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Gravitational waves induced by a spinning particle falling into a rotating black hole

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Using the formalisms of Teukolsky and of Sasaki and Nakamura for the perturbation around a Kerr black hole we calculate the energy flux and the waveform of gravitational waves induced by a *spinning* particle of mass μ and spin S falling from infinity with zero in-fall velocity into a rotating black hole of mass $M \gg \mu$ and spin a along the z axis. The calculations are performed combining the Teukolsky formalism with the equations of motion of a spinning particle derived by Papapetrou and the energy-momentum tensor of a spinning particle derived by Dixon. Thus, there appear two additional effects due to the spin of the particle: one is due to the spin-spin interaction force which appears in the equations of motion and the other is due to the contribution of the energy-momentum tensor of the spinning particle. From numerical calculations, it is found that these spin effects are very important: In the case of $a = S = 0.99M$, the total energy flux becomes $0.0106(\mu/M)^2 M$, which is almost the same as that obtained by Davis *et al.* for $a = S = 0$, while in the case of $a = -S = 0.99M$, it becomes $0.0298(\mu/M)^2 M$, i.e., about three times larger. We also show that the contribution of the energy-momentum tensor of the spinning particle dominates over that of the spin-spin interaction term in the equations of motion. The results obtained in this paper will be an important guideline to quantitative estimates of gravitational waves in numerical relativistic simulations of the head-on collision of two spinning black holes.

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I. INTRODUCTION

The last stage of coalescing compact binaries composed of neutron stars and/or black holes is one of the most promising sources of gravitational waves for the kilometer size laser-interferometric detectors such as the Laser Interferometric Gravitational Wave Observatory (LIGO) [1], VIRGO [2], and the laser-interferometric detector in space such as the Laser Interferometer Space Antenna (LISA) [3]. The evolution of such a binary can be classified into two phases. One is the inspiraling phase [4]. In this phase, the binary evolves in the adiabatic manner radiating periodic gravitational waves. It has been recently realized that if we can detect the signal of gravitational waves in such a phase, and compare it with a theoretical template, we may be able to obtain not only a variety of parameters of binary such as the mass, spin, etc. [5,6], but also the cosmological parameters [5,7]. Hence theoretical research on this phase is a potentially important field of relativity [8]. On the other hand, research on the merging phase is also very important. When the separation of the binary becomes very small, the centrifugal force cannot be balanced with the strong gravity of general relativity [4] or the tidal force [9], so that the binary cannot maintain the bound orbit and finally merges. In such a phase, the binary is in the strongly nonlinear gravity because the characteristic length scale of the system is less than ten times of the Schwarzschild radius of the system. Therefore, by the detection of gravitational waves from merging of compact binary, we will be able to obtain the information of a highly general relativistic spacetime.

In order not to fail to detect such a signal and in order to extract maximum information about the strong gravitational field, however, we need to make a great deal of theoretical efforts. The reason is that the expected signal is very weak [the typical signal-to-noise ratio (SNR) is less than 10]. To confirm the detection of such a signal and to extract information from the signal of such a SNR, we have to predict the waveform of gravitational waves theoretically with a sufficient accuracy. In order to predict the waveform accurately, a fully general relativistic three-dimensional (3D) simulation is necessary. However, it is extremely difficult to perform the simulation with a sufficient accuracy throughout the large dynamic range from the initial inspiral with a large separation to the final merging. In order to carry it out, we have to solve many problems, i.e., to find appropriate gauge and slice conditions for 3D numerical relativity. Moreover, it will be very time consuming to perform many simulations for several possible parameters of binary systems. Although much effort is focused on this field and much progress can be expected [10,11], it would be very helpful if we could adopt a more economical and sufficiently reliable approximate method to calculate the waveform and the energy flux of gravitational waves.

The perturbation calculation around a black hole is expected to be an appropriate approximation method to treat coalescing binary black holes, in which we calculate gravitational waves induced by a particle of mass μ orbiting around a black hole of mass $M \gg \mu$ and spin a. Although we neglect the nonlinear effect of gravity induced by the small mass particle [i.e., we neglect $O((\mu/M)^2)$ terms in the metric], it is still possible to investigate that of the background rotating black hole. In fact, it is well known that the perturbation calculation gives a fairly good approximation to numerical relativistic calculations; the waveform and the total energy flux of gravitational waves in the head-on collision of two equal mass black holes [12] agree with the extrapolated value $[\mu/(M+\mu) \to 1/2]$ of those obtained by perturbation calculation [13] with a fairly good accuracy ($\lesssim 15\%$) [12]. The waveforms obtained in the simulations of the stellar core collapse [14] also agree with those calculated by an approximate perturbation study of a rotating ring inspiraling into a Schwarzschild black hole [15]. Furthermore, the results obtained in the perturbation calculations are almost exact in the case when the mass of a companion μ is much smaller than the black hole mass M. Thus in this paper, we pursue the perturbation calculation in order to make an approximate model of coalescence of spinning black holes. In particular, we here consider the head-on collision of two *spinning* black holes as a first step.

When we calculate gravitational waves from merging of binary black holes, we should specify their masses and spins in general. Hence, to make an approximate model of coalescence of the rotating binary black holes by the perturbation calculation, we need to take into account the spin of the small mass particle as well as that of the background black hole. In the previous works the latter effect was incorporated but the former has not been considered [15]. To incorporate the former effect, one must know (1) the equations of motion and (2) the energy-momentum tensor of the small mass particle. Fortunately, we know that (1) have already been derived by Papapetrou [16], Dixon [17], and Wald [18], and so on, and (2) has also already been derived by Dixon [17]. Hence, by using the energy-momentum tensor of a spinning particle by Dixon as the source term in the Teukolsky formalism [19], we can calculate the waveform and the energy flux of gravitational waves by a spinning small mass particle falling and/or orbiting around a rotating black hole. Here, a word of caution is appropriate. A Kerr black hole of mass μ and spin S, where S is defined so that $|S|\mu$ represents the spin angular momentum, has the quadrupole moment which becomes μS^2 and higher multipole moments ($\propto \mu S^l$) as well. Since we neglect the contribution of these higher multipole moments in this paper, our treatment is not complete to model the Kerr black hole. Incorporation of the higher multipole moments to represent the Kerr black hole is a future problem to be investigated. However, we stress that in our treatment, the terms related to the spin are taken into account, and they always appear even in the head-on collision of two equal mass black holes. Furthermore, the spin effect remarkably affects the waveform and the energy flux of gravitational waves as shown below. This means that the spin effect must be important for the head-on collision of two spinning black holes of nearly equal masses. Thus the results obtained in this paper will be an important quantitative guideline to a future coming analysis of gravitational waves in realistic numerical simulations of the head-on collision of two spinning black holes, which have not been performed yet.

The paper is organized as follows. In Sec. II, we review the energy-momentum tensor of a spinning particle. Although there exists an excellent review by Dixon [17], his consideration is restricted to the case in which the spinning particle neither is a strongly self-gravitating body, nor possess the event horizon. Since we want to use his results extensively to examine the two black hole collision, some justification is necessary. We discuss this point at the end of this section. In Sec. III, by making use of this energy-momentum tensor in the context of Teukolsky formalism [19] which is a method to handle the metric perturbations on a Kerr background, we perform numerical calculations of the energy flux and the waveform of gravitational waves induced by a spinning particle falling along the z axis. This is an extension of the perturbation calculations by Davis *et al.* [13], in which they considered gravitational waves by a nonspinning particle falling into a Schwarzschild black hole and that by Sasaki and Nakamura [20], in which they considered gravitational waves by a nonspinning particle falling along the z axis into a Kerr black hole. Calculating the energy flux and the waveform of gravitational waves for a wide range of the spin parameters of the particle and the Kerr black hole, we show the importance of the spin effect in the head-on collision of two spinning black holes.

Throughout this paper, we use the unit of $c = G = 1$. We define the signature of the metric as $(-,+,+,+)$ and the Riemann tensor as

$$R_{\mu\nu\lambda}{}^{\sigma} u_{\sigma} = 2\nabla_{[\mu}\nabla_{\nu]}u_{\lambda},$$

where ∇_μ and u^σ denote the covariant derivative and an arbitrary vector, respectively. The square bracket means the antisymmetrization.

II. THE ENERGY-MOMENTUM TENSOR OF A SPINNING PARTICLE

The energy-momentum tensor and the equations of motion of a particle with multipole moments in a curved spacetime have been discussed by many authors [16–18].

