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Conservative form of Boltzmann's equation in general relativity

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We derive a conservative form of Boltzmann's equation in general relativity, which is concisely written. Several explicit forms of this equation are written for black-hole spacetime with several coordinate conditions in real spacetime and momentum-space coordinates.

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I. INTRODUCTION

Radiation fields and their interaction with matter fields often play a crucial role in general-relativistic astrophysical phenomena; e.g., the neutrino cooling and heating play a special role in core-collapse supernova. The reason is that general-relativistic phenomena are usually accompanied by the high-density and high-temperature matter with which neutrinos strongly interact. For a physical simulation in numerical relativity, we are often required to take into account the neutrino radiation transfer effects. For this, it is necessary to solve radiation transfer equations.

For strictly handling radiation transfer effects, it is necessary to numerically solve Boltzmann's equation, taking into account the absorption, emission, and scattering source terms. This equation has a $3+3+1$-dimensional form (three dimensions in real and momentum space, respectively, and one dimension in time); hence, the computational domain has to cover six-dimensional space for a simulation (unless we impose any spatial symmetry). It is an extremely challenging task to perform a well-resolved numerical simulation with a sufficient grid resolution for this equation, unless a high spatial symmetry such as spherical symmetry is imposed (e.g., see [1–4] for formulations based on the S_N schemes, [5,6] for results in spherically symmetric and general-relativistic simulations, and also [7] for an alternative approach based on long characteristics and its application to protoneutron star evolution in full general relativity by means of a tangent-ray scheme). To date, no challenge has been made on this issue (but see [8] in Newtonian gravity).

Indeed, even for formulations suitable for a numerical simulation, only a few attempts [9,10] have been reported, and there is no established formulation to date (see Ref. [10] for a review in this field). Cardall and Mezzacappa [9] gave a conservative formulation of Boltzmann's equation but not in a $3+1$ form applicable to numerical-relativity simulations. A $3+1$ formulation of Boltzmann's equation was first derived by Cardall, Endeve and Mezzacappa [10] with laboratory frame in spacetime coordinates and fluid rest frame in momentum-space coordinates. As a consequence of adopting a fluid-rest-frame momentum-space coordinate basis, the resulting equations are rather complicated. An attempt to solve Boltzmann's equation using spectral methods with the conformally flat approximation was done by [11], but their formulation was not in a conservative form. In this paper, we derive a more concise and general formulation for the conservative form of Boltzmann's equation in general relativity.

The paper is organized as follows. In Sec. II, after we briefly review the basics of Boltzmann's equation, we describe its conservative form in a concise way. We also give conservative forms in black-hole spacetime with several coordinate conditions in real-spacetime and momentum-space coordinates. Section III is devoted to a summary. Throughout this paper, we employ the units in which the Planck constant, h, speed of light, c, and gravitational constant, G, are unity. Latin indices a, b, c, and d denote the abstract index while greek ones $\alpha, \beta, \gamma, \ldots$ and latin ones i, j, k, and l denote the spacetime and spatial components, respectively.

II. BOLTZMANN'S EQUATION

A. Basics

First, we review Boltzmann's equation in the context of the radiation transfer for massless particles in general relativity. Let p^a be a null vector of massless particles, and $f(x^\alpha, p^i)$ $(x^\alpha = (t, x^i))$ be their distribution function. Then, Boltzmann's equation is written in the form (e.g., [12–14])

$$\frac{dx^\alpha}{d\tau}\frac{\partial f}{\partial x^\alpha}\bigg|_{p^i} + \frac{dp^i}{d\tau}\frac{\partial f}{\partial p^i}\bigg|_{x^\mu} = (-p^\alpha \hat{u}_\alpha) S_{\rm rad}(p^\mu, x^\mu, f), \quad (1)$$

or

$$p^\alpha \frac{\partial f}{\partial x^\alpha}\bigg|_{p^i} - \Gamma^i{}_{\alpha\beta} p^\alpha p^\beta \frac{\partial f}{\partial p^i}\bigg|_{x^\mu} = (-p^\alpha \hat{u}_\alpha) S_{\rm rad}(p^\mu, x^\mu, f), \quad (2)$$

where $S_{\rm rad}$ is a source term which is determined by interaction processes between the radiation and matter fields, τ is the affine parameter for a trajectory of radiation particles (i.e., $p^a = dx^a/d\tau$), and $\hat{u}^a$ may be in general any timelike unit vector (e.g., the four-velocity of a fluid). Here, $(\partial f/\partial x^\alpha)|_{p^i}$ and $(\partial f/\partial p^i)|_{x^\mu}$ are partial derivatives of f fixing p^i and x^μ, respectively.

Instead of p^i, we may use other sets of variables for describing the momentum space: let $q_{(i)}$ ($i = 1, 2, 3$) be a set of the momentum-space variables. Then, Boltzmann's equation is rewritten as

$$p^\alpha \frac{\partial f}{\partial x^\alpha}\bigg|_{q_{(j)}} + \sum_{i=1}^{3} \frac{dq_{(i)}}{d\tau} \frac{\partial f}{\partial q_{(i)}}\bigg|_{x^\alpha} = (-p^\alpha \hat{u}_\alpha) S_{\rm rad}. \quad (3)$$

We will mainly employ this form in this paper.

As described by Lindquist [12] and Ehlers [13], the number of world lines crossing an invariant three volume dV_x with four-momenta in the range of an invariant momentum-space volume dV_p is

$$dN = f(x^\alpha, p^i)(-p^a \hat{u}_a) dV_x dV_p, \quad (4)$$

where

$$dV_x = \hat{u}^a \epsilon_{abcd} dx_1^b dx_2^c dx_3^d, \quad (5)$$

$$dV_p = \frac{1}{(-p^e \hat{v}_e)} \hat{v}^a \epsilon_{abcd} dp_1^b dp_2^c dp_3^d, \quad (6)$$

and $x^\mu = (t, x_1, x_2, x_3)$ and $p^\mu = (p_0, p_1, p_2, p_3)$ denote real-spacetime and momentum-space coordinates, respectively. Again, $\hat{u}^a$ and $\hat{v}^a$ are arbitrary timelike unit vectors ($\hat{v}^a$ may be equal to $\hat{u}^a$). ϵ_{abcd} is completely antisymmetric tensor with $\epsilon_{0123} = \sqrt{-g}$ where g is the determinant of the spacetime metric g_{ab}. dV_p may be defined by an integral with the on-shell condition as [15]

$$2\int \epsilon_{abcd} dp_0^a dp_1^b dp_2^c dp_3^d \delta(p^e p_e) = \frac{\sqrt{-g}}{-p_0} dp_1 dp_2 dp_3, \quad (7)$$

where $p^e p_e = 0$ for massless particles and $\delta(x)$ is the delta function.

In any local orthonormal frame, dV_p is written as [16]

$$dV_p = \frac{d\hat{p}^1 d\hat{p}^2 d\hat{p}^3}{\hat{p}^0}, \quad (8)$$

where $\hat{p}^\alpha$ denotes four-momenta of a radiation particle in a local orthonormal frame. dV_p may be written as

$$dV_p = \nu d\nu d\bar{\Omega}, \quad (9)$$

where ν is a frequency (energy) of radiation particles measured in the local orthonormal frame, $\nu = -p_a e^a_{(0)}$ with $e^a_{(0)}$ being the timelike unit vector in this frame: In numerical relativity, one of the simplest choices would be $e^a_{(0)} = n^a$ where n^a is the timelike unit normal to spatial hypersurfaces (but see Sec. II C for examples that with this choice, the equations are usually complicated). If $e^a_{(0)}$ denotes the four-velocity of a fluid, ν is a frequency measured by the fluid rest frame. $\bar{\Omega}$ denotes the surface element over the solid angle of a unit sphere. The specific definition of a local orthonormal frame will be given below.

In numerical astrophysics, Boltzmann's equation is often rewritten for the distribution function as a function of (t, x^i) and momentum-space argument variables defined in a local orthonormal frame. This method is in particular robust in spherically symmetric spacetime [12,17]. Basic equations along this line are derived by setting

$$p^a = \nu \left(e^a_{(0)} + \sum_{i=1}^{3} \ell_{(i)} e^a_{(i)} \right), \quad (10)$$

where $e^a_{(\mu)}$ ($\mu = 0, 1, 2, 3$) denotes a set of the tetrad basis for a local orthonormal frame satisfying $g_{ab} e^a_{(\alpha)} e^b_{(\beta)} = \eta_{\alpha\beta}$ and $\eta^{\alpha\beta} e^a_{(\alpha)} e^b_{(\beta)} = g^{ab}$, with $\eta_{\alpha\beta}$ being the Minkowski metric. Following Lindquist [12], we write $\ell_{(i)}$ as

$$\ell_{(1)} = \cos\bar{\theta}, \qquad \ell_{(2)} = \sin\bar{\theta}\cos\bar{\varphi}, \qquad \ell_{(3)} = \sin\bar{\theta}\sin\bar{\varphi}, \quad (11)$$

where $\bar{\theta}$ and $\bar{\varphi}$ denote angles of radiation rays in the momentum space at each spatial position (see Fig. 1) and in this context, $e^a_{(1)}$, $e^a_{(2)}$, and $e^a_{(3)}$ should be unit vectors pointing approximately to r, θ, and φ directions, respectively. Using $\bar{\theta}$ and $\bar{\varphi}$, the area element is written as $d\bar{\Omega} = \sin\bar{\theta} d\bar{\theta} d\bar{\varphi}$.

