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Gravitational waves from a spinning particle in circular orbits around a rotating black hole

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Using the Teukolsky and Sasaki-Nakamura formalisms for the perturbations around a Kerr black hole, we calculate the energy flux of gravitational waves induced by a *spinning* particle of mass μ and spin S moving in circular orbits near the equatorial plain of a rotating black hole of mass $M(\gg\mu)$ and spin Ma. The calculations are performed by using the recently developed post-Newtonian expansion technique of the Teukolsky equation. To evaluate the source terms of perturbations caused by a *spinning* particle, we use the equations of motion of a spinning particle derived by Papapetrou and the energy-momentum tensor of a spinning particle derived by Dixon. We present the post-Newtonian formula of the gravitational wave luminosity up to the order $(v/c)^5$ beyond the quadrupole formula including the linear order of particle spin. The results obtained in this paper will be an important guideline to the post-Newtonian calculation of the inspiral of two spinning compact objects. [S0556-2821(96)02016-4]

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I. INTRODUCTION

One of the most promising sources of gravitational waves for kilometer size laser-interferometric detectors such as the Laser Interferometric Gravitational Wave Observatory (LIGO) [1], VIRGO [2], and future laser-interferometric detectors in space such as the Laser Interferometer Space Antenna (LISA) [3] is the coalescing compact binary of neutron stars and/or black holes. Since it is a highly general relativistic event, detection of gravitational waves from those binaries will bring us very fruitful information about relativistic astrophysical objects if we know the physics of the final phase of the coalescence. Thus, there have been continuous efforts made by many authors to understand this phase [4,5].

Recently, it has been recognized that detection of the signal from a binary in the inspiraling phase is particularly very important because it can tell us a variety of parameters of the binary, i.e., mass, spin, etc [6,7]. Furthermore, it may provide some knowledge about the cosmological parameters [6,8]. In order to extract these important parameters from the data, we need an accurate theoretical template of the waveform. Especially, the accumulated phase of the emitted gravitational waves is very sensitive to the binary parameters. Thus, the rate of change in the frequency of the orbital rotation due to the radiation reaction of gravitational waves must be evaluated accurately.

The post-Newtonian expansion is the standard method to calculate the waveform of gravitational waves. Recently, the energy loss rate to second post-Newtonian (2PN) order beyond the quadrupole formula has been derived by Blanchet *et al.* [9] and to 2.5PN order by Blanchet [10] for a binary composed of nonspinning compact bodies. The leading order effect of spin, which appears at 1.5PN and 2PN orders, has been evaluated by Kidder, Will, and Wiseman [11] and Kidder [12].

However, difficulties and complications increase exponentially as one goes to higher orders with the standard post-Newtonian calculation technique. Hence it will be very useful if we have different approaches to the higher order approximation and if we are able to provide some guiding principle prior to the standard post-Newtonian calculations. The black hole perturbation approach is the only method known that is independent of the standard post-Newtonian approach and that can handle (not all but some important portion of) higher order post-Newtonian effects in a relatively straightforward way. In this approach, we consider a particle orbiting a black hole and assume that the mass of the black hole M is much larger than that of the particle μ.

The black hole perturbation approach is based on the perturbation equation derived by Teukolsky [13], which applies to a general rotating (Kerr) black hole. One of great advantages of this approach is that it takes full account of relativistic effects by construction and numerical methods can be easily implimented to treat very general orbits. Moreover, it has been shown that one can formulate an analytical post-Newtonian expansion scheme in this approach as well. Poisson first noticed this fact and calculated the energy flux in the case of a particle in circular orbits around a nonrotating black hole to 1.5PN order [15]. A technically important point was to deal with the Regge-Wheeler equation, which is equivalent to the Teukolsky equation but has a much nicer property than it. Along this line, Sasaki developed a systematic method to proceed to the higher orders in the case of a nonrotating black hole [17], and Tagoshi and Sasaki [18] gave the analytical expression for the energy flux up to 4PN order. Meanwhile Poisson calculated the case of a rotating black hole to 1.5PN order [16], but directly dealing with the Teukolsky equation. Hence it seemed formidable to go beyond this order. Then, extending the method of Ref. [17], a better method was developed by Shibata *et al.* [19] to treat the case of a rotating black hole and the energy flux up to 2.5PN order was calculated. This was made possible by using the Sasaki-Nakamura equation [14], which is a generalization of the Regge-Wheeler equation for a nonrotating black hole. Recently the calculation was extended to 4PN order by Tagoshi *et al.* [20].

In all of these previous papers, the small mass particle was assumed to be spinless. However, apart from the interest in its own right, for the purpose of providing a better theoretical template or at least a better guideline for higher order post-Newtonian calculations, it is desirable to take into account the spin of the particle. To incorporate this effect, we must know the energy-momentum tensor of a spinning particle as well as the equations of motion. Fortunately, we can find them in the literature. The equations of motion of a spinning particle were first derived by Papapetrou [21], and they were put into more elaborate form by Dixon [22] and Wald [23]. In particular, Wald [23] clarified all the conserved quantities along the general particle trajectory. On the other hand, Dixon [22] succeeded in giving the general form of the energy-momentum tensor of a particle with multipole moments, which of course includes that of a spinning particle as a limit. Hence, by using them, we can calculate the waveform and the energy flux of gravitational waves by a spinning particle orbiting a rotating black hole.

Here, a word of caution is appropriate. Usually, we regard the small mass particle to be a model of a neutron star or a black hole. However, if we regard the spinning small mass particle as a Kerr black hole with mass μ and spin angular momentum μS, it should have definite multipole moments ($\propto \mu S^{\ell}$). Since we neglect the contribution of these higher multipole moments in this paper, the particle in our treatment is not an adequate model for a (rapidly rotating) Kerr black hole. To incorporate the contribution of all higher multipole moments to represent a Kerr black hole is a future issue. Here, we concentrate on the leading order effect due to the spin of the small mass particle.

