the final results.

(i) Fluid part (see, e.g., Refs. [4,32,33]):

$$\delta\int[\sqrt{-g}\rho(1+\varepsilon) - \rho h(u^a n_a)u_i v^i\sqrt{\gamma}]d^3x = \int\Big[\frac{\rho\bar{T}}{u^t}(\Delta s)u^a d\Sigma_a + \Big(\frac{h}{u^t} + hu_i v^i\Big)\Delta(\rho u^a d\Sigma_a) + v^i\Delta(hu_i)\rho u^a d\Sigma_a + \Big(-\frac{1}{2}\sqrt{-g}T^{ab}\delta g_{ab} + \sqrt{-g}\zeta_a\nabla_b T^{ab} - \sqrt{\gamma}D_a(\alpha\rho h q^{ab}\zeta_b)\Big)\Big]d^3x, \tag{105}$$

where $\bar{T}$ is temperature, s is the specific entropy, $d\Sigma_a$ is the volume element on the spatial hypersurface, ζ^a is the Lagrangian displacement, and $q^{ab} = g^{ab} + u^a u^b$. Δ denotes the Lagrangian perturbation. When rewritten to a surface integral, the last term—assuming that $\zeta^a = 0$ at spatial infinity and on the horizon—is vanishing.

(ii) Geometric part:

$$\delta\int \phi R\sqrt{-g}d^3x = \int \delta(\phi R\sqrt{-g})d^3x = \int[(G_{ab}\phi - \nabla_a\nabla_b\phi + g_{ab}\Box\phi) \times\sqrt{-g}\delta g^{ab} + R\sqrt{-g}\delta\phi + \partial_a\{\sqrt{-g}\phi g^{ad}g^{bc}(\nabla_c\delta g_{bd} - \nabla_d\delta g_{bc})\}]d^3x. \tag{106}$$

Here the surface terms associated with the derivative of the scalar field are vanishing because of the asymptotic condition for $r\to\infty$ and because $\alpha n^a\nabla_a\phi = \chi^a\nabla_a\phi = 0$ on the horizon. Assuming the presence of a timelike Killing vector, the last term is written as the surface integral, which is known to subsequently yield [6]

$$8\pi\phi_0(2\delta M_{\rm ADM} - \delta M_{\rm K}) \tag{107}$$

for the surface integral at spatial infinity and

$$\phi_{\mathcal{H}}(-2\delta\kappa A_{\mathcal{H}} - 16\pi\delta\Omega_{\mathcal{H}} J_{\mathcal{H}}) \tag{108}$$

for the surface integral on the horizon. Here, we again assumed that ϕ is constant on the horizon.

(iii) Scalar part: It is straightforward to obtain

$$\delta\int\frac{\omega}{\phi}(\nabla_c\phi)\nabla^c\phi\sqrt{-g}d^3x = \int\Big[\delta\phi\Big\{\Big(\frac{\omega}{\phi^2} - \frac{1}{\phi}\frac{d\omega}{d\phi}\Big)(\nabla_c\phi)\nabla^c\phi - \frac{2\omega}{\phi}\Box\phi\Big\} + \frac{\omega}{\phi}\delta g^{ab}\Big(\nabla_a\phi\nabla_b\phi - \frac{1}{2}g_{ab}(\nabla_c\phi)\nabla^c\phi\Big)\Big]\sqrt{-g}d^3x. \tag{109}$$

Here, the surface-integral term at spatial infinity is vanishing because $\nabla_a\phi = O(r^{-2})$ and $\delta\phi = O(r^{-1})$ for $r\to\infty$. The surface-integral term on the horizon is also vanishing because of the condition $\chi^a\nabla_a\phi = 0$.