Choosing the argument variables as $q_{(1)} = \nu$, $q_{(2)} = \bar{\theta}$, and $q_{(3)} = \bar{\varphi}$, the second term of Eq. (3) is written as

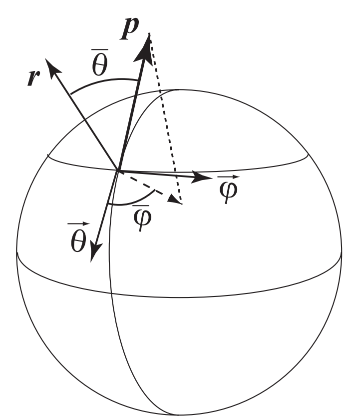

FIG. 1. The locally defined angles $(\bar{\theta}, \bar{\varphi})$ of null rays, $\boldsymbol{p}(= p^a)$. $\boldsymbol{r}$, $\vec{\theta}$, $\vec{\varphi}$ denote unit vectors pointing to radial, θ, and φ directions, corresponding to $e^a_{(1)}$, $e^a_{(2)}$, and $e^a_{(3)}$, respectively.

$$\sum_{i=1}^{3}\frac{dq_{(i)}}{d\tau}\frac{\partial f}{\partial q_{(i)}}=\frac{d\nu}{d\tau}\frac{\partial f}{\partial\nu}+\frac{d\bar{\theta}}{d\tau}\frac{\partial f}{\partial\bar{\theta}}+\frac{d\bar{\varphi}}{d\tau}\frac{\partial f}{\partial\bar{\varphi}}. \tag{12}$$

To proceed further, we should remember the following relations,

$$\nu=-p_a e^a_{(0)}=:-p_{(0)},$$
$$\tan\bar{\theta}=\frac{\sqrt{p^2_{(2)}+p^2_{(3)}}}{p_{(1)}},$$
$$\tan\bar{\varphi}=\frac{p_{(3)}}{p_{(2)}}, \tag{13}$$

where $p_{(j)}=p_a e^a_{(j)}$ and thus, $\sum_{i=1}^3 p^2_{(i)}=\nu^2=p^2_{(0)}$. Using the geodesic equation for p^a, $p^b\nabla_b p^a=0$, and Eq. (13), we obtain

$$\frac{d\nu}{d\tau}=-p^a p_b\nabla_a e^b_{(0)}, \tag{14}$$

$$\frac{d\bar{\theta}}{d\tau}=\frac{1}{\nu}\sum_{j=1}^{3}\frac{\partial\ell_{(j)}}{\partial\bar{\theta}}\frac{dp_{(j)}}{d\tau}=\frac{1}{\nu}\sum_{j=1}^{3}\frac{\partial\ell_{(j)}}{\partial\bar{\theta}}p^a p_b\nabla_a e^b_{(j)}, \tag{15}$$

$$\frac{d\bar{\varphi}}{d\tau}=\frac{1}{\nu\sin^2\bar{\theta}}\sum_{j=2}^{3}\frac{\partial\ell_{(j)}}{\partial\bar{\varphi}}\frac{dp_{(j)}}{d\tau}=\frac{1}{\nu\sin^2\bar{\theta}}\sum_{j=2}^{3}\frac{\partial\ell_{(j)}}{\partial\bar{\varphi}}p^a p_b\nabla_a e^b_{(j)}. \tag{16}$$

Here, ∇_a is the covariant derivative with respect to g_{ab}.

In the general curved spacetime, we have to constitute $e^a_{(\mu)}$ for a general geometry. However, in the local orthonormal frame with the choice $e^a_{(0)}=n^a$, the procedure is quite straightforward because $e^a_{(i)}$ should have only spatial components (remember $n_a=-\alpha\nabla_a t$ where α is the lapse function). For example, for the spherical polar coordinates (r,θ,φ), it is easy to find the following set as the tetrad basis,

$$e^\alpha_{(1)}=(0,\gamma_{rr}^{-1/2},0,0),$$
$$e^\alpha_{(2)}=\left(0,-\frac{\gamma_{r\theta}}{\sqrt{\gamma_{rr}(\gamma_{rr}\gamma_{\theta\theta}-\gamma^2_{r\theta})}},\sqrt{\frac{\gamma_{rr}}{\gamma_{rr}\gamma_{\theta\theta}-\gamma^2_{r\theta}}},0\right),$$
$$e^\alpha_{(3)}=\left(0,\frac{\gamma^{r\varphi}}{\sqrt{\gamma^{\varphi\varphi}}},\frac{\gamma^{\theta\varphi}}{\sqrt{\gamma^{\varphi\varphi}}},\sqrt{\gamma^{\varphi\varphi}}\right), \tag{17}$$

where $\gamma_{ab}=g_{ab}+n_a n_b$ is the induced metric on the spatial hypersurface.

In numerical astrophysics, $e^a_{(0)}$ is often chosen as the four-velocity of a matter field, $e^a_{(0)}=u^a$. In this case, all the momentum-space argument variables are defined in the local rest frame moving with the matter field. This choice of the momentum-space basis vector has an advantage that we do not need to perform a transformation of the frame for evaluating the source term of Boltzmann's equation which should be evaluated in the local rest frame. However, there is a serious drawback in this scheme that we have to take derivatives of the four-velocity in general relativity, as shown in Eqs. (14)–(16), for which accurate numerical computation is not an easy task in numerical hydrodynamics. In the following, therefore, we will take the other basis than $e^a_{(0)}=u^a$. With such general argument variables for the momentum space, we then have to perform a transformation of the frame to adjust the variables, for which the reader may refer to Appendix B.

B. Conservative form

In numerical astrophysics, it is often desirable to write Boltzmann's equation in a conservative form, in which accurate conservations of the particle number and total energy can be numerically guaranteed in the equation level [5,9]. The existence of the conservative forms itself can be shown in a purely geometrical and covariant manner [13] (see Appendix A for a brief discussion). Explicit expressions of Boltzmann's equation in its conservative form are also given by a straightforward procedure, remembering the fact that dV_x and dV_p are invariant real- and momentum-space volume elements, as follows.

Starting from Eq. (2), a conservation form is derived as

$$\frac{p_t}{(-g)}\left[\frac{\partial}{\partial x^\alpha}\left(\frac{f(-g)p^\alpha}{p_t}\right)-\frac{\partial}{\partial p^i}\left(\frac{f(-g)}{p_t}\Gamma^i{}_{\alpha\beta}p^\alpha p^\beta\right)\right]=(-p_a\hat{u}^a)S_{\rm rad}. \tag{18}$$

This equation was also derived in a straightforward calculation using the following relations,

$$p^\alpha\frac{\partial f}{\partial x^\alpha}\bigg|_{p^j}=\frac{1}{(-g)}\frac{\partial[f(-g)p^\alpha]}{\partial x^\alpha}\bigg|_{p^j}-2p^\alpha\Gamma^\beta{}_{\alpha\beta}f-f\frac{\partial p^t}{\partial t}\bigg|_{p^j},$$
$$\frac{\partial p^t}{\partial t}\bigg|_{p^j}=-\frac{p^\alpha p^\beta}{2p_t}\partial_t g_{\alpha\beta}=-\frac{1}{p_t}\Gamma^\beta{}_{t\alpha}p^\alpha p_\beta=-\frac{1}{p_t}\frac{dp_t}{d\tau},$$
$$-\Gamma^i{}_{\alpha\beta}p^\alpha p^\beta\frac{\partial f}{\partial p^i}\bigg|_{x^\mu}=-\frac{\partial(f\Gamma^i{}_{\alpha\beta}p^\alpha p^\beta)}{\partial p^i}\bigg|_{x^\mu}+2fp^\alpha\left(\Gamma^i{}_{i\alpha}+\Gamma^i{}_{t\alpha}\frac{\partial p^t}{\partial p^i}\bigg|_{x^\mu}\right),$$
$$\Gamma^i{}_{t\alpha}\frac{\partial p^t}{\partial p^i}\bigg|_{x^\mu}=-\Gamma^i{}_{t\alpha}\frac{p_i}{p_t}=-\Gamma^\beta{}_{t\alpha}\frac{p_\beta}{p_t}+\Gamma^t{}_{t\alpha},$$

and remembering the definition

$$\frac{dp_t}{d\tau} = \frac{dx^\alpha}{\partial\tau}\frac{\partial p_t}{\partial x^\alpha}\bigg|_{p^i} + \frac{dp^i}{d\tau}\frac{\partial p_t}{\partial p^i}\bigg|_{x^\mu} = p^\alpha\frac{\partial p_t}{\partial x^\alpha}\bigg|_{p^i} - \Gamma^i{}_{\alpha\beta}p^\alpha p^\beta\frac{\partial p_t}{\partial p^i}\bigg|_{x^\mu}. \tag{19}$$

