Post-Newtonian expansion of gravitational waves from a particle in circular orbit around a rotating black hole: Up to $O(v^8)$ beyond the quadrupole formula

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(Received 6 February 1996)

Extending a method developed by Sasaki in the Schwarzschild case and by Shibata, Sasaki, Tagoshi, and Tanaka in the Kerr case, we calculate the post-Newtonian expansion of the gravitational wave luminosities from a test particle in circular orbit around a rotating black hole up to $O(v^8)$ beyond the quadrupole formula. The orbit of a test particle is restricted on the equatorial plane. We find that spin-dependent terms appear in each post-Newtonian order, and that at $O(v^6)$ they have a significant effect on the orbital phase evolution of coalescing compact binaries. By comparing the post-Newtonian formula of the luminosity with numerical results we find that, for $30M \lesssim r \lesssim 100M$, the spin-dependent terms at $O(v^6)$ and $O(v^7)$ improve the accuracy of the post-Newtonian formula significantly, but those at $O(v^8)$ do not improve. [S0556-2821(96)02714-2]

PACS number(s): 04.25.Nx, 04.30.Db, 97.60.Lf

I. INTRODUCTION

Among the possible sources of gravitational waves, coalescing compact binaries are considered to be the most promising candidates for detection by near-future, ground-based laser interferometric detectors such as the Laser Interferometric Gravitational Wave Observatory (LIGO) [1], VIRGO [2], GEO600, TAMA, and AIGO. There are two reasons for this: First, we can expect a sufficiently large amplitude of gravitational waves from these systems. Second, the estimated event rate, for neutron star binaries, is several/yr within 200 Mpc [3]. Furthermore, the observations of coalescing compact binaries are potentially important because they bring us new physical and astronomical information. They can be used to test general relativity [4], and to measure cosmological parameters [5] and neutron star radii. It may even be possible to obtain information about the equation of state of neutron stars [6]. If a neutron star or a small black hole spirals into a massive black hole with mass $<300M_{\odot}$, the inspiral waveform will be detected by the above detectors. Such waveforms carry detailed information about the spacetime geometry around the black hole, and, therefore, may be used to test the black hole no-hair theorem [7].

When a gravitational wave signal is detected, matched filtering will be used to extract the binary's parameters (i.e., masses, spins, etc.) [6]. In this method, the parameters are determined by cross correlating the noisy signal from the detectors with theoretical templates. If the signal and the templates lose phase with each other by one cycle over $\sim 10^3$–10^4 cycles as the waves sweep through the LIGO-VIRGO band, their cross correlation will be significantly reduced. This means that we need to construct theoretical templates which are accurate to better than one cycle during entire sweep through the LIGO-VIRGO band [6]. If we have accurate templates, we can, in principle, determine the mass of the systems within 1% error [8]. Thus, much effort has been expended to construct accurate theoretical templates [9].

The standard method to calculate inspiraling waveforms from coalescing binaries is the post-Newtonian expansion of the Einstein equations, in which the orbital velocity v of the binaries is assumed to be small compared to the speed of light. Since, for coalescing binaries, the orbital velocity is not so small when the frequency of gravitational waves is in LIGO-VIRGO band, it is necessary to carry the post-Newtonian expansion up to extremely high order in v. A post-Newtonian wave generation formalism which can handle the high order calculation has been developed by Blanchet and Damour [10] and Damour and Iyer [11]. Based on this formalism, calculations have been carried out up to post$^{5/2}$-Newtonian order or $O(v^5)$ beyond the leading order quadrupole formula [12–20]. Another formalism is also developed up to $O(v^4)$ by Will and Wiseman [16,20] which is based on the Epstein-Wagoner formalism [21,22].

Although the post-Newtonian calculation technique will be developed and applied to the higher order calculation, it will become more difficult and complicated. Thus, it would be very helpful if we could have another reliable method to calculate the higher order post-Newtonian corrections. Recently the post-Newtonian expansion based on black hole perturbation formalism is developed. In this analysis, one considers gravitational waves from a particle of mass μ orbiting a black hole of mass M when $\mu \ll M$. Although this method is restricted to the case when $\mu \ll M$, one can calculate very high order post-Newtonian corrections to gravitational waves using a relatively simple algorithm in contrast with the standard post-Newtonian analysis. This direction of research was first done analytically by Poisson [23] who worked to $O(v^3)$ and numerically by Cutler *et al.* [24] to $O(v^5)$. Subsequently, a highly accurate numerical calculation was carried out by Tagoshi and Nakamura [25] to

$O(v^8)$ in which they found the appearance of $\ln v$ terms in the energy flux at $O(v^6)$ and at $O(v^8)$. They also clarified that the accuracy of the energy flux to at least $O(v^6)$ is needed to construct template waveforms for coalescing binaries. Tagoshi and Sasaki [26], using the formulation built up by Sasaki [27], performed analytic calculations which confirmed the result of Tagoshi and Nakamura. These calculations were extended to a rotating black hole case by Shibata, Sasaki, Tagoshi, and Tanaka (SSTT) [28] to $O(v^5)$. They calculated gravitational waves from a particle in circular orbit with small inclination from the equatorial plane to see the effect of spin at high post-Newtonian orders. They found that the effect of spin on the orbital phase is important at $O(v^5)$ order when one of the stars is a rapidly rotating neutron star with its pulse period less than 2 ms or a rapidly rotating black hole with $q=J_{\rm BH}/M^2\geqslant 0.2$. This analysis was extended to the case of slightly eccentric orbits by Tagoshi [29]. The absorption of gravitational waves into the black hole horizon, appearing at $O(v^8)$, was also calculated by Poisson and Sasaki in the case when a test particle is in a circular orbit around a Schwarzschild black hole [30].

In this paper, we extend these analyses in the rotating black hole case to $O(v^8)$ order. Once again, the calculation is based on the formalism developed by Sasaki [27] to treat a Schwarzschild black hole. Based on the post-Newtonian expansion of the luminosity in the test particle limit when the central body is a Schwarzschild black hole [25,26], Cutler and Flanagan [31] estimated that we will have to calculate post-Newtonian expansion of gravitational wave luminosity at least up to $O(v^6)$ in order to obtain the theoretical templates which cause less systematic errors than statistical errors for the LIGO detector. Further, in a previous paper [28], we suggested that the effect of spin at $O(v^6)$ to the orbital phase of coalescing binaries would not be negligible if spin of the black hole was large (i.e., $|q|\sim 1$). Also, the perturbation study can provide accurate templates for binaries with $M\gg\mu$ (for example, binaries of $100M_\odot$ black hole, $1.4M_\odot$ neutron star). Since LIGO and VIRGO will be able to detect gravitational wave signals from binaries with masses less than $\sim 300M_\odot$, it is important to construct templates for such binaries. The frequency of gravitational waves from such a massive binary, however, comes into the frequency band for LIGO and VIRGO at $r/M\sim 16(100M_\odot/M)^{2/3}$, i.e., highly relativistic region. We do not know whether the convergence property of the post-Newtonian approximation is good or not in such a highly relativistic motion. Hence, it is an urgent problem to clarify at what point the convergence property of the post-Newtonian expansion is good. For these purposes, we study the effect of spin beyond $O(v^6)$ order in this paper.

The paper is organized as follows. In Sec. II, we present the basic formalism to perform the post-Newtonian expansion in our perturbative approach. First, we perform the post-Newtonian expansion of the Teukolsky radial function using the Sasaki-Nakamura equation. We also show the post-Newtonian expansion of the angular equation, which is given in Appendix F. In Sec. III, we first describe the post-Newtonian expansion of the source terms. We consider circular orbits in the equatorial plane around a Kerr black hole. Then the gravitational wave luminosities to $O(v^8)$ beyond the quadrupole formula are derived. In Sec. IV, we compare post-Newtonian formulas with numerical data which gives the exact value of gravitational wave luminosity and investigate the convergence property of the post-Newtonian expansion. Section V is devoted to a summary and discussion.

Throughout this paper we use the units of $c=G=1$.

II. GENERAL FORMULATION

A. Teukolsky equation

We consider the case when a test particle of mass μ travels in a circular orbit around a Kerr black hole of mass $M\gg\mu$. We follow the notation used by SSTT [28], but for definiteness, we recapitulate necessary formulas and definitions.

To calculate gravitational radiation from a particle orbiting a Kerr black hole, we start with the Teukolsky equation [32,33]. We focus on the radiation going out to infinity described by the fourth Newman-Penrose quantity ψ_4 [34], which may be expressed as

$$\psi_4=(r-ia\cos\theta)^{-4}\int d\omega e^{-i\omega t}\sum_{\ell,m}\frac{e^{im\varphi}}{\sqrt{2\pi}}{}_{-2}S^{a\omega}_{\ell m}(\theta)R_{\ell m\omega}(r), \tag{2.1}$$

where ${}_{-2}S^{a\omega}_{\ell m}$ is the spheroidal harmonic function of spin weight $s=-2$, which is normalized as

$$\int_0^\pi|{}_{-2}S^{a\omega}_{\ell m}|^2\sin\theta d\theta=1. \tag{2.2}$$

The radial function $R_{\ell m\omega}(r)$ obeys the Teukolsky equation with spin weight $s=-2$:

$$\Delta^2\frac{d}{dr}\left(\frac{1}{\Delta}\frac{dR_{\ell m\omega}}{dr}\right)-V(r)R_{\ell m\omega}=T_{\ell m\omega}(r), \tag{2.3}$$

where $T_{\ell m\omega}(r)$ is the source term whose explicit form will be shown later, and $\Delta=r^2-2Mr+a^2$. The potential $V(r)$ is given by

$$V(r)=-\frac{K^2+4i(r-M)K}{\Delta}+8i\omega r+\lambda, \tag{2.4}$$

where $K=(r^2+a^2)\omega-ma$ and λ is the eigenvalue of ${}_{-2}S^{a\omega}_{\ell m}$.

The solution of the Teukolsky equation at infinity $(r\rightarrow\infty)$ is expressed as

$$R_{\ell m\omega}(r)\rightarrow\frac{r^3e^{i\omega r^*}}{2i\omega B^{\rm in}_{\ell m\omega}}\int_{r_+}^\infty dr'\frac{T_{\ell m\omega}(r')R^{\rm in}_{\ell m\omega}(r')}{\Delta^2(r')}$$
$$\equiv\tilde{Z}_{\ell m\omega}r^3e^{i\omega r^*}, \tag{2.5}$$

where $r_+=M+\sqrt{M^2-a^2}$ denotes the radius of the event horizon and $R^{\rm in}_{\ell m\omega}$ is the homogeneous solution which satisfies the ingoing-wave boundary condition at the horizon,

$$R^{\rm in}_{\ell m\omega}\to\begin{cases} D_{\ell m\omega}\Delta^2 e^{-ikr^*} & \text{for } r^*\to-\infty,\\ r^3 B^{\rm out}_{\ell m\omega}e^{i\omega r^*}+r^{-1}B^{\rm in}_{\ell m\omega}e^{-i\omega r^*} & \text{for } r^*\to+\infty,\end{cases} \tag{2.6}$$

where $k=\omega-ma/2Mr_+$ and r^* is the tortoise coordinate defined by

$$\frac{dr^*}{dr}=\frac{r^2+a^2}{\Delta}. \tag{2.7}$$

For definiteness, we fix the integration constant such that r^* is given explicitly by

$$r^*=\int\frac{dr^*}{dr}dr=r+\frac{2Mr_+}{r_+-r_-}\ln\frac{r-r_+}{2M}-\frac{2Mr_-}{r_+-r_-}\ln\frac{r-r_-}{2M}, \tag{2.8}$$

where $r_\pm=M\pm\sqrt{M^2-a^2}$.

B. Post-Newtonian expansion of the homogeneous solution

In the previous papers [27,28], the post-Newtonian expansion of the homogeneous solution was performed to $O(\epsilon^2)$ in the Schwarzschild case and $O(\epsilon)$ in the Kerr case, where $\epsilon\equiv 2M\omega$. In this section, we extend those methods, performing the expansion of homogeneous solutions up to $O(\epsilon^2)$.

In order to calculate gravitational waves emitted to infinity from a particle in a circular orbit, we need to know the explicit form of the source term $T_{\ell m\omega}(r)$, which has support only at $r=r_0$ where r_0 is orbital radius in the Boyer-Lindquist coordinate, the ingoing-wave Teukolsky function $R^{\rm in}_{\ell m\omega}(r)$ at $r=r_0$, and its incident amplitude $B^{\rm in}_{\ell m\omega}$ at infinity. We consider the expansion of these quantities in terms of a small parameter $v^2\equiv M/r_0$. In addition, we need to expand those quantity in terms of $\epsilon\equiv 2M\omega$ since $\omega=O(\Omega)$ where Ω is the orbital angular velocity of the particle and $M\omega=O(v^3)$. In the case of a Kerr black hole, other combination of parameters $a\omega$ appears in the Teukolsky equation. We define $q\equiv a/M$ and we have $a\omega=q\epsilon/2=O(v^3)$.

First we perform the expansion of the spheroidal harmonics ${}_{-2}S^{a\omega}_{\ell m}$ and their eigenvalues λ in terms of $a\omega$. Since $a\omega=O(v^3)$, we have to calculate ${}_{-2}S^{a\omega}_{\ell m}$ and λ up to $O((a\omega)^2)$. The eigenvalue λ has already been evaluated up to $O((a\omega)^2)$ in a previous paper [28]. We calculate the expansion of ${}_{-2}S^{a\omega}_{\ell m}$ at $O((a\omega)^2)$ in the Appendix F. As a result, the spheroidal harmonics ${}_{-2}S^{a\omega}_{\ell m}$ are given by

$${}_{-2}S^{a\omega}_{\ell m}={}_{-2}P_{\ell m}+a\omega S^{(1)}_{\ell m}+(a\omega)^2S^{(2)}_{\ell m}+O((a\omega)^3), \tag{2.9}$$

where ${}_{-2}P_{\ell m}$ are the spherical harmonics of spin weight $s=-2$ [35] and

$$S^{(1)}_{\ell m}=\sum_{\ell'}c^{\ell'}_{\ell m}\,{}_{-2}P_{\ell' m}. \tag{2.10}$$

Here $c^{\ell'}_{\ell m}$ are nonzero only for $\ell'=\ell\pm1$, explicitly,

$$c^{\ell+1}_{\ell m}=\frac{2}{(\ell+1)^2}\times\left[\frac{(\ell+3)(\ell-1)(\ell+m+1)(\ell-m+1)}{(2\ell+1)(2\ell+3)}\right]^{1/2},$$

$$c^{\ell-1}_{\ell m}=-\frac{2}{\ell^2}\left[\frac{(\ell+2)(\ell-2)(\ell+m)(\ell-m)}{(2\ell+1)(2\ell-1)}\right]^{1/2}.$$

$S^{(2)}_{\ell m}$ is given by

$$S^{(2)}_{\ell m}=\sum_{\ell'}d^{\ell'}_{\ell m}\,{}_{-2}P_{\ell' m}, \tag{2.11}$$

where the nonzero components of $d^{\ell'}_{\ell m}$ are given by

$$d^{\ell}_{\ell m}=-\frac{1}{2}[(c^{\ell+1}_{\ell m})^2+(c^{\ell-1}_{\ell m})^2], \tag{2.12}$$

for any ℓ, by

$$d^{\ell+1}_{\ell m}=\frac{m}{324\sqrt{7}}(3-m)^{1/2}(3+m)^{1/2},$$

$$d^{\ell+2}_{\ell m}=\frac{11}{1764\sqrt{3}}(3-m)^{1/2}(3+m)^{1/2}(4-m)^{1/2}(4+m)^{1/2},$$

for $\ell=2$, and by

$$d^{\ell+1}_{\ell m}=\frac{m}{120\sqrt{21}}(4-m)^{1/2}(4+m)^{1/2},$$

$$d^{\ell+2}_{\ell m}=\frac{1}{180\sqrt{11}}(4-m)^{1/2}(4+m)^{1/2}(5-m)^{1/2}(5+m)^{1/2},$$

$$d^{\ell-1}_{\ell m}=-\frac{m}{324\sqrt{7}}(3-m)^{1/2}(3+m)^{1/2},$$

for $\ell=3$. We do not need $S^{(2)}_{\ell m}$ for $\ell=4$ in this paper.