The equations of motion of a spinning particle were first derived by Papapetrou [16], and then reformulated by Dixon [17], in which he discussed a general situation where the particle has arbitrary higher multipole moments. Here, we follow the Dixon's approach neglecting the higher multipole moments other than mass monopole and spin dipole moments. In this case, the equations of motion become

$$\frac{D}{d\tau}u^\mu(\tau) = -\frac{1}{2}R^\mu{}_{\nu\gamma\delta}(z(\tau))v^\nu(\tau)S^{\gamma\delta}(\tau),$$
$$\frac{D}{d\tau}S^{\mu\nu}(\tau) = 2u^{[\mu}v^{\nu]}(\tau), \tag{2.1}$$

where the vector $\mu u^\mu(\tau)$ and the antisymmetric tensor $\mu S^{\mu\nu}(\tau)$ represent the total momentum and the angular momentum, respectively, and $v^\mu(\tau) := dz^\mu(\tau)/d\tau$. Here we use the notation $D/d\tau$ to represent $v^\mu\nabla_\mu$, and ∇_μ denotes the covariant derivative with respect to the background metric. Since we do not have the evolution equation of $v^\mu(\tau)$, in order to close this set of equations, we need to impose a supplementary condition which determines the center of mass of the spinning particle [17]:

$$S^{\mu\nu}(\tau)u_\nu(\tau) = 0. \tag{2.2}$$

Noting that the above equations of motion are reparametrization invariant, we can fix the orbital parameter τ to satisfy

$$u^\mu(\tau)v_\mu(\tau) = -1. \tag{2.3}$$

Then using Eq. (2.3) with Eq. (2.2), the relation between u^μ and v^μ becomes

$$v^\mu(\tau) - u^\mu(\tau) = -\frac{1}{2}\left(1 - \frac{1}{4}R_{\zeta\eta\theta\iota}(z(\tau))S^{\zeta\eta}(\tau)S^{\theta\iota}(\tau)\right)^{-1} S^{\mu\nu}(\tau)R_{\nu\gamma\sigma\lambda}(\tau)u^\gamma(\tau)S^{\sigma\lambda}(\tau). \tag{2.4}$$

Thus, Eqs. (2.1) determine the orbital evolution of a spinning particle.

This system has several conserved quantities. Regardless of the symmetry of the background spacetime, we can show that $u^\mu u_\mu$ and $S^{\mu\nu}S_{\mu\nu}$ conserve along the orbit [18]. Hence we can normalize them as

$$u^\mu u_\mu = -1,$$
$$S^{\mu\nu}S_{\mu\nu} = 2S^2. \tag{2.5}$$

In the case when there is a Killing vector field ξ_μ which satisfies the Killing equation

$$\nabla_{(\mu}\xi_{\nu)} := \frac{1}{2}(\nabla_\mu\xi_\nu + \nabla_\nu\xi_\mu) = 0, \tag{2.6}$$

the quantity

$$Q := u^\mu\xi_\mu + \frac{1}{2}S^{\mu\nu}\nabla_\mu\xi_\nu, \tag{2.7}$$

becomes a conserved quantity. From Eqs. (2.1), it can be easily verified that Q conserves along the orbit.

Although it is originally given in a slightly different form, the energy-momentum tensor of the spinning particle is described by Dixon as [17]

$$T^{\mu\nu}(x) = \mu\int d\tau\left\{\frac{\delta^{(4)}(x-z(\tau))}{\sqrt{-g}}u^{(\mu}(x,\tau)v^{\nu)}(x,\tau) - \nabla_\sigma\left(S^{\sigma(\mu}(x,\tau)v^{\nu)}(x,\tau)\frac{\delta^{(4)}(x-z(\tau))}{\sqrt{-g}}\right)\right\}, \tag{2.8}$$

where

$$\delta^{(4)}(x-z(\tau)) = \delta(x^0-z^0(\tau))\delta(x^1-z^1(\tau))\delta(x^2-z^2(\tau))\delta(x^3-z^3(\tau)). \tag{2.9}$$

Here, we introduced bitensors $v^\mu(x,\tau), u^\mu(x,\tau), S^{\mu\nu}(x,\tau)$ which are the extensions of variables $v^\alpha(\tau), u^\alpha(\tau), S^{\alpha\beta}(\tau)$ defined only along the world line[1] $z(\tau)$. To define $v^\mu(x,\tau), u^\mu(x,\tau), S^{\mu\nu}(x,\tau)$, we introduce a bitensor $\bar{g}^\mu{}_\alpha(x,z)$ [21] which satisfies

$$\lim_{x\to z}\bar{g}^\mu{}_\alpha(x,z) = \delta^\mu{}_\alpha,$$
$$\lim_{x\to z}\nabla_\nu\bar{g}^\mu{}_\alpha(x,z) = 0. \tag{2.10}$$

[1]From now on, we use μ,ν as the tensor indices associated with x and α,β as that with z.

For the present purpose, further specification of $\bar{g}^{\mu}{}_{\alpha}(x,z)$ is not needed. Using $\bar{g}^{\mu}{}_{\alpha}(x,z)$, we define $u^{\mu}(x,\tau)$, $v^{\mu}(x,\tau)$, and $S^{\mu\nu}(x,\tau)$ as

$$\begin{aligned} u^{\mu}(x,\tau) &= \bar{g}^{\mu}{}_{\alpha}(x,z(\tau))u^{\alpha}(\tau), \\ v^{\mu}(x,\tau) &= \bar{g}^{\mu}{}_{\alpha}(x,z(\tau))v^{\alpha}(\tau), \\ S^{\mu\nu}(x,\tau) &= \bar{g}^{\mu}{}_{\alpha}(x,z(\tau))\bar{g}^{\nu}{}_{\beta}(x,z(\tau))S^{\alpha\beta}(\tau). \end{aligned} \tag{2.11}$$

It is easy to see that the divergence free condition of the energy-momentum tensor in Eq. (2.8) gives Eqs. (2.1). Noting the relations

$$\begin{aligned} &\nabla_{\nu}\bar{g}^{\mu}{}_{\alpha}(x,z)\delta^{(4)}(x,z) = 0, \\ &v^{\mu}(x)\nabla_{\mu}\left(\frac{\delta^{(4)}(x,z(\tau))}{\sqrt{-g}}\right) = -\frac{d}{d\tau}\left(\frac{\delta^{(4)}(x,z(\tau))}{\sqrt{-g}}\right), \end{aligned} \tag{2.12}$$

the divergence of Eq. (2.8) becomes

$$\begin{aligned} \nabla_{\nu}T^{\mu\nu}(x) &= \mu\int d\tau\Bigg[-u^{\mu}(x,\tau)\frac{d}{d\tau}\frac{\delta^{(4)}(x-z(\tau))}{\sqrt{-g}} - \nabla_{\nu}\left(u^{[\mu}(x,\tau)v^{\nu]}(x,\tau)\frac{\delta^{(4)}(x-z(\tau))}{\sqrt{-g}}\right) \\ &\quad +\nabla_{\nu}\left(-\frac{1}{2}S^{\mu\nu}(x,\tau)\frac{d}{d\tau}\frac{\delta^{(4)}(x-z(\tau))}{\sqrt{-g}}\right) + \frac{1}{2}R^{\mu}{}_{\nu\lambda\sigma}(x)v^{\nu}(x,\tau)S^{\lambda\sigma}(x,\tau)\frac{\delta^{(4)}(x-z(\tau))}{\sqrt{-g}}\Bigg] \\ &= \mu\int d\tau\,\bar{g}^{\mu}{}_{\alpha}(x,z(\tau))\frac{\delta^{(4)}(x-z(\tau))}{\sqrt{-g}}\left(\frac{d}{d\tau}u^{\alpha}(\tau) + \frac{1}{2}R^{\alpha}{}_{\beta\gamma\delta}(z(\tau))v^{\beta}(\tau)S^{\gamma\delta}(\tau)\right) \\ &\quad +\frac{\mu}{2}\int d\tau\,\nabla_{\nu}\left(\bar{g}^{\mu}{}_{\alpha}(x,z(\tau))\bar{g}^{\nu}{}_{\beta}(x,z(\tau))\frac{\delta^{(4)}(x-z(\tau))}{\sqrt{-g}}\right)\left(\frac{d}{d\tau}S^{\alpha\beta}(\tau) - 2u^{[\alpha}(\tau)v^{\beta]}(\tau)\right). \end{aligned} \tag{2.13}$$

Since the third and fourth lines must vanish separately, we obtain Eqs. (2.1).

All above results are obtained under the assumption that the particle is not a strongly self-gravitating body. Since we dare to use the Dixon's formalism to the case of a rotating black hole, we need to give a certain justification of this extensive use. One crude argument is that if we believe that the strong equivalence principle holds, we will not be able to distinguish the black hole from an ordinary extended body of the same mass and spin. This might be also stated in the following way.