Gathering all the terms and using the conditions that Einstein's equation $G_{ab} = 8\pi\tau_{ab}$ is satisfied, the scalar field equation is satisfied, the stress-energy conservation law is satisfied $\nabla_a T^{ab} = 0$, the entropy is conserved $\Delta s = 0$, the baryon rest mass is conserved $\Delta(\rho u^a d\Sigma_a) = 0$, and the specific angular momentum or vorticity is conserved $\Delta(hu_a) = 0$ (see, e.g., Ref. [4] for the conservation laws), we finally obtain

$$\phi_0\delta M_{\rm ADM} = \frac{1}{8\pi}\kappa\delta(\phi_{\mathcal{H}}A_{\mathcal{H}}) + \Omega_{\mathcal{H}}\delta(\phi_{\mathcal{H}}J_{\mathcal{H}}) - \delta M_{\rm S}, \tag{110}$$

or—by using the so-called tensor mass [17,18,21], $M_{\rm T} := M_{\rm ADM} + M_{\rm S}\phi_0^{-1}$—we may write it as

$$\phi_0\delta M_{\rm T} = \frac{1}{8\pi}\kappa\delta(\phi_{\mathcal{H}}A_{\mathcal{H}}) + \Omega_{\mathcal{H}}\delta(\phi_{\mathcal{H}}J_{\mathcal{H}}). \tag{111}$$

In the absence of the scalar field ($M_{\rm S} = 0$ and $\phi_{\mathcal{H}} = \phi_0$), which is the case in vacuum, the relation is the same as that in general relativity [6]. However, in the presence of the matter together with a black hole the relation is modified because a nontrivial profile of the scalar field may be induced (unless $T = 0$). In particular, it is worthy to note that even in the case that the scalar charge is conserved, $\delta M_{\rm S} = 0$, the relation is modified if the value of ϕ on the horizon varies along the sequence.

For binaries in quasicircular orbits, we only need to repeat the calculation performed in Ref. [4] starting from Eq. (56) and using the same gauge. The final result is

$$\phi_0 \delta M_{\rm ADM} = \frac{1}{8\pi}\kappa\delta(\phi_{\mathcal{H}} A_{\mathcal{H}}) + \phi_0 \Omega \delta J - \delta M_{\rm S}, \quad (112)$$

or

$$\phi_0(\delta M_{\rm T} - \Omega\delta J) = \frac{1}{8\pi}\kappa\delta(\phi_{\mathcal{H}} A_{\mathcal{H}}). \quad (113)$$

Again, in the presence of the matter together with a black hole, the relation is different from that in general relativity.

For both Eqs. (111) and (113), we may say that for the quantities defined in the Einstein frame in which the metric is ϕg_{ab}, the relation is the same as that in general relativity, e.g., the area of the black hole horizon in the Einstein frame should be $\phi_{\mathcal{H}} A_{\mathcal{H}}$, and hence $\phi_{\mathcal{H}}$ is absorbed in the expression.

We note that the first-law relations derived here are satisfied irrespective of the virial relation, and thus in the final results the variation of $M_{\rm K}$ is absent. The relation for $\delta M_{\rm K}$ is obtained if we use the virial relations.

VI. DISCUSSION

Now we show that Eq. (113) is a reasonable result for the evolution of binary compact objects in quasicircular orbits. In the following, we denote the angular velocity of the circular orbit by Ω.

To consider the evolution due to the radiation reaction by the emission of gravitational waves and scalar waves, it is quite useful to rewrite Einstein's equation (using the method of Ref. [29]) as

$$2G^{ab} = 16\pi\tau^{ab} = -16\pi t_{\rm LL}^{ab} + (-g)^{-1}\partial_c\partial_d H^{acbd}, \quad (114)$$

where H^{abcd} is a spacetime superpotential defined by

$$H^{abcd} := (-g)(g^{ac}g^{bd} - g^{ad}g^{bc}), \quad (115)$$

and $t_{\rm LL}^{ab}$ is the pseudotensor of Landau and Lifshitz. Then, the it and $i\varphi$ components of Eq. (114) yield the conservation law of $M_{\rm T}$ and J, as shown by Lee [17]. The rates of decrease of $M_{\rm T}$ and J due to the emission of gravitational waves and scalar waves are then written as