The conservative form is also derived for a local orthonormal frame. Starting from Eq. (3) with the choice of $\hat{u}^a = e^a_{(0)}$ and Eqs. (12), (14)–(16), we obtain

$$\frac{1}{\sqrt{-g}}\frac{\partial(\sqrt{-g}\nu^{-1}p^\alpha f)}{\partial x^\alpha}\bigg|_{q_{(i)}} + \frac{1}{\nu^2}\frac{\partial}{\partial\nu}(-\nu f p^\alpha p_\beta \nabla_\alpha e^\beta_{(0)}) + \frac{1}{\sin\bar\theta}\frac{\partial}{\partial\bar\theta}\left(\nu^{-2}\sin\bar\theta f\sum_{j=1}^{3}p^\alpha p_\beta\nabla_\alpha e^\beta_{(j)}\frac{\partial\ell_{(j)}}{\partial\bar\theta}\right) + \frac{1}{\sin^2\bar\theta}\frac{\partial}{\partial\bar\varphi}\left(\nu^{-2}f\sum_{j=2}^{3}p^\alpha p_\beta\nabla_\alpha e^\beta_{(j)}\frac{\partial\ell_{(j)}}{\partial\bar\varphi}\right) = S_{\rm rad}, \tag{20}$$

or a practical form

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\alpha}\bigg|_{q_{(i)}}\left[\left(e^\alpha_{(0)} + \sum_{i=1}^{3}\ell_{(i)}e^\alpha_{(i)}\right)\sqrt{-g}f\right] - \frac{1}{\nu^2}\frac{\partial}{\partial\nu}(\nu^3 f\omega_{(0)}) + \frac{1}{\sin\bar\theta}\frac{\partial}{\partial\bar\theta}(\sin\bar\theta f\omega_{(\bar\theta)}) + \frac{1}{\sin^2\bar\theta}\frac{\partial}{\partial\bar\varphi}(f\omega_{(\bar\varphi)}) = S_{\rm rad}, \tag{21}$$

where

$$\omega_{(0)} \coloneqq \nu^{-2}p^\alpha p_\beta\nabla_\alpha e^\beta_{(0)} = \sum_{i=1}^{3}\ell_{(i)}\left(\gamma_{i00} + \sum_{j=1}^{3}\gamma_{i0j}\ell_{(j)}\right),$$
$$\omega_{(\bar\theta)} \coloneqq \sum_{j=1}^{3}\omega_{(j)}\frac{\partial\ell_{(j)}}{\partial\bar\theta},$$
$$\omega_{(\bar\varphi)} \coloneqq \sum_{j=2}^{3}\omega_{(j)}\frac{\partial\ell_{(j)}}{\partial\bar\varphi}, \tag{22}$$

and

$$\omega_{(j)} \coloneqq \nu^{-2}p^\alpha p_\beta\nabla_\alpha e^\beta_{(j)} = \gamma_{0j0} + \sum_{i=1}^{3}\ell_{(i)}\left\{(\gamma_{0ji} + \gamma_{ij0}) + \sum_{k=1}^{3}\gamma_{ijk}\ell_{(k)}\right\}. \tag{23}$$

$\gamma_{\alpha\beta\gamma} = -\gamma_{\beta\alpha\gamma}$ is the Ricci rotation coefficients defined by $\gamma_{\alpha\beta\gamma} \coloneqq e^a_{(\alpha)}e^b_{(\gamma)}\nabla_b(e_{(\beta)})_a$. We also used

$$\nabla_a\left(e^a_{(0)} + \sum_{i=1}^{3}\ell_{(i)}e^a_{(i)}\right) = \sum_{i=1}^{3}\left(\gamma_{i0i} - \gamma_{0i0}\ell_{(i)} + \sum_{k=1}^{3}\gamma_{iki}\ell_{(k)}\right),$$
$$-\cot\bar\theta\frac{\partial\ell_{(j)}}{\partial\bar\theta} - \frac{1}{\sin^2\bar\theta}\frac{\partial^2\ell_{(j)}}{\partial\bar\varphi^2} = \ell_{(j)},$$
$$\frac{\partial\ell_{(i)}}{\partial\bar\theta}\frac{\partial\ell_{(j)}}{\partial\bar\theta} + \frac{1}{\sin^2\bar\theta}\frac{\partial\ell_{(i)}}{\partial\bar\varphi}\frac{\partial\ell_{(j)}}{\partial\bar\varphi} = \delta_{ij} - \ell_{(i)}\ell_{(j)}.$$

Note that the partial derivative with respect to x^α that appears in the first term for Eqs. (20) and (21) has to be taken fixing ν, $\bar\theta$, and $\bar\varphi$ (not fixing p^i). For Eq. (21), it is trivially seen that $N = \int dN$ is the conserved quantity [see Eqs. (4) and (9)].

It is soon found that $\omega_{(0)}$ is related to $\omega_{(i)}$ by

$$\omega_{(0)} = -\sum_{i=1}^{3}\omega_{(i)}\ell_{(i)}. \tag{24}$$

Since $\ell_{(i)}$, $\partial\ell_{(i)}/\partial\bar\theta$, and $(\partial\ell_{(i)}/\partial\bar\varphi)/\sin\bar\theta$ constitute an orthonormal set of the unit vector in the local three-momentum space of subscript (i), we find that $\omega_{(0)}$, $\omega_{(\bar\theta)}$, and $\omega_{(\bar\varphi)}$ are the independent components of $\omega_{(i)}$. $[\omega_{(0)}, \omega_{(\bar\theta)}, \omega_{(\bar\varphi)}]$ are independent projection components of the $\omega_{(i)}$ vector, satisfying

$$\omega^2_{(0)} + \omega^2_{(\bar\theta)} + \frac{1}{\sin^2\bar\theta}\omega^2_{(\bar\varphi)} = \sum_{i=1}^{3}\omega^2_{(i)}. \tag{25}$$

We note that $\omega_{(0)}$ and $\omega_{(j)}$ are composed of nine basis functions of $Y_{lm}(\bar\theta, \bar\varphi)$ with $0 \le l \le 2$ and $0 \le |m| \le 2$, where Y_{lm} is the spherical harmonics function. Also, $\omega_{(\bar\theta)}\sin\theta$ and $\omega_{(\bar\varphi)}$ are composed of fourteen basis functions of $Y_{lm}(\bar\theta, \bar\varphi)$ with $0 \le l \le 3$ and $0 \le |m| \le 2$. Thus, in general, $[\omega_{(0)}, \omega_{(\bar\theta)}, \omega_{(\bar\varphi)}]$ are written as functions of these basis functions, although with a good choice of the tetrad, they can be written in a simple form in particular for spacetime of a special symmetry (see below).

C. Explicit form in black hole spacetime

1. Schwarzschild black hole

As an illustration, we explicitly describe the conservative form of Boltzmann's equation in black-hole spacetime. As the simplest case, first, we choose the Schwarzschild background for which the line element is written as

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2), \tag{26}$$

where (t, r, θ, φ) are usual Schwarzschild coordinates. In this case, one of the natural choices of the tetrad components is

$$\begin{aligned} e^a_{(0)} &= \left(1 - \frac{2M}{r}\right)^{-1/2} \left(\frac{\partial}{\partial t}\right)^a, \\ e^a_{(1)} &= \left(1 - \frac{2M}{r}\right)^{1/2} \left(\frac{\partial}{\partial r}\right)^a, \\ e^a_{(2)} &= \frac{1}{r} \left(\frac{\partial}{\partial \theta}\right)^a, \\ e^a_{(3)} &= \frac{1}{r \sin\theta} \left(\frac{\partial}{\partial \varphi}\right)^a, \end{aligned} \tag{27}$$

and thus, $dV_x = (1 - 2M/r)^{-1/2} r^2 \sin\theta dr d\theta d\varphi$. This choice is valid only for $r > 2M$ because for $r \leq 2M$, $e^a_{(0)}$ is not timelike and $e^a_{(1)}$ is not spacelike, respectively. The nonzero components of $\gamma_{\alpha\beta\gamma}$ for this tetrad are

$$\begin{aligned} &\gamma_{122} = -\gamma_{212} = \gamma_{133} = -\gamma_{313} = -\frac{1}{r}\left(1 - \frac{2M}{r}\right)^{1/2}, \\ &\gamma_{233} = -\gamma_{323} = -\frac{\cot\theta}{r}, \\ &\gamma_{100} = -\gamma_{010} = \frac{M}{r^2}\left(1 - \frac{2M}{r}\right)^{-1/2}. \end{aligned} \tag{28}$$

Hence,

$$\omega_{(0)} = \frac{M}{r^2}\left(1 - \frac{2M}{r}\right)^{-1/2} \cos\bar{\theta}, \tag{29}$$

$$\omega_{(1)} = -\frac{M}{r^2}\left(1 - \frac{2M}{r}\right)^{-1/2} + \frac{1}{r}\left(1 - \frac{2M}{r}\right)^{1/2} \sin^2\bar{\theta}, \tag{30}$$

$$\omega_{(2)} = -\frac{1}{r}\left(1 - \frac{2M}{r}\right)^{1/2} \sin\bar{\theta}\cos\bar{\theta}\cos\bar{\varphi} + \frac{\cot\theta}{r} \sin^2\bar{\theta} \sin^2\bar{\varphi}, \tag{31}$$

$$\omega_{(3)} = -\frac{1}{r}\left(1 - \frac{2M}{r}\right)^{1/2} \sin\bar{\theta}\cos\bar{\theta}\sin\bar{\varphi} - \frac{\cot\theta}{r} \sin^2\bar{\theta} \sin\bar{\varphi}\cos\bar{\varphi}, \tag{32}$$

and

$$\omega_{(\bar{\theta})} = \frac{3M - r}{r^2}\left(1 - \frac{2M}{r}\right)^{-1/2} \sin\bar{\theta}, \tag{33}$$

$$\omega_{(\bar{\varphi})} = -\frac{\cot\theta}{r} \sin^3\bar{\theta} \sin\bar{\varphi}. \tag{34}$$