We consider to put the spinning particle on a flat spacetime. We denote the metric perturbation from the flat spacetime by $h^{\mu\nu} = \sqrt{-g}g^{\mu\nu} - \eta^{\mu\nu}$, where g is the determinant of $g_{\mu\nu}$. Setting the harmonic coordinate condition

$$h^{\mu\nu}{}_{,\nu} = 0, \tag{2.14}$$

the Einstein equation becomes

$$\Box h^{\mu\nu} = 16\pi(-g)T^{\mu\nu} + \Lambda^{\mu\nu}[h], \tag{2.15}$$

where we denote the flat d'Alembertian operator by $\Box$ and the energy-momentum tensor of the spinning particle by $T^{\mu\nu}$. The second term of the right-hand side represents the nonlinear terms with respect to $h^{\mu\nu}$. For simplicity, we choose t axis along the direction of the motion and z axis along the direction of spin. In the linear approximation, we obtain the solution of Eq. (2.15) as

$$h_{\mu\nu}dx^{\mu}dx^{\nu} = -\frac{4\mu}{r}dt^2 - \frac{4\mu S}{r^2}\frac{x\,dy - y\,dx}{r}dt, \tag{2.16}$$

where the indices of $h^{\mu\nu}$ are lowered by the flat metric $\eta_{\mu\nu}$. On the other hand, the Kerr metric in the harmonic coordinate is written as

$$\begin{aligned} h_{\mu\nu}dx^{\mu}dx^{\nu} = &-\left(\frac{4\mu}{r} + \frac{7\mu^2}{r^2}\right)dt^2 - \frac{\mu^2}{r^2}dr^2 \\ &-\frac{4\mu S}{r^2}\frac{x\,dy - y\,dx}{r}dt + \delta h_{\mu\nu}dx^{\mu}dx^{\nu}, \end{aligned} \tag{2.17}$$

where $\delta h_{\mu\nu}$ is of order $(\mu/r)^3$ or $\mu S^2/r^3$. As long as $M \gg \mu$ and $M \gg S$ hold, these residual terms can be neglected because the background curvature scale is $\geq M$. Thus we can say that the metric perturbation caused by the energy-momentum tensor Eq. (2.8) denotes a correct leading asymptotic behavior of Kerr black holes with respect to the spin term.

However, there may be several objections to this argument. First, as already mentioned before, it might be impossible to justify to neglect these residual terms when we extrapolate μ and S to the order of M. We only expect that the extrapolation is meaningful as was so in the nonrotating particle case.

Secondly, even in the case $M \gg \mu$, there is a subtle issue. The first two terms of Eq. (2.17) indicate the monopole contribution of the particle which also appears in the asymptotic metric of the Schwarzschild black hole. The third term of Eq. (2.17) comes from the spin effect of the particle. Comparing Eq. (2.16) with Eq. (2.17), we find that the same order or larger terms (i.e., μ^2 terms) than the spin terms are neglected when we use the energy-momentum tensor Eq. (2.8). These terms are derived from Eq. (2.15) as a consequence of the next iteration including the nonlinear source term $\Lambda^{\mu\nu}$. Thus our following analysis using the energy-momentum tensor Eq. (2.8) might be only a qualitative estimate of the spin effect. However, we would like to remind again that the neglected terms did not bring any serious quantitative error in the nonspinning particle case. Therefore, we expect that the present analysis also gives a fairly good quantitative estimate of gravitational waves from the head-on collision of two spinning black holes.

III. GRAVITATIONAL WAVES INDUCED BY A SPINNING PARTICLE FALLING INTO A KERR BLACK HOLE ALONG THE z AXIS

In this section, we first briefly review the perturbation formalism of a Kerr black hole which was originated by Teukolsky [19] and developed by Sasaki and Nakamura [20]. Then, substituting the energy-momentum tensor discussed in the preceding section into the source term for the perturbation formalism, we evaluate gravitational waves induced by the spinning particle.

A. Teukolsky formalism

In the Teukolsky formalism, the waveform and the energy flux of gravitational waves are calculated from the fourth Newman-Penrose quantity [22], which is expanded as

$$\psi_4 = (r - ia\cos\theta)^{-4} \int d\omega\, e^{-i\omega t} \times \sum_{l,m} e^{im\varphi} \frac{{}_{-2}S^{a\omega}_{lm}(\theta)}{\sqrt{2\pi}} R_{lm\omega}(r). \qquad (3.1)$$

Here, ${}_{-2}S^{a\omega}_{lm}(\theta)$ is the spheroidal harmonics normalized by

$$\int_0^\pi |{}_{-2}S^{a\omega}_{lm}(\theta)|^2 \sin\theta\, d\theta = 1, \qquad (3.2)$$

and its eigenvalue is λ. $R_{lm\omega}$ obeys the Teukolsky equation as

$$\Delta^2 \frac{d}{dr}\left(\frac{1}{\Delta}\frac{dR_{lm\omega}}{dr}\right) - V(r)R_{lm\omega} = T_{lm\omega}(r), \qquad (3.3)$$

and

$$V(r) = -\frac{K^2 + 4i(r-M)K}{\Delta} + 8i\omega r + \lambda, \qquad (3.4)$$

where $\Delta = r^2 - 2Mr + a^2$ and $K = (r^2 + a^2)\omega - ma$. The source term $T_{lm\omega}(r)$ is constructed from the energy-moment tensor of the matter, and its explicit form is given later. By using the Green's function method, the solution of the Teukolsky equation at infinity is written as

$$R_{lm\omega} = R^{(0)}_{\rm out}(\infty) W_r^{-1} \int_{r_+}^\infty T_{lm\omega} R^{(0)}_{\rm in} \Delta^{-2} dr, \qquad (3.5)$$

where r_+ denotes the radius of the event horizon. $R^{(0)}_{\rm out}$ and $R^{(0)}_{\rm in}$ are two homogeneous solutions, which satisfy the following boundary conditions at infinity:

$$R^{(0)}_{\rm out} \to r^3 e^{i\omega r^*}, \qquad R^{(0)}_{\rm in} \to r^3 C_{\rm out} e^{i\omega r^*} + r^{-1} C_{\rm in} e^{-i\omega r^*}, \qquad (3.6)$$

where $C_{\rm out}$ and $C_{\rm in}$ are two complex constants and r^* is the tortoise coordinate defined by

$$r^* = r + M \ln\frac{\Delta}{(2M)^2} + \frac{M^2}{\sqrt{M^2 - a^2}} \ln\frac{r - r_+}{r - r_-}, \qquad (3.7)$$

with $r_\pm = M \pm \sqrt{M^2 - a^2}$. The Wronskian factor is determined by

$$W_r := -\Delta^{-1} R^{(0)}_{\rm out} \frac{dR^{(0)}_{\rm in}}{dr} + \Delta^{-1} R^{(0)}_{\rm in} \frac{dR^{(0)}_{\rm out}}{dr} = 2i\omega C_{\rm in}. \qquad (3.8)$$

In order to calculate gravitational waves at infinity, we must know accurate values of $R^{(0)}_{\rm in}$ and $C_{\rm in}$ numerically, but it is difficult to obtain $C_{\rm in}$ accurately because of the bad behavior of $R^{(0)}_{\rm in}$ at infinity. Also, since $T_{lm\omega}$ for large r diverges as $\to r^3$ in the present case (see below), the convergence property of the integral in Eq. (3.5) is not guaranteed [20]. Thus, as was done in the previous numerical works [23,24], we use the Sasaki-Nakamura (SN) equation [20], which is obtained by changing the variables from the Teukolsky equation. In their formalism, the equation becomes

$$\left[\frac{d^2}{dr^{*2}} - F(r)\frac{d}{dr^*} - U(r)\right] X_{lm\omega}(r) = S_{lm\omega}(r), \qquad (3.9)$$

and the potentials $F(r) = \Delta(\ln\gamma)_{,r}/(r^2 + a^2)$ and $U(r)$ behave,[2] respectively, as $O(r^{-n})$ and $(iK/(r^2+a^2))^2 + O(r^{-n}), n \geq 2$, at $r^* \to \pm\infty$. Hence the boundary conditions of the homogeneous equation at $r^* \to \pm\infty$ are well behaved as

$$\begin{aligned} &X^{(0)}_{\rm out} \to e^{i\omega r^*}, \\ &X^{(0)}_{\rm in} \to A_{\rm out} e^{i\omega r^*} + A_{\rm in} e^{-i\omega r^*}, \quad r^* \to \infty, \\ &X^{(0)}_{\rm out} \to B_{\rm out} e^{ikr^*} + B_{\rm in} e^{-ikr^*}, \\ &X^{(0)}_{\rm in} \to e^{-ikr^*}, \quad r^* \to -\infty, \end{aligned} \qquad (3.10)$$

where $k = \omega - ma/2Mr_+$. Thus accurate values of $X^{(0)}_{\rm in}$ and $A_{\rm in}$ are easily obtained numerically in the SN equation contrary to the Teukolsky one.