The eigenvalue λ is given by

$$\lambda=\lambda_0+a\omega\lambda_1+a^2\omega^2\lambda_2+O((a\omega)^3), \tag{2.13}$$

where $\lambda_0=(\ell-1)(\ell+2)$, $\lambda_1=-2m(\ell^2+\ell+4)/(\ell^2+\ell)$, and

$$\lambda_2=-2(\ell+1)(c^{\ell+1}_{\ell m})^2+2\ell(c^{\ell-1}_{\ell m})^2+\frac{2}{3}-\frac{2}{3}\frac{(\ell+4)(\ell-3)(\ell^2+\ell-3m^2)}{\ell(\ell+1)(2\ell+3)(2\ell-1)}. \tag{2.14}$$

Next we calculate the homogeneous solution $R^{\rm in}_{\ell m\omega}$. Here we only consider the case when $\omega>0$. We must treat the case $\omega\le 0$ separately. The Teukolsky equation is transformed into the Sasaki-Nakamura equation [36], which is given by

$$\left[\frac{d^2}{dr^{*2}}-F(r)\frac{d}{dr^*}-U(r)\right]X_{\ell m\omega}=0. \tag{2.15}$$

The explicit forms of $F(r)$ and $U(r)$ are given in Appendix A. The relation between $R_{\ell m\omega}$ and $X_{\ell m\omega}$ is

$$R_{\ell m\omega}=\frac{1}{\eta}\left\{\left(\alpha+\frac{\beta_{,r}}{\Delta}\right)\chi_{\ell m\omega}-\frac{\beta}{\Delta}\chi_{\ell m\omega,r}\right\}, \qquad (2.16)$$

where $\chi_{\ell m\omega}=X_{\ell m\omega}\Delta/(r^2+a^2)^{1/2}$, and the functions α, β, and η are shown in Appendix A. Conversely, we can express $X_{\ell m\omega}$ in terms of $R_{\ell m\omega}$ as

$$X_{\ell m\omega}=(r^2+a^2)^{1/2}r^2J_-J_-\left[\frac{1}{r^2}R_{\ell m\omega}\right], \qquad (2.17)$$

where $J_-=(d/dr)-i(K/\Delta)$. Then the asymptotic behavior of the ingoing-wave solution $X^{\rm in}_{\ell m\omega}$ which corresponds to Eq. (2.6) is

$$X^{\rm in}_{\ell m\omega}\to\begin{cases}A^{\rm out}_{\ell m\omega}e^{i\omega r^*}+A^{\rm in}_{\ell m\omega}e^{-i\omega r^*} & \text{for } r^*\to\infty,\\ C_{\ell m\omega}e^{-ikr^*} & \text{for } r^*\to-\infty.\end{cases} \qquad (2.18)$$

The coefficient $A^{\rm in}_{\ell m\omega}$, $A^{\rm out}_{\ell m\omega}$, and $C_{\ell m\omega}$ are, respectively, related to $B^{\rm in}_{\ell m\omega}$, $B^{\rm out}_{\ell m\omega}$, and $D_{\ell m\omega}$, defined in Eq. (2.6), by

$$B^{\rm in}_{\ell m\omega}=-\frac{1}{4\omega^2}A^{\rm in}_{\ell m\omega},$$

$$B^{\rm out}_{\ell m\omega}=-\frac{4\omega^2}{c_0}A^{\rm out}_{\ell m\omega},$$

$$D_{\ell m\omega}=\frac{1}{d_{\ell m\omega}}C_{\ell m\omega}, \qquad (2.19)$$

where c_0 is given in Eq. (A3) of Appendix A and

$$\begin{aligned}d_{\ell m\omega}=\sqrt{2Mr_+}[&(8-24iM\omega-16M^2\omega^2)r_+^2\\&+(12iam-16M+16amM\omega+24iM^2\omega)r_+\\&-4a^2m^2-12iamM+8M^2].\end{aligned}$$

Now we introduce the variable $z=\omega r$ and

$$\begin{aligned}z^*&=z+\epsilon\left[\frac{z_+}{z_+-z_-}\ln(z-z_+)-\frac{z_-}{z_+-z_-}\ln(z-z_-)\right]\\&=\omega r^*+\epsilon\ln\epsilon,\end{aligned} \qquad (2.20)$$

where $z_\pm=\omega r_\pm$. To solve $X^{\rm in}_{\ell m\omega}$ by expanding it in terms of ϵ, we set

$$X^{\rm in}_{\ell m\omega}=\sqrt{z^2+a^2\omega^2}\,\xi_{\ell m}(z)\exp[-i\phi(z)], \qquad (2.21)$$

where

$$\phi(z)=\int dr\left(\frac{K}{\Delta}-\omega\right)=z^*-z-\frac{\epsilon}{2}mq\frac{1}{z_+-z_-}\ln\frac{z-z_+}{z-z_-}, \qquad (2.22)$$

which generalizes the phase function $\omega(r^*-r)$ of the Schwarzschild case. This prescription makes it easy to implement the ingoing-wave boundary condition on $X^{\rm in}_{\ell m\omega}$.

Inserting Eq. (2.21) into Eq. (2.15) and expanding it in powers of $\epsilon=2M\omega$, we obtain

$$\begin{aligned}L^{(0)}[\xi_{\ell m}]=&\epsilon L^{(1)}[\xi_{\ell m}]+\epsilon Q^{(1)}[\xi_{\ell m}]+\epsilon^2Q^{(2)}[\xi_{\ell m}]\\&+\epsilon^3Q^{(3)}[\xi_{\ell m}]+\epsilon^4Q^{(4)}[\xi_{\ell m}]+O(\epsilon^5),\end{aligned} \qquad (2.23)$$

where $L^{(0)}$, $L^{(1)}$, $Q^{(1)}$, and $Q^{(2)}$ are differential operators given by

$$L^{(0)}=\frac{d^2}{dz^2}+\frac{2}{z}\frac{d}{dz}+\left(1-\frac{\ell(\ell+1)}{z^2}\right), \qquad (2.24)$$

$$L^{(1)}=\frac{1}{z}\frac{d^2}{dz^2}+\left(\frac{1}{z^2}+\frac{2i}{z}\right)\frac{d}{dz}-\left(\frac{4}{z^3}-\frac{i}{z^2}+\frac{1}{z}\right), \qquad (2.25)$$

$$Q^{(1)}=\frac{iq\lambda_1}{2z^2}\frac{d}{dz}-\frac{4imq}{l(l+1)z^3}, \qquad (2.26)$$

and $Q^{(2)}$, $Q^{(3)}$, and $Q^{(4)}$ are given in Appendix C. Note that the real part of $Q^{(1)}$ vanishes when we insert the expression for λ_1. There are λ_3 or λ_4 in the formulas for $Q^{(3)}$ and $Q^{(4)}$. However, it is straghtforward to show that both λ_3 and λ_4 do not influence the results in this paper.

By expanding $\xi_{\ell m}$ in terms of ϵ as

$$\xi_{\ell m}=\sum_{n=0}^{\infty}\epsilon^n\xi^{(n)}_{\ell m}(z), \qquad (2.27)$$

we obtain from Eq. (2.23) the iterative equations

$$L^{(0)}[\xi^{(0)}_{\ell m}]=0, \qquad (2.28)$$

$$L^{(0)}[\xi^{(1)}_{\ell m}]=L^{(1)}[\xi^{(0)}_{\ell m}]+Q^{(1)}[\xi^{(0)}_{\ell m}]\equiv W^{(1)}_{\ell m}, \qquad (2.29)$$

$$L^{(0)}[\xi^{(2)}_{\ell m}]=L^{(1)}[\xi^{(1)}_{\ell m}]+Q^{(1)}[\xi^{(1)}_{\ell m}]+Q^{(2)}[\xi^{(0)}_{\ell m}]\equiv W^{(2)}_{\ell m}, \qquad (2.30)$$

$$\begin{aligned}L^{(0)}[\xi^{(3)}_{\ell m}]=&L^{(1)}[\xi^{(2)}_{\ell m}]+Q^{(1)}[\xi^{(2)}_{\ell m}]+Q^{(2)}[\xi^{(1)}_{\ell m}]+Q^{(3)}[\xi^{(0)}_{\ell m}]\\\equiv&W^{(3)}_{\ell m},\end{aligned} \qquad (2.31)$$

$$\begin{aligned}L^{(0)}[\xi^{(4)}_{\ell m}]=&L^{(1)}[\xi^{(3)}_{\ell m}]+Q^{(1)}[\xi^{(3)}_{\ell m}]+Q^{(2)}[\xi^{(2)}_{\ell m}]+Q^{(3)}[\xi^{(1)}_{\ell m}]\\&+Q^{(4)}[\xi^{(0)}_{\ell m}]\equiv W^{(4)}_{\ell m}.\end{aligned} \qquad (2.32)$$

The general solution to Eq. (2.28) is immediately obtained as

$$\xi^{(0)}_{\ell m}=\alpha^{(0)}_{\ell}j_\ell+\beta^{(0)}_{\ell}n_\ell, \qquad (2.33)$$

where j_ℓ and n_ℓ are the usual spherical Bessel functions. As we discuss later, the boundary condition for $n\leqslant 2$ is that $\xi^{(n)}_{\ell m}$ is regular at $z=0$. Hence $\beta^{(0)}_{\ell}=0$ and we set $\alpha^{(0)}_{\ell}=1$ for convenience.

To calculate $\xi^{(n)}_{\ell m}$ for $n\geqslant 1$, we rewrite Eqs. (2.29)–(2.32) in the indefinite integral form by using the spherical Bessel functions as

$$\xi^{(n)}_{\ell m}=n_\ell\int^z dz z^2 j_\ell W^{(n)}_{\ell m}-j_\ell\int^z dz z^2 n_\ell W^{(n)}_{\ell m} \quad (n=1,2). \qquad (2.34)$$

The calculation for $n=1$ was done in a previous paper [28] and we have

$$\xi^{(1)}_{\ell m}=\alpha^{(1)}_{\ell}j_\ell+\frac{(\ell-1)(\ell+3)}{2(\ell+1)(2\ell+1)}j_{\ell+1}-\frac{\ell^2-4}{2\ell(2\ell+1)}j_{\ell-1}+z^2(n_\ell j_0-j_\ell n_0)j_0+\sum_{k=1}^{\ell-1}\left(\frac{1}{k}+\frac{1}{k+1}\right)z^2(n_\ell j_k-j_\ell n_k)j_k$$
$$+n_\ell[\mathrm{Ci}(2z)-\gamma-\ln 2z]-j_\ell\mathrm{Si}(2z)+ij_\ell\ln z+\frac{imq}{2}\left(\frac{\ell^2+4}{\ell^2(2\ell+1)}\right)j_{\ell-1}+\frac{imq}{2}\left(\frac{(\ell+1)^2+4}{(\ell+1)^2(2\ell+1)}\right)j_{\ell+1}, \qquad (2.35)$$

where $\mathrm{Ci}(x)=-\int_x^\infty dt\cos t/t$ and $\mathrm{Si}(x)=\int_0^x dt\sin t/t$ are cosine and sine integral functions, γ is the Euler constant, and $\alpha^{(1)}_\ell$ is an integration constant which represents the arbitrariness of the normalization of $X^{\mathrm{in}}_{\ell m\omega}$. We set $\alpha^{(1)}_\ell=0$ for simplicity.

Next we consider $\xi^{(2)}_{\ell m}$. From Eqs. (2.34) and (2.35), and by using formulas in the paper [27], we obtain $\xi^{(2)}_{\ell m}$ as

$$\xi^{(2)}_{\ell m}=f^{(2)}_\ell+ig^{(2)}_\ell+k^{(2)}_{\ell m}(q)+\alpha^{(2)}_{\ell m}j_\ell+\beta^{(2)}_{\ell m}n_\ell, \qquad (2.36)$$

where $f^{(2)}_\ell$ and $g^{(2)}_\ell$ are the real and imaginary part of $\xi^{(2)}_{\ell m}$ in the Schwarzschild case, respectively, $k^{(2)}_{\ell m}(q)$ exists only in Kerr case, and $\alpha^{(2)}_{\ell m}$ and $\beta^{(2)}_{\ell m}$ are arbitrary constants. The explicit forms of $f^{(2)}_\ell$ and $g^{(2)}_\ell$ are given in a previous paper [27]. The term $k^{(2)}_{\ell m}(q)$ is given for $\ell=2$ by

$$k^{(2)}_{2m}=\frac{191i}{180}mqj_0-\frac{m^2q^2j_0}{30}-\frac{mqj_1}{10}-\frac{68i}{63}mqj_2-\frac{q^2}{392}j_2-\frac{73m^2q^2}{1764}j_2+\frac{7mqj_3}{180}-\frac{i}{72}q^2j_3+\frac{i}{324}m^2q^2j_3+\frac{11i}{420}mqj_4-\frac{q^2j_4}{392}$$
$$-\frac{71m^2q^2j_4}{8820}+\frac{13i}{6}mqn_1+\left(-\frac{mqj_1}{5}-\frac{13mqj_3}{90}\right)\ln z+\left(\frac{-i}{5}mqj_1-\frac{13i}{90}mqj_3\right)\mathrm{S}(z)+\left(\frac{i}{5}mqn_1+\frac{13i}{90}mqn_3\right)\mathrm{C}(z) \qquad (2.37)$$

and, for $\ell=3$,

$$k^{(2)}_{3m}=\frac{3527i}{840}mqj_1-\frac{2m^2q^2j_1}{315}-\frac{mqj_2}{36}-\frac{5i}{504}q^2j_2+\frac{5i}{2268}m^2q^2j_2-\frac{379i}{360}mqj_3-\frac{q^2j_3}{360}-\frac{7m^2q^2j_3}{720}+\frac{3mqj_4}{160}-\frac{i}{140}q^2j_4$$
$$+\frac{i}{1120}m^2q^2j_4+\frac{97i}{5040}mqj_5-\frac{q^2j_5}{360}-\frac{17m^2q^2j_5}{5040}-\frac{103i}{48}mqn_0+\frac{25i}{8}mqn_2-\left(\frac{13mqj_2}{126}+\frac{5mqj_4}{56}\right)\ln z$$
$$+\left(\frac{-13i}{126}mqj_2-\frac{5i}{56}mqj_4\right)\mathrm{S}(z)+\left(\frac{13i}{126}mqn_2+\frac{5i}{56}mqn_4\right)\mathrm{C}(z), \qquad (2.38)$$

where $\mathrm{C}(z)=\mathrm{Ci}2z-\gamma-\ln 2z$ and $\mathrm{S}(z)=\mathrm{Si}2z$. Note that to obtain the above two formulas, we have added terms proportional to j_ℓ to simplify the formulas of $A^{\mathrm{in}}_{\ell m\omega}$ below. As noted previously, the source term $T_{\ell m\omega}$ has support only at $r=r_0$ and $\omega r_0=O(v)$. Hence we only need $X^{\mathrm{in}}_{\ell m\omega}$ at $z=O(v)\ll 1$ to evaluate the source integral, apart from the value of the incident amplitude $A^{\mathrm{in}}_{\ell m\omega}$. Hence the post-Newtonian expansion of $X^{\mathrm{in}}_{\ell m\omega}$ corresponds to the expansion not only in terms of $\epsilon=O(v^3)$, but also z by assuming $\epsilon\ll z\ll 1$. In order to evaluate the gravitational wave luminosity to $O(v^8)$ beyond the leading order, we must calculate the series expansion of $\xi^{(n)}_{\ell m}$ in powers of z for $n=0$ to $\ell=6$, for $n=1$ to $\ell=5$, for $n=2$ to $\ell=4$, for $n=3$ to $\ell=3$, and for $n=4$ to $\ell=2$ (see Appendix C of SSTT).