This paper is organized as follows. In Sec. II we briefly review the Teukolsky formalism. In Sec. III we discuss the equations of motion and the energy-momentum tensor of a spinning particle. In Sec. IV we solve the equations of motion to the linear order of the amplitude of spin. There we obtain a family of ''circular'' orbits which have vanishing radial velocity and stay close to the equatorial plane. In Sec. V we evaluate gravitational waves from the spinning particle in circular orbits and give the formula for the energy loss rate to 2.5PN order. In Sec. VI we consider the problem of the radiation reaction. There we show that the assumption that the orbit remains circular under a radiation reaction is consistent with the energy and angular momentum loss rates in the linear order of spin. Then we evaluate the rate of change in the orbital frequency under this assumption. In Sec. VII, a brief summary and discussion are given.

We use the units $G=c=1$ and the metric signature $(-,+,+,+)$. The round (square) brackets on the indices denote (anti) symmetrization, e.g.,

$$\Phi_{(\mu\nu)}=\tfrac{1}{2}(\Phi_{\mu\nu}+\Phi_{\nu\mu}), \quad \Phi_{[\mu\nu]}=\tfrac{1}{2}(\Phi_{\mu\nu}-\Phi_{\nu\mu}).$$

II. TEUKOLSKY FORMALISM

In this section we briefly review the Teukolsky formalism. For details, see, e.g., Ref. [24] and references cited therein. In the Teukolsky formalism, the waveform and the energy flux of gravitational waves are calculated from the fourth Newman-Penrose quantity [25], which is expanded as

$$\psi_4=(r-ia\cos\theta)^{-4}\int d\omega e^{-i\omega t}\sum_{\ell,m} e^{im\varphi}\frac{{}_{-2}S^{a\omega}_{\ell m}(\theta)}{\sqrt{2\pi}} \times R_{\ell m\omega}(r). \tag{2.1}$$

Here, ${}_{-2}S^{a\omega}_{\ell m}(\theta)$ is the spin-weighted spheroidal harmonics normalized by

$$\int_0^{\pi}|{}_{-2}S^{a\omega}_{\ell m}(\theta)|^2\sin\theta\, d\theta=1, \tag{2.2}$$

and its eigen value is λ. Then $R_{\ell m\omega}$ obeys the Teukolsky equation

$$\Delta^2\frac{d}{dr}\left(\frac{1}{\Delta}\frac{dR_{\ell m\omega}}{dr}\right)-V(r)R_{\ell m\omega}=T_{\ell m\omega}(r) \tag{2.3}$$

and

$$V(r)=-\frac{K^2+4i(r-M)K}{\Delta}+8i\omega r+\lambda, \tag{2.4}$$

where $\Delta=r^2-2Mr+a^2$ and $K=(r^2+a^2)\omega-ma$. The source term $T_{\ell m\omega}(r)$ is constructed from the energy-momentum tensor of the particle, and its explicit form is given later.

The solution of the Teukolsky equation at infinity $(r\to\infty)$ is expressed as

$$R_{\ell m\omega}(r)\to\frac{r^3e^{i\omega r^*}}{2i\omega B^{\rm in}_{\ell m\omega}}\int_{r_+}^{\infty}dr'\frac{T_{\ell m\omega}(r')R^{\rm in}_{\ell m\omega}(r')}{\Delta^2(r')} \equiv\tilde{Z}_{\ell m\omega}r^3e^{i\omega r^*}, \tag{2.5}$$

where $r_+=M+\sqrt{M^2-a^2}$ denotes the radius of the event horizon and $R^{\rm in}_{\ell m\omega}$ is the homogeneous solution which satisfies the ingoing-wave boundary condition at the horizon:

$$R^{\rm in}_{\ell m\omega}\to\begin{cases} D_{\ell m\omega}\Delta^2e^{-ikr^*}, & r^*\to-\infty,\\ r^3B^{\rm out}_{\ell m\omega}e^{i\omega r^*}+r^{-1}B^{\rm in}_{\ell m\omega}e^{-i\omega r^*}, & r^*\to+\infty,\end{cases} \tag{2.6}$$

where $k=\omega-ma/2Mr_+$ and r^* is the tortoise coordinate defined by

$$\frac{dr^*}{dr}=\frac{r^2+a^2}{\Delta}. \tag{2.7}$$

For definiteness, we fix the integration constant such that r^* is given explicitly by

$$r^*=\int\frac{dr^*}{dr}dr=r+\frac{2Mr_+}{r_+-r_-}\ln\frac{r-r_+}{2M}-\frac{2Mr_-}{r_+-r_-}\ln\frac{r-r_-}{2M}, \tag{2.8}$$

where $r_\pm=M\pm\sqrt{M^2-a^2}$.

Thus, in order to calculate gravitational waves emitted to infinity from a particle in circular orbits, we need to know the explicit form of the source term $T_{\ell m\omega}(r)$, which has support only at $r=r_0$ where r_0 is the orbital radius in the Boyer-Lindquist coordinate, the ingoing-wave Teukolsky function $R^{\rm in}_{\ell m\omega}(r)$ at $r=r_0$, and its incident amplitude $B^{\rm in}_{\ell m\omega}$ at infinity. We consider the expansion of these quantities in terms of a small parameter $v^2\equiv M/r_0$. Note that v is approximately equal to the orbital velocity, but not strictly equal to it in the case of $a\neq0$ or $S\neq0$. A systematic expansion method to calculate these necessary quantities has been developed in Refs. [19,20], by considering the Sasaki-Nakamura equation first and then transforming the result to Teukolsky equation.

In addition to these, we need to expand the spheroidal harmonics and their eigenvalues in powers of $a\omega$. Since $\omega=O(\Omega)$, where Ω is the orbital angular velocity of the particle, we have $a\omega=O(M\omega)=O(v^3)$. Thus the expansion in powers of $a\omega$ is also a part of the post-Newtonian expansion. Note also that the spin parameter of the black hole a does not have to be small but can be of order M.