$S_{lm\omega}(r)$ is related to the source term of the Teukolsky equation by

$$S_{lm\omega} = \frac{\Delta\gamma W_{lm\omega}}{r^2(r^2+a^2)^{3/2}} \exp\left(-i\int \frac{K}{\Delta} dr\right), \qquad (3.11)$$

[2]The detailed descriptions of F, U, and γ are shown in Refs. [23] and/or [25].

where

$$\frac{d^2W_{lm\omega}}{dr^2} = \frac{T_{lm\omega}r^2}{\Delta^2}\exp\left(i\int\frac{K}{\Delta}dr\right). \qquad (3.12)$$

This equation is integrated under the boundary condition $W_{lm\omega} \to r^{1/2}$ at $r \to \infty$, so that, $S_{lm\omega} \to r^{-5/2}$ at $r \to \infty$. The solution of gravitational waves at infinity is obtained by the Green's function method:

$$X_{lm\omega} = X^{(0)}_{\rm out}(\infty)\frac{c_0}{2i\omega A_{\rm in}}\int_{-\infty}^{\infty}\frac{S_{lm\omega}X^{(0)}_{\rm in}}{\gamma}dr^*, \qquad (3.13)$$

where c_0 is a constant, $\lim_{r\to\infty}\gamma(r)$ [20]. Thus the convergence property of the integration is guaranteed.

Once $X_{lm\omega}$ is obtained, the waveform of gravitational waves at infinity is calculated from

$$h_+ - ih_\times = \frac{8}{r}\sum_{l,m}\int_{-\infty}^{\infty}d\omega\frac{X_{lm\omega}}{c_0}\frac{{}_{-2}S^{a\omega}_{lm}}{\sqrt{2\pi}}e^{i\omega(r^*-t)+im\varphi}. \qquad (3.14)$$

From Eq. (3.14), the energy flux and the total energy flux, respectively, become

$$\frac{dE}{dt} = \frac{4}{\pi}\sum_{l,m}\left|\int_{-\infty}^{\infty}d\omega\,\omega\frac{X_{lm\omega}}{c_0}e^{i\omega(r^*-t)}\right|^2, \qquad (3.15)$$

and

$$\Delta E = 8\sum_{l,m}\int_{-\infty}^{\infty}d\omega\,\omega^2\left|\frac{X_{lm\omega}}{c_0}\right|^2. \qquad (3.16)$$

B. Equations of motion and source term

In solving Eqs. (2.1) for u^μ and $S^{\mu\nu}$ on the Kerr spacetime, it is convenient to introduce the tetrad frame as

$$\begin{aligned} e^0_\mu &= \left(\sqrt{\frac{\Delta}{\Sigma}},0,0,-a\sin^2\theta\sqrt{\frac{\Delta}{\Sigma}}\right),\\ e_\mu &= \left(0,\sqrt{\frac{\Sigma}{\Delta}},0,0\right),\\ e^2_\mu &= (0,0,\sqrt{\Sigma},0),\\ e^3_\mu &= \left(-\frac{a}{\sqrt{\Sigma}}\sin\theta,0,0,\frac{r^2+a^2}{\sqrt{\Sigma}}\sin\theta\right), \end{aligned} \qquad (3.17)$$

where $\Sigma = r^2 + a^2\cos^2\theta$, and $e^a_\mu = (e^a_t, e^a_r, e^a_\theta, e^a_\varphi)$ for $a = 0 \sim 3$. Here we use the latin indices to denote the tetrad component.

In the tetrad component, Eqs. (2.1) are rewritten as

$$\begin{aligned} \frac{d}{d\tau}u^a &= \omega_{bc}{}^a v^b u^c - \frac{1}{2}R^a{}_{bcd}v^b S^{cd},\\ \frac{d}{d\tau}S^{ab} &= -2v^e\omega_{ec}{}^{[a}S^{b]c} + 2u^{[a}v^{b]}, \end{aligned} \qquad (3.18)$$

where $\omega_{ab}{}^c := e^\mu_a e^\nu_b \nabla_\mu e^c_\nu$ is the Ricci rotation coefficient. For the convenience, we write the explicit expressions of $\omega_{ab}{}^c$ and R_{abcd} in the Appendix.

Since we now consider the case where the spinning particle falls along the z axis (i.e., $\theta = 0$) and its spin is parallel to the z axis, it is very natural to set that $u^2 = u^3 = 0$, and the only nonvanishing component of spin tensor is $S_{23} =: S$. Then, $u^a = v^a$ follows from Eq. (2.4). Under these conditions, we find

$$\begin{aligned} \frac{d}{d\tau}u^0 &= -\frac{M}{\sqrt{\Sigma_0\Delta}}u^0u^1 + \frac{2aM(3r^2-a^2)}{\Sigma_0^3}u^1S,\\ \frac{d}{d\tau}u^1 &= -\frac{M}{\sqrt{\Sigma_0\Delta}}u^0u^0 + \frac{2aM(3r^2-a^2)}{\Sigma_0^3}u^0S,\\ \frac{d}{d\tau}u^2 &= \frac{d}{d\tau}u^3 = \frac{d}{d\tau}S^{ab} = 0, \end{aligned} \qquad (3.19)$$

where $\Sigma_0 = r^2 + a^2$. From these equations, we can see that the spin-spin interaction acts as the repulsive (attractive) force in the case of parallel (antiparallel) spin, i.e., $aS > 0$ ($aS < 0$).

To solve the above equations, we can utilize the conservation law which is obtained by setting ξ^μ in Eq. (2.7) as the timelike Killing vector of the Kerr spacetime. The tetrad components of the timelike Killing vector and its covariant derivative are given by

$$\xi^\mu = -\sqrt{\frac{\Delta}{\Sigma}}e^\mu_0 + \frac{a\sin\theta}{\sqrt{\Sigma}}e^\mu_3, \qquad (3.20)$$

$$\nabla_\mu\xi_\nu = -\frac{2M(r^2-a^2\cos^2\theta)}{\Sigma^2}e^0_{[\mu}e^1_{\nu]} + \frac{4aMr\cos\theta}{\Sigma^2}e^2_{[\mu}e^3_{\nu]}.$$

Thus in the present case, we obtain the conserved quantity E_0 as

$$\frac{E_0}{\mu} = \sqrt{\frac{\Delta}{\Sigma_0}}u^0 + \frac{2aMr}{\Sigma_0^2}S. \qquad (3.21)$$

Thus we get

$$\begin{aligned} u^0 &= \sqrt{\frac{\Sigma_0}{\Delta}}\left(\frac{E_0}{\mu} - \frac{2aMr}{\Sigma_0^2}S\right),\\ u^1 &= -\sqrt{\frac{\Sigma_0}{\Delta}\left(\frac{E_0}{\mu} - \frac{2aMr}{\Sigma_0^2}S\right)^2 - 1}. \end{aligned} \qquad (3.22)$$

It can be easily verified that these u^0 and u^1 solve the Eqs. (3.19) with the aid of the relations

$$u^0 = \sqrt{\frac{\Delta}{\Sigma_0}}\frac{dt}{d\tau}, \quad u^1 = \sqrt{\frac{\Sigma_0}{\Delta}}\frac{dr}{d\tau}. \qquad (3.23)$$

Hereafter, by setting $E_0 = \mu$, we concentrate on the case where the particle is at rest at infinity.

For the later convenience, we define $t(r)$ regarding the coordinate time of a particle as a function of r. For this purpose, we numerically solve the equation,

$$\frac{dt}{dr} = -\frac{\Sigma_0}{\Delta}\left(1-\frac{2aMSr}{\Sigma_0^2}\right) \times\left[\left(1-\frac{2aMSr}{\Sigma_0^2}\right)^2-\frac{\Delta}{\Sigma_0}\right]^{-1/2}, \tag{3.24}$$

which is derived from the Eqs. (3.22) and (3.23), under the boundary condition $t(r) = 0$ at $r = r_+ + \epsilon$, where ϵ is an appropriately small constant $\sim 10^{-14}M$.