When we evaluate $A^{\mathrm{in}}_{\ell m\omega}$, we examine the asymptotic behavior of $\xi^{(n)}_{\ell m}$ at infinity. Since the accuracy of $A^{\mathrm{in}}_{\ell m\omega}$ we need is $O(\epsilon^2)$, we do not have to calculate $\xi^{(3)}_{\ell m}$ and $\xi^{(4)}_{\ell m}$ in closed analytic form. We need only the series expansion formulas for $\xi^{(3)}_{\ell m}$ and $\xi^{(4)}_{\ell m}$ around $z=0$, which is easily obtained by Eq. (2.34). Inserting $\xi^{(n)}_{\ell m}$ into Eq. (2.21) and expanding it by z and ϵ assuming $\epsilon\ll z\ll 1$, we obtain

$$\xi^{(3)}_{2m}=\frac{-q^2}{30z}-\frac{i}{30z}mq^3+\frac{-i}{30}+\frac{7mq}{180}-\frac{i}{60}m^2q^2+\frac{mq^3}{36}-\frac{m^3q^3}{90}-\frac{mq\ln z}{30}-\frac{i}{30}m^2q^2\ln z$$
$$+z\left(\frac{319}{6300}+\frac{100637i}{441000}mq-\frac{q^2}{180}+\frac{17m^2q^2}{1134}+\frac{83i}{5880}mq^3-\frac{61i}{13230}m^3q^3+\frac{\ln z}{15}-\frac{106i}{1575}mq\ln z-\frac{i}{30}mq(\ln z)^2\right)$$
$$+O(z^2)+\alpha^{(3)}_{2m}j_2+\beta^{(3)}_{2m}n_2, \qquad (2.39)$$

$$\xi^{(4)}_{2m}=\frac{q^4}{80\,z^2}+O(z^{-1})+\alpha^{(4)}_{2m}j_2+\beta^{(4)}_{2m}n_2, \qquad (2.40)$$

$$\xi^{(3)}_{3m}=\frac{-i}{1260}mq+\frac{m^2q^2}{1890}-\frac{i}{1260}mq^3-\frac{i}{3780}m^3q^3+O(z)+\alpha^{(3)}_{3m}j_3+\beta^{(3)}_{3m}n_3. \tag{2.41}$$

The boundary condition of $\xi^{(n)}_{\ell m}$ that correctly represent the boundary condition of $X^{\rm in}_{\ell m\omega}$ [Eq. (2.18)] is that $z\xi^{(n)}_{\ell m}$ must be no more singular than $z^{(\ell+1-n)}$ at $z\to 0$. Since we need $\xi^{(n)}_{\ell m}$ only up to $n=4$, we set $\beta^{(n)}_{\ell m}=0$ for all of ℓ and n in this paper. As for $\alpha^{(n)}_{\ell m}$, they still remain arbitrary and we set $\alpha^{(n)}_{\ell m}=0$ for all of ℓ, m, and $n=3,4$.

Inserting $\xi^{(n)}_{\ell m}$ into Eq. (2.22) and expanding it in terms of $\epsilon=2M\omega$, we obtain $X^{\rm in}_{\ell m\omega}$ which are shown in Appendix C. Using the transformation of Eq. (2.16), we obtain $R^{\rm in}_{\ell m\omega}$ which are also shown in Appendix D.

Next, we consider $A^{\rm in}_{\ell m\omega}$ at $O(\epsilon^2)$. Using the relation $j_{\ell+1}\sim -j_{\ell-1}\sim (-1)^{\ell+n}n_{2n-\ell}$ at $z\sim\infty$, etc., we obtain the asymptotic behavior of $\xi^{(1)}_{\ell m}$ and $\xi^{(2)}_{\ell m}$ at $z\sim\infty$ as

$$\xi^{(1)}_{\ell m}\sim p^{(1)}_{\ell m}j_\ell+(q^{(1)}_{\ell m}-\ln z)n_\ell+ij_\ell\ln z, \tag{2.42}$$

$$\xi^{(2)}_{\ell m}\sim[p^{(2)}_{\ell m}+q^{(1)}_{\ell m}\ln z-(\ln z)^2]j_\ell+(q^{(2)}_{\ell m}-p^{(1)}_{\ell m}\ln z)n_\ell+ip^{(1)}_{\ell m}j_\ell\ln z+i(q^{(1)}_{\ell m}-\ln z)n_\ell\ln z, \tag{2.43}$$

where

$$p^{(1)}_{\ell m}=-\frac{\pi}{2}, \tag{2.44}$$

$$q^{(1)}_{\ell m}=\frac{1}{2}\left[\psi(\ell)+\psi(\ell+1)+\frac{(\ell-1)(\ell+3)}{\ell(\ell+1)}\right]-\ln 2-\frac{2imq}{\ell^2(\ell+1)^2}, \tag{2.45}$$

$$\psi(\ell)=\sum_{k=1}^{\ell-1}\frac{1}{k}-\gamma \tag{2.46}$$

for any ℓ and

$$p^{(2)}_{2m}=\frac{457\gamma}{210}-\frac{\gamma^2}{2}+5\frac{\pi^2}{24}-\frac{i}{18}\gamma mq+\frac{457\ln 2}{210}-\gamma\ln 2-\frac{i}{18}mq\ln 2-\frac{(\ln 2)^2}{2}, \tag{2.47}$$

$$q^{(2)}_{2m}=\frac{-457\pi}{420}+\frac{\gamma\pi}{2}+\frac{5mq}{36}+\frac{i}{36}m\pi q-\frac{i}{72}q^2+\frac{i}{324}m^2q^2+\frac{\pi\ln 2}{2}, \tag{2.48}$$

$$p^{(2)}_{3m}=\frac{52\gamma}{21}-\frac{\gamma^2}{2}+5\frac{\pi^2}{24}-\frac{i}{72}\gamma mq+\frac{52\ln 2}{21}-\gamma\ln 2-\frac{i}{72}mq\ln 2-\frac{(\ln 2)^2}{2}, \tag{2.49}$$

$$q^{(2)}_{3m}=\frac{-26\pi}{21}+\frac{\gamma\pi}{2}+\frac{67mq}{1440}+\frac{i}{144}m\pi q+\frac{i}{360}q^2-\frac{17i}{12960}m^2q^2+\frac{\pi\ln 2}{2}. \tag{2.50}$$

Then noting that $\exp(-i\phi)\sim\exp[-i(z^*-z)]$ at $z\sim\infty$, the asymptotic form of $X^{\rm in}_{\ell m\omega}$ is expressed as

$$\begin{aligned}X^{\rm in}_{\ell m\omega}&=\sqrt{z^2+a^2\omega^2}\exp(-i\phi)\{f^{(0)}_{\ell m}+\epsilon\xi^{(1)}_{\ell m}+\epsilon^2\xi^{(2)}_{\ell m}+\cdots\}\sim e^{-iz^*}(zh^{(2)}_\ell e^{iz})[1+\epsilon(p^{(1)}_{\ell m}+iq^{(1)}_{\ell m})+\epsilon^2(p^{(2)}_{\ell m}+iq^{(2)}_{\ell m})]\\&\quad+e^{iz^*}(zh^{(1)}_\ell e^{-iz})[1+\epsilon(p^{(1)}_{\ell m}-iq^{(1)}_{\ell m})+\epsilon^2(p^{(2)}_{\ell m}-iq^{(2)}_{\ell m})],\end{aligned} \tag{2.51}$$

where $h^{(1)}_\ell$ and $h^{(2)}_\ell$ are the spherical Hankel functions of the first and second kinds, respectively, which are given by

$$h^{(1)}_\ell=j_\ell+in_\ell\to(-1)^{\ell+1}\frac{e^{iz}}{z},$$

$$h^{(2)}_\ell=j_\ell-in_\ell\to(-1)^{\ell+1}\frac{e^{-iz}}{z}. \tag{2.52}$$

From these equations, noting $\omega r^*=z^*-\epsilon\ln\epsilon$, we obtain

$$\begin{aligned}A^{\rm in}_{\ell m\omega}&=\frac{1}{2}i^{\ell+1}e^{-i\epsilon\ln\epsilon}[1+\epsilon(p^{(1)}_{\ell m}+iq^{(1)}_{\ell m})+\epsilon^2(p^{(2)}_{\ell m}+iq^{(2)}_{\ell m})\\&\quad+\cdots].\end{aligned} \tag{2.53}$$

The corresponding incident amplitude $B^{\rm in}_{\ell m\omega}$ for the Teukolsky function are obtained from Eq. (2.19).

III. GRAVITATIONAL WAVE LUMINOSITY TO $O(V^8)$

A. Geodesic equations

In this section, we solve the geodesic equation for circular motion in the equatorial plane. The geodesic equations in the Kerr geometry are given by

$$\Sigma\frac{d\theta}{d\tau}=\pm\left[C-\cos^2\theta\left\{a^2(1-E^2)+\frac{l_z^2}{\sin^2\theta}\right\}\right]^{1/2}\equiv\Theta(\theta),$$

$$\Sigma\frac{d\varphi}{d\tau}=-\left(aE-\frac{l_z}{\sin^2\theta}\right)+\frac{a}{\Delta}[E(r^2+a^2)-al_z]\equiv\Phi,$$

$$\Sigma\frac{dt}{d\tau}=-\left(aE-\frac{l_z}{\sin^2\theta}\right)a\sin^2\theta+\frac{r^2+a^2}{\Delta}[E(r^2+a^2)-al_z]\equiv T,$$

$$\Sigma\frac{dr}{d\tau}=\pm\sqrt{R}, \tag{3.1}$$

where E, l_z, and C are the energy, the z component of the angular momentum, and the Carter constant of a test particle, respectively. $\Sigma=r^2+a^2\cos^2\theta$ and

$$R=[E(r^2+a^2)-al_z]^2-\Delta[(Ea-l_z)^2+r^2+C]. \tag{3.2}$$

Since we consider a motion of a particle in the equatorial plane $\theta=\pi/2$, we can set $C=0$. We define the orbital radius as $r=r_0$. Then E and l_z are determined by $R(r_0)=0$ and $\partial R/\partial r|_{r=r_0}=0$ as

$$E=\frac{1-2v^2+qv^3}{(1-3v^2+2qv^3)^{1/2}},$$

$$l_z=\frac{r_0v(1-2qv^3+q^2v^4)}{(1-3v^2+2qv^3)^{1/2}},$$

where $v=(M/r_0)^{1/2}$. After these preparations, we can easily obtain $\varphi(t)$ as

$$\varphi(t)=\Omega t,$$

$$\Omega=\frac{M^{1/2}}{r_0^{3/2}}[1-qv^3+q^2v^6+O(v^9)]. \tag{3.3}$$

B. Integration of the source term

Using results of the previous section, we can now derive the source term of the Teukolsky equation and integrate it to give the amplitude of the Teukolsky function at infinity.

The energy-momentum tensor of a test particle of mass μ is given by

$$T^{\mu\nu}=\frac{\mu}{\Sigma\sin\theta dt/d\tau}\frac{dz^\mu}{d\tau}\frac{dz^\nu}{d\tau}\delta(r-r_0)\delta(\theta-\pi/2)\delta(\varphi-\varphi(t)). \tag{3.4}$$

The source term of the Teukolsky equation is given by

$$T_{\ell m\omega}=4\int d\Omega dt\rho^{-5}\bar{\rho}^{-1}(B_2'+B_2'^*)e^{-im\varphi+i\omega t}\frac{{}_{-2}S_{\ell m}^{a\omega}}{\sqrt{2\pi}}, \tag{3.5}$$

where

$$B_2'=-\frac{1}{2}\rho^8\bar{\rho}L_{-1}[\rho^{-4}L_0(\rho^{-2}\bar{\rho}^{-1}T_{nn})]$$
$$-\frac{1}{2\sqrt{2}}\rho^8\bar{\rho}\Delta^2L_{-1}[\rho^{-4}\bar{\rho}^2J_+(\rho^{-2}\bar{\rho}^{-2}\Delta^{-1}T_{\bar{m}n})],$$

$$B_2'^*=-\frac{1}{4}\rho^8\bar{\rho}\Delta^2J_+[\rho^{-4}J_+(\rho^{-2}\bar{\rho}T_{\bar{m}\bar{m}})]$$
$$-\frac{1}{2\sqrt{2}}\rho^8\bar{\rho}\Delta^2J_+[\rho^{-4}\bar{\rho}^2\Delta^{-1}L_{-1}(\rho^{-2}\bar{\rho}^{-2}T_{\bar{m}n})], \tag{3.6}$$

with

$$\rho=(r-ia\cos\theta)^{-1},$$

$$L_s=\partial_\theta+\frac{m}{\sin\theta}-a\omega\sin\theta+s\cot\theta,$$

$$J_+=\partial_r+iK/\Delta, \tag{3.7}$$

and $\bar{\rho}$ denotes the complex conjugate of ρ.