The expressions of these quantities required to calculate the energy loss rate up to 2.5PN order are already obtained in Ref. [19]. We summarize the results, omitting all the derivations. The homogeneous solutions of the Teukolsky equation with the ingoing boundary condition for $\ell=2,3,4$ are

$$\omega R^{\rm in}_{2m\omega}=\frac{z^4}{30}+\frac{i}{45}z^5-\frac{11\,z^6}{1260}-\frac{i}{420}z^7+\frac{23\,z^8}{45\,360}+\frac{i}{11\,340}z^9 +\epsilon\left(\frac{-z^3}{15}-\frac{i}{60}mqz^3-\frac{i}{60}z^4+\frac{mqz^4}{45}-\frac{41\,z^5}{3780}+\frac{277\,i}{22\,680}mqz^5-\frac{31\,i}{3780}z^6-\frac{7\,mqz^6}{1620}\right) +\epsilon^2\left(\frac{z^2}{30}+\frac{i}{40}mqz^2+\frac{q^2z^2}{60}-\frac{m^2q^2z^2}{240}-\frac{i}{60}z^3-\frac{mqz^3}{30}+\frac{i}{90}q^2z^3-\frac{i}{120}m^2q^2z^3\right), \tag{2.9}$$

$$\omega R^{\rm in}_{3m\omega}=\frac{z^5}{630}+\frac{i}{1260}z^6-\frac{z^7}{3780}-\frac{i}{16\,200}z^8+\epsilon\left(\frac{-z^4}{252}-\frac{i}{1890}mqz^4-\frac{i}{756}z^5+\frac{11\,mqz^5}{22\,680}\right), \tag{2.10}$$

$$\omega R^{\rm in}_{4m\omega}=\frac{z^6}{11\,340}+\frac{iz^7}{28\,350}, \tag{2.11}$$

where $\epsilon:=2M\omega$, $z:=\omega r$, and $q:=a/M$. The incident amplitudes are

$$B^{\rm in}_{2m\omega}=\frac{i}{8\omega^2}\left\{1-\epsilon\frac{\pi}{2}+i\epsilon\left(\frac{5}{3}-\gamma-\ln 2\right)+\frac{mq}{18}\epsilon+O(\epsilon^2)\right\}, \tag{2.12}$$

$$B^{\rm in}_{3m\omega}=-\frac{1}{8\omega^2}\left\{1-\epsilon\frac{\pi}{2}+i\epsilon\left(\frac{13}{6}-\gamma-\ln 2\right)+\frac{mq}{72}\epsilon+O(\epsilon^2)\right\}, \tag{2.13}$$

$$B^{\rm in}_{4m\omega}=-\frac{i}{8\omega^2}\{1+O(\epsilon)\}. \tag{2.14}$$

III. SPINNING PARTICLE

To give the source term of the Teukolsky equation, we need to solve the motion of the spinning particle and also to give the expression of the energy-momentum tensor. In this section we give the necessary expressions, following Refs. [22,23,26].

Neglecting the effect of the higher multipole moments, the equations of motion of a spinning particle are given by

$$\frac{D}{d\tau}p^\mu(\tau)=-\tfrac{1}{2}R^\mu{}_{\nu\rho\sigma}(z(\tau))v^\nu(\tau)S^{\rho\sigma}(\tau),$$

$$\frac{D}{d\tau}S^{\mu\nu}(\tau)=2p^{[\mu}(\tau)v^{\nu]}(\tau), \tag{3.1}$$

where $v^\mu(\tau)=dz^\mu(\tau)/d\tau$, τ is a parameter which is not necessarily the proper time of the particle, and, as we will see later, the vector $p^\mu(\tau)$ and the antisymmetric tensor $S^{\mu\nu}(\tau)$ represent the linear and spin angular momenta of the particle, respectively. Here $D/d\tau$ denotes the covariant derivative along the particle trajectory.

We do not have the evolution equation for $v^\mu(\tau)$ yet. In order to determine $v^\mu(\tau)$, we need to impose a supplementary condition which determines the center of mass of the particle [22]:

$$S^{\mu\nu}(\tau)p_\nu(\tau)=0. \tag{3.2}$$

Then one can show that $p_\mu p^\mu=\text{const}$ and $S_{\mu\nu}S^{\mu\nu}=\text{const}$ along the particle trajectory [23]. Therefore, we may set

$$p^\mu=\mu u^\mu,\quad u_\mu u^\mu=-1,$$

$$S^{\mu\nu}=\epsilon^{\mu\nu}{}_{\rho\sigma}p^\rho S^\sigma,\quad p_\mu S^\mu=0,$$

$$S^2=S_\mu S^\mu=\frac{1}{2\mu^2}S_{\mu\nu}S^{\mu\nu}, \tag{3.3}$$

where μ is the mass of the particle, u^μ is the specific linear momentum, and S^μ is the specific spin vector with S its magnitude. Note that if we use S^μ instead of $S^{\mu\nu}$ in the equations of motion, the center-of-mass condition (3.2) will be replaced by the condition $p_\mu S^\mu=0$ (see Sec. IV).

Since the above equations of motion are invariant under reparametrization of the orbital parameter τ, we can fix τ to satisfy

$$u^{\mu}(\tau)v_{\mu}(\tau)=-1. \tag{3.4}$$

Then, from Eqs. (3.1), (3.2), and (3.4), $v^{\mu}(\tau)$ is given as [22]

$$v^{\mu}(\tau)-u^{\mu}(\tau)=\frac{1}{2}\left(\mu^2+\frac{1}{4}R_{\chi\xi\zeta\eta}(z(\tau))S^{\chi\xi}(\tau)S^{\zeta\eta}(\tau)\right)^{-1}\times S^{\mu\nu}(\tau)R_{\nu\rho\sigma\kappa}z^{(\tau)}u^{\rho}(\tau)S^{\sigma\kappa}(\tau). \tag{3.5}$$

With this equation, the equations of motion (3.1) completely determine the evolution of the orbit and the spin.