Since the system is axially symmetric in the present case, we have only to consider the axially symmetric mode, i.e., $m = 0$. Then, by using the components of the energy momentum tensor with respect to the null tetrad which is defined by

$$\begin{aligned} \ell^\mu &= \sqrt{\frac{\Sigma}{\Delta}}(e_0^\mu + e_1^\mu),\\ n^\mu &= \frac{1}{2}\sqrt{\frac{\Delta}{\Sigma}}(e_0^\mu - e_1^\mu),\\ m^\mu &= (r+ia\cos\theta)^{-1}\sqrt{\frac{\Sigma}{2}}(e_2^\mu + ie_3^\mu), \end{aligned} \tag{3.25}$$

the source term of the Teukolsky equation is given by

$$T_{l0\omega} = 4\int dt\,\sin\theta\,d\theta\,d\varphi\,\rho^{-5}\bar\rho^{-1}(B_2' + B_2'^*)e^{i\omega t}\frac{{}_{-2}S_{l0}^{a\omega}}{\sqrt{2\pi}}, \tag{3.26}$$

where

$$\begin{aligned} B_2' &= -\frac{1}{2}\rho^8\bar\rho L_{-1}[\rho^{-4}L_0(\rho^{-2}\bar\rho^{-1}T_{nn})] - \frac{1}{2\sqrt2}\rho^8\bar\rho\Delta^2 L_{-1}[\rho^{-4}\bar\rho^2 J_+(\rho^{-2}\bar\rho^{-2}\Delta^{-1}T_{\bar m n})],\\ B_2'^* &= -\frac{1}{4}\rho^8\bar\rho\Delta^2 J_+[\rho^{-4}J_+(\rho^{-2}\bar\rho T_{\bar m\bar m})] - \frac{1}{2\sqrt2}\rho^8\bar\rho\Delta^2 J_+[\rho^{-4}\bar\rho^2\Delta^{-1}L_{-1}(\rho^{-2}\bar\rho^{-2}T_{\bar m n})], \end{aligned} \tag{3.27}$$

and

$$\begin{aligned} \rho &= (r-ia\cos\theta)^{-1},\\ L_s &= \partial_\theta - a\omega\sin\theta + s\cot\theta,\\ J_+ &= \partial_r + \frac{i\omega\Sigma_0}{\Delta}. \end{aligned} \tag{3.28}$$

In the above equations, an overbar denotes the complex conjugate. Substituting Eq. (3.27) into Eq. (3.26) and performing integral by part, we obtain

$$\begin{aligned} T_{l0\omega} = \frac{4}{\sqrt{2\pi}}\int dt\,\sin\theta\,d\theta\,d\varphi\,e^{i\omega t}\Bigg[&-\frac{1}{2\rho^2\bar\rho}L_1^\dagger\{\rho^{-4}L_2^\dagger({}_{-2}S_{l0}^{a\omega}\rho^3)\}T_{nn} + \frac{\Delta^2\bar\rho}{\sqrt2\rho^2}L_2^\dagger({}_{-2}S_{l0}^{a\omega}\rho\bar\rho)J_+\left(\frac{T_{\bar m n}}{\rho^2\bar\rho^2\Delta}\right)\\ &+\frac{\Delta}{2\sqrt2\rho^2\bar\rho^2}L_2^\dagger\{\rho^3{}_{-2}S_{l0}^{a\omega}(\bar\rho^2\rho^{-4})_{,r}\}T_{\bar m n} - \frac{1}{4}\rho^3\Delta^2{}_{-2}S_{l0}^{a\omega}J_+\{\rho^{-4}J_+(\bar\rho\rho^{-2}T_{\bar m\bar m})\}\Bigg], \end{aligned} \tag{3.29}$$

where $L_s^\dagger = \partial_\theta + a\omega\sin\theta + s\cot\theta$. Using the energy-momentum tensor of the spinning particle Eq. (2.8), we can calculate the source term of Teukolsky equation $T_{l0\omega}$. After a somewhat long calculation, we find that $T_{l0\omega}$ becomes $T_{l0\omega} = T_{nn}^0 + T_{nn}^1 + T_{\bar m n}^1$, which are defined by

$$\begin{aligned} T_{nn}^0 &= -4\mu\hat S_l^{a\omega}\Delta^2\bar\rho_0^2\frac{dr}{d\tau}\left(\frac{dv}{dr}\right)^2 e^{i\omega t(r)},\\ T_{nn}^1 &= 8\mu S\hat S_l^{a\omega}a\frac{\Delta^2}{\Sigma_0}\bar\rho_0^2\frac{dv}{dr}e^{i\omega t(r)},\\ T_{\bar m n}^1 &= 4i\mu S\hat S_l^{a\omega}\Delta^2\bar\rho_0^2\left[\left(\frac{2ia}{\Sigma_0}+i\omega\frac{dv}{dr}\right)\frac{dv}{dr}+\frac{d^2v}{dr^2}\right]e^{i\omega t(r)}, \end{aligned} \tag{3.30}$$

where $v(r) = t(r) + r^*$, $\bar\rho_0 = (r+ia)^{-1}$, and

$$\hat S_l^{a\omega} = \lim_{\theta\to0}\frac{{}_{-2}S_{l0}^{a\omega}}{\sqrt{2\pi}\sin^2\theta}. \tag{3.31}$$

In total,

$$T_{l0\omega} = -4\mu\hat S_l^{a\omega}\Delta^2\bar\rho_0^2\left[\left(\frac{dr}{d\tau}+\omega S\right)\left(\frac{dv}{dr}\right)^2 - iS\frac{d^2v}{dr^2}\right] \times e^{i\omega t(r)}. \tag{3.32}$$

We note that in the case $\theta = 0, T_{\bar m\bar m}$ does not contribute to both the spin-dependent and -independent terms, and also $T_{\bar m n}$ does not contribute to the spin-independent term. T_{nn}^0 does not depend on S. (This term has already been obtained by Sasaki and Nakamura [20].) On the other hand, T_{nn}^1 and $T_{\bar m n}^1$ are spin dependent. (Note that T_{nn}^0, T_{nn}^1, and $T_{\bar m n}^1$ diverge as $r^{5/2}$, $r^{1/2}$, and r^3 at $r\to\infty$.) T_{nn}^1 is zero when $aS = 0$, so that it represents the spin-spin coupling term. On the other hand, $T_{\bar m n}^1$ involves not only the spin-spin coupling terms, but also the other contributions which do not vanish even if the spin of black hole is zero. Hence we may consider that it involves a contribution from the energy momentum of the spinning particle. Once the source term of the Teukolsky equation is obtained, $W_{l0\omega}$ in Eq. (3.12) is integrated from a large radius, $r_{\rm max}$, to r_+ under the boundary condition $W_{l0\omega} \to r^{1/2}$ at $r = r_{\rm max}$ [20]. Then, from Eq. (3.11), we obtain a source term of the SN equation $S_{l0\omega}(r)$ which behaves well at $r \to \infty$.

Finally, we briefly explain the numerical strategies. To obtain the waveform and the energy flux of gravitational waves, i.e., to obtain $X_{l0\omega}$, we need to calculate the following quantities, numerically: (1) the coordinate time of the particle trajectory as a function of r, $t(r)$, (2) the homogeneous radial wave function with the ingoing boundary condition $X_{\rm in}^{(0)}$ and $A_{\rm in}$, (3) the spheroidal harmonics ${}_{-2}S_{l0}^{a\omega}$, and (4) the source term of the SN equation $S_{l0\omega}$ (or $W_{l0\omega}$). Once the above quantities are calculated, we can obtain $X_{l0\omega}$ by performing the numerical integration in Eq. (3.13). The numerical methods are the same as those adopted in Ref. [23]. For each model of a and S, $X_{l0\omega}$ is solved from $l = 2$ to $l = 6$. Error caused by the neglect of higher multipole modes is found to be at most $\sim 10^{-4}$. For each l, $X_{l0\omega}$ is solved for $-1.4M^{-1} \leq \omega \leq 1.4M^{-1}$ following Sasaki and Nakamura [20]. In calculating the energy flux, we adopt the step size of ω to be $\Delta\omega = 0.02M^{-1}$, and in calculating the waveform, we adopt $\Delta\omega = 0.005M^{-1}$ because if we use a step size as large as $\Delta\omega = 0.02M^{-1}$, the spurious period $2\pi/\Delta\omega$ stands out in the waveform. To check the dependence of $\Delta\omega$ on the energy flux, we also perform the calculation using the other $\Delta\omega$. We found that the energy flux changes 0.2% for the case when the spin vectors of the black hole and the particle are antiparallel and $|aS|$ is large $\sim M^2$, but the error is typically less than 0.1%.