$$\dot{M}_{\rm T} = -(\dot{E}_{\rm GW} + \dot{E}_{\rm S}), \quad (116)$$

$$\dot{J} = -(\dot{J}_{\rm GW} + \dot{J}_{\rm S}), \quad (117)$$

where $\dot{E}_{\rm GW}$ and $\dot{E}_{\rm S}$ are the energy emission rates of gravitational waves and scalar waves, respectively, and $\dot{J}_{\rm GW}$ and $\dot{J}_{\rm S}$ are the corresponding dissipation rates of the angular momentum. For the circular orbits, $\dot{E}_{\rm GW} = \Omega\dot{J}_{\rm GW}$ and $\dot{E}_{\rm S} = \Omega\dot{J}_{\rm S}$. Therefore, Eq. (113) implies that $\phi_{\mathcal{H}} A_{\mathcal{H}}$ has to be constant along the sequence of quasicircular states in the Jordan-Brans-Dicke frame. Thus, $A_{\mathcal{H}}$ is not conserved in general. In particular, if $\phi_{\mathcal{H}}$ *increases* along the sequence, $A_{\mathcal{H}}$ *decreases*. This may occur because the null energy condition may not be satisfied in the Jordan-Brans-Dicke frame [21].

$\phi_{\mathcal{H}} A_{\mathcal{H}} =$ const yields the relation

$$\frac{d\ln A_{\mathcal{H}}}{d\ln \phi_{\mathcal{H}}} = -1. \quad (118)$$

For $A_{\mathcal{H}} \propto M_{\mathcal{H}}^2$, where $M_{\mathcal{H}}$ is the mass of the black hole, we have the well-known relation of the sensitivity of black holes [18,21],

$$\frac{d\ln M_{\mathcal{H}}}{d\ln \phi_{\mathcal{H}}} = -\frac{1}{2}. \quad (119)$$

Next we consider the implication of the virial relation, $M_{\rm K} = M_{\rm T} + \phi_0^{-1} M_{\rm S}$. Here, $M_{\rm K}$ is the gravitational mass [18] that primarily determines the strength of the gravitational force $\propto M_{\rm K}^2$. As mentioned above, $M_{\rm T}$ is a conserved mass in the absence of radiation, whereas $M_{\rm S}$ does not have to be. This implies that $M_{\rm K}$ may vary if $M_{\rm S}$ does even in the absence of the radiation reaction. (In addition to this effect, the curve of $M_{\rm T}$ as a function of Ω should also be modified in the presence of a large value of $M_{\rm S}$). For example, in the scalar-tensor theory of Ref. [23] with the maximum allowed value $C_{\rm DEF} \sim 9$–10 [34], the value of $M_{\rm S}$ for neutron stars in binary neutron stars with close orbits can steeply grow depending on the background value of ϕ in the vicinity of the neutron stars [35]. In the inspiral orbits of binary neutron stars, the background value of ϕ of each neutron star increases with a decrease of the orbital separation due to the effect of the companion neutron star, and hence $M_{\rm S}$ monotonically increases with the time evolution [36]. For such a case, the magnitude of the attractive force increases monotonically with a decrease of the orbital separation: in other words, the chirp mass increases with an increase of Ω [37]. In addition, the luminosity of gravitational waves will be enhanced because the orbital velocity is increased, as indicated from the post-Newtonian results (e.g., Ref. [38]). Thus, the rate of increase of the orbital frequency $\dot{f}$ is increased. This effect depends on the structure of the neutron stars and hence on the equation of state [36]. If this effect is large enough to change the orbital phase by more than one cycle, it will be found in the data analysis of gravitational waves from inspiraling binary neutron stars.

VII. SUMMARY

We have derived virial relations and the first law for stationary spacetimes and for quasiequilibrium spacetimes in a class of scalar-tensor theories of gravity in the Jordan-Brans-Dicke frame. In the presence of nontrivial scalar fields—that is, in the general case for nonvacuum spacetimes—these relations are different from those in general relativity. These relations will be useful for checking the accuracy of numerical solutions for equilibrium and quasiequilibrium states. We also found that (i) the first-law relation implies that we have to construct a sequence of black hole-neutron star binaries in quasicircular orbits fixing $\phi_{\mathcal{H}}A_{\mathcal{H}}$ for the black hole in the Jordan-Brans-Dicke frame, and (ii) in the scalar-tensor theory of Ref. [23], the attractive force between two neutron stars in binary neutron stars could be enhanced in the late inspiral phase.

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