As for the energy-momentum tensor, Dixon [22] gives it in terms of the Dirac δ function on the tangent space at $x^{\mu}=z^{\mu}(\tau)$. For later convenience, in this paper we use an equivalent but alternative form of the energy-momentum tensor, given in terms of the Dirac δ function on coordinate space [26]:

$$T^{\alpha\beta}(x)=\int d\tau\left\{p^{(\alpha}(x,\tau)v^{\beta)}(x,\tau)\frac{\delta^{(4)}(x-z(\tau))}{\sqrt{-g}}-\nabla_{\gamma}\left(S^{\gamma(\alpha}(x,\tau)v^{\beta)}(x,\tau)\frac{\delta^{(4)}(x-z(\tau))}{\sqrt{-g}}\right)\right\}, \tag{3.6}$$

where $v^{\alpha}(x,\tau)$, $p^{\alpha}(x,\tau)$, and $S^{\alpha\beta}(x,\tau)$ are bitensors which are spacetime extensions of $v^{\mu}(\tau)$, $p^{\mu}(\tau)$, and $S^{\mu\nu}(\tau)$, which are defined only along the world line,[1] $x^{\mu}=z^{\mu}(\tau)$. To define $v^{\alpha}(x,z(\tau))$, $p^{\alpha}(x,z(\tau))$, and $S^{\alpha\beta}(x,z(\tau))$ we introduce a bitensor $\overline{g^{\alpha}_{\mu}}(x,z)$ which satisfies

$$\lim_{x\to z}\overline{g^{\alpha}_{\mu}}(x,z(\tau))=\delta^{\alpha}_{\mu},$$

$$\lim_{x\to z}\nabla_{\beta}\overline{g^{\alpha}_{\mu}}(x,z(\tau))=0. \tag{3.7}$$

For the present purpose, further specification of $\overline{g^{\alpha}_{\mu}}(x,z)$ is not necessary. Using this bi-tensor $\overline{g^{\alpha}_{\mu}}(x,z)$, we define $p^{\alpha}(x,\tau)$, $v^{\alpha}(x,\tau)$, and $S^{\alpha\beta}(x,\tau)$ as

$$p^{\alpha}(x,\tau)=\overline{g^{\alpha}_{\mu}}(x,z(\tau))p^{\mu}(\tau),$$

$$v^{\alpha}(x,\tau)=\overline{g^{\alpha}_{\mu}}(x,z(\tau))v^{\mu}(\tau),$$

$$S^{\alpha\beta}(x,\tau)=\overline{g^{\alpha}_{\mu}}(x,z(\tau))\overline{g^{\beta}_{\nu}}((x,z(\tau))S^{\mu\nu}(\tau). \tag{3.8}$$

It is easy to see that the divergence-free condition of this energy-momentum tensor gives the equations of motion (3.1). Noting the relations

$$[\nabla_{\beta}\overline{g^{\alpha}_{\mu}}(x,z(\tau))]\delta^{(4)}(x-z(\tau))=0,$$

$$v^{\alpha}(x)\nabla_{\alpha}\left(\frac{\delta^{(4)}(x-z(\tau))}{\sqrt{-g}}\right)=-\frac{d}{d\tau}\left(\frac{\delta^{(4)}(x-z(\tau))}{\sqrt{-g}}\right), \tag{3.9}$$

the divergence of Eq. (3.6) becomes

$$\nabla_{\beta}T^{\alpha\beta}(x)=\int d\tau\overline{g^{\alpha}_{\mu}}(x,z(\tau))\frac{\delta^{(4)}(x-z(\tau))}{\sqrt{-g}}\times\left(\frac{D}{d\tau}p^{\mu}(\tau)+\frac{1}{2}R^{\mu}{}_{\nu\sigma\kappa}(z(\tau))v^{\nu}(\tau)S^{\sigma\kappa}(\tau)\right)+\frac{1}{2}\int d\tau\nabla_{\beta}\left(\overline{g^{\alpha}_{\mu}}(x,z(\tau))\overline{g^{\beta}_{\nu}}(x,z(\tau))\times\frac{\delta^{(4)}(x-z(\tau))}{\sqrt{-g}}\right)\left(\frac{D}{d\tau}S^{\mu\nu}(\tau)-2p^{[\mu}(\tau)v^{\nu]}(\tau)\right). \tag{3.10}$$

Since the first and second terms on the right-hand side must vanish separately, we obtain the equations of motion (3.1).

In order to clarify the meaning of p^{μ} and $S^{\mu\nu}$, we consider the volume integral of this energy-momentum tensor such as $\int_{\Sigma(\tau_0)}\overline{g^{\mu}_{\alpha}}T^{\alpha\beta}d\Sigma_{\beta}$, where we take the surface $\Sigma(\tau_0)$ to be perpendicular to $u^{\alpha}(\tau_0)$. It is convenient to introduce a scalar function $\tau(x)$, which determines the surface $\Sigma(\tau_0)$ by the equation $\tau(x)=\tau_0$, and $\partial\tau/\partial x^{\beta}=-u_{\beta}$ at $x=z(\tau_0)$. Then we have

$$\int_{\Sigma(\tau_0)}\overline{g^{\mu}_{\alpha}}T^{\alpha\beta}d\Sigma_{\beta}=\int d^4x\sqrt{-g}\frac{\partial\tau}{\partial x^{\beta}}\delta(\tau(x)-\tau_0)\overline{g^{\mu}_{\alpha}}T^{\alpha\beta}(x)=\int d\tau'\left\{\delta(\tau'-\tau_0)\left[p^{\mu}+p^{[\mu}v^{\nu]}u_{\nu}-\frac{1}{2}\frac{Du_{\nu}}{d\tau}S^{\nu\mu}\right]\right\}=p^{\mu}(\tau_0), \tag{3.11}$$

where we used the center-of-mass condition and the equation of motion for $S^{\mu\nu}$. We clearly see that p^{μ} indeed represents the linear momentum of the particle.