We note that $\omega_{(0)}$, $\omega_{(\bar{\theta})}$, and $\omega_{(\bar{\varphi})}$ are composed only of one basis function of $(\bar{\theta}, \bar{\varphi})$, respectively, although they may have more functions in general. Hence, the equation for f in the Schwarzschild background is written in a quite simple form:

$$\begin{aligned} &\left(1 - \frac{2M}{r}\right)^{-1/2} \frac{\partial f}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}\left[f \cos\bar{\theta} r^2 \left(1 - \frac{2M}{r}\right)^{1/2}\right] + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(f \sin\theta\sin\bar{\theta}\cos\bar{\varphi}) \\ &+ \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}(f\sin\bar{\theta}\sin\bar{\varphi}) - \frac{1}{\nu^2}\frac{\partial}{\partial\nu}\left[f\nu^3\cos\bar{\theta}\frac{M}{r^2}\left(1 - \frac{2M}{r}\right)^{-1/2}\right] \\ &- \frac{1}{\sin\bar{\theta}}\frac{\partial}{\partial\bar{\theta}}\left[f\sin^2\bar{\theta}\frac{r - 3M}{r^2}\left(1 - \frac{2M}{r}\right)^{-1/2}\right] \\ &- \frac{\partial}{\partial\bar{\varphi}}\left(f\frac{\cot\theta}{r}\sin\bar{\theta}\sin\bar{\varphi}\right) = S_{\mathrm{rad}}. \end{aligned} \tag{35}$$

It is found that the transport term associated with ν in Eq. (35) is present only for the curved spacetime; hence, this term is related to the gravitational redshift (for $\cos\bar{\theta} > 0$) and blueshift (for $\cos\bar{\theta} < 0$). It is also interesting to point out that the transport term associated with $\bar{\theta}$ changes the sign at the so-called photon sphere $r = 3M$: for $r > 3M$, the direction of outgoing rays tends to converge toward $\bar{\theta} \to 0$ as usual in the flat spacetime, while for $r < 3M$, rays are dragged by the gravity of the black hole.

By setting $M = 0$, we can also obtain Boltzmann's equation in the flat spacetime (e.g. [8]):

$$\begin{aligned} &\frac{\partial f}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(f\cos\bar{\theta}r^2) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(f\sin\theta\sin\bar{\theta}\cos\bar{\varphi}) \\ &+ \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}(f\sin\bar{\theta}\sin\bar{\varphi}) - \frac{1}{r}\frac{1}{\sin\bar{\theta}}\frac{\partial}{\partial\bar{\theta}}(f\sin^2\bar{\theta}) \\ &- \frac{\partial}{\partial\bar{\varphi}}\left(f\frac{\cot\theta}{r}\sin\bar{\theta}\sin\bar{\varphi}\right) = S_{\mathrm{rad}}. \end{aligned} \tag{36}$$

This equation together with Eq. (35) shows that for $\omega_{(0)}$, $\omega_{(\bar{\theta})}$, and $\omega_{(\bar{\varphi})}$, $\cos\bar{\theta}$, $\sin\bar{\theta}$, and $\sin^3\bar{\theta}\sin\bar{\varphi}$ are the primary basis functions, respectively.

Nonrotating black holes may be written in the Eddington-Finkelstein form (e.g., [18]) as

$$ds^2 = -\left(1 - \frac{2M}{r}\right)d\bar{t}^2 + \frac{4M}{r}d\bar{t}dr + \left(1 + \frac{2M}{r}\right)dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2), \quad (37)$$

where $\bar{t} = t + 2M\log|r/2M - 1|$. In this case, the lapse function, α, is always positive for $r > 0$,

$$\alpha = \left(\frac{r}{r + 2M}\right)^{1/2}, \quad (38)$$

and $n_a = -\alpha\nabla_a\bar{t}$ is well defined for $r > 0$. This coordinate has a horizon-penetrating property (i.e., $\bar{t}$ is a timelike coordinate even for $r < 2M$), and thus, it could be useful for numerical simulations covering the region up to $r \lesssim 2M$. If we choose

$$e^a_{(0)} = n^a, \qquad e^a_{(1)} = \left(1 + \frac{2M}{r}\right)^{-1/2}\left(\frac{\partial}{\partial r}\right)^a,$$
$$e^a_{(2)} = \frac{1}{r}\left(\frac{\partial}{\partial\theta}\right)^a, \qquad e^a_{(3)} = \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\varphi}\right)^a, \quad (39)$$

$e^a_{(0)}$ and $e^a_{(1)}$ are timelike and spacelike for the entire region with $r > 0$, respectively, and $dV_x = (1 + 2M/r)^{1/2}\sin\theta r^2 dr d\theta d\varphi$ is not singular for $r > 0$. For this case, the equation for f becomes

$$\left(\frac{r}{r+2M}\right)^{-1/2}\frac{\partial f}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}\left[fr^2\left(\frac{r}{r+2M}\right)^{1/2}\left(\cos\bar{\theta} - \frac{2M}{r}\right)\right]$$
$$+ \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(f\sin\theta\sin\bar{\theta}\cos\bar{\varphi})$$
$$+ \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}(f\sin\bar{\theta}\sin\bar{\varphi})$$
$$- \frac{1}{\nu^2}\frac{\partial}{\partial\nu}[f\nu^3\omega_{(0)}] + \frac{1}{\sin\bar{\theta}}\frac{\partial}{\partial\bar{\theta}}(f\sin\bar{\theta}\omega_{(\bar{\theta})})$$
$$- \frac{\partial}{\partial\bar{\varphi}}\left(f\frac{\cot\theta}{r}\sin\bar{\theta}\sin\bar{\varphi}\right) = S_{\rm rad}, \quad (40)$$

where

$$\omega_{(0)} = \frac{M}{r^3}\left(1 + \frac{2M}{r}\right)^{-3/2} \times [-M + r\cos\bar{\theta} + (2r + 3M)\cos(2\bar{\theta})], \quad (41)$$

$$\omega_{(\bar{\theta})} = \frac{1}{r^3}\left(1 + \frac{2M}{r}\right)^{-3/2}\sin\bar{\theta} \times [-r(r + M) + 2M(2r + 3M)\cos\bar{\theta}]. \quad (42)$$

$\omega_{(0)}$ and $\omega_{(\bar{\theta})}$ in this case are different from Eqs. (30) and (33), respectively: They are composed of three and two functions of $\bar{\theta}$, respectively, and hence, it seems that they might contain redundant unphysical components, although they are regular functions for the entire region with $r > 0$ and hence for the numerical simulation, this has a tractable form. On the other hand, if we choose

$$e^a_{(0)} = \left(1 - \frac{2M}{r}\right)^{-1/2}\left(\frac{\partial}{\partial\bar{t}}\right)^a,$$
$$e^a_{(1)} = \left(1 - \frac{2M}{r}\right)^{-1/2}\left[\left(1 - \frac{2M}{r}\right)\left(\frac{\partial}{\partial r}\right)^a + \frac{2M}{r}\left(\frac{\partial}{\partial\bar{t}}\right)^a\right],$$
$$e^a_{(2)} = \frac{1}{r}\left(\frac{\partial}{\partial\theta}\right)^a, \qquad e^a_{(3)} = \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\varphi}\right)^a, \quad (43)$$

the resulting form of $\omega_{(0)}$, $\omega_{(\bar{\theta})}$, and $\omega_{(\bar{\varphi})}$ agree with Eqs. (30), (33), and (34), respectively (although $e^a_{(0)}$ is timelike only for $r > 2M$). In this case, the resulting equation for f is the same as Eq. (35) except for the term $(1 - 2M/r)^{-1/2}\partial f/\partial t$ which is replaced by $(1 - 2M/r)^{-1}(1 + 2M\cos\bar{\theta}/r)\partial f/\partial\bar{t}$. These examples show that the functions included in the transport term of the momentum-space variables depend in general on the choice of the tetrad basis. This implies that choice of $e^a_{(\mu)}$ corresponds to the "gauge choice" of the momentum space composed of ν, $\bar{\theta}$, and $\bar{\varphi}$. For a physical interpretation of the momentum-space variables, the appropriate choice of the tetrad basis as well as the spacetime gauge would be necessary.

We note that the resulting equation for f often becomes singular due to the presence of the term $(1 - 2M/r)^{-1/2}$ even in the Eddington-Finkelstein coordinates for a choice that $e^a_{(0)}$ is timelike only for $r > 2M$. Thus, to avoid the appearance of the singular term, a horizon penetrating coordinate together with an appropriate choice of the tetrad is necessary.