C. Numerical results

We have calculated the total energy flux and the waveform of gravitational waves induced by a spinning particle of mass μ and spin S falling into a Kerr black hole of mass M and spin a for a wide variety of spin parameters $-1 \leq S/M \leq 1$ and $0 \leq a/M \leq 0.99$. In Fig. 1, we show the total energy flux $\Delta E_{a,S}$ for $0 \leq a/M \leq 0.9$ and $-0.9 \leq S/M \leq 0.9$. We here note that when we regard the spinning particle as a compact object, the spin of the particle S must be at most of order μ. Thus, if we consider the case $\mu \ll M$, the spin must be very small ($S \ll M$) and its effect is negligible. However, our purpose is to see the spin effect in the case $\mu \lesssim M$ and $S \lesssim M$ extrapolating $\mu \to O(M)$, so that we regard S as a quantity of order M. From Fig. 1, we will soon recognize that in the case of the antiparallel spin, a larger amount of the energy flux is emitted: For the case $a = S = 0$, the total energy flux is $0.0104(\mu/M)^2M$ [13], and for $a = S = 0.9M$, it is almost the same (see Fig. 2). On the other hand, for $a = -S = 0.9M$, the total energy flux becomes $0.0249(\mu/M)^2M$ and it becomes $0.0298(\mu/M)^2M$ for $a = 0.99M$ and $S = -0.99M$. This is consistent with the interpretation of Wald [18] about the radiation efficiency expected from the Hawking's area

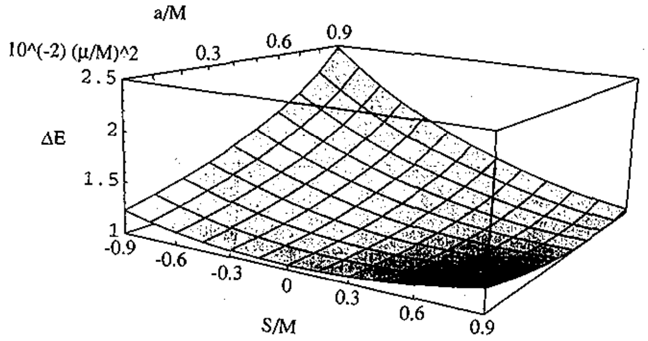

FIG. 1. The total energy flux $\Delta E_{a,S}/M$ of gravitational waves by a spinning particle falling into a Kerr black hole for a variety of spin parameters a and S.

theorem [26], in which he mentioned that the energy of the spinning particle [i.e., $(\Delta/\Sigma_0)^{1/2}u^0$] is gained due to the spin-spin interaction for the antiparallel spin case and lost for the parallel spin case during the in-fall into a black hole [see Eq. (3.21)], so that $\Delta E_{a,S}$ for an antiparallel collision may well be larger than that for a parallel collision. However, the detailed mechanism for this property seems due to a somewhat different reason from that he considered.

To investigate the reason for this property of the total energy flux in detail, let us pay close attention to the total energy flux as a function of a and S. In Fig. 3(a), we show the total energy flux for $a = 0$ as a function of S/M, $\Delta E_{0,S}$, and that for $S = 0$ as a function of a/M, $\Delta E_{a,0}$. In Fig. 3(b), we also show $\log_{10}(\Delta E_{0,S}/\Delta E_{0,0}-1)$ (open circle) and $\log_{10}(\Delta E_{a,0}/\Delta E_{0,0}-1)$ (filled circle) as a function of $\log_{10}(S/M)$ and $\log_{10}(a/M)$, respectively. This figure shows that both $\Delta E_{0,S}$ and $\Delta E_{a,0}$ are well approximated by the formula $\Delta E_{0,S} = \Delta E_{0,0} + C_1S^2$ and $\Delta E_{a,0} = \Delta E_{0,0} + C_2a^2$, respectively, except for large $a/M \gtrsim 0.7$ (C_1 and C_2 are constants.) We should note that in the case $a = 0$, there does not appear the spin-

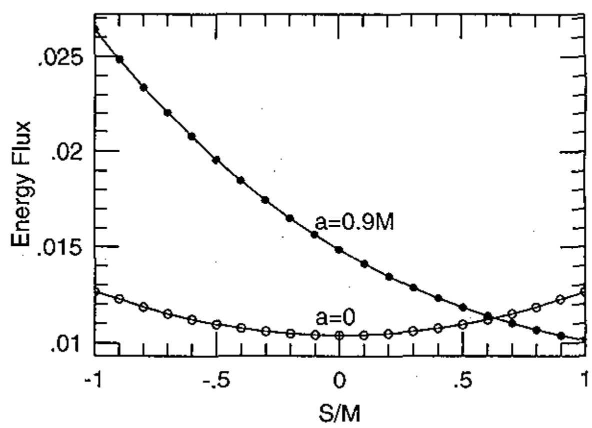

FIG. 2. The total energy flux of gravitational waves in units of $(\mu/M)^2M$ as a function of S/M in the case $a = 0$ (open circle) and $a = 0.9M$ (filled circle).

spin interaction term in the equations of motion, so that the motion of the spinning particle is not affected by its spin. Nevertheless, the total energy flux increases with the increase of the spin of the particle. This means that the contribution of the energy-momentum tensor of the spinning particle to the total energy flux is significant. Figures 3(a) and 3(b) also show that the two curves do not coincide (i.e., $C_1 < C_2$). This seems mainly due to the difference of the frequencies of the quasinormal model (QNM) between the Schwarzschild and Kerr black holes: For $a = 0$ and $S \neq 0$, the difference of the total energy flux among different S is only caused by the energy-momentum tensor of the spinning particle. On the other hand, for $S = 0$ and $a \neq 0$, both the energy-momentum tensor of the Kerr black hole and the change of the QNM frequency of the Kerr black hole [15] contribute to the difference of the total energy flux among different a. Since the frequency of the fundamental QNM ($l = 2, m = 0$) of a Kerr black hole is always larger than that of the Schwarzschild black hole [27] and the total energy flux is roughly proportional to square of the frequency, C_2 may well become larger than C_1. In any case, the total energy flux of gravitational waves is changed by the following two reasons in the case when we include the spin of the particle. One is due to the spin-spin interaction which appears in the equations of motion, and the other is due to the contribution of the energy-momentum tensor of the spinning particle to the source term of the Teukolsky equation. In the following, let us explain that the latter contribution is larger.

In Fig. 4, we show the waveforms of gravitational waves observed on the equatorial plane for $a = 0$, and $S/M = 0$, 0.45, and 0.9 (the solid, dashed, and dotted lines, respectively). It is very interesting to note that the + mode of gravitational waves is not affected by the spin, while the × mode is highly affected. This indicates that the increase of the energy flux with the increase of $|S|$ is due to the excitation of the × mode. This feature can also be seen even for $a \neq 0$. In Figs. 5 and 6, we show the waveforms for $a = 0.6M$ and $a = 0.9M$ with various values of S/M. In Fig. 5, the solid, dashed, dotted, long dashed, and dotted-dashed lines show the waveforms for $S/M = 0, -0.6, -0.9, 0.6$, and 0.9, respectively, and in Fig. 6, solid, dotted, and dashed lines show the waveforms for $S/M = 0$, -0.9, and 0.9. Like in the case $a = 0$, the + mode is not affected so much by the spin in all cases, but the × mode is highly affected. Thus by

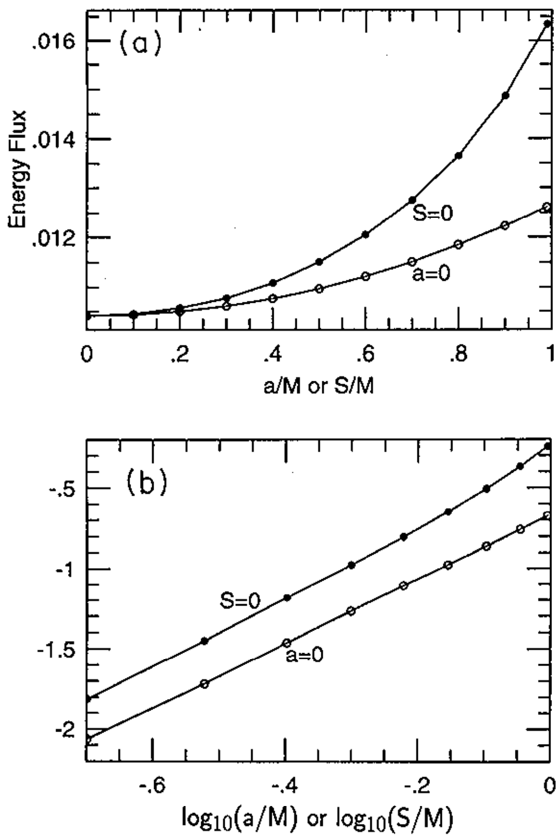

FIG. 3. (a) The total energy flux of gravitational waves in units of $(\mu/M)^2M$ for the case $a = 0$, $S \neq 0$ (open circle) and $a \neq 0$, $S = 0$ (filled circle). (b) The behavior of the total energy flux as a function of the spin parameters a or S in the case $a = 0$, $S \neq 0$ and $a \neq 0$, $S = 0$. The horizontal axis shows $\log_{10}(a/M)$ or $\log_{10}(S/M)$, and the vertical axis shows $\log_{10}(\Delta E_{0,S}/\Delta E_{0,0} - 1)$ (open circle) or $\log_{10}(\Delta E_{a,0}/\Delta E_{0,0} - 1)$ (filled circle). We can see that both $\Delta E_{0,S}$ and $\Delta E_{a,0}$ are well approximated by formulas $\Delta E_{0,S} = \Delta E_{0,0} + C_1 S^2$ and $\Delta E_{a,0} = \Delta E_{0,0} + C_2 a^2$, respectively. Here, C_1 and C_2 are constants, and $C_1 < C_2$.