In order to clarify the meaning of $S^{\mu\nu}$, following Dixon [22], we introduce the relative position vector

$$X^{\mu}:=-g^{\mu\nu}\partial_{\nu}\sigma(x,z), \tag{3.12}$$

where $\sigma(x,z)$ is the geodetic interval between z and x defined by using the parametric form of a geodesic $y(u)$ joining $z=y(0)$ and $x=y(1)$ as

$$\sigma(x,z):=\frac{1}{2}\int_0^1 g_{\alpha\beta}\frac{dy^{\alpha}}{du}\frac{dy^{\beta}}{du}du. \tag{3.13}$$

Then noting the relations

$$\lim_{x\to z}X^{\mu}=0,\quad \lim_{x\to z}X^{\mu}{}_{\beta}=\delta^{\mu}_{\beta}, \tag{3.14}$$

it is easy to see that

[1]In the rest of this section, we use $\mu,\nu,\sigma,\ldots$ as the tensor indices associated with the world line $z(\tau)$ and $\alpha,\beta,\gamma,\ldots$ as those with a field point x, and suppress the coordinate indices of $z(\tau)$ and x for notational simplicity.

$$S^{\mu\nu}=2\int_{\Sigma_{\tau_0}} X^{[\mu}\overline{g^{\nu]}_{\alpha}}T^{\alpha\beta}d\Sigma_{\beta}. \tag{3.15}$$

Now that the meaning of $S^{\mu\nu}$ is manifest. From the above equation, it is also easy to see that the center-of-mass condition (3.2) is the generalization of the Newtonian counterpart,

$$\int d^3x\rho(x)x^i=0, \tag{3.16}$$

where ρ is the matter density.

Before closing this section, we mention several conserved quantities of the present system. We have already noted that $p_\mu p^\mu=-\mu^2$ and $S_\mu S^\mu=S^2$ are constant along the particle trajectory on an arbitrary spacetime. There will be an additional conserved quantity if the spacetime admits a Killing vector field ξ_μ:

$$\xi_{(\mu;\nu)}=0. \tag{3.17}$$

Namely, the quantity

$$Q_\xi:=p^\mu\xi_\mu-\tfrac{1}{2}S^{\mu\nu}\xi_{\mu;\nu} \tag{3.18}$$

is conserved along the particle trajectory [22]. It is easy to verify that Q_ξ is conserved by directly using the equations of motion.

IV. CIRCULAR ORBITS

Let us consider ''circular'' orbits in Kerr spacetime with a fixed Boyer-Lindquist radial coordinate, $r=r_0$. We consider a class of orbits that would stay on the equatorial plane if the particle were spinless. Hence we assume that $\tilde{\theta}:=\theta-\pi/2=O(S/M)\ll 1$. Under this assumption, we write down the equations of motion and solve them up to linear order in S. In the Appendix we give a further analysis in the case in which the spin vector (see below) is parallel or antiparallel to the rotation axis of the black hole.

In order to find a solution representing a circular orbit, it is convenient to introduce the tetrad frame defined by

$$e^0_\mu=\left(\sqrt{\frac{\Delta}{\Sigma}},0,0,-a\sin^2\theta\sqrt{\frac{\Delta}{\Sigma}}\right),$$

$$e^1_\mu=\left(0,\sqrt{\frac{\Sigma}{\Delta}},0,0\right),$$

$$e^2_\mu=(0,0,\sqrt{\Sigma},0),$$

$$e^3_\mu=\left(-\frac{a}{\sqrt{\Sigma}}\sin\theta,0,0,\frac{r^2+a^2}{\sqrt{\Sigma}}\sin\theta\right), \tag{4.1}$$

where $\Sigma=r^2+a^2\cos^2\theta$ and $e^a_\mu=(e^a_t,e^a_r,e^a_\theta,e^a_\varphi)$ for $a=0-3$. Hereafter, we use Latin letters to denote the tetrad indices.

For convenience, we introduce $\omega_1-\omega_6$ to represent the tetrad components of the spin coefficients near the equatorial plane:

$$\omega_{01}{}^0=\omega_{00}{}^1=\omega_1+O(\tilde{\theta}^2),\quad \omega_1=\frac{a^2-Mr}{r^2\Delta^{1/2}},$$

$$\omega_{31}{}^0=\omega_{30}{}^1=\omega_{13}{}^0=\omega_{10}{}^3=\omega_{03}{}^1=-\omega_{01}{}^3=\omega_2+O(\tilde{\theta}^2),\quad \omega_2:=\frac{a}{r^2},$$

$$\omega_{22}{}^1=-\omega_{21}{}^2=\omega_{33}{}^1=-\omega_{31}{}^3=\omega_3+O(\tilde{\theta}^2),\quad \omega_3:=\frac{\Delta^{1/2}}{r^2},$$

$$\omega_{02}{}^0=\omega_{00}{}^2=\omega_{12}{}^1=-\omega_{11}{}^2=\tilde{\theta}\omega_4+O(\tilde{\theta}^2),\quad \omega_4:=-\frac{a^2}{r^3},$$

$$\omega_{32}{}^0=\omega_{30}{}^2=-\omega_{23}{}^0=-\omega_{20}{}^3=\omega_{03}{}^2=-\omega_{02}{}^3=\tilde{\theta}\omega_5+O(\tilde{\theta}^2),\quad \omega_5:=-\frac{a\Delta^{1/2}}{r^3},$$

$$\omega_{33}{}^2=-\omega_{32}{}^3=\tilde{\theta}\omega_6+O(\tilde{\theta}^2),\quad \omega_6:=-\frac{(r^2+a^2)}{r^3}, \tag{4.2}$$

where $\omega_{ab}{}^{c}=e^{\mu}_{a}e^{\nu}_{b}e^{c}_{\nu;\mu}$. Since the following relation holds

$$e^{a}_{\mu}\frac{D}{d\tau}f^{\mu}=\frac{d}{d\tau}f^{a}-\omega_{bc}{}^{a}v^{b}f^{c},$$

the tetrad components of $Df^{\mu}/d\tau$ along a circular orbit are given explicitly as

$$e^{0}_{\mu}\frac{D}{d\tau}f^{\mu}=\dot{f}^{0}-(Af^{1}+\tilde{\theta}Cf^{2})+O(\tilde{\theta}^{2}),$$

$$e^{1}_{\mu}\frac{D}{d\tau}f^{\mu}=\dot{f}^{1}-(Af^{0}+Bf^{3}+Ef^{2})+O(\tilde{\theta}^{2}),$$

$$e^{2}_{\mu}\frac{D}{d\tau}f^{\mu}=\dot{f}^{2}-(\tilde{\theta}Cf^{0}+\tilde{\theta}Df^{3}-Ef^{1})+O(\tilde{\theta}^{2}),$$

$$e^{3}_{\mu}\frac{D}{d\tau}f^{\mu}=\dot{f}^{3}-(-Bf^{1}-\tilde{\theta}Df^{2})+O(\tilde{\theta}^{2}), \qquad (4.3)$$

where A, B, C, D, and E are defined by[2]

$$A:=\omega_{1}v^{0}+\omega_{2}v^{3},$$

$$B:=\omega_{2}v^{0}+\omega_{3}v^{3},$$

$$C:=\omega_{4}v^{0}+\omega_{5}v^{3},$$

$$D:=\omega_{5}v^{0}+\omega_{6}v^{3},$$

$$E:=\omega_{3}v^{2}, \qquad (4.4)$$

and we have assumed that $v^{1}=0$ and $v^{2}=O(\tilde{\theta})$.