In the present case, the tetrad components of the energy-momentum tensor, T_{nn}, $T_{\bar{m}n}$, and $T_{\bar{m}\bar{m}}$, take the form

$$T_{nn}=\frac{C_{nn}}{\sin\theta}\delta(r-r_0)\delta(\theta-\pi/2)\delta(\varphi-\varphi(t)),$$

$$T_{\bar{m}n}=\frac{C_{\bar{m}n}}{\sin\theta}\delta(r-r_0)\delta(\theta-\pi/2)\delta(\varphi-\varphi(t)),$$

$$T_{\bar{m}\bar{m}}=\frac{C_{\bar{m}\bar{m}}}{\sin\theta}\delta(r-r_0)\delta(\theta-\pi/2)\delta(\varphi-\varphi(t)), \tag{3.8}$$

where

$$C_{nn}=\frac{\mu}{4\Sigma^3\dot{t}}[E(r^2+a^2)-al_z]^2,$$

$$C_{\bar{m}n}=-\frac{\mu\rho}{2\sqrt{2}\Sigma^2\dot{t}}[E(r^2+a^2)-al_z]\left[i\sin\theta\left(aE-\frac{l_z}{\sin^2\theta}\right)\right],$$

$$C_{\bar{m}\bar{m}}=\frac{\mu\rho^2}{2\Sigma\dot{t}}\left[i\sin\theta\left(aE-\frac{l_z}{\sin^2\theta}\right)\right]^2, \tag{3.9}$$

and $\dot{t}=dt/d\tau$.

Substituting Eq. (3.6) into Eq. (3.5) and integrating by parts, we obtain

$$T_{\ell m\omega}=\frac{4}{\sqrt{2\pi}}\int_{-\infty}^{\infty}dt\int d\theta e^{i\omega t-im\varphi(t)}\Bigg[-\frac{1}{2}L_1^{\dagger}\{\rho^{-4}L_2^{\dagger}(\rho^3 S)\}C_{nn}\rho^{-2}\overline{\rho}^{-1}\delta(r-r_0)\delta(\theta-\pi/2)+\frac{\Delta^2\overline{\rho}^2}{\sqrt{2}\rho}[L_2^{\dagger}S+ia(\overline{\rho}-\rho)\sin\theta S]$$
$$\times J_+\{C_{\overline{m}n}\rho^{-2}\overline{\rho}^{-2}\Delta^{-1}\delta(r-r_0)\delta(\theta-\pi/2)\}+\frac{1}{2\sqrt{2}}L_2^{\dagger}\{\rho^3 S(\overline{\rho}^2\rho^{-4})_{,r}\}C_{\overline{m}n}\Delta\rho^{-2}\overline{\rho}^{-2}\delta(r-r_0)\delta(\theta-\pi/2)$$
$$-\frac{1}{4}\rho^3\Delta^2 SJ_+\{\rho^{-4}J_+[\overline{\rho}\rho^{-2}C_{\overline{m}\overline{m}}\delta(r-r_0)\delta(\theta-\pi/2)]\}\Bigg], \tag{3.10}$$

where

$$L_s^{\dagger}=\partial_\theta-\frac{m}{\sin\theta}+a\omega\sin\theta+s\cot\theta, \tag{3.11}$$

and S denotes ${}_{-2}S_{\ell m}^{a\omega}(\theta)$ for simplicity.

We further rewrite Eq. (3.10) as

$$T_{\ell m\omega}=\int_{-\infty}^{\infty}dte^{i\omega t-im\varphi(t)}\Delta^2[(A_{nn\,0}+A_{\overline{m}n\,0}+A_{\overline{m}\overline{m}\,0})\delta(r-r_0)$$
$$+\{(A_{\overline{m}n\,1}+A_{\overline{m}\overline{m}\,1})\delta(r-r_0)\}_{,r}$$
$$+\{A_{\overline{m}\overline{m}\,2}\delta(r-r_0)\}_{,rr}]_{\theta=\pi/2}, \tag{3.12}$$

where $A_{nn\,0}$, etc., are given in Appendix B. Inserting Eq. (3.12) into Eq. (2.5), we obtain $\tilde{Z}_{\ell m\omega}$ as

$$\tilde{Z}_{\ell m\omega}=\frac{2\pi\delta(\omega-m\Omega)}{2i\omega B_{\ell m\omega}^{\rm in}}\Bigg[R_{\ell m\omega}^{\rm in}\{A_{nn\,0}+A_{\overline{m}n\,0}+A_{\overline{m}\overline{m}\,0}\}$$
$$-\frac{dR_{\ell m\omega}^{\rm in}}{dr}\{A_{\overline{m}n\,1}+A_{\overline{m}\overline{m}\,1}\}$$
$$+\frac{d^2R_{\ell m\omega}^{\rm in}}{dr^2}A_{\overline{m}\overline{m}\,2}\Bigg]_{r=r_0,\theta=\pi/2}$$
$$\equiv\delta(\omega-m\Omega)Z_{\ell m\omega}. \tag{3.13}$$

Using characters of ${}_{-2}S_{\ell m}^{a\omega}(\theta)$ at $\theta=\pi/2$, it is straightforward to show that $\overline{T}_{\ell,-m,-\omega}=(-1)^{\ell}T_{\ell,m,\omega}$ where $\overline{T}_{\ell,m,\omega}$ is the complex conjugate of $T_{\ell,m,\omega}$. Since the homogeneous Teukolsky equation is invariant under the complex conjugate followed by $m\to-m$ and $\omega\to-\omega$, we have $\overline{Z}_{\ell,-m,-\omega}=(-1)^{\ell}Z_{\ell,m,\omega}$.

C. Results

In this section, we calculate the gravitational wave luminosity up to $O(v^8)$ beyond the quadrupole formula. From Eq. (2.1), ψ_4 at $r\to\infty$ takes a form

$$\psi_4=\frac{1}{r}\sum_{\ell=2}^{6}\sum_{m=-\ell}^{\ell}Z_{\ell m\omega_0}\frac{{}_{-2}S_{\ell m}^{a\omega_0}}{\sqrt{2\pi}}e^{i\omega_0(r^*-t)+im\varphi}, \tag{3.14}$$

where $\omega_0=m\Omega$. At infinity, ψ_4 is related to the two independent modes of gravitational waves h_+ and $h_\times$ as

$$\psi_4=\frac{1}{2}(\ddot{h}_+-i\ddot{h}_\times). \tag{3.15}$$

From Eqs. (3.13), (3.14), and (3.15), the gravitational wave luminosity is given by

$$\left\langle\frac{dE}{dt}\right\rangle=\sum_{\ell,m}\frac{|Z_{\ell m\omega_0}|^2}{4\pi\omega_0^2}\equiv\sum_{\ell,m}\left(\frac{dE}{dt}\right)_{\ell m}. \tag{3.16}$$

In order to express the post-Newtonian corrections to the luminosity, we define $\eta_{\ell m}$ as

$$\left(\frac{dE}{dt}\right)_{\ell m}\equiv\frac{1}{2}\left(\frac{dE}{dt}\right)_N\eta_{\ell,m}, \tag{3.17}$$

where $(dE/dt)_N$ is the Newtonian quadrupole luminosity:

$$\left(\frac{dE}{dt}\right)_N=\frac{32\mu^2M^3}{5r_0^5}=\frac{32}{5}\left(\frac{\mu}{M}\right)^2v^{10}.$$

We only show $\eta_{\ell m}$ for the $m>0$ mode since $\eta_{\ell,m}=\eta_{\ell,-m}$:

$$\eta_{2,2}=1-\frac{107v^2}{21}+(4\pi-6q)v^3+\left(\frac{4784}{1323}+2q^2\right)v^4+\left(\frac{-428\pi}{21}+\frac{4216q}{189}\right)v^5+\left(\frac{99210071}{1091475}-\frac{1712\gamma}{105}+\frac{16\pi^2}{3}-28\pi q+\frac{8830q^2}{567}\right.$$
$$\left.-\frac{3424\ln 2}{105}-\frac{1712\ln v}{105}\right)v^6+\left(\frac{19136\pi}{1323}+\frac{163928q}{11907}+8\pi q^2-12q^3\right)v^7+\left(-\frac{27956920577}{81265275}+\frac{183184\gamma}{2205}-\frac{1712\pi^2}{63}\right.$$
$$\left.+\frac{20716\pi q}{189}-\frac{456028q^2}{9261}+q^4+\frac{366368\ln 2}{2205}+\frac{183184\ln v}{2205}\right)v^8, \tag{3.18}$$

$$\eta_{2,1}=\frac{v^2}{36}-\frac{qv^3}{12}+\left(-\frac{17}{504}+\frac{q^2}{16}\right)v^4+\left(\frac{\pi}{18}-\frac{793q}{9072}\right)v^5+\left(-\frac{2215}{254016}-\frac{\pi q}{6}+\frac{859q^2}{1512}\right)v^6+\left(\frac{-17\pi}{252}+\frac{11861q}{190512}+\frac{\pi q^2}{8}-\frac{7q^3}{12}\right)v^7+\left(\frac{15707221}{26195400}-\frac{107\gamma}{945}+\frac{\pi^2}{27}-\frac{1045\pi q}{4536}+\frac{118943q^2}{190512}+\frac{q^4}{16}-\frac{107\ln 2}{945}-\frac{107\ln v}{945}\right)v^8. \tag{3.19}$$

Putting together the above results, we obtain $(dE/dt)_{\ell}\equiv\Sigma_m(dE/dt)_{\ell m}$ for $\ell=2$ as

$$\left(\frac{dE}{dt}\right)_2=\left(\frac{dE}{dt}\right)_N\left\{1-\frac{1277v^2}{252}+\left(4\pi-\frac{73q}{12}\right)v^3+\left(\frac{37915}{10584}+\frac{33q^2}{16}\right)v^4+\left(\frac{-2561\pi}{126}+\frac{201575q}{9072}\right)v^5+\left(\frac{2116278473}{23284800}-\frac{1712\gamma}{105}+\frac{16\pi^2}{3}-\frac{169\pi q}{6}+\frac{73217q^2}{4536}-\frac{3424\ln 2}{105}-\frac{1712\ln v}{105}\right)v^6+\left(\frac{76187\pi}{5292}+\frac{376387q}{27216}+\frac{65\pi q^2}{8}-\frac{151q^3}{12}\right)v^7+\left(-\frac{2455920939443}{7151344200}+\frac{548803\gamma}{6615}-\frac{5129\pi^2}{189}+\frac{70877\pi q}{648}-\frac{64835431q^2}{1333584}+\frac{17q^4}{16}+\frac{219671\ln 2}{1323}+\frac{548803\ln v}{6615}\right)v^8\right\}. \tag{3.20}$$

For $\ell=3$, we obtain

$$\eta_{3,3}=\frac{1215v^2}{896}-\frac{1215v^4}{112}+\left(\frac{3645\pi}{448}-\frac{1215q}{112}\right)v^5+\left(\frac{243729}{9856}+\frac{3645q^2}{896}\right)v^6+\left(\frac{-3645\pi}{56}+\frac{131949q}{1792}\right)v^7+\left(\frac{25037019729}{125565440}-\frac{47385\gamma}{1568}+\frac{3645\pi^2}{224}-\frac{32805\pi q}{448}+\frac{346275q^2}{14336}-\frac{47385\ln 2}{1568}-\frac{47385\ln 3}{1568}-\frac{47385\ln v}{1568}\right)v^8, \tag{3.21}$$

$$\eta_{3,2}=\frac{5v^4}{63}-\frac{40qv^5}{189}+\left(-\frac{193}{567}+\frac{80q^2}{567}\right)v^6+\left(\frac{20\pi}{63}+\frac{352q}{1701}\right)v^7+\left(\frac{86111}{280665}-\frac{160\pi q}{189}+\frac{40q^2}{27}\right)v^8, \tag{3.22}$$

$$\eta_{3,1}=\frac{v^2}{8064}-\frac{v^4}{1512}+\left(\frac{\pi}{4032}-\frac{17q}{9072}\right)v^5+\left(\frac{437}{266112}+\frac{17q^2}{24192}\right)v^6+\left(\frac{-\pi}{756}+\frac{3601q}{435456}\right)v^7+\left(-\frac{1137077}{50854003200}-\frac{13\gamma}{42336}+\frac{\pi^2}{6048}-\frac{145\pi q}{36288}+\frac{41183q^2}{3483648}-\frac{13\ln 2}{42336}-\frac{13\ln v}{42336}\right)v^8. \tag{3.23}$$

Then we obtain

$$\left(\frac{dE}{dt}\right)_3=\left(\frac{dE}{dt}\right)_N\left\{\frac{1367v^2}{1008}-\frac{32567v^4}{3024}+\left(\frac{16403\pi}{2016}-\frac{896q}{81}\right)v^5+\left(\frac{152122}{6237}+\frac{341q^2}{81}\right)v^6+\left(\frac{-13991\pi}{216}+\frac{4019665q}{54432}\right)v^7+\left(\frac{5712521850527}{28605376800}-\frac{79963\gamma}{2646}+\frac{6151\pi^2}{378}-\frac{192005\pi q}{2592}+\frac{11168371q^2}{435456}-\frac{79963\ln 2}{2646}-\frac{47385\ln 3}{1568}-\frac{79963\ln v}{2646}\right)v^8\right\}. \tag{3.24}$$

For $\ell=4$, we have

$$\eta_{4,4}=\frac{1280v^4}{567}-\frac{151808v^6}{6237}+\left(\frac{10240\pi}{567}-\frac{12800q}{567}\right)v^7+\left(\frac{560069632}{6243237}+\frac{5120q^2}{567}\right)v^8, \tag{3.25}$$

$$\eta_{4,3}=\frac{729v^6}{4480}-\frac{729qv^7}{1792}+\left(-\frac{28431}{24640}+\frac{3645q^2}{14336}\right)v^8, \tag{3.26}$$

$$\eta_{4,2}=\frac{5v^4}{3969}-\frac{437v^6}{43659}+\left(\frac{20\pi}{3969}-\frac{80q}{3969}\right)v^7+\left(\frac{7199152}{218513295}+\frac{200q^2}{27783}\right)v^8, \tag{3.27}$$

$$\eta_{4,1}=\frac{v^6}{282240}-\frac{qv^7}{112896}+\left(-\frac{101}{4656960}+\frac{5q^2}{903168}\right)v^8. \tag{3.28}$$

Then we obtain

$$\left(\frac{dE}{dt}\right)_4=\left(\frac{dE}{dt}\right)_N\left\{\frac{8965v^4}{3969}-\frac{84479081v^6}{3492720}+\left(\frac{23900\pi}{1323}-\frac{59621q}{2592}\right)v^7+\left(\frac{51619996697}{582702120}+\frac{66084895q^2}{7112448}\right)v^8\right\}. \tag{3.29}$$