2. Kerr black hole

Next, we consider spinning black holes (Kerr black holes). In the Boyer-Lindquist coordinates, the line element is written as (e.g., [18])

$$ds^2 = -\left(1 - \frac{2Mr}{\Sigma}\right)dt^2 - \frac{4Mar\sin^2\theta}{\Sigma}dtd\varphi + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 + \frac{\Xi}{\Sigma}\sin^2\theta d\varphi^2, \quad (44)$$

where a is the spin parameter from which the angular momentum of the black hole is derived as $J = Ma$. Σ, Δ, and Ξ are defined by

$$\Sigma := r^2 + a^2\cos^2\theta,$$
$$\Delta := r^2 - 2Mr + a^2,$$
$$\Xi := (r^2 + a^2)\Sigma + 2Ma^2 r\sin^2\theta, \tag{45}$$

and $\sqrt{-g} = \Sigma \sin\theta$. The canonical orthonormal tetrad introduced by Carter [19] is

$$e^a_{(0)} = \frac{1}{\sqrt{\Sigma\Delta}}\left[(r^2+a^2)\left(\frac{\partial}{\partial t}\right)^a + a\left(\frac{\partial}{\partial\varphi}\right)^a\right],$$
$$e^a_{(1)} = \sqrt{\frac{\Delta}{\Sigma}}\left(\frac{\partial}{\partial r}\right)^a, \qquad e^a_{(2)} = \frac{1}{\sqrt{\Sigma}}\left(\frac{\partial}{\partial\theta}\right)^a,$$
$$e^a_{(3)} = \frac{1}{\sqrt{\Sigma}\sin\theta}\left[a\sin^2\theta\left(\frac{\partial}{\partial t}\right)^a + \left(\frac{\partial}{\partial\varphi}\right)^a\right], \tag{46}$$

and thus, $dV_x = (r^2+a^2)\sqrt{\Sigma/\Delta}\sin\theta dr d\theta d\varphi$. This choice is valid only outside the horizon for which $\Delta > 0$ because $e^a_{(0)}$ is timelike only for this region. For this case, $\gamma_{\alpha\beta\gamma}$ has many zero components and, furthermore, the nonzero components are described probably in the simplest way as

$$\gamma_{121} = -\gamma_{211} = \gamma_{200} = -\gamma_{020} = -\frac{a^2\sin\theta\cos\theta}{\Sigma^{3/2}},$$
$$\gamma_{122} = -\gamma_{212} = \gamma_{133} = -\gamma_{313} = -\frac{r\Delta^{1/2}}{\Sigma^{3/2}},$$
$$\gamma_{130} = -\gamma_{310} = \gamma_{103} = -\gamma_{013} = \gamma_{301} = -\gamma_{031} = -\frac{ar\sin\theta}{\Sigma^{3/2}},$$
$$\gamma_{233} = -\gamma_{323} = -\frac{(r^2+a^2)\cot\theta}{\Sigma^{3/2}},$$
$$\gamma_{230} = -\gamma_{320} = \gamma_{203} = -\gamma_{023} = -\gamma_{302} = \gamma_{032} = -\frac{a\Delta^{1/2}\cos\theta}{\Sigma^{3/2}},$$
$$\gamma_{100} = -\gamma_{010} = \frac{Mr^2 - a^2 r\sin^2\theta - Ma^2\cos^2\theta}{\Sigma^{3/2}\Delta^{1/2}}, \tag{47}$$

and thus,

$$\begin{aligned}&\left[\frac{r^2+a^2}{\sqrt{\Sigma\Delta}} + \frac{a\sin\theta}{\sqrt{\Sigma}}\sin\bar\theta\sin\bar\varphi\right]\frac{\partial f}{\partial t} + \frac{1}{\Sigma}\frac{\partial}{\partial r}(f\sqrt{\Sigma\Delta}\cos\bar\theta)\\
&+ \frac{1}{\Sigma\sin\theta}\frac{\partial}{\partial\theta}(f\sqrt{\Sigma}\sin\theta\sin\bar\theta\cos\bar\varphi)\\
&+ \left[\frac{\sin\bar\theta\sin\bar\varphi}{\sqrt{\Sigma}\sin\theta} + \frac{a}{\sqrt{\Sigma\Delta}}\right]\frac{\partial f}{\partial\varphi}\\
&- \frac{1}{\nu^2}\frac{\partial}{\partial\nu}(\nu^3 f\omega_{(0)}) + \frac{1}{\sin\bar\theta}\frac{\partial}{\partial\bar\theta}(\sin\bar\theta f\omega_{(\bar\theta)})\\
&+ \frac{1}{\sin^2\bar\theta}\frac{\partial}{\partial\bar\varphi}(f\omega_{(\bar\varphi)}) = S_{\rm rad},\end{aligned} \tag{48}$$

where

$$\begin{aligned}\omega_{(0)} = &\frac{1}{\Sigma^{3/2}\Delta^{1/2}}[\cos\bar\theta(Mr^2 - a^2 r + a^2(r-M)\cos^2\theta)\\
&- 2ar\Delta^{1/2}\sin\theta\sin\bar\theta\cos\bar\theta\sin\bar\varphi\\
&- a^2\Delta^{1/2}\sin\theta\cos\theta\sin\bar\theta\cos\bar\varphi],\end{aligned} \tag{49}$$

$$\begin{aligned}\omega_{(\bar\theta)} = -&\frac{\sin\bar\theta}{\Sigma^{3/2}\Delta^{1/2}}[r(r^2 - 3Mr + 2a^2)\\
&- a^2(r-M)\cos^2\theta\\
&+ 2ar\Delta^{1/2}\sin\theta\sin\bar\theta\sin\bar\varphi],\end{aligned} \tag{50}$$

$$\begin{aligned}\omega_{(\bar\varphi)} = -&\frac{\sin^2\bar\theta}{\Sigma^{3/2}}[(r^2 + a^2 + a^2\sin^2\theta)\cot\theta\sin\bar\theta\sin\bar\varphi\\
&+ 2a\cos\theta\Delta^{1/2}].\end{aligned} \tag{51}$$

Note that for $a \to 0$, $\omega_{(0)}$, $\omega_{(\bar\theta)}$, and $\omega_{(\bar\varphi)}$ agree with Eqs. (30), (33), and (34), respectively.

It is worthy to note that $\omega_{(\bar\theta)}$ and $\omega_{(\bar\varphi)}$ are composed only of two basis functions of $(\bar\theta, \bar\varphi)$: only one component is increased compared to the Schwarzschild case. Here, the linear order term of a in the above functions are qualitatively new terms associated with the frame-dragging effect of the rotating body. They couple or decouple with $\sin\bar\varphi$ in a unique manner. Namely, even if a swarm of light rays is emitted locally in the isotropic manner with respect to $\bar\varphi$, the isotropy is violated by the frame-dragging effect. Besides such a physical term, no additional function of $(\bar\theta, \bar\varphi)$ appears. $\omega_{(0)}$ also has only three terms, one of which is associated with the frame dragging effect.

Other tetrad sets such as the locally nonrotating frame [20] may be chosen as a tetrad basis,

$$e^a_{(0)} = \sqrt{\frac{\Xi}{\Delta\Sigma}}\left[\left(\frac{\partial}{\partial t}\right)^a + \frac{2Mar}{\Xi}\left(\frac{\partial}{\partial\varphi}\right)^a\right],$$
$$e^a_{(1)} = \sqrt{\frac{\Delta}{\Sigma}}\left(\frac{\partial}{\partial r}\right)^a, \qquad e^a_{(2)} = \frac{1}{\sqrt{\Sigma}}\left(\frac{\partial}{\partial\theta}\right)^a,$$
$$e^a_{(3)} = \frac{\sqrt{\Sigma}}{\sqrt{\Xi}\sin\theta}\left(\frac{\partial}{\partial\varphi}\right)^a, \tag{52}$$

where $e_{(0)a} = -\alpha\nabla_a t (= n_a)$, $\alpha = \sqrt{\Delta\Sigma/\Xi}$, and thus, $dV_x = \sqrt{\Xi\Sigma/\Delta}\sin\theta dr d\theta d\varphi$. Again the choice of this tetrad is valid only outside the event horizon due to the choice of the spacetime coordinates. For this case, the equation becomes a bit complicated because $\gamma_{\alpha\beta\gamma}$ has more non-zero components and moreover the non-zero components are not written as simply as those in Eq. (47) (see Appendix C). For such a case, the physical meaning of the transport terms with respect to the momentum-space variables are slightly obscured.

In the Kerr-Schild coordinates (e.g., [18]),

$$ds^2 = -\left(1-\frac{2Mr}{\Sigma}\right)d\bar{t}^2 + \frac{4Mr}{\Sigma}\left[\frac{\Sigma}{r^2+a^2}dr - a\sin^2\theta d\varphi\right]d\bar{t} + \left(\frac{\Sigma}{r^2+a^2}+\frac{2Mr\Sigma}{(r^2+a^2)^2}\right)dr^2 + \Sigma d\theta^2 + \frac{\Xi}{\Sigma}\sin^2\theta d\varphi^2 - \frac{4Mra}{r^2+a^2}\sin^2\theta dr d\varphi, \tag{53}$$

the equation is even more complicated except for the choice of the Carter's tetrad [19]. First, we simply choose a locally nonrotating frame as

$$e^a_{(0)} = n^a = \frac{1}{\alpha}\left[\left(\frac{\partial}{\partial\bar{t}}\right)^a - \beta^r\left(\frac{\partial}{\partial r}\right)^a - \beta^\varphi\left(\frac{\partial}{\partial\varphi}\right)^a\right],$$
$$e^a_{(1)} = \frac{1}{\sqrt{g_{rr}}}\left(\frac{\partial}{\partial r}\right)^a, \qquad e^a_{(2)} = \frac{1}{\sqrt{\Sigma}}\left(\frac{\partial}{\partial\theta}\right)^a,$$
$$e^a_{(3)} = \frac{\gamma^{r\varphi}}{\sqrt{\gamma^{\varphi\varphi}}}\left(\frac{\partial}{\partial r}\right)^a + \sqrt{\gamma^{\varphi\varphi}}\left(\frac{\partial}{\partial\varphi}\right)^a, \tag{54}$$

where $n_a = -\alpha\nabla_a\bar{t}$ and

$$\alpha = \sqrt{\frac{\Sigma}{\Sigma+2Mr}}, \qquad \beta^r = \frac{2Mr}{(\Sigma+2Mr)},$$
$$\beta^\varphi = -\frac{2Mar}{(\Sigma+2Mr)(r^2+a^2)}, \qquad \gamma^{rr} = \frac{\Xi}{\Sigma(\Sigma+2Mr)},$$
$$\gamma^{r\varphi} = \frac{2Mar}{(\Sigma+2Mr)(r^2+a^2)},$$
$$\gamma^{\varphi\varphi} = \frac{\Sigma(r^2+2Mr+a^2)}{(r^2+a^2)^2(\Sigma+2Mr)\sin^2\theta}.$$