Now we rewrite the equations of motion, changing the spin variable. We replace the spin tensor with the unit spin vector ζ^{a}, which is defined by

$$\zeta^{a}:=\frac{S^{a}}{S}=-\frac{1}{2\mu S}\epsilon^{a}{}_{bcd}u^{b}S^{cd} \qquad (4.5)$$

or, equivalently, by

$$S^{ab}=\mu S\epsilon^{ab}{}_{cd}u^{c}\zeta^{d}, \qquad (4.6)$$

where ϵ_{abcd} is the completely antisymmetric symbol with the convention of $\epsilon_{0123}=1$. As noted in the previous section, if we use the spin vector as an independent variable, the center-of-mass condition is automatically satisfied, while it becomes necessary to impose another supplementary condition

$$\zeta^{a}u_{a}=0. \qquad (4.7)$$

Then the equations of motion reduce to

$$\frac{du^{a}}{d\tau}=\omega_{bc}{}^{a}v^{b}u^{c}-SR^{a},$$

$$\frac{d\zeta^{a}}{d\tau}=\omega_{bc}{}^{a}v^{b}\zeta^{c}-Su^{a}\zeta^{b}R_{b}, \qquad (4.8)$$

where

$$R^{a}:=R^{*a}{}_{bcd}v^{b}u^{c}\zeta^{d}=\frac{1}{2\mu S}R^{a}{}_{bcd}v^{b}S^{cd} \qquad (4.9)$$

and $R^{*}_{abcd}=\frac{1}{2}R_{abef}\epsilon^{ef}{}_{cd}$ is the right dual of the Riemann tensor. It will be convenient to write explicitly the tetrad components of R^{*}_{abcd} Since we only need R^{*}_{abcd} at $O(\tilde{\theta}^{0})$, the nonvanishing components of R^{*}_{abcd} are given by

$$-\frac{1}{2}R^{*}_{0123}=-R^{*}_{0213}=R^{*}_{0312}=R^{*}_{1203}=-R^{*}_{1302}=-\frac{1}{2}R^{*}_{2301}$$

$$=-\frac{M}{r^{3}}+O(\tilde{\theta}^{2}). \qquad (4.10)$$

Although we do not need them, we note that the following components are not identically zero but are of $O(\tilde{\theta})$:

$$R^{*}_{1212},\ R^{*}_{1313},\ R^{*}_{1010},\ R^{*}_{2323},\ R^{*}_{2020},\ \text{and}\ R^{*}_{3030}.$$

A. Lowest order in *S*

We first solve the equations of motion for a circular orbit at $r=r_{0}$ at the lowest order in S. For notational simplicity, we omit the suffix 0 of r_{0} in the following. For the class of orbits we have assumed, we have $v^{1}=0$ and $v^{2}=O(\tilde{\theta})$.Then the nontrivial equations are

$$\frac{d}{d\tau}v^{1}=Av^{0}+Bv^{3}=0, \qquad (4.11)$$

$$\frac{d}{d\tau}\zeta^{2}=0,\quad \frac{d}{d\tau}\begin{pmatrix}\zeta^{0}\\ \zeta^{1}\\ \zeta^{3}\end{pmatrix}=\begin{pmatrix}0 & A & 0\\ A & 0 & B\\ 0 & -B & 0\end{pmatrix}\begin{pmatrix}\zeta^{0}\\ \zeta^{1}\\ \zeta^{3}\end{pmatrix}. \qquad (4.12)$$

Equation (4.11) determines the rotation velocity of the orbital motion. By setting $x:=v^{3}/v^{0}$, we obtain the equation

$$\omega_{1}+2\omega_{2}x+\omega_{3}x^{2}=0, \qquad (4.13)$$

which is solved to give

$$x=\frac{\pm\sqrt{Mr}-a}{\sqrt{\Delta}}. \qquad (4.14)$$

The upper (lower) sign corresponds to the case that v^{3} is positive (negative). Then, with the aid of the normalization condition of the four-momentum, $v^{\mu}v_{\mu}=-1+O(S^{2})$, we find

$$v^{0}=\frac{1}{\sqrt{1-x^{2}}},\quad v^{3}=\frac{x}{\sqrt{1-x^{2}}}. \qquad (4.15)$$

[2]The symbols A–E used here to define the auxiliary variable are applicable only in this section, and not to be confused with quantities defined with the same symbols, such as E for energy, in the later sections.