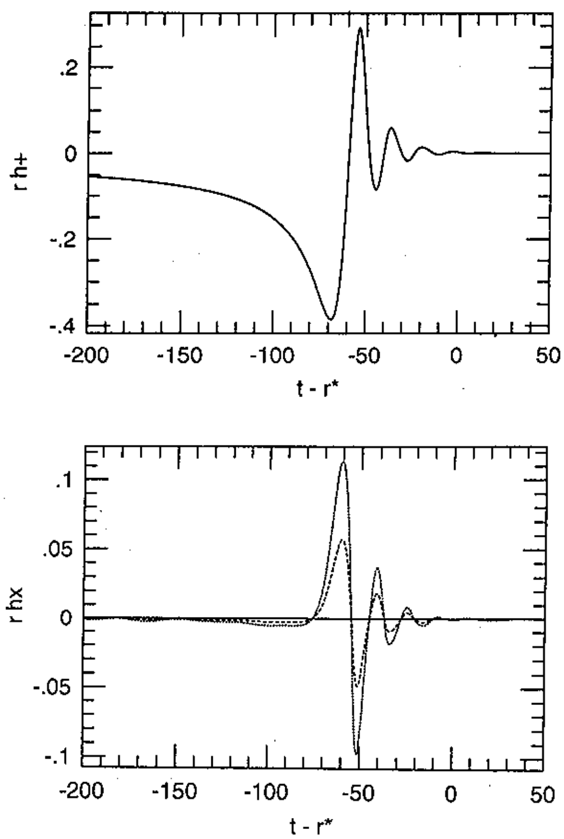

FIG. 4. Waveforms of gravitational waves in units of μ for $a = 0$ and $S/M = 0$ (solid line), 0.45 (dashed line), and 0.9 (dotted line). Note that for h_+, three lines agree.

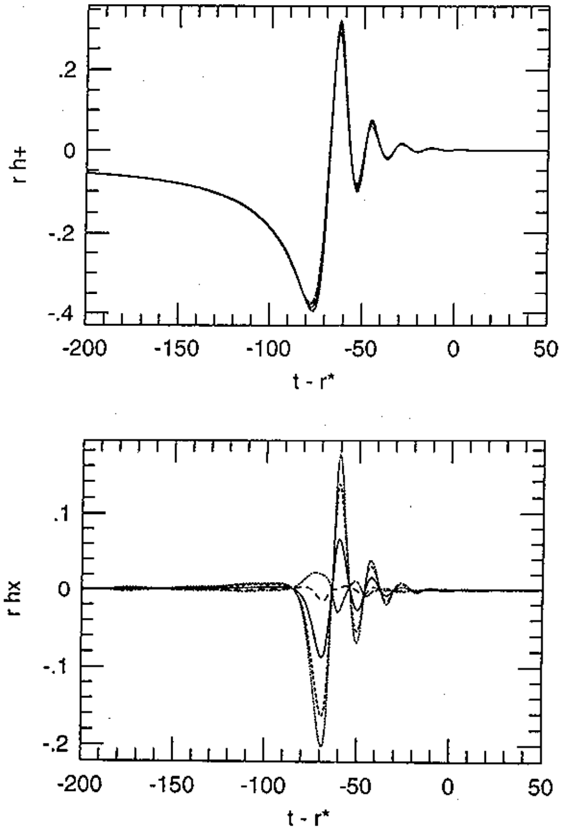

FIG. 5. Waveforms of gravitational waves in units of μ for $a/M = 0.6$ and $S/M = 0$ (solid line), -0.6 (dashed line), -0.9 (dotted line), 0.6 (long dashed line), and 0.9 (dotted-dashed line).

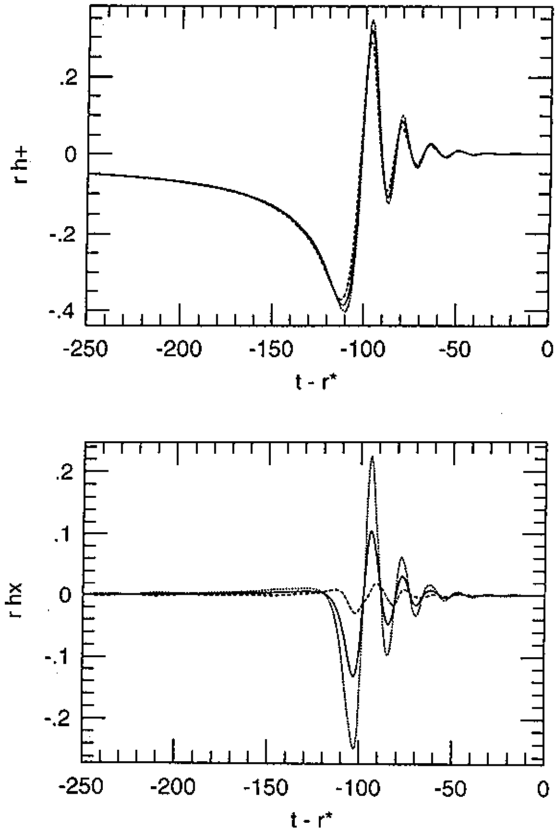

FIG. 6. Waveforms of gravitational waves in units of μ for $a/M = 0.9$ and $S/M = 0$ (solid line), -0.9 (dotted line), and 0.9 (dashed line).

the inclusion of the spin of the particle, the $\times$ mode is excited so much, but the + mode is not almost altered by the spin of the particle. This fact suggests that the effect of the energy-momentum tensor of the spinning particle dominates over that of the spin-spin interaction which appears in the equations of motion. In fact, the effect of the spin-spin interaction is small. In Fig. 7, we show the in-fall velocity $|dr/dt|$ as a function of r for $a = 0.9M$ and $S/M = -0.9$ (dotted line), 0 (solid line), and 0.9 (dotted-dashed line). We note that for $a = 0.9M$, the event horizon locates at $r \simeq 1.436M$. Figure 7 indicates that the spin-spin interaction does not alter the in-fall velocity so much even just before the collision. If the spin-spin interaction were effective, the in-fall velocity of the spinning particle should be largely altered by the effect, and the amplitude of the + mode of gravitational waves would change. Hence we can conclude that the spin-spin interaction is not effective compared with the effect of the energy-momentum tensor of the spinning particle.

It is worth noting that for the antiparallel spin case, the amplitude of the $\times$ mode monotonically increases with the increase of $|S|$. On the other hand, for the parallel spin case, its amplitude becomes minimum around $S \sim a$. Also, the amplitude of the + mode is not altered so much when the parameters of spin a and S are changed. This means that the total energy flux is almost the same as that for $a = S = 0$ in the case $a = S$. In reality, the total energy fluxes for $a/M = S/M = 0.1$ to 0.99 coincide with that for $a = S = 0$ within 2% error (see Fig. 8). Therefore, we may conjecture that in the head-on collision of two spinning black holes of masses M_1 and

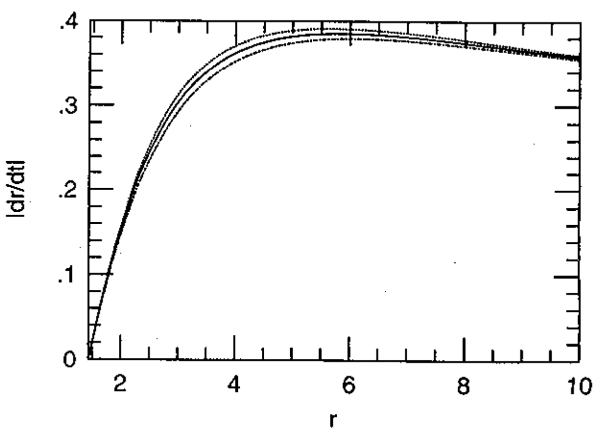

FIG. 7. The absolute value of the in-fall velocity of the spinning particle, $|dr/dt|$, as a function of r for $a = 0.9M$ and $S/M = -0.9$ (dotted line), 0 (solid line), and 0.9 (dotted-dashed line). The location of the event horizon for $a = 0.9M$ is at $r \simeq 1.436M$.

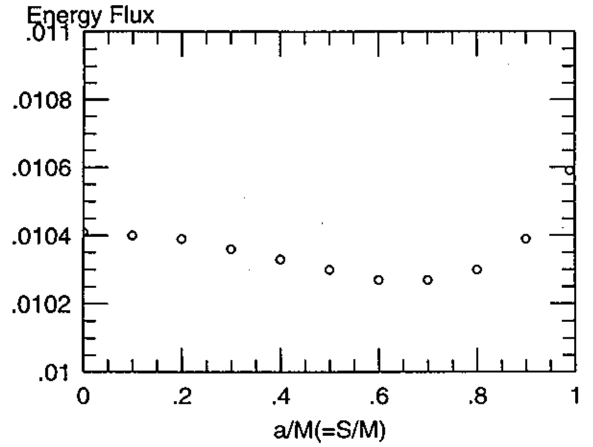

FIG. 8. The total energy flux in units of $(\mu/M)^2 M$ in the case of $a = S$.