For $\ell=5$ we have

$$\eta_{5,5}=\frac{9765625v^6}{2433024}-\frac{2568359375v^8}{47443968}, \tag{3.30}$$

$$\eta_{5,4}=\frac{4096v^8}{13365}, \tag{3.31}$$

$$\eta_{5,3}=\frac{2187v^6}{450560}-\frac{150903v^8}{2928640}, \tag{3.32}$$

$$\eta_{5,2}=\frac{4v^8}{40095}, \tag{3.33}$$

$$\eta_{5,1}=\frac{v^6}{127733760}-\frac{179v^8}{2490808320}. \tag{3.34}$$

Then we have

$$\left(\frac{dE}{dt}\right)_5=\left(\frac{dE}{dt}\right)_N\left\{\frac{1002569v^6}{249480}-\frac{3145396841v^8}{58378320}\right\}. \tag{3.35}$$

For $\ell=6$ we have

$$\eta_{6,6}=\frac{26244v^8}{3575}, \tag{3.36}$$

$$\eta_{6,4}=\frac{131072v^8}{9555975}, \tag{3.37}$$

$$\eta_{6,2}=\frac{4v^8}{5733585}, \tag{3.38}$$

and $\eta_{6,5}$, $\eta_{6,3}$, $\eta_{6,1}$ become $O(v^9)$. Then we have

$$\left(\frac{dE}{dt}\right)_6=\left(\frac{dE}{dt}\right)_N\frac{210843872v^8}{28667925}. \tag{3.39}$$

Finally, gathering all the above results, the total luminosity up to $O(v^8)$ is expressed as

$$\begin{aligned}\left\langle\frac{dE}{dt}\right\rangle=\left(\frac{dE}{dt}\right)_N&\left\{1-\frac{1247v^2}{336}+\left(4\pi-\frac{73q}{12}\right)v^3+\left(-\frac{44711}{9072}+\frac{33q^2}{16}\right)v^4+\left(\frac{-8191\pi}{672}+\frac{3749q}{336}\right)v^5\right.\\&+\left(\frac{6643739519}{69854400}-\frac{1712\gamma}{105}+\frac{16\pi^2}{3}-\frac{169\pi q}{6}+\frac{3419q^2}{168}-\frac{3424\ln 2}{105}-\frac{1712\ln v}{105}\right)v^6\\&+\left(\frac{-16285\pi}{504}+\frac{83819q}{1296}+\frac{65\pi q^2}{8}-\frac{151q^3}{12}\right)v^7+\left(-\frac{323105549467}{3178375200}+\frac{232597\gamma}{4410}-\frac{1369\pi^2}{126}+\frac{3389\pi q}{96}\right.\\&\left.\left.-\frac{124091q^2}{9072}+\frac{17q^4}{16}+\frac{39931\ln 2}{294}-\frac{47385\ln 3}{1568}+\frac{232597\ln v}{4410}\right)v^8\right\}.\end{aligned} \tag{3.40}$$

In Appendix G, we present formulas for $\eta_{\ell,m}$ and dE/dt in terms of $v' \equiv (M\Omega)^{1/3}$ for the sake of convenience to calculate the phase function for an inspiraling waveform [8].

Setting $q=0$, above reproduces the previous results [25, 26] in a Schwarzschild case. Up to $O(v^5)$, the results agree with those obtained by SSTT [28] in the case when the test particle moves a circular orbit in the equatorial plane. For $\ell=5$ and 6, there are no contributions due to the black hole spin and the results are identical to the Schwarzschild case.

In Eq. (3.40), the numerical value of terms at order $O(v^6)$ is given by $v^6(115.7-88.48q+20.35q^2-16.30\ln v)$. We find that the spin-dependent terms are not so small compared to the other two terms if $|q|$ is of order unity. Thus, we see that spin-dependent terms at $O(v^6)$ will give a significant effect to template waveforms of coalescing binaries when spin of a black hole is large.

Finally we note that the angular momentum flux can be easily calculated from

$$\left\langle \frac{dJ}{dt} \right\rangle = \frac{1}{\Omega} \left\langle \frac{dE}{dt} \right\rangle. \tag{3.41}$$

IV. COMPARISON WITH NUMERICAL RESULTS

As discussed in Sec. I, it is important to investigate the detailed convergence property of the post-Newtonian approximation. Therefore we compare the formula for dE/dt, derived above, with numerical results and investigate the accuracy of the post-Newtonian expansion of dE/dt.

In this section, we consider the total mass of the binary systems including black holes $\sim(2-300)M_\odot$ because gravitational waves from such binaries can be detected by LIGO and VIRGO. In particular, we pay attention to the accuracy of post-Newtonian formula for dE/dt when $r \leqslant 100M$ (or $v \geqslant 0.1$), because gravitational waves from these binary systems will be detected when the orbital separation becomes less than $r \simeq 100M$. Here, we ignore the effect of absorption of gravitational waves by the black hole. We will briefly discuss its effect in the next section.

A numerical study of dE/dt from a particle in a circular orbit in the equatorial plane around a Kerr black hole has been performed by Shibata [39]. Since nothing was assumed about the velocity of a test particle, those results are correct relativistically in the limit $\mu \ll M$. In that work, dE/dt was calculated with accuracy $\lesssim 10^{-4}$. However, we found that this accuracy is not sufficient to compare it with the post-Newtonian formula for dE/dt including terms up to $O(v^8)$. Thus, in this paper, we calculate dE/dt again requiring the accuracy to be $\sim 10^{-5}$. In the numerical calculations, we have taken into account the contribution from the $\ell=2$–6 modes in dE/dt which is consistent with the post-Newtonian formula.

In Figs. 1(a)–1(e), we show the error in the post-Newtonian formulas as a function of the Boyer-Lindquist coordinate radius when $q=-0.9$, -0.5, 0, 0.5, and 0.9. In these figures, we show the error for $6 \leqslant r/M \leqslant 100$. Since the radius of the inner stable circular orbit for $q=0.9$ is $r_{\rm lso} \simeq 2.32M$ and a stable circular orbit is possible for $r > r_{\rm lso}$, we also show the errors in the case when $q=0.9$ for $2.5 \leqslant r/M \leqslant 12$ in Fig. 2. The error in the post-Newtonian formula is defined as

$$\text{error} = \left| 1 - \left(\frac{dE}{dt} \right)_{\rm PN} \bigg/ \left(\frac{dE}{dt} \right)_{\rm NR} \right|, \tag{4.1}$$

where $(dE/dt)_{\rm PN}$ and $(dE/dt)_{\rm NR}$ denote the post-Newtonian (PN) formula and the numerical results, respectively. As for $(dE/dt)_{\rm PN}$, we have used 2-PN, 2.5-PN, 3-PN, 3.5-PN, and 4-PN formulas. Here, we define n-PN formula as the expression for dE/dt which includes post-Newtonian terms up to $O(v^{2n})$ beyond the quadrupole formula. In each figure, the open square, solid triangle, open triangle, solid circle, and open circle denote the error of 2-PN, 2.5-PN, 3-PN, 3.5-PN, and 4-PN formulas, respectively. We note that in Fig. 2, the errors in the 2.5-, 3-, and 4-PN formulas become greater than unity for very small radius, because in such a region, dE/dt for those PN formulas becomes negative.

From these figures, we find the following.

(1) If we use the 2-PN or 2.5-PN formula, the error is always greater than 10^{-4} when $r \lesssim 100M$ irrespective of q. If we use the 3-PN formula, however, the error decreases significantly, and it becomes less than 10^{-4} for $r > 60M$, and less than 10^{-3} for $r > 30M$ irrespective of q.

(2) If we adopt the 3.5-PN formula, the accuracy becomes better than that of 3-PN formula. The error is always less than 10^{-4} when r is greater than $\sim 30M$ and less than 10^{-5} when r is greater than $\sim 60M$. This feature does not depend on q. However, if we use the 4-PN formula, the accuracy is not improved compared with the 3.5-PN formula. In particular, this tendency is remarkable for smaller radius.

(3) The accuracy of the 3.5-PN or 4-PN formula is not always better than that of the lower-PN one inside r_c, where $r_c \lesssim 5M$ for $q=0.5$ and 0.9, $r_c \sim 10M$ for $q=0$ and -0.5, and $r_c \sim 15M$ for $q=-0.9$. Thus, the convergence of the post-Newtonian expansion seems rather poor around r_c.

Using the above results, we investigate the accuracy of the post-Newtonian formulas as templates for various binary systems. As explained in Sec. I, to investigate the accuracy of the post-Newtonian formulas as templates, it is useful to check if they can predict the number of cycles of the gravitational waves, N, with accuracy less than 1. compact binary systems, the cycles are mainly accumulated around ~ 10 Hz which is the lowest-frequency region in the LIGO band, and N is approximately given by

$$N \sim 1.9 \times 10^3 \left(\frac{10M_\odot}{M} \right)^{5/3} \left(\frac{M}{4\mu} \right), \tag{4.2}$$

where M and μ are the total mass and reduced mass, respectively. This means that the template must have an accuracy less than

$$\sim 5 \times 10^{-4} \left(\frac{M}{10M_\odot} \right)^{5/3} \left(\frac{4\mu}{M} \right), \tag{4.3}$$

when the frequency of gravitational wave becomes 10 Hz.

First we consider equal mass binary systems, that is, $M=4\mu$. At 10 Hz, the orbital separation of a binary of total mass M is approximately given by $r/M \simeq 347(M_\odot/M)^{2/3}$. We find that the 2-PN and 2.5-PN formulas are insufficient if

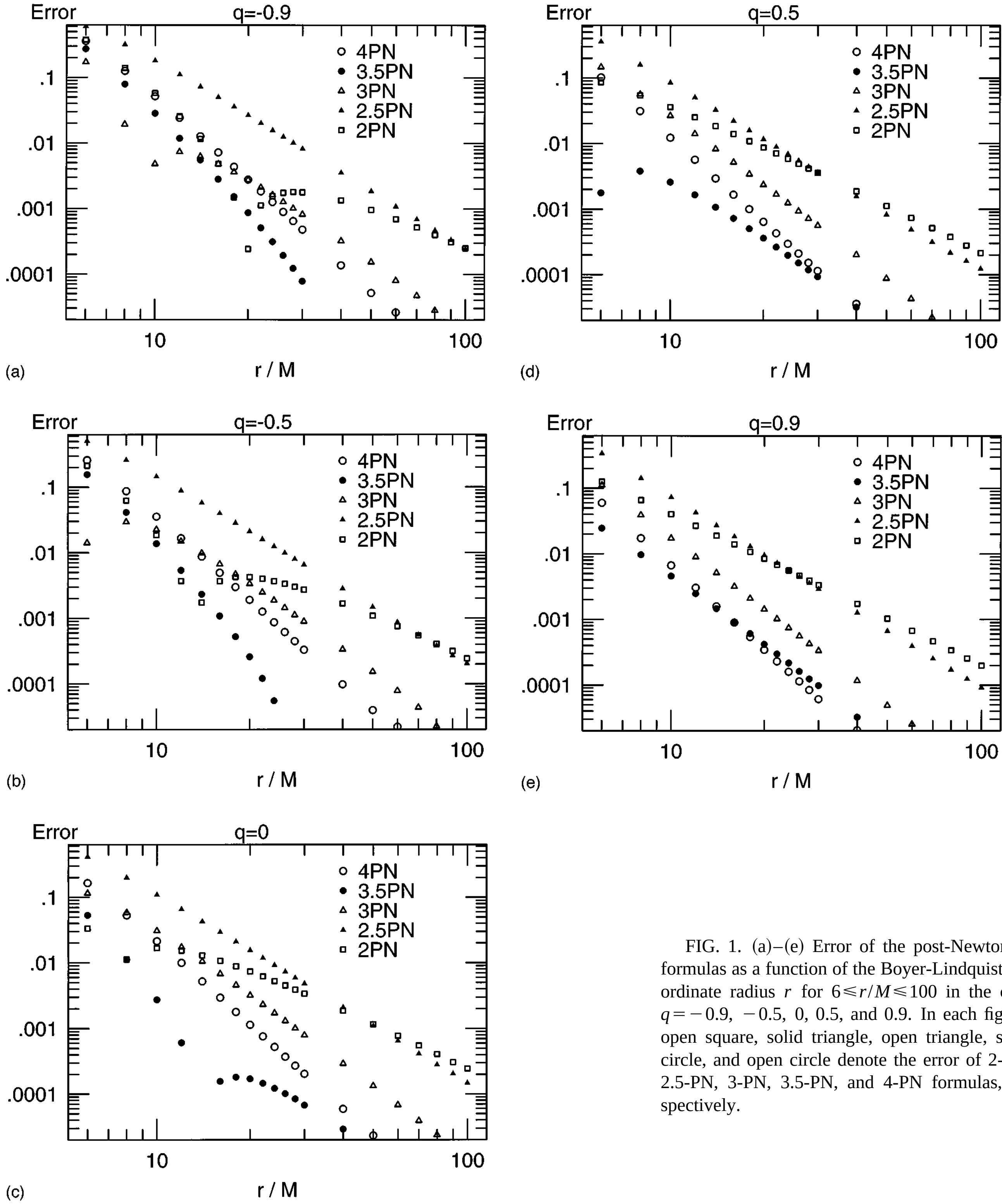

FIG. 1. (a)–(e) Error of the post-Newtonian formulas as a function of the Boyer-Lindquist coordinate radius r for $6 \leq r/M \leq 100$ in the case $q = -0.9$, -0.5, 0, 0.5, and 0.9. In each figure, open square, solid triangle, open triangle, solid circle, and open circle denote the error of 2-PN, 2.5-PN, 3-PN, 3.5-PN, and 4-PN formulas, respectively.

$M \gtrsim 5 M_\odot$, and the 3-PN formula is needed. The 3-PN formula seems adequate irrespective of q.

On the other hand, the situation is slightly different in the case when a neutron star of mass $\sim 1.4 M_\odot$ spirals into a larger black hole. In such a case, the number of the cycles of the gravitational waves is large compared with the equal mass case when the total mass is the same. Thus, it seems that we need at least the 3.5-PN formula for binaries of mass greater than $\sim 30 M_\odot$ to obtain the required accuracy. Also, for binaries of mass greater than $\sim 70 M_\odot$, we need higher post-Newtonian corrections beyond 4-PN order.

Binary systems of total mass greater than $\sim 100 M_\odot$ can be detected when r is smaller than $\sim 15M$. However, as mentioned in Eq. (3) above, the convergence property of the post-Newtonian expansion becomes bad for small orbital separations. In particular, for $q \sim -1$, the accuracy of the post-Newtonian expansion seems bad at $r \sim 15M$. Thus, it may not be appropriate to use the post-Newtonian approxi-

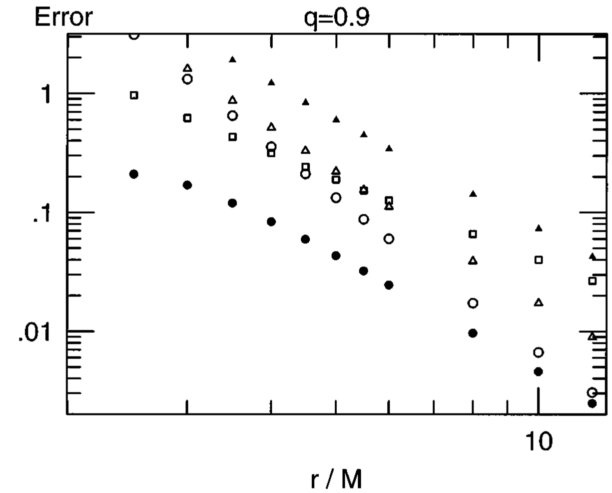

FIG. 2. Error of the post-Newtonian formula of $q=0.9$ for $2.5\leqslant r/M\leqslant 12$. The open square, solid triangle, open triangle, solid circle, and open circle denote the error of 2-PN, 2.5-PN, 3-PN, 3.5-PN, and 4-PN formulas, respectively.

mation for binaries of total mass $\sim 100 M_{\odot}$ with large mass ratio $\mu \ll M$. A more detailed investigations of the convergence of the post-Newtonian expansion will require the calculation to be carried beyond 4-PN order.