Note that, in this case, the orbital angular frequency Ω is given by a well-known formula

$$\Omega=\frac{\pm\sqrt{M}}{r^{3/2}\pm\sqrt{M}a}. \tag{4.16}$$

On the other hand, the equations of spin (4.12) are solved to give

$$\zeta^2=-\zeta_\perp,\begin{pmatrix}\zeta^0\\ \zeta^1\\ \zeta^3\end{pmatrix}=\zeta_\parallel\begin{pmatrix}\alpha\sin(\phi+c_1)+\beta c_2\\ \cos(\phi+c_1)\\ -\beta\sin(\phi+c_1)-\alpha c_2\end{pmatrix}, \tag{4.17}$$

where $\zeta_\perp$, $\zeta_\parallel$, c_1, and c_2 are constants and

$$\alpha=\frac{A}{\sqrt{B^2-A^2}}=\mp v^3,\quad \beta=\frac{B}{\sqrt{B^2-A^2}}=\pm v^0,$$

$$\phi=\Omega_p\tau,\quad \Omega_p=\sqrt{B^2-A^2}=\sqrt{\frac{M}{r^3}}. \tag{4.18}$$

The supplementary condition $v^a\zeta_a=0$ requires that $c_2=0$. The condition $\zeta_a\zeta^a=1$ implies $\zeta_\perp^2+\zeta_\parallel^2=1$. Further, since the origin of the time τ can be chosen arbitrarily, we set $c_1=0$. Thus, we obtain

$$\zeta^2=-\zeta_\perp,\quad \begin{pmatrix}\zeta^0\\ \zeta^1\\ \zeta^3\end{pmatrix}=\zeta_\parallel\begin{pmatrix}\alpha\sin\phi\\ \cos\phi\\ -\beta\sin\phi\end{pmatrix}. \tag{4.19}$$

Here, we should note that $\Omega_p\neq\Omega$ in general if $a\neq 0$ or $S\neq 0$ (see below).

B. Next order

Having obtained the leading order solution with respect to S, we now turn to the equations of motion up to the linear order in S. We assume that the spin vector components are expressed in the same form as were in the leading order but consider corrections to the coefficients α, β, and Ω_p of order S. As long as we are working only up to linear order in S, Eq. (3.5) tells us that v^a can be identified with u^a. In order to write down the equations of motion up to linear order in S, we need the explicit form of R^a, which can be evaluated by using the knowledge of the lowest order solution as

$$R^0=R^3=O(\widetilde{\theta}),$$

$$R^1=3\frac{M}{r^3}v^0v^3\zeta^2+O(\widetilde{\theta}),$$

$$R^2=3\frac{M}{r^3}v^0v^3\zeta^1+O(\widetilde{\theta}). \tag{4.20}$$

First we consider the orbital equations of motion. With the assumption that $v^1=0$ and $v^2=O(\widetilde{\theta})$, the nontrivial equations of the orbital motion are

$$\dot{v}^1=Av^0+Bv^3-SR^1=0, \tag{4.21}$$

$$\dot{v}^2=(Cv^0+Dv^3)\widetilde{\theta}-SR^2. \tag{4.22}$$

The first equation gives the rotation velocity as before, while the second equation determines the motion in the θ direction.

Again, using the variable $x=v^3/v^0$, Eq. (4.21) is rewritten as

$$\omega_1+2\omega_2x+\omega_3x^2+3\frac{S_\perp M}{r^3}x=0, \tag{4.23}$$

where $S_\perp:=S\zeta_\perp$. The solution of this equation is

$$x=\left(\frac{\pm\sqrt{Mr}-a}{\sqrt{\Delta}}\right)\left(1\mp\frac{3S_\perp\sqrt{M}}{2r^{3/2}}\right)+O(S^2). \tag{4.24}$$

Using the relations (4.15), it immediately gives v^0 and v^3. From the definition of the tetrad, we have the relations

$$v^0=\sqrt{\frac{\Delta}{\Sigma}}\left[\frac{dt}{d\tau}-a\sin^2\theta\frac{d\varphi}{d\tau}\right],$$

$$v^3=\frac{\sin\theta}{\sqrt{\Sigma}}\left[-a\frac{dt}{d\tau}+(r^2+a^2)\frac{d\varphi}{d\tau}\right]. \tag{4.25}$$

Thus, the orbital angular velocity observed at infinity is calculated to be

$$\begin{aligned}\Omega:&=\frac{d\varphi}{dt}=\frac{a+x\sqrt{\Delta}}{r^2+a^2+ax\sqrt{\Delta}}+O(\widetilde{\theta}^2)\\ &=\pm\frac{\sqrt{M}}{r^{3/2}\pm a\sqrt{M}}\left[1-\frac{3S_\perp}{2}\frac{\pm\sqrt{Mr}-a}{r^2\pm a\sqrt{Mr}}\right]+O(\widetilde{\theta}^2).\end{aligned} \tag{4.26}$$

In order to solve the second equation (4.22), we note that $v^2=\sqrt{\Sigma}\,\dot{\theta}\simeq r\dot{\widetilde{\theta}}$ and

$$Cv^0+Dv^3=-\frac{M}{r^2}\frac{1+2x^2}{1-x^2}+O(S). \tag{4.27}$$

Then we find that Eq. (4.22) reduces to

$$r\ddot{\widetilde{\theta}}=-\frac{M}{r^2}\frac{1+2x^2}{1-x^2}\widetilde{\theta}-3\frac{S_\parallel M}{r^3}\frac{x}{1-x^2}\cos\phi, \tag{4.28}$$

where $S_\parallel=S\zeta_\parallel$. This equation can be solved easily by setting $\widetilde{\theta}=\theta_0\cos\phi$. Recalling that $\Omega_p^2=M/r^3+O(S)$, we obtain

$$\theta_0=-\frac{S_\parallel}{rx}. \tag{4.29}$$

Thus we see that the orbit will remain in the equatorial plane if $S_\parallel=0$, but deviates from it if $S_\parallel\neq 0$. We note that there exists a degree of freedom to add a homogeneous solution of Eq. (4.28), whose frequency

$$\Omega_\theta=\sqrt{\frac{M}{r^3}\frac{1+2x^2}{1-x^2}}$$

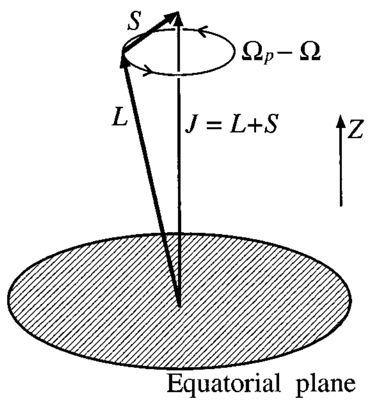

FIG. 1. A schematic picture of the precession of orbit and spin vector, to leading order in S. The vector **J** represents the total angular momentum of the particle. The vector **L** is orthogonal to the orbital plane and reduces to the orbital angular momentum in the Newtonian limit. In the relativistic case, however, these vectors should not be regarded as well defined.

is different from Ω_p and which corresponds to giving a small inclination angle to the orbit, indifferent to the spin. Here, for simplicity, we only consider the case when this homogeneous solution to $\tilde{\theta}$ is zero. Schematically speaking, the orbits under consideration are those with the total angular momentum **J** being parallel to the z direction, which is the sum of the orbital and spin angular momenta, $\mathbf{J}=\mathbf{L}+\mathbf{S}$ (see Fig. 1).