Then the timelike nature of $e^a_{(0)}$ is guaranteed for the entire region with $r>0$, and also $dV_x = \sqrt{\gamma}drd\theta d\varphi = \sqrt{\Sigma(\Sigma+2Mr)}\sin\theta drd\theta$ is guaranteed to be nonsingular for $r>0$. However, in this case, all the components of $\gamma_{\alpha\beta\gamma}$ are nonzero (see Appendix C). As a result, $\omega_{(0)}$, $\sin\bar{\theta}\omega_{(\bar{\theta})}$, $\omega_{(\bar{\varphi})}$ are written by many basis functions of $Y_{lm}(\bar{\theta},\bar{\varphi})$ (see also Appendix C), although all these functions in the equation for f are regular for $r>0$ and hence in numerical computation, this choice of the tetrad will be useful. Indeed, the following equation for f does not have any term that is the singular at the horizon:

$$\left(1+\frac{2Mr}{\Sigma}\right)^{1/2}\frac{\partial f}{\partial\bar{t}} + \frac{1}{\Sigma}\frac{\partial}{\partial r}\left[f\sqrt{\Sigma}\left(-\frac{2Mr}{\sqrt{\Sigma+2Mr}} + \frac{(r^2+a^2)\cos\bar{\theta}}{\sqrt{r^2+2Mr+a^2}} + \frac{2Mar\sin\theta\sin\bar{\theta}\sin\bar{\varphi}}{\sqrt{(\Sigma+2Mr)(r^2+2Mr+a^2)}}\right)\right]$$
$$+\frac{1}{\Sigma\sin\theta}\frac{\partial}{\partial\theta}(f\sqrt{\Sigma}\sin\theta\sin\bar{\theta}\cos\bar{\varphi}) + \left[\frac{2Mar}{\sqrt{\Sigma(\Sigma+2Mr)}(r^2+a^2)} + \frac{\sqrt{\Sigma(r^2+2Mr+a^2)}\sin\bar{\theta}\sin\bar{\varphi}}{(r^2+a^2)\sqrt{\Sigma+2Mr}\sin\theta}\right]\frac{\partial f}{\partial\varphi}$$
$$-\frac{1}{\nu^2}\frac{\partial}{\partial\nu}(\nu^3 f\omega_{(0)}) + \frac{1}{\sin\bar{\theta}}\frac{\partial}{\partial\bar{\theta}}(\sin\bar{\theta}f\omega_{(\bar{\theta})}) + \frac{1}{\sin^2\bar{\theta}}\frac{\partial}{\partial\bar{\varphi}}(f\omega_{(\bar{\varphi})}) = S_{\rm rad}. \tag{55}$$

On the other hand, if we choose the Carter's tetrad which is written in this case as

$$e^a_{(0)} = \frac{1}{\sqrt{\Sigma\Delta}}\left[(r^2+a^2)\left(\frac{\partial}{\partial\bar{t}}\right)^a + a\left(\frac{\partial}{\partial\varphi}\right)^a\right],$$
$$e^a_{(1)} = \sqrt{\frac{\Delta}{\Sigma}}\left(\frac{\partial}{\partial r}\right)^a + \frac{2Mr}{\sqrt{\Sigma\Delta}}\left(\frac{\partial}{\partial\bar{t}}\right)^a + \frac{2Mar}{\sqrt{\Sigma\Delta}(r^2+a^2)}\left(\frac{\partial}{\partial\varphi}\right)^a,$$
$$e^a_{(2)} = \frac{1}{\sqrt{\Sigma}}\left(\frac{\partial}{\partial\theta}\right)^a,$$
$$e^a_{(3)} = \frac{1}{\sqrt{\Sigma}\sin\theta}\left[a\sin^2\theta\left(\frac{\partial}{\partial\bar{t}}\right)^a + \left(\frac{\partial}{\partial\varphi}\right)^a\right], \tag{56}$$

the resulting forms of $\gamma_{\alpha\beta\gamma}$, $\omega_{(0)}$, $\omega_{(\bar{\theta})}$, and $\omega_{(\bar{\varphi})}$ are the same as Eqs. (47), (49), (50), and (51). Thus, the equation for f is only slightly modified from Eq. (48) and is simple as

$$\left[\frac{(r^2+a^2)+2Mr\cos\bar{\theta}}{\sqrt{\Sigma\Delta}} + \frac{a\sin\theta}{\sqrt{\Sigma}}\sin\bar{\theta}\sin\bar{\varphi}\right]\frac{\partial f}{\partial\bar{t}}$$
$$+\frac{1}{\Sigma}\frac{\partial}{\partial r}(f\sqrt{\Sigma\Delta}\cos\bar{\theta}) + \frac{1}{\Sigma\sin\theta}\frac{\partial}{\partial\theta}(f\sqrt{\Sigma}\sin\theta\sin\bar{\theta}\cos\bar{\varphi})$$
$$+\left[\frac{\sin\bar{\theta}\sin\bar{\varphi}}{\sqrt{\Sigma}\sin\theta} + \frac{a}{\sqrt{\Sigma\Delta}}\left(1+\frac{2Mr\cos\bar{\theta}}{r^2+a^2}\right)\right]\frac{\partial f}{\partial\varphi}$$
$$-\frac{1}{\nu^2}\frac{\partial}{\partial\nu}(\nu^3 f\omega_{(0)}) + \frac{1}{\sin\bar{\theta}}\frac{\partial}{\partial\bar{\theta}}(\sin\bar{\theta}f\omega_{(\bar{\theta})})$$
$$+\frac{1}{\sin^2\bar{\theta}}\frac{\partial}{\partial\bar{\varphi}}(f\omega_{(\bar{\varphi})}) = S_{\rm rad}. \tag{57}$$

Again, the choice of this tetrad is valid only for $\Delta>0$, and the resulting equation for f becomes singular due to the presence of the term $\Delta^{-1/2}$. This illustrates the fact that only for the horizon penetrating coordinates together with an appropriate choice of the tetrad, the appearance of the coordinate singularities at the horizon are avoided.

It is worthy to note that at the horizon where $\Delta = r^2+a^2-2Mr=0$ with $\bar{\theta}=\pi$, the coefficient of term

$\partial\tilde{f}/\partial\bar{t}$ (and $\partial\tilde{f}/\partial\varphi$) in Eq. (57) vanishes. For this point, we have to impose a particular condition for $\partial(f\nu^3)/\partial\nu^3$ based on the consistency, and this gives $f \propto \nu$ irrespective of the value of a/M.

III. SUMMARY

We derived a conservative form of Boltzmann's equation in general relativity, which is concisely written. As an illustration, we described explicit forms of Boltzmann's equation in black-hole spacetime with several coordinate conditions in real spacetime and momentum space, and showed that the conservative forms can be indeed written in a concise form. It is also found that the meaning of the transport term in the momentum-space variables is clearly understood in the black-hole spacetime for a suitable choice of the tetrad basis, while for less suitable choices, the meaning is obscured. This indicates that an appropriate choice of the tetrad basis may be necessary for understanding the physical meaning of the numerical results of Boltzmann's equation in general relativity. We showed the properties of several tetrad sets in the Schwarzschild, Eddington-Finkelstein, Boyer-Lindquist, and Kerr-Schild coordinates.

In numerical computation, by contrast, the absence of the singular terms, which could appear at the horizon in the equation, is a more serious requirement for the basic equation. For this, Eddington-Finkelstein and Kerr-Schild coordinates together with the locally nonrotating-frame tetrad with $e^a_{(0)} = n^a$ have the suitable property. We described the equations of f for such choices in an explicit manner for the future numerical simulations.

For the practical implementation, the finite differentiation has to be carefully chosen so that the physical principles, such as conservation laws and Fermi statistics of neutrinos, and the other numerical issues, such as the correct limiting behavior of numerical fluxes in free-streaming and diffusion regimes, should be fulfilled. In particular, ensuring the correct diffusion limit in a solution of the discretized Boltzmann's equation is a difficult problem, as detailed in [4]. However, discussions of this are beyond the scope of the present paper.

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APPENDIX A: ON THE EXISTENCE OF THE CONSERVATIVE FORMS

The existence of the conservative forms is guaranteed by the following mathematical fact described in [13]. We here outline it.