V. SUMMARY AND DISCUSSION

In this paper, we have performed a post-Newtonian expansion of gravitational waves from a particle in a circular orbit around a Kerr black hole. The orbit lies in the equatorial plane and the calculations are accurate to $O(v^8)$ beyond the quadrupole level. We have performed the post-Newtonian expansion of the Sasaki-Nakamura equation and obtained the Green function of the radial Teukolsky equation up to $O(\epsilon^2)$ using methods developed previously. Then we obtained all the necessary radial functions to the required accuracy. We have also calculated the spin-weighted spheroidal harmonics up to $O((a\omega)^2)$. The outgoing wave amplitude of the Teukolsky function and the gravitational wave luminosities were derived up to $O(v^8)$ beyond the quadrupole formula.

It is worth noting that in the formula for $\eta_{2,2}$ in Appendix G, there are terms such as $(-8/3)qv'^3$, $2q^2v'^4$, $(-8/3)q^3v'^7$, and $q^4v'^8$. In a previous paper [28], we pointed out that the term $2q^2v'^4$ can be explained in terms of the quadrupole formula as the contribution of the quadrupole moment of the Kerr black hole to the orbit of the test particle. A similar explanation is possible for $(-8/3)q^3v'^7$ and $q^4v'^8$. We can derive those terms by using the quadrupole formula $dE/dt=32/5\mu^2\hat{r}^4\Omega^6$, where $\hat{r}$ is the orbital radius of a test particle in de Donder coordinates. If multipole moments of the black hole exist, the orbital radius is changed due to the influence of those multipole moments (or if we fix the orbital radius, Ω is changed due to the multipole moments of black hole). We can calculate the leading order effect of the multipole moments to the orbital radius by using multipole expansion of the Kerr metric [Eq. (10.6) of Ref. [40]]. In this way, we find that the dominant effect of the multipole moments of a Kerr black hole to dE/dt can be expressed as [41]

$$\frac{dE}{dt}=\frac{32}{5}\left(\frac{\mu}{M}\right)^2 v'^{10}\left\{1-\frac{8}{3}S_1v'^3-2M_2v'^4+4S_3v'^7 +\left(-\frac{3}{2}M_2^2+\frac{5}{2}M_4\right)v'^8\right\}, \tag{5.1}$$

where M_ℓ and S_ℓ are mass and current multipole moments of a Kerr black hole given by $M_\ell+iS_\ell=M(ia)^\ell$. Now we can interpret the term $-12q^3v^7$ as the effect of the current octopole moment of a black hole and the term q^4v^8 as the effect of both the mass quadrupole moment and $\ell=4$ mass multipole moment of a black hole.

As for $\ell=2$ and $m=1$ mode, there are terms $-qv'^3/12$, $q^2v'^4/16$, $-7q^3v'^7/24$, and $q^4v'^8/16$. The terms $-qv'^3/12$ and $q^2v'^4/16$ can be explained as the correction to the radiative current quadrupole moment [12,42]. We expect that the terms $-7q^3v'^7/24$ and $q^4v'^8/16$ can also be derived simply in a similar way.

In Sec. IV, by comparing post-Newtonian formulas for dE/dt with numerical data, we indicated that the convergence of the post-Newtonian expansion seems bad when orbital radii of binaries become less than $\sim 15M$. This suggests that the post-Newtonian expansion may not be appropriate to construct theoretical templates for large mass ratio binaries where the total mass is greater than $\sim 100M_{\odot}$ because gravitational waves from such binaries enter the LIGO-VIRGO frequency band when $r \lesssim 15M$. Nevertheless, the higher order post-Newtonian terms gradually improve the accuracy of the templates. Hence, it is very natural to ask whether the post-Newtonian expansion is always appropriate or not, and if appropriate, up to what order do we need the post-Newtonian terms to construct accurate templates. Fortunately, it is possible to obtain the formulas for dE/dt which include post-Newtonian order terms beyond $O(v^8)$ by extending techniques developed in this paper. Extension of the present work up to the higher post-Newtonian order, beyond $O(v^8)$, is very important and that is our future work.

The analysis, in this paper, has been restricted to the case when a test particle moves in a circular orbit on the equatorial plane. However, as shown in a previous paper [28], inclination of the orbital plane from the equatorial plane will significantly affect the orbital phase evolution. Hence, the present work should be considered as a first step toward the complete calculation of the energy and angular momentum luminosities including the orbital inclination.

Finally, we comment on the effect of absorption of gravitational waves by the black hole event horizon which should be taken into account when we consider the orbital evolution of black hole binaries. According to Gal'tsov [43], the lowest order contribution of the gravitational wave absorption to dE/dt is given by

$$\frac{dE}{dt}=\left(\frac{dE}{dt}\right)_N\frac{v^5}{2}\left\{v^3(1+\sqrt{1-q^2})-\frac{q}{2}\right\}(1+3q^2). \tag{5.2}$$

Thus, the effect of absorption appears from $O(v^5)$ if $q\neq 0$. Although the coefficient is small compared with that of dE/dt for the outgoing wave even in the case $|q|\sim 1$, we need the expression for dE/dt due to the black hole absorption to obtain an accurate template up to $O(v^8)$. Therefore, to obtain the higher order post-Newtonian corrections to the black hole absorption is a problem for the future.

ACKNOWLEDGMENTS

H.T. thanks F. Ryan for useful comments and Kip Thorne for continuous encouragement. We thank P. R. Brady for carefully reading the manuscript. H.T. was supported by the Japan Society for the Promotion of Science and by NSF Grant No. AST-9417371 and NASA Grant No. NAGW-4268. This work was also supported in part by Japanese Grants-in-Aid for Scientific Research of the Ministry of Education, Science and Culture, No. 07740355 and No. 04234104.

APPENDIX A: THE FORMULAS OF *F* AND *U*

In this appendix we show the potential functions F and U of the SN equation (2.15). Details of the derivation are given in Ref. [36].

The function $F(r)$ is given by

$$F(r)=\frac{\eta_{,r}}{\eta}\frac{\Delta}{r^2+a^2}, \tag{A1}$$

where

$$\eta=c_0+c_1/r+c_2/r^2+c_3/r^3+c_4/r^4, \tag{A2}$$

with

$$c_0=-12i\omega M+\lambda(\lambda+2)-12a\omega(a\omega-m),$$

$$c_1=8ia[3a\omega-\lambda(a\omega-m)],$$

$$c_2=-24iaM(a\omega-m)+12a^2[1-2(a\omega-m)^2],$$

$$c_3=24ia^3(a\omega-m)-24Ma^2,$$

$$c_4=12a^4. \tag{A3}$$

The function $U(r)$ is given by

$$U(r)=\frac{\Delta U_1}{(r^2+a^2)^2}+G^2+\frac{\Delta G_{,r}}{r^2+a^2}-FG, \tag{A4}$$

where

$$G=-\frac{2(r-M)}{r^2+a^2}+\frac{r\Delta}{(r^2+a^2)^2},$$

$$U_1=V+\frac{\Delta^2}{\beta}\left[\left(2\alpha+\frac{\beta_{,r}}{\Delta}\right)_{,r}-\frac{\eta_{,r}}{\eta}\left(\alpha+\frac{\beta_{,r}}{\Delta}\right)\right],$$

$$\alpha=-i\frac{K\beta}{\Delta^2}+3iK_{,r}+\lambda+\frac{6\Delta}{r^2},$$

$$\beta=2\Delta\left(-iK+r-M-\frac{2\Delta}{r}\right). \tag{A5}$$

APPENDIX B: FUNCTIONS IN THE SOURCE TERM

In this appendix, we show the A's in Eq. (3.5):

$$A_{nn\,0}=\frac{-2}{\sqrt{2\pi}\Delta^2}C_{nn}\rho^{-2}\bar{\rho}^{-1}L_1^+\{\rho^{-4}L_2^+(\rho^3S)\},$$

$$A_{\bar{m}n\,0}=\frac{2}{\sqrt{\pi}\Delta}C_{\bar{m}n}\rho^{-3}\left[(L_2^+S)\left(\frac{iK}{\Delta}+\rho+\bar{\rho}\right)\right.$$
$$\left.-a\sin\theta S\frac{K}{\Delta}(\bar{\rho}-\rho)\right],$$

$$A_{\bar{m}\bar{m}\,0}=-\frac{1}{\sqrt{2\pi}}\rho^{-3}\bar{\rho}C_{\bar{m}\bar{m}}S\left[-i\left(\frac{K}{\Delta}\right)_{,r}-\frac{K^2}{\Delta^2}+2i\rho\frac{K}{\Delta}\right],$$

$$A_{\bar{m}n\,1}=\frac{2}{\sqrt{\pi}\Delta}\rho^{-3}C_{\bar{m}n}[L_2^+S+ia\sin\theta(\bar{\rho}-\rho)S],$$

$$A_{\bar{m}\bar{m}\,1}=-\frac{2}{\sqrt{2\pi}}\rho^{-3}\bar{\rho}C_{\bar{m}\bar{m}}S\left(i\frac{K}{\Delta}+\rho\right),$$

$$A_{\bar{m}\bar{m}\,2}=-\frac{1}{\sqrt{2\pi}}\rho^{-3}\bar{\rho}C_{\bar{m}\bar{m}}S,$$

where S denotes ${}_{-2}S^{a\omega}_{\ell m}$.

APPENDIX C: $Q^{(2)},Q^{(3)},Q^{(4)}$

$$
\begin{aligned}
Q^{(2)}=\Bigg[\Bigg(&-28imq-\frac{32imq}{\ell}+8i\ell mq+4i\ell^2mq-13q^2-\frac{6q^2}{\ell}-12\ell q^2-\ell^2q^2+6\ell^3q^2+2\ell^4q^2\\
&+8m^2q^2+\frac{32m^2q^2}{\ell^2}+\frac{8m^2q^2}{\ell}\Bigg)\frac{1}{z^4}+\Bigg(16mq+\frac{24mq}{\ell^2}+\frac{20mq}{\ell}-8\ell mq-4\ell^2mq-14iq^2-\frac{16iq^2}{\ell}+4i\ell q^2\\
&+2i\ell^2q^2+2i\lambda_1mq^2-\frac{4i\lambda_1mq^2}{\ell^2}+\frac{2i\lambda_1mq^2}{\ell}-4im^2q^2+\frac{56im^2q^2}{\ell^2}-\frac{4im^2q^2}{\ell}\Bigg)\frac{1}{z^3}\\
&+\Bigg(\frac{24imq}{\ell^2}+\frac{17q^2}{2}+\frac{10q^2}{\ell}-\frac{13\ell q^2}{4}-\frac{9\ell^2q^2}{4}-\frac{3\ell^3q^2}{4}-\frac{\ell^4q^2}{4}-\frac{\lambda_2q^2}{2}-\frac{3\ell\lambda_2q^2}{4}+\frac{\ell^2\lambda_2q^2}{4}+\frac{3\ell^3\lambda_2q^2}{4}\\
&+\frac{\ell^4\lambda_2q^2}{4}-2\lambda_1mq^2+\frac{4\lambda_1mq^2}{\ell^2}-\frac{2\lambda_1mq^2}{\ell}-\frac{24m^2q^2}{\ell^2}\Bigg)\frac{1}{z^2}\Bigg]\frac{1}{(\ell+1)^2(\ell^2+\ell-2)}\\
&+\Bigg[(-24i\lambda_0mq-4i\lambda_0^2mq+4i\lambda_0^3mq-12\lambda_0q^2-6\lambda_0^2q^2+24\lambda_0m^2q^2-4\lambda_0^2m^2q^2)\frac{1}{z^3}+(24\lambda_0mq-12i\lambda_0q^2\\
&-2i\lambda_0^2q^2+2i\lambda_0^3q^2+2i\lambda_0^2\lambda_1mq^2+24i\lambda_0m^2q^2)\frac{1}{z^2}\Bigg]\frac{1}{(\lambda_0+2)^2\lambda_0^2}\frac{d}{dz}-\frac{q^2}{4z^2}\frac{d^2}{dz^2},
\end{aligned}
$$

$$
\begin{aligned}
Q^{(3)}_{\ell=2}=\Bigg[\Bigg(&\frac{i}{2}mq+\frac{5q^2}{8}-\frac{5m^2q^2}{9}+\frac{11i}{24}mq^3-\frac{11i}{54}m^3q^3\Bigg)\frac{1}{z^4}+\Bigg(\frac{-(mq)}{24}+\frac{5i}{48}q^2-\frac{65i}{216}m^2q^2-\frac{mq^3}{2}+\frac{16m^3q^3}{81}\Bigg)\frac{1}{z^3}\\
&+\Bigg(\frac{i}{24}mq-\frac{q^2}{48}+\frac{17m^2q^2}{216}-\frac{65i}{378}mq^3+\frac{17i}{252}m^3q^3\Bigg)\frac{1}{z^2}\Bigg]\frac{d}{dz}+\Bigg[\Bigg(imq+\frac{7q^2}{4}-\frac{25m^2q^2}{18}+\frac{4i}{3}mq^3-\frac{29i}{54}m^3q^3\Bigg)\frac{1}{z^5}+\Bigg(\frac{-5mq}{12}\\
&+\frac{19i}{24}q^2-\frac{103i}{108}m^2q^2-\frac{19mq^3}{12}+\frac{101m^3q^3}{162}\Bigg)\frac{1}{z^4}+\Bigg(\frac{-q^2}{8}+\frac{41m^2q^2}{108}-\frac{127i}{189}mq^3+\frac{601i}{2268}m^3q^3\Bigg)\frac{1}{z^3}+\Bigg(\frac{-(mq)}{24}-\frac{i}{48}q^2\\
&+\frac{17i}{216}m^2q^2+\frac{\lambda_3q^3}{8}+\frac{65mq^3}{378}-\frac{17m^3q^3}{252}\Bigg)\frac{1}{z^2}\Bigg],
\end{aligned}
$$