M_2, the total energy flux would be almost independent of their spins if $a_1 = a_2$ holds.

Finally, we mention the point where the QNM begins to be excited. In Fig. 9, we show the retarded time $t - r^*$ as a function of r for $a/M = 0$, 0.6, 0.9, and $S = 0$. We note that the figure hardly depends on S because the trajectories do not depend on the spin-spin interaction effect so much (see Fig. 7). Comparing Figs. 4, 5, and 6 with Fig. 9, we can find that the points where the QNM begins to be excited are $r \sim 3 - 4M$ irrespective of a. These coincide with the points where the spin-spin interaction is most effective (see Fig. 7). Nevertheless, the spin-spin interaction effect is still small to affect gravitational waves so much. Therefore, we can conclude again that the effect of the spin-spin interaction is small.

IV. SUMMARY AND DISCUSSION

In this paper, to investigate the features of gravitational waves in the head-on collision of two spinning black holes, we have calculated the energy flux and the waveform of gravitational waves induced by a spinning particle falling into a rotating black hole along the z axis. Calculations are performed incorporating the equations of motion of the spinning particle derived by Papapetrou [16] and its energy-momentum tensor derived originally by Dixon [17] into the formalisms of Teukolsky and of Sasaki and Nakamura. We obtained the following results.

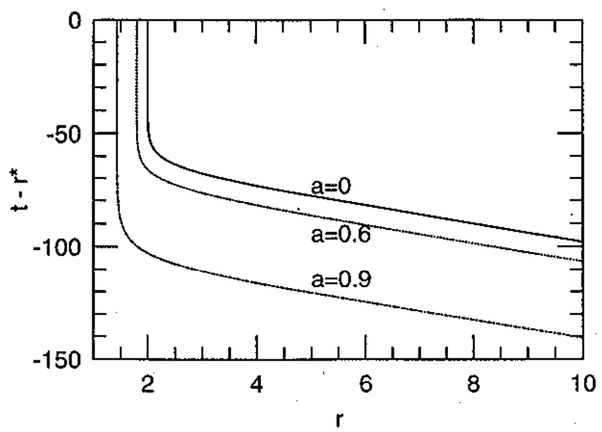

FIG. 9. The retarded time $t - r^*$ as a function of r for $a/M = 0$, 0.6, 0.9, and $S = 0$.

(A) In a head-on collision where a small mass particle of mass μ and spin S collides with a black hole of mass M and spin a, the total energy flux for the antiparallel spin collision is larger than that for the parallel one.

(B) There are two spin effects of the particle to gravitational waves; one is due to the contribution of the energy-momentum tensor of the spinning particle to the source term of the Teukolsky equation, and the other is due to the spin-spin interaction between the black hole and the spinning particle which appears in the equations of motion.

(C) The above two spin effects mainly affect the $h_\times$ mode of gravitational waves, while the h_+ mode is hardly affected. This suggests that the former effect in (B) dominates over the latter effect.

An actual numerical simulation of the head-on collision of black holes could take into account other effects which are not included in the present perturbation calculation: (1) the effects by the higher multipole moments ($l \geq 2$) of the spinning particle $\propto \mu S^l$ which were not taken into account in the equations of motion and the energy-momentum tensor of the spinning particle, (2) the tidal heating of the black hole event horizon [12], and (3) the final spin parameter of a merged black hole, which will become about $(Ma + \mu S)/(M + \mu)$. Since gravitational waves for the head-on collision of two black holes will be mainly radiated by the excitation of the quasinormal mode (QNM) of the merged black hole, (3) must be correctly considered.

As for (1), we will not know details until we perform an analysis including the effects of the higher multipole moments of the spinning particle. However, we can expect the following effect of the quadrupole moment which is the leading term ($l = 2$) of the higher multipole moments from a simple analysis. In the Newtonian limit, the equation of motion for a test body falling straightly toward another body of mass M and quadrupole moment Ma^2 becomes

$$\frac{d^2 r}{dt^2} = -\left(\frac{M}{r} - \frac{3Ma^2}{r^3}\right).$$

Thus the quadrupole moment reduces the attractive force. This means that if we include the contribution of the quadrupole moment of the spinning particle, the velocity will decrease and the total energy flux may reduce. However, this effect seems to be small except for the case $a \lesssim M$ because the excitation of the QNM occurs around $r \sim 3 - 4M$ and at this point, the repulsive force of the quadrupole moment term is small compared with the leading term. As for (2), the kinetic energy of black holes is dissipated by the tidal heating of the event horizons, so that the total energy flux may reduce about a few $\times 10\%$ [28,12]. As for (3), let us consider the cases

$M = \mu$ and $S = \pm a$ as examples. For $S = a$, the spin parameter of the merged black hole becomes a, so that results by the perturbation calculation can be applied without change. For $S = -a$, the final spin becomes zero, so that the frequency of the QNM ($f_{\rm QNM}$) reduces 0–12 % depending on a [27]. This means that the wavelength of gravitational waves becomes long. Also, since the amplitude may be expected not to change so much, the total energy flux (proportional to $f_{\rm QNM}^2$) is expected to reduce about 0–25 %.

In this way, the total energy flux may reduce about a few ×10% in total compared with that obtained by extrapolating the results of the perturbation calculation [$\mu/(M+\mu) \to 1/2$]. However, the effects (1)–(3) may be regarded as small corrections, and we may expect that the essential feature of the output on gravitational waves would not change so much. Therefore, the present results will be an important guideline to quantitative results of gravitational waves in the numerical simulation of the head-on collision of two spinning black holes, which has not been performed yet.

So far, we have focused on the head-on collision between the spinning black hole and the spinning particle. However, the method developed in the present paper will be useful to calculate gravitational waves for general three-dimensional coalescence case or for the case when the particle is orbiting around the black hole. These calculations are very important for checking results in 3D numerical relativistic simulations or the post-Newtonian calculations [4,8] of gravitational wave induced by two spinning bodies. As for the post-Newtonian formula on the energy flux of gravitational waves by a spinning particle in circular orbit, we will show the results in another paper [29].

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APPENDIX

In this Appendix, we write down the explicit tetrad components of the Ricci rotation coefficient and the Riemann tensor for the convenience.

The nonvanishing components of the Ricci rotation coefficient of the Kerr spacetime are

$$\begin{aligned}
\omega_{01}{}^{0} &= \omega_{00}{}^{1} = \frac{1}{\Sigma^{3/2}\Delta^{1/2}}(ra^2\sin^2\theta - Mr^2 \\
&\quad + Ma^2\cos^2\theta), \\
\omega_{31}{}^{0} &= \omega_{30}{}^{1} = \omega_{13}{}^{0} = \omega_{10}{}^{3} = \omega_{03}{}^{1} = -\omega_{01}{}^{3} = \frac{ar\sin\theta}{\Sigma^{3/2}}, \\
\omega_{02}{}^{0} &= \omega_{00}{}^{2} = \omega_{12}{}^{1} = -\omega_{11}{}^{2} = \frac{a^2\cos\theta\sin\theta}{\Sigma^{3/2}}, \\
\omega_{32}{}^{0} &= \omega_{30}{}^{2} = -\omega_{23}{}^{0} = -\omega_{20}{}^{3} = \omega_{03}{}^{2} \\
&= -\omega_{02}{}^{3} = \frac{a\cos\theta\Delta^{1/2}}{\Sigma^{3/2}}, \\
\omega_{22}{}^{1} &= -\omega_{21}{}^{2} = \omega_{33}{}^{1} = -\omega_{31}{}^{3} = \frac{r\Delta^{1/2}}{\Sigma^{3/2}}, \\
\omega_{33}{}^{2} &= -\omega_{32}{}^{3} = \frac{(r^2+a^2)\cos\theta}{\Sigma^{3/2}\sin\theta}.
\end{aligned} \tag{A1}$$

The nonvanishing components of the Riemann tensor are

$$\begin{aligned}
R_{1212} &= R_{1313} = \frac{1}{2}R_{1010} = -\frac{1}{2}R_{2323} = -R_{2020} = -R_{3030} = \frac{Mr(3a^2\cos^2\theta - r^2)}{\Sigma^3}, \\
R_{1230} &= -R_{1320} = -\frac{1}{2}R_{1023} = \frac{aM\cos\theta(3r^2 - a^2\cos^2\theta)}{\Sigma^3}.
\end{aligned} \tag{A2}$$

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