The so-called Liouville vector field is defined as

$$\mathcal{L} = p^\mu \frac{\partial}{\partial x^\mu} - \Gamma^\mu_{\nu\sigma} p^\nu p^\sigma \frac{\partial}{\partial p^\mu} \tag{A1}$$

in the eight-dimensional tangent bundle of the four-dimensional spacetime. It is easy to confirm that the vector field is tangential to the seven-dimensional closed submanifold of the tangent bundle defined by

$$p^\mu p_\mu = m^2 \tag{A2}$$

at each point where m is the mass of particles described by Boltzmann's equation ($m = 0$ for radiations). We can hence regard the Liouville vector field as a vector field on this seven-dimensional submanifold or mass shell, which is nothing but the differential operator on the left-hand side of Eq. (2). It is then confirmed by direct calculations that this seven-dimensional version of the Liouville vector field is divergence free:

$$\mathrm{div}\mathcal{L} = 0. \tag{A3}$$

The tangent bundle is naturally endowed with a (pseudo-) Riemannian metric, which is inherited from the four-dimensional spacetime. The metric is then pulled back to the mass shell, making it also a (pseudo-) Riemannian manifold. The volume element of the mass shell may be written as

$$\Omega = \eta \wedge \pi_m, \tag{A4}$$

with the volume element $\eta = \sqrt{-g} dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3$ of four-dimensional space-time and the volume element $\pi_m = (\sqrt{-g}/|p_0|) dp^1 \wedge dp^2 \wedge dp^3$ of the three-dimensional fiber space. Here the wedge product of forms is denoted by $\wedge$. If we define a volume element ω on an arbitrary six-dimensional momentum space by the inner product of the Liouville vector field with the volume element Ω as

$$\omega = \mathcal{L} \cdot \Omega, \tag{A5}$$

then we find that its exterior derivative vanishes,

$$d\omega = 0, \tag{A6}$$

as a consequence of Eq. (A3). It is also confirmed that ω is invariant with respect to the Liouville flows, i.e. integral curves of the Liouville vector field.

The momentum space is composed of six-dimensional hypersurfaces in the mass shell, which are traversed by the Liouville flows. Recalling that Boltzmann's equation can be cast in the following integral form for an arbitrary domain in the mass shell:

$$\int_{\partial D} f\omega = \int_D \left(\frac{\delta f}{\delta \tau}\right)_{\mathrm{coll}} \Omega, \tag{A7}$$

and using the Stokes's theorem,

$$\int_{\partial D} f\omega = \int_D d(f\omega), \tag{A8}$$

as well as Eq. (A6), we obtain the following relation:

$$\int_{\partial D} f\omega = \int_D df\wedge\omega = \int_D df\wedge(\mathcal{L}\cdot\Omega) = \int_D \mathcal{L}(f)\Omega = \int_D \left(\frac{\delta f}{\delta\tau}\right)_{\rm coll}\Omega. \tag{A9}$$

Since the domain D is arbitrary, this relation gives Boltzmann's equation in the ordinary form. Here the collision term is denoted by $\int_D (\delta f/\delta\tau)_{\rm coll}\Omega$, which is identical to $-p^\alpha\hat{u}_\alpha S_{\rm rad}$ of Eq. (1). If we use instead the following relation in Eq. (A8),

$$d(f\omega) = d((f\mathcal{L})\cdot\Omega) = {\rm div}(f\mathcal{L})\Omega, \tag{A10}$$

then we obtain the conservative form of Boltzmann's equation as follows:

$${\rm div}(f\mathcal{L}) = \left(\frac{\delta f}{\delta\tau}\right)_{\rm coll}. \tag{A11}$$

Note that the mass shell is a (pseudo-) Riemannian manifold, and the divergence in the above equation is identical to the one defined by the covariant derivatives for the Levi-Civita connection, which is then recast into the conservative form such as Eqs. (18) and (20) with ordinary partial derivatives. The above derivation is completely covariant and hence applicable to any local coordinates on the mass shell including the ones considered in this paper.

APPENDIX B: TRANSFORATION TO THE LOCAL REST FRAME

In Sec. II, we focus only on the left-hand side of Boltzmann's equation. When solving Boltzmann's equation, however, we have to consider how to handle the right-hand side, i.e., collision terms, which is associated with the interaction of the radiation field f with matters and should be calculated in the matter rest frame. For this, we need to perform a coordinate transformation for the momentum-space variables. Specifically, we have to obtain the momentum-space variables in the matter rest frame, $(\hat{\nu},\hat{\theta},\hat{\varphi})$. In our formulation in which a local orthonormal frame with respect to a tetrad is prepared, however, the procedure is quite straightforward, as shown in the following.

First, $\hat{\nu}$ is simply obtained by

$$\hat{\nu} = -p^a u_a = -\nu\left(u_{(0)} + \sum_{i=1}^{3}\ell_{(i)}u_{(i)}\right), \tag{B1}$$

where u^a denotes the four-velocity of the matter as before, and

$$u_{(0)} = u_a e^a_{(0)} \quad \text{and} \quad u_{(i)} = u_a e^a_{(i)}, \tag{B2}$$

Here $u^{(\alpha)}$ denotes the spacetime component of the four-velocity in the local orthonormal frame of a tetrad basis, and $u^{(0)}=-u_{(0)}$ and $u^{(i)}=u_{(i)}$ which satisfy $\eta_{\alpha\beta}u^{(\alpha)}u^{(\beta)}=-1$. We also define the spacetime components of p^a in the local orthonormal frame of the tetrad basis as $p_{(\mu)} \coloneqq p_a e^a_{(\mu)} = \nu(-1,\ell^i)$ and $p^{(\mu)} = \nu(1,\ell^i)$.

The next task is to determine $\hat{\theta}$ and $\hat{\varphi}$. For this, we first determine three spatial unit vectors in the local orthonormal frame, $q_1^{(\alpha)}$, $q_2^{(\alpha)}$, and $q_3^{(\alpha)}$, which are perpendicular to $u^{(\alpha)}$. Because these vectors are independent of each other, $\eta_{\alpha\beta}q_i^{(\alpha)}q_j^{(\beta)} = \delta_{ij}$. Now, we suppose that $q_1^{(\alpha)}$ points to the radial direction in the matter rest frame, and $q_2^{(\alpha)}$ and $q_3^{(\alpha)}$ are unit vectors in the two angular directions as in $\ell_{(2)}$ and $\ell_{(3)}$. Then, we can define the angles in the matter rest frame by

$$\cos\hat{\theta} = \frac{q_1^{(\alpha)}p_{(\alpha)}}{\hat{\nu}}, \tag{B3}$$

$$\tan\hat{\varphi} = \frac{q_3^{(\alpha)}p_{(\alpha)}}{\hat{q}_2^{(\alpha)}p_{(\alpha)}}. \tag{B4}$$

Here, for example, we can choose

$$q_1^{(\alpha)} = \frac{1}{\sqrt{(u^{(0)})^2-(u^{(1)})^2}}[u^{(1)},u^{(0)},0,0], \tag{B5}$$

$$q_2^{(\alpha)} = \frac{\sqrt{(u^{(0)})^2-(u^{(1)})^2}}{\sqrt{(u^{(0)})^2-(u^{(1)})^2-(u^{(2)})^2}} \times\left[\frac{u^{(0)}u^{(2)}}{(u^{(0)})^2-(u^{(1)})^2},\frac{u^{(1)}u^{(2)}}{(u^{(0)})^2-(u^{(1)})^2},1,0\right], \tag{B6}$$

$$q_3^{(\alpha)} = \frac{1}{\sqrt{(u^{(0)})^2-(u^{(1)})^2-(u^{(2)})^2}} \times[u^{(0)}u^{(3)},u^{(1)}u^{(3)},u^{(2)}u^{(3)},(u^{(0)})^2-(u^{(1)})^2-(u^{(2)})^2]. \tag{B7}$$

Therefore, the transformation from $(\nu,\bar{\theta},\bar{\varphi})$ to $(\hat{\nu},\hat{\theta},\hat{\varphi})$ can be done only with several projection procedures (we do not have to solve any equation).

APPENDIX C: RICCI ROTATION COEFFICIENTS FOR ROTATING BLACK HOLES

In this appendix, we list Ricci rotation coefficients and resulting forms for $\omega_{(0)}$, $\omega_{(\bar{\theta})}$, and $\omega_{(\bar{\varphi})}$ for the zero angular momentum frame in the Boyer-Lindquist and Kerr-Schild coordinates of rotating black holes.