$$
\begin{aligned}
Q^{(3)}_{\ell=3}=\Bigg[\Bigg(&\frac{i}{10}mq+\frac{q^2}{8}-\frac{7m^2q^2}{180}+\frac{i}{60}mq^3-\frac{7i}{540}m^3q^3\Bigg)\frac{1}{z^4}+\Bigg(\frac{-7mq}{600}-\frac{83i}{1200}q^2-\frac{7i}{1350}m^2q^2-\frac{23mq^3}{720}+\frac{19m^3q^3}{3240}\Bigg)\frac{1}{z^3}\\
&+\Bigg(\frac{i}{1200}mq-\frac{7q^2}{1200}+\frac{23m^2q^2}{5400}-\frac{7i}{2160}mq^3+\frac{11i}{12960}m^3q^3\Bigg)\frac{1}{z^2}\Bigg]\frac{d}{dz}\\
&+\Bigg[\Bigg(\frac{i}{5}mq+\frac{q^2}{2}-\frac{31m^2q^2}{180}+\frac{23i}{120}mq^3-\frac{5i}{108}m^3q^3\Bigg)\frac{1}{z^5}+\Bigg(\frac{-8mq}{75}+\frac{109i}{1200}q^2-\frac{79i}{1350}m^2q^2-\frac{25mq^3}{144}+\frac{31m^3q^3}{810}\Bigg)\frac{1}{z^4}\\
&+\Bigg(\frac{-13i}{1200}mq+\frac{19q^2}{300}+\frac{17m^2q^2}{1800}-\frac{19i}{540}mq^3+\frac{29i}{4320}m^3q^3\Bigg)\frac{1}{z^3}\\
&+\Bigg(\frac{-(mq)}{1200}-\frac{7i}{1200}q^2+\frac{23i}{5400}m^2q^2+\frac{\lambda_3q^3}{8}+\frac{7mq^3}{2160}-\frac{11m^3q^3}{12960}\Bigg)\frac{1}{z^2}\Bigg],
\end{aligned}
$$

$$
\begin{aligned}
Q^{(4)}_{\ell=2}=\Bigg[\Bigg(&\frac{-3q^2}{8}+\frac{3m^2q^2}{8}-\frac{23i}{24}mq^3+\frac{43i}{108}m^3q^3-\frac{5q^4}{32}+\frac{4m^2q^4}{9}-\frac{79m^4q^4}{648}\Bigg)\frac{1}{z^5}\\
&+\Bigg(\frac{-(mq)}{8}-\frac{3i}{32}q^2+\frac{5i}{72}m^2q^2+\frac{77mq^3}{96}-\frac{263m^3q^3}{648}-\frac{19i}{96}q^4+\frac{79i}{108}m^2q^4-\frac{137i}{648}m^4q^4\Bigg)\frac{1}{z^4}\\
&+\Bigg(\frac{-i}{96}mq-\frac{5q^2}{192}+\frac{73m^2q^2}{864}+\frac{13i}{108}mq^3-\frac{i}{9}m^3q^3+\frac{25q^4}{252}-\frac{109m^2q^4}{252}+\frac{680m^4q^4}{5103}\Bigg)\frac{1}{z^3}\\
&+\Bigg(\frac{-(mq)}{96}-\frac{i}{192}q^2+\frac{11i}{288}m^2q^2+\frac{5mq^3}{108}-\frac{m^3q^3}{3888}+\frac{7i}{432}q^4+\frac{i}{72}\lambda_3mq^4-\frac{1349i}{13608}m^2q^4+\frac{509i}{15309}m^4q^4\Bigg)\frac{1}{z^2}\Bigg]\frac{d}{dz}
\end{aligned}
$$

$$+\left[\left(\frac{-3q^2}{4}+\frac{7m^2q^2}{12}-\frac{47i}{24}mq^3+\frac{19i}{27}m^3q^3-\frac{7q^4}{16}+\frac{65m^2q^4}{72}-\frac{71m^4q^4}{324}\right)\frac{1}{z^6}+\left(\frac{-(mq)}{4}-\frac{7i}{16}q^2+\frac{3i}{8}m^2q^2+\frac{29mq^3}{12}\right.\right.$$
$$\left.-\frac{659m^3q^3}{648}-\frac{29i}{48}q^4+\frac{49i}{27}m^2q^4-\frac{53i}{108}m^4q^4\right)\frac{1}{z^5}$$
$$+\left(\frac{-5i}{48}mq+\frac{5q^2}{96}+\frac{41m^2q^2}{432}+\frac{2287i}{3024}mq^3-\frac{2039i}{4536}m^3q^3+\frac{101q^4}{288}-\frac{3991m^2q^4}{3024}+\frac{15853m^4q^4}{40824}\right)\frac{1}{z^4}$$
$$+\left(\frac{-i}{32}q^2+\frac{53i}{432}m^2q^2-\frac{2mq^3}{27}+\frac{431m^3q^3}{3888}+\frac{349i}{3024}q^4+\frac{i}{72}\lambda_3 mq^4-\frac{7235i}{13608}m^2q^4+\frac{2549i}{15309}m^4q^4\right)\frac{1}{z^3}$$
$$\left.+\left(\frac{-i}{96}mq+\frac{q^2}{192}-\frac{11m^2q^2}{288}+\frac{5i}{108}mq^3-\frac{i}{3888}m^3q^3-\frac{7q^4}{432}+\frac{\lambda_4q^4}{16}-\frac{\lambda_3mq^4}{72}+\frac{1349m^2q^4}{13608}-\frac{509m^4q^4}{15309}\right)\frac{1}{z^2}\right].$$

APPENDIX D: $X^{\rm in}_{\ell M\omega}$

(a) $\ell=2$:

$$X^{\rm in}_{2m\omega}=\frac{z^3}{15}-\frac{z^5}{210}+\frac{z^7}{7560}-\frac{z^9}{498960}+\frac{z^{11}}{51891840}+\epsilon\left(\frac{i}{30}mqz^2-\frac{13z^4}{630}-\frac{11i}{3780}mqz^4+\frac{z^6}{810}\right.$$
$$\left.+\frac{13i}{136080}mqz^6-\frac{53z^8}{1782000}-\frac{i}{598752}mqz^8\right)+\epsilon^2\left[\left(\frac{i}{60}mq+\frac{q^2}{120}-\frac{m^2q^2}{120}\right)z-\frac{mqz^2}{30}\right.$$
$$+z^3\left(\frac{26743}{110250}-\frac{433i}{22680}mq-\frac{3q^2}{3920}+\frac{79m^2q^2}{105840}-\frac{107\ln z}{3150}\right)+\left(\frac{mq}{270}-\frac{i}{7560}q^2+\frac{i}{34020}m^2q^2\right)z^4$$
$$\left.+z^5\left(-\frac{140953}{9261000}+\frac{17i}{12960}mq+\frac{11q^2}{423360}-\frac{19m^2q^2}{762048}+\frac{107\ln z}{44100}\right)\right]+\epsilon^3\left[\frac{-i}{90}mq-\frac{q^2}{30}+\frac{m^2q^2}{40}-\frac{19i}{720}mq^3+\frac{7i}{720}m^3q^3\right.$$
$$+\left(\frac{-(mq)}{36}-\frac{i}{120}q^2+\frac{mq^3}{36}-\frac{m^3q^3}{90}\right)z$$
$$\left.+z^2\left(\frac{319}{6300}+\frac{2074i}{18375}mq-\frac{41q^2}{5040}+\frac{m^2q^2}{648}+\frac{2887i}{211680}mq^3-\frac{1153i}{211680}m^3q^3-\frac{107i}{6300}mq\ln z\right)\right]$$
$$+\epsilon^4\left(\frac{-i}{120}mq+\frac{17m^2q^2}{1440}+\frac{11i}{480}mq^3-\frac{i}{480}m^3q^3+\frac{23q^4}{1920}-\frac{11m^2q^4}{576}+\frac{17m^4q^4}{5760}\right)\frac{1}{z}.$$

(b) $\ell=3$:

$$X^{\rm in}_{3m\omega}=\frac{z^4}{105}-\frac{z^6}{1890}+\frac{z^8}{83160}-\frac{z^{10}}{6486480}+\epsilon\left(\frac{-z^3}{126}+\frac{2i}{945}mqz^3-\frac{z^5}{630}-\frac{i}{7560}mqz^5+\frac{221z^7}{2494800}+\frac{i}{299376}mqz^7\right)$$
$$+\epsilon^2\left[\left(\frac{-i}{630}mq+\frac{q^2}{840}+\frac{m^2q^2}{7560}\right)z^2+\left(\frac{-mq}{540}-\frac{i}{1512}q^2+\frac{i}{6804}m^2q^2\right)z^3\right.$$
$$\left.+z^4\left(\frac{76369}{1852200}-\frac{299i}{453600}mq-\frac{q^2}{10800}-\frac{m^2q^2}{75600}-\frac{13\ln z}{4410}\right)\right]$$
$$+\epsilon^3\left(\left[\frac{-i}{2520}mq-\frac{q^2}{1008}+\frac{m^2q^2}{2160}-\frac{i}{7560}mq^3+\frac{i}{7560}m^3q^3\right)z\right].$$

(c) $\ell=4$:

$$X^{\rm in}_{4m\omega}=\frac{z^5}{945}-\frac{z^7}{20790}+\frac{z^9}{1081080}+\epsilon\left(\frac{-z^4}{630}+\frac{i}{7560}mqz^4-\frac{z^6}{9900}-\frac{i}{154000}mqz^6\right)$$
$$+\epsilon^2\left(\frac{z^3}{1764}-\frac{i}{5040}mqz^3+\frac{q^2z^3}{4410}+\frac{m^2q^2z^3}{52920}\right).$$

(d) $\ell=5$:

$$X^{\rm in}_{5m\omega}=\frac{z^6}{10395}-\frac{z^8}{270270}+\epsilon\left(\frac{-z^5}{4950}+\frac{2i}{259875}mqz^5\right).$$

(e) $\ell=6$:

$$X^{\rm in}_{6m\omega}=\frac{z^7}{135135}.$$

APPENDIX E: $R^{\rm in}_{\ell M\omega}$

(a) $\ell=2$:

$$\begin{aligned}\omega R^{\rm in}_{2m\omega}=&\frac{z^4}{30}+\frac{i}{45}z^5-\frac{11z^6}{1260}-\frac{i}{420}z^7+\frac{23z^8}{45360}+\frac{i}{11340}z^9-\frac{13z^{10}}{997920}-\frac{i}{598752}z^{11}+\frac{59z^{12}}{311351040}\\&+\epsilon\left(\frac{-z^3}{15}-\frac{i}{60}mqz^3-\frac{i}{60}z^4+\frac{mqz^4}{45}-\frac{41z^5}{3780}+\frac{277i}{22680}mqz^5-\frac{31i}{3780}z^6-\frac{7mqz^6}{1620}+\frac{17z^7}{5670}-\frac{61i}{54432}mqz^7+\frac{41i}{54432}z^8\right.\\&\left.+\frac{47mqz^8}{204120}-\frac{1579z^9}{10692000}+\frac{703i}{17962560}mqz^9\right)+\epsilon^2\left(\frac{z^2}{30}+\frac{i}{40}mqz^2+\frac{q^2z^2}{60}-\frac{m^2q^2z^2}{240}-\frac{i}{60}z^3-\frac{mqz^3}{30}+\frac{i}{90}q^2z^3\right.\\&-\frac{i}{120}m^2q^2z^3+\frac{7937z^4}{55125}-\frac{53i}{9072}mqz^4-\frac{101q^2z^4}{35280}+\frac{4213m^2q^2z^4}{635040}+\frac{4673i}{55125}z^5-\frac{13mqz^5}{2835}-\frac{5i}{63504}q^2z^5\\&\left.+\frac{3503i}{1143072}m^2q^2z^5-\frac{1665983z^6}{55566000}-\frac{1777i}{544320}mqz^6-\frac{q^2z^6}{5040}-\frac{643m^2q^2z^6}{653184}-\frac{107z^4\ln z}{6300}-\frac{107i}{9450}z^5\ln z+\frac{1177z^6\ln z}{264600}\right)\\&+\epsilon^3\left[\left(\frac{-i}{180}mq-\frac{q^2}{60}+\frac{m^2q^2}{240}-\frac{i}{144}mq^3+\frac{i}{1440}m^3q^3\right)z+\left(\frac{i}{120}+\frac{2mq}{135}-\frac{i}{360}q^2+\frac{19i}{1440}m^2q^2+\frac{11mq^3}{1080}-\frac{m^3q^3}{540}\right)z^2\right.\\&\left.+z^3\left(-\frac{10933}{49000}-\frac{578569i}{7938000}mq-\frac{677q^2}{52920}-\frac{529m^2q^2}{63504}+\frac{317i}{63504}mq^3-\frac{167i}{84672}m^3q^3+\frac{107\ln z}{3150}+\frac{107i}{12600}mq\ln z\right)\right]\\&+\epsilon^4\left(\frac{-i}{720}mq+\frac{m^2q^2}{2880}+\frac{i}{288}mq^3-\frac{i}{2880}m^3q^3+\frac{q^4}{480}-\frac{m^2q^4}{720}+\frac{m^4q^4}{11520}\right).\end{aligned}$$

(b) $\ell=3$:

$$\begin{aligned}\omega R^{\rm in}_{3m\omega}=&\frac{z^5}{630}+\frac{i}{1260}z^6-\frac{z^7}{3780}-\frac{i}{16200}z^8+\frac{29z^9}{2494800}+\frac{i}{554400}z^{10}-\frac{47z^{11}}{194594400}\\&+\epsilon\left(\frac{-z^4}{252}-\frac{i}{1890}mqz^4-\frac{i}{756}z^5+\frac{11mqz^5}{22680}+\frac{19i}{90720}mqz^6-\frac{i}{9450}z^7-\frac{mqz^7}{16200}+\frac{647z^8}{14968800}-\frac{247i}{17962560}mqz^8\right)\\&+\epsilon^2\left(\frac{z^3}{315}+\frac{i}{945}mqz^3+\frac{q^2z^3}{1260}-\frac{m^2q^2z^3}{15120}+\frac{i}{2520}z^4-\frac{17mqz^4}{15120}+\frac{i}{2160}q^2z^4-\frac{31i}{272160}m^2q^2z^4+\frac{81409z^5}{11113200}\right.\\&\left.-\frac{313i}{907200}mqz^5-\frac{41q^2z^5}{226800}+\frac{617m^2q^2z^5}{8164800}-\frac{13z^5\ln z}{26460}\right)+\epsilon^3\left(\frac{-z^2}{1260}-\frac{i}{1680}mqz^2-\frac{q^2z^2}{840}+\frac{m^2q^2z^2}{10080}-\frac{i}{5040}mq^3z^2\right).\end{aligned}$$

(c) $\ell=4$:

$$\begin{aligned}\omega R^{\rm in}_{4m\omega}=&\frac{z^6}{11340}+\frac{i}{28350}z^7-\frac{13z^8}{1247400}-\frac{i}{467775}z^9+\frac{71z^{10}}{194594400}\\&+\epsilon\left(\frac{-z^5}{3780}-\frac{i}{45360}mqz^5-\frac{11i}{136080}z^6+\frac{mqz^6}{64800}+\frac{131z^7}{18711000}+\frac{697i}{124740000}mqz^7\right)\\&+\epsilon^2\left(\frac{z^4}{3528}+\frac{i}{18144}mqz^4+\frac{q^2z^4}{21168}-\frac{m^2q^2z^4}{635040}\right).\end{aligned}$$

(d) $\ell=5$:

$$\omega R^{\rm in}_{5m\omega}=\frac{z^7}{207900}+\frac{i}{623700}z^8-\frac{z^9}{2316600}+\epsilon\left(\frac{-z^6}{59400}-\frac{i}{1039500}mqz^6\right).$$

(e) $\ell=6$:

$$\omega R^{\rm in}_{6m\omega}=\frac{z^8}{4054050}.$$

APPENDIX F: SPHEROIDAL HARMONICS

In this appendix, we describe the expansion of the spheroidal harmonics ${}_{-2}S^{a\omega}_{\ell m}$ at order $O((a\omega)^2)$.

The spheroidal harmonics of spin weight $s=-2$ obey the equation

$$\left[\frac{1}{\sin\theta}\frac{d}{d\theta}\left\{\sin\theta\frac{d}{d\theta}\right\}-a^2\omega^2\sin^2\theta-\frac{(m-2\cos\theta)^2}{\sin^2\theta}+4a\omega\cos\theta-2+2ma\omega+\lambda\right]{}_{-2}S^{a\omega}_{\ell m}=0. \tag{F1}$$

We expand ${}_{-2}S^{a\omega}_{\ell m}$ and λ as

$$\begin{aligned}{}_{-2}S^{a\omega}_{\ell m}&={}_{-2}P_{\ell m}+a\omega S^{(1)}_{\ell m}+(a\omega)^2S^{(2)}_{\ell m}+O((a\omega)^3),\\ \lambda&=\lambda_0(\ell)+a\omega\lambda_1(\ell)+a^2\omega^2\lambda_2(\ell)+O((a\omega)^3),\end{aligned} \tag{F2}$$

where ${}_{-2}P_{\ell m}$ are the spherical harmonics of spin weight $s=-2$ and λ_n are given in Sec. II B. Here we explicitly represent the ℓ dependence of λ_n for later convenience. We set the normalization of ${}_{-2}P_{\ell m}$ as

$$\int_0^\pi |{}_{-2}P_{\ell m}|^2\sin\theta d\theta=1. \tag{F3}$$

Inserting Eq. (6.2) into Eq. (6.1) and collecting the terms of order $(a\omega)^2$, we obtain

$$\mathcal{L}_0S^{(2)}_{\ell m}+\lambda_0(\ell)S^{(2)}_{\ell m}=-[4\cos\theta+2m+\lambda_1(\ell)]S^{(1)}_{\ell m}-[\lambda_2(\ell)-\sin^2\theta]{}_{-2}P_{\ell m}, \tag{F4}$$

where $\mathcal{L}_0$ is the operator for the spin-weighted spherical harmonics,

$$\mathcal{L}_0[{}_{-2}P_{\ell m}]\equiv\left[\frac{1}{\sin\theta}\frac{d}{d\theta}\left\{\sin\theta\frac{d}{d\theta}\right\}-\frac{(m-2\cos\theta)^2}{\sin^2\theta}-2\right]{}_{-2}P_{\ell m} \tag{F5}$$

$$=-\lambda_0\,{}_{-2}P_{\ell m}. \tag{F6}$$

By setting

$$\begin{aligned}S^{(1)}_{\ell m}&=\sum_{\ell'}c^{\ell'}_{\ell m}\,{}_{-2}P_{\ell' m},\\ S^{(2)}_{\ell m}&=\sum_{\ell'}d^{\ell'}_{\ell m}\,{}_{-2}P_{\ell' m},\end{aligned} \tag{F7}$$

we insert it into Eq. (6.4), multiply it by ${}_{-2}P_{\ell' m}$, and integrate it over θ. Then we have

$$\begin{aligned}d^{\ell'}_{\ell m}=\frac{1}{\lambda_0(\ell)-\lambda_0(\ell')}\Big[&-[2m+\lambda_1(\ell)](c^{\ell+1}_{\ell m}\delta_{\ell',\ell+1}+c^{\ell-1}_{\ell m}\delta_{\ell',\ell-1})-\delta_{\ell',\ell}\lambda_2(\ell)\\ &-4c^{\ell+1}_{\ell m}\int d(\cos\theta)\,{}_{-2}P_{\ell' m}\,{}_{-2}P_{\ell+1 m}\cos\theta-4c^{\ell-1}_{\ell m}\int d(\cos\theta)\,{}_{-2}P_{\ell' m}\,{}_{-2}P_{\ell-1 m}\cos\theta\\ &+\int d(\cos\theta)\,{}_{-2}P_{\ell' m}\,{}_{-2}P_{\ell m}\sin^2\theta\Big].\end{aligned} \tag{F8}$$

The integrals in this equation are given by [37,38]

$$\int d(\cos\theta)\,{}_{-2}P_{\ell' m}\,{}_{-2}P_{\ell m}\cos\theta=\sqrt{\frac{2\ell+1}{2\ell'+1}}\langle \ell,1,m,0|\ell',m\rangle\langle \ell,1,2,0|\ell',2\rangle,$$

$$\int d(\cos\theta)\,{}_{-2}P_{\ell' m}\,{}_{-2}P_{\ell m}\sin^2\theta=\frac{2}{3}\delta_{\ell',\ell}-\frac{2}{3}\sqrt{\frac{2\ell+1}{2\ell'+1}}\langle \ell,2,m,0|\ell',m\rangle\langle \ell,2,2,0|\ell',2\rangle,$$

where $\langle j_1,j_2,m_1,m_2|J,M\rangle$ is a Clebsch-Gordan coefficient. Then, for $\ell=2$ and 3, we obtain $d^{\ell'}_{\ell m}$ $(\ell'\neq\ell)$ which are given in Sec. II B. As for $d^{\ell}_{\ell m}$, we consider the normalization of ${}_{-2}P_{\ell m}$ [Eq. (2.2)]. Inserting Eq. (6.2) into Eq. (2.2), and using the orthogonality of ${}_{-2}P_{\ell m}$, we obtain

$$\begin{aligned}1=\int_0^\pi d\theta\sin\theta|{}_{-2}S_{\ell m}|^2&=\int_0^\pi d\theta\sin\theta\Bigg\{({}_{-2}P_{\ell m})^2+2a\omega\sum_{\ell'}c^{\ell'}_{\ell m}\,{}_{-2}P_{\ell' m}\,{}_{-2}P_{\ell m}+(a\omega)^2\sum_{\ell'\ell''}c^{\ell'}_{\ell m}c^{\ell''}_{\ell m}\,{}_{-2}P_{\ell' m}\,{}_{-2}P_{\ell'' m}\\&\quad+2(a\omega)^2\sum_{\ell'}d^{\ell'}_{\ell m}\,{}_{-2}P_{\ell' m}\,{}_{-2}P_{\ell m}+O((a\omega)^3)\Bigg\}\\&=1+(a\omega)^2\sum_{\ell'}(c^{\ell'}_{\ell m})^2+2(a\omega)^2d^{\ell}_{\ell m}+O((a\omega)^3).\end{aligned}$$

Then we have

$$d^{\ell}_{\ell}=-\frac{1}{2}\{(c^{\ell+1}_{\ell m})^2+(c^{\ell-1}_{\ell m})^2\}. \tag{F9}$$

APPENDIX G: THE EXPRESSION OF THE LUMINOSITY BY MEANS OF THE ORBITAL ANGULAR FREQUENCY

For the sake of convenience to calculate the orbital phase error, we describe the formula of gravitational wave luminosity by means of $v'\equiv(M\Omega)^{1/3}$. In this appendix, we define $\eta_{\ell,m}$ as

$$\left(\frac{dE}{dt}\right)_{\ell m}\equiv\frac{16}{5}\left(\frac{\mu}{M}\right)^2 v'^{10}\eta_{\ell,m}, \tag{G1}$$

$$\begin{aligned}\eta_{2,2}=&1-\frac{107v'^2}{21}+\left(4\pi-\frac{8q}{3}\right)v'^3+\left(\frac{4784}{1323}+2q^2\right)v'^4+\left(\frac{-428\pi}{21}+\frac{52q}{27}\right)v'^5+\left(\frac{99210071}{1091475}-\frac{1712\gamma}{105}+\frac{16\pi^2}{3}-\frac{32\pi q}{3}\right.\\&\left.-\frac{1817q^2}{567}-\frac{3424\ln2}{105}-\frac{1712\ln v'}{105}\right)v'^6+\left(\frac{19136\pi}{1323}+\frac{364856q}{11907}+8\pi q^2-\frac{8q^3}{3}\right)v'^7\\&+\left(-\frac{27956920577}{81265275}+\frac{183184\gamma}{2205}-\frac{1712\pi^2}{63}+\frac{208\pi q}{27}+\frac{105022q^2}{9261}+q^4+\frac{366368\ln2}{2205}+\frac{183184\ln v'}{2205}\right)v'^8,\end{aligned} \tag{G2}$$

$$\begin{aligned}\eta_{2,1}=&\frac{v'^2}{36}-\frac{qv'^3}{12}+\left(-\frac{17}{504}+\frac{q^2}{16}\right)v'^4+\left(\frac{\pi}{18}+\frac{215q}{9072}\right)v'^5+\left(-\frac{2215}{254016}-\frac{\pi q}{6}+\frac{313q^2}{1512}\right)v'^6\\&+\left(\frac{-17\pi}{252}-\frac{18127q}{190512}+\frac{\pi q^2}{8}-\frac{7q^3}{24}\right)v'^7+\left(\frac{15707221}{26195400}-\frac{107\gamma}{945}+\frac{\pi^2}{27}+\frac{215\pi q}{4536}\right.\\&\left.+\frac{44299q^2}{95256}+\frac{q^4}{16}-\frac{107\ln2}{945}-\frac{107\ln v'}{945}\right)v'^8,\end{aligned} \tag{G3}$$

$$\begin{aligned}\eta_{3,3}=&\frac{1215v'^2}{896}-\frac{1215v'^4}{112}+\left(\frac{3645\pi}{448}-\frac{1215q}{224}\right)v'^5+\left(\frac{243729}{9856}+\frac{3645q^2}{896}\right)v'^6+\left(\frac{-3645\pi}{56}+\frac{41229q}{1792}\right)v'^7\\&+\left(\frac{25037019729}{125565440}-\frac{47385\gamma}{1568}+\frac{3645\pi^2}{224}-\frac{3645\pi q}{112}-\frac{236925q^2}{14336}-\frac{47385\ln2}{1568}-\frac{47385\ln3}{1568}-\frac{47385\ln v'}{1568}\right)v'^8,\end{aligned} \tag{G4}$$

$$\eta_{3,2}=\frac{5v'^4}{63}-\frac{40qv'^5}{189}+\left(-\frac{193}{567}+\frac{80q^2}{567}\right)v'^6+\left(\frac{20\pi}{63}+\frac{982q}{1701}\right)v'^7+\left(\frac{86111}{280665}-\frac{160\pi q}{189}+\frac{80q^2}{189}\right)v'^8, \tag{G5}$$

$$\eta_{3,1}=\frac{v'^2}{8064}-\frac{v'^4}{1512}+\left(\frac{\pi}{4032}-\frac{25q}{18144}\right)v'^5+\left(\frac{437}{266112}+\frac{17q^2}{24192}\right)v'^6+\left(\frac{-\pi}{756}+\frac{2257q}{435456}\right)v'^7$$
$$+\left(-\frac{1137077}{50854003200}-\frac{13\gamma}{42336}+\frac{\pi^2}{6048}-\frac{25\pi q}{9072}+\frac{12863q^2}{3483648}-\frac{13\ln 2}{42336}-\frac{13\ln v'}{42336}\right)v'^8, \tag{G6}$$

$$\eta_{4,4}=\frac{1280v'^4}{567}-\frac{151808v'^6}{6237}+\left(\frac{10240\pi}{567}-\frac{20480q}{1701}\right)v'^7+\left(\frac{560069632}{6243237}+\frac{5120q^2}{567}\right)v'^8, \tag{G7}$$

$$\eta_{4,3}=\frac{729v'^6}{4480}-\frac{729qv'^7}{1792}+\left(-\frac{28431}{24640}+\frac{3645q^2}{14336}\right)v'^8, \tag{G8}$$

$$\eta_{4,2}=\frac{5v'^4}{3969}-\frac{437v'^6}{43659}+\left(\frac{20\pi}{3969}-\frac{170q}{11907}\right)v'^7+\left(\frac{7199152}{218513295}+\frac{200q^2}{27783}\right)v'^8, \tag{G9}$$

$$\eta_{4,1}=\frac{v'^6}{282240}-\frac{qv'^7}{112896}+\left(-\frac{101}{4656960}+\frac{5q^2}{903168}\right)v'^8, \tag{G10}$$

$$\eta_{5,5}=\frac{9765625v'^6}{2433024}-\frac{2568359375v'^8}{47443968}, \tag{G11}$$

$$\eta_{5,4}=\frac{4096v'^8}{13365}, \tag{G12}$$

$$\eta_{5,3}=\frac{2187v'^6}{450560}-\frac{150903v'^8}{2928640}, \tag{G13}$$

$$\eta_{5,2}=\frac{4v'^8}{40095}, \tag{G14}$$

$$\eta_{5,1}=\frac{v'^6}{127733760}-\frac{179v'^8}{2490808320}, \tag{G15}$$

$$\eta_{6,6}=\frac{26244v'^8}{3575}, \tag{G16}$$

$$\eta_{6,4}=\frac{131072v'^8}{9555975}, \tag{G17}$$

$$\eta_{6,2}=\frac{4v'^8}{5733585}. \tag{G18}$$

In total,

$$\left\langle\frac{dE}{dt}\right\rangle=\frac{32}{5}\left(\frac{\mu}{M}\right)v'^{10}\Bigg[1-\frac{1247v'^2}{336}+\left(4\pi-\frac{11q}{4}\right)v'^3+\left(-\frac{44711}{9072}+\frac{33q^2}{16}\right)v'^4+\left(\frac{-8191\pi}{672}-\frac{59q}{16}\right)v'^5$$
$$+\left(\frac{6643739519}{69854400}-\frac{1712\gamma}{105}+\frac{16\pi^2}{3}-\frac{65\pi q}{6}+\frac{611q^2}{504}-\frac{3424\ln 2}{105}-\frac{1712\ln v'}{105}\right)v'^6$$
$$+\left(\frac{-16285\pi}{504}+\frac{162035q}{3888}+\frac{65\pi q^2}{8}-\frac{71q^3}{24}\right)v'^7+\left(-\frac{323105549467}{3178375200}+\frac{232597\gamma}{4410}-\frac{1369\pi^2}{126}-\frac{359\pi q}{14}\right.$$
$$\left.+\frac{22667q^2}{4536}+\frac{17q^4}{16}+\frac{39931\ln 2}{294}-\frac{47385\ln 3}{1568}+\frac{232597\ln v'}{4410}\right)v'^8\Bigg]. \tag{G19}$$

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