For the choice of the zero angular momentum frame in the Boyer-Lindquist coordinates,

$$\begin{aligned}
\gamma_{121} &= -\gamma_{211} = -\frac{a^2\sin\theta\cos\theta}{\Sigma^{3/2}},\\
\gamma_{122} &= -\gamma_{212} = -\frac{r\Delta^{1/2}}{\Sigma^{3/2}},\\
\gamma_{133} &= -\gamma_{313} = \frac{\Delta^{1/2}}{\Sigma^{3/2}\Xi}[-r\Sigma^2 + (r^2 - a^2\cos^2\theta)a^2M\sin^2\theta],\\
\gamma_{130} &= -\gamma_{310} = \gamma_{103} = -\gamma_{013} = \gamma_{301} = -\gamma_{031}\\
&= -\frac{aM\sin\theta}{\Sigma^{3/2}\Xi}[r^2(3r^2+a^2) + a^2(r^2-a^2)\cos^2\theta],\\
\gamma_{100} &= -\gamma_{010} = \frac{M}{\Sigma^{3/2}\Delta^{1/2}\Xi}\times[(r^2+a^2)^2(r^2-a^2\cos^2\theta)\\
&\quad - 4a^2Mr^3\sin^2\theta],\\
\gamma_{233} &= -\gamma_{323} = -\frac{\cot\theta}{\Sigma^{3/2}\Xi}[r^4(r^2+a^2) + 2Mra^2(2r^2+a^2)\\
&\quad + a^2\cos^2\theta(2r^2+a^2\cos^2\theta)\Delta],\\
\gamma_{230} &= -\gamma_{320} = \gamma_{203} = -\gamma_{023} = \gamma_{302} = -\gamma_{032}\\
&= \frac{2a^3Mr\Delta^{1/2}\sin^2\theta\cos\theta}{\Sigma^{3/2}\Xi},\\
\gamma_{200} &= -\gamma_{020} = -\frac{2Ma^2r(r^2+a^2)\sin\theta\cos\theta}{\Sigma^{3/2}\Xi},
\end{aligned}\tag{C1}$$

and thus,

$$\begin{aligned}
\omega_{(0)} &= \gamma_{100}\cos\bar\theta + \gamma_{200}\sin\bar\theta\cos\bar\varphi + 2\gamma_{130}\sin\bar\theta\cos\bar\theta\sin\bar\varphi\\
&\quad + 2\gamma_{230}\sin^2\bar\theta\sin\bar\varphi\cos\bar\varphi,
\end{aligned}\tag{C2}$$

$$\begin{aligned}
\omega_{(\bar\theta)} &= \sin\bar\theta(\gamma_{100} + \gamma_{122}\cos^2\bar\varphi + \gamma_{133}\sin^2\bar\varphi)\\
&\quad + (\gamma_{121} + \gamma_{020})\cos\bar\theta\cos\bar\varphi + 2\gamma_{130}\sin^2\bar\theta\sin\bar\varphi\\
&\quad - 2\gamma_{230}\sin\bar\theta\cos\bar\theta\sin\bar\varphi\cos\bar\varphi,
\end{aligned}\tag{C3}$$

$$\begin{aligned}
\omega_{(\bar\varphi)} &= \sin\bar\theta\sin\bar\varphi[\gamma_{200} - \gamma_{121}\cos^2\bar\theta + \gamma_{233}\sin^2\bar\theta\\
&\quad + 2\gamma_{230}\sin\bar\theta\sin\bar\varphi\\
&\quad + (\gamma_{133} - \gamma_{122})\sin\bar\theta\cos\bar\theta\cos\bar\varphi].
\end{aligned}\tag{C4}$$

For the choice of the zero angular momentum frame in the Kerr-Schild coordinates,

$$\begin{aligned}
\gamma_{121} &= -\gamma_{211} = -\frac{a^2\sin\theta\cos\theta}{\Sigma^{3/2}},\\
\gamma_{122} &= -\gamma_{212} = -\frac{r(r^2+a^2)}{\Sigma^{3/2}(r^2+2Mr+a^2)^{1/2}},\\
\gamma_{123} &= -\gamma_{213} = -\gamma_{231} = \gamma_{321} = -\gamma_{132} = \gamma_{312}\\
&= -\frac{2Mar\cos\theta}{\Sigma^{3/2}(\Sigma+2Mr)^{1/2}},\\
\gamma_{120} &= -\gamma_{210} = -\gamma_{102} = \gamma_{012} = -\gamma_{201} = \gamma_{021}\\
&= \frac{2Ma^2r\sin\theta\cos\theta}{\Sigma^{3/2}(r^2+2Mr+a^2)^{1/2}(\Sigma+2Mr)^{1/2}},\\
\gamma_{131} &= -\gamma_{311} = -\frac{2Ma\sin\theta(r^4+Mr^3-a^2(Mr+a^2)\cos^2\theta)}{\Sigma^{3/2}(r^2+2Mr+a^2)^{3/2}(\Sigma+2Mr)^{1/2}},\\
\gamma_{133} &= -\gamma_{313} = \frac{1}{\Sigma^{3/2}(r^2+2Mr+a^2)^{3/2}(\Sigma+2Mr)}\\
&\quad\times[-r(r^2+2Mr+a^2)\Sigma^2 + M\{a^4(-r^2+a^2)\cos^4\theta\\
&\quad - a^2\cos^2\theta(5r^4+8Mr^3+2a^2r^2+a^4)\\
&\quad + r^2(-2r^4-4Mr^3+a^2r^2+4Ma^2r+a^4)\}],\\
\gamma_{130} &= -\gamma_{310} = \gamma_{103} = -\gamma_{013} = \gamma_{301} = -\gamma_{031}\\
&= -\frac{aM\sin\theta[r^2(3r^2+4Mr+a^2)+a^2(r^2-a^2)\cos^2\theta]}{\Sigma^{3/2}(r^2+2Mr+a^2)(\Sigma+2Mr)},\\
\gamma_{101} &= -\gamma_{011} = \frac{2M[r^4+Mr^3-a^2(Mr+a^2)\cos^2\theta]}{\Sigma^{3/2}(r^2+2Mr+a^2)(\Sigma+2Mr)^{1/2}},\\
\gamma_{100} &= -\gamma_{010} = \frac{M(r^2+a^2)(r^2-a^2\cos^2\theta)}{\Sigma^{3/2}(r^2+2Mr+a^2)^{1/2}(\Sigma+2Mr)},\\
\gamma_{232} &= -\gamma_{322} = \frac{2Mar^2\sin\theta}{\Sigma^{3/2}(r^2+2Mr+a^2)^{1/2}(\Sigma+2Mr)^{1/2}},\\
\gamma_{233} &= -\gamma_{323} = -\frac{\cot\theta}{\Sigma^{3/2}(\Sigma+2Mr)}[r^3(r+2M)\\
&\quad + a^2(2r^2\cos^2\theta + 2Mr + a^2\cos^4\theta)],\\
\gamma_{230} &= -\gamma_{320} = \gamma_{203} = -\gamma_{023} = \gamma_{302} = -\gamma_{032}\\
&= \frac{2Ma^3r\sin^2\theta\cos\theta}{\Sigma^{3/2}(r^2+2Mr+a^2)^{1/2}(\Sigma+2Mr)},\\
\gamma_{202} &= -\gamma_{022} = -\frac{2Mr^2}{\Sigma^{3/2}(\Sigma+2Mr)^{1/2}},\\
\gamma_{200} &= -\gamma_{020} = -\frac{2Ma^2r\sin\theta\cos\theta}{\Sigma^{3/2}(\Sigma+2Mr)},\\
\gamma_{303} &= -\gamma_{033} = -\frac{2Mr}{\Sigma^{3/2}(\Sigma+2Mr)^{3/2}(r^2+2Mr+a^2)}\\
&\quad\times[(M+r)\Sigma^2 + M(r^2-a^2)\Sigma\\
&\quad + 2Mr^2(r^2+2Mr+a^2)],\\
\gamma_{300} &= -\gamma_{030} = \frac{2M^2ar(r^2-a^2\cos^2\theta)\sin\theta}{\Sigma^{3/2}(r^2+2Mr+a^2)^{1/2}(\Sigma+2Mr)^{3/2}}.
\end{aligned}\tag{C5}$$

Then,

$$\omega_{(0)} = \gamma_{100}\cos\bar\theta + \gamma_{200}\sin\bar\theta\cos\bar\varphi + \gamma_{300}\sin\bar\theta\sin\bar\varphi + \gamma_{101}\cos^2\bar\theta + \gamma_{202}\sin^2\bar\theta\cos^2\bar\varphi + \gamma_{303}\sin^2\bar\theta\sin^2\bar\varphi \\ + 2\gamma_{102}\cos\bar\theta\sin\bar\theta\cos\bar\varphi + 2\gamma_{103}\cos\bar\theta\sin\bar\theta\sin\bar\varphi + 2\gamma_{203}\sin^2\bar\theta\sin\bar\varphi\cos\bar\varphi, \tag{C6}$$

$$\omega_{(\bar\theta)} = \sin\bar\theta[\gamma_{100} + \gamma_{122}\cos^2\bar\varphi + \gamma_{133}\sin^2\bar\varphi] + (\gamma_{121} - \gamma_{200})\cos\bar\theta\cos\bar\varphi + (\gamma_{131} - \gamma_{300})\cos\bar\theta\sin\bar\varphi \\ + \sin\bar\theta\cos\bar\theta(-\gamma_{011} + \gamma_{022}\cos^2\bar\varphi + \gamma_{033}\sin^2\bar\varphi + 2\gamma_{023}\sin\bar\varphi\cos\bar\varphi) \\ + 2\gamma_{021}\cos^2\bar\theta\cos\bar\varphi - 2\gamma_{031}\sin^2\bar\theta\sin\bar\varphi, \tag{C7}$$

$$\omega_{(\bar\varphi)} = \sin\bar\theta(\gamma_{200}\sin\bar\varphi - \gamma_{300}\cos\bar\varphi) + 2\gamma_{201}\sin\bar\theta\cos\bar\theta\sin\bar\varphi + \sin^2\bar\theta[(\gamma_{202} - \gamma_{303})\sin\bar\varphi\cos\bar\varphi + 2\gamma_{203}\sin^2\bar\varphi] \\ + \sin\bar\theta\cos^2\bar\theta(\gamma_{211}\sin\bar\varphi - \gamma_{311}\cos\bar\varphi) - \sin^3\bar\theta(\gamma_{322}\cos\bar\varphi + \gamma_{323}\sin\bar\varphi) \\ + \sin^2\bar\theta\cos\bar\theta[(\gamma_{212} - \gamma_{313})\sin\bar\varphi\cos\bar\varphi - 2\gamma_{123}]. \tag{C8}$$

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