Next we consider the evolution of the spin vector. To linear order in S, the equations to be solved are

$$\begin{aligned}\dot{\zeta}^0&=A\zeta^1+C\zeta^2\tilde{\theta}-Sv^0\zeta^aR_a,\\ \dot{\zeta}^1&=A\zeta^0+B\zeta^3+E\zeta^2,\\ \dot{\zeta}^2&=(C\zeta^0+D\zeta^3)\tilde{\theta}-E\zeta^1,\\ \dot{\zeta}^3&=-B\zeta^1-D\zeta^2\tilde{\theta}-Sv^3\zeta^aR_a.\end{aligned}\tag{4.30}$$

The third equation is written down explicitly as

$$\dot{\zeta}^2=-\overline{\theta\zeta_\parallel}\kappa\sin\phi\cos\phi,\tag{4.31}$$

with

$$\kappa:=\alpha D-\beta C-\Omega_p\omega_3 r.\tag{4.32}$$

Thus we find that

$$\zeta^2=-\zeta_\perp+\frac{\theta_0\zeta_\parallel\kappa}{4\Omega_p}\cos2\phi.\tag{4.33}$$

Since the spin vector S^a is itself of $O(S)$ already, the effect of the second term is always unimportant as long as we neglect corrections of $O(S^2)$ to the orbit.

The remaining three equations determine α, β, and Ω_p. Corrections to α and β of $O(S)$ are less interesting because they remain to be small, however long the time passes. On the other hand, the correction to Ω_p will cause a large effect after a sufficiently long lapse of time because it appears in the combination of $\Omega_p\tau$. The small phase correction will accumulate and become large. Hence, we solve Ω_p alone to next leading order. Eliminating ζ^0 and ζ^3 from these three equations, we obtain

$$[(B^2-A^2)-\Omega_p^2]=\frac{S_\perp}{x}\left(\frac{AC-BD}{r}-\Omega_p^2\omega_3\right).\tag{4.34}$$

Then, after a straightforward calculation, we find

$$\Omega_p^2=\frac{M}{r^3}\left\{1-\frac{3S_\perp}{r^{3/2}}\frac{\pm\sqrt{M}(2r^2-3Mr+a^2)+ar^{1/2}(M-r)}{r^2-3Mr\pm2a\sqrt{Mr}}\right\}.\tag{4.35}$$

As noted above, Ω_p is different from Ω for $S_\perp\neq0$. The difference $\Omega_p-\Omega$ gives the angular velocity of the precession of the spin vector (see Fig. 1).

V. GRAVITAIONAL WAVES AND ENERGY LOSS RATE

We now proceed to the calculation of the source term in the Teukolsky equation and evaluate the gravitational wave flux. For this purpose, we must write down the expression of the energy-momentum tensor of the spinning particle explicitly. We rewrite the tetrad components of the energy-momentum tensor in the following way:

$$\begin{aligned}T^{ab}&=\int d\tau\left\{p^{(a}v^{b)}\frac{\delta^{(4)}(x-z(\tau))}{\sqrt{-g}}\right.\\ &\qquad\left.-e_\nu^{(a}e_\rho^{b)}\nabla_\mu S^{\mu\nu}v^\rho\frac{\delta^{(4)}(x-z(\tau))}{\sqrt{-g}}\right\}\\ &=\int d\tau\left\{[p^{(a}v^{b)}+\omega_{dc}{}^{(a}v^{b)}S^{dc}\right.\\ &\qquad-\omega_{dc}{}^{(a}S^{b)d}v^c]\frac{\delta^{(4)}(x-z(\tau))}{\sqrt{-g}}\\ &\qquad\left.-\frac{1}{\sqrt{-g}}\partial_\mu[S^{\mu(a}v^{b)}\delta^{(4)}(x-z(\tau))]\right\}\\ &=:\mu\int d\tau\left\{A^{ab}\frac{\delta^{(4)}(x-z(\tau))}{\sqrt{-g}}\right.\\ &\qquad\left.+\frac{1}{\sqrt{-g}}\partial_\mu[B^{\mu ab}\delta^{(4)}(x-z(\tau))]\right\}.\end{aligned}\tag{5.1}$$

The last line is the definition of A^{ab} and $B^{\mu ab}$.

The source term of the Teukolsky equation is expressed in terms of the components of the energy-momentum tensor projected with respect to the complex null tetrad defined as

$$l^\mu = \sqrt{\frac{\Sigma}{\Delta}}(e_0^\mu + e_1^\mu),$$

$$n^\mu = \frac{1}{2}\sqrt{\frac{\Delta}{\Sigma}}(e_0^\mu - e_1^\mu),$$

$$m^\mu = (r + ia\cos\theta)^{-1}\sqrt{\frac{\Sigma}{2}}(e_2^\mu + ie_3^\mu). \tag{5.2}$$

We adopt notation such as $T_{nn} := n^\mu n^\nu T_{\mu\nu}$ to denote the tetrad components. Then the source term is given by [19]

$$T_{\ell m\omega} = 4\int d\Omega dt \rho^{-5}\overline{\rho}^{-1}(B_2' + B_2'^*)e^{-im\varphi + i\omega t}\frac{{}_{-2}S_{\ell m}^{a\omega}}{\sqrt{2\pi}}, \tag{5.3}$$

where

$$B_2' = -\frac{1}{2}\rho^8\overline{\rho}L_{-1}[\rho^{-4}L_0(\rho^{-2}\overline{\rho}^{-1}T_{nn})]$$
$$-\frac{1}{2\sqrt{2}}\rho^8\overline{\rho}\Delta^2 L_{-1}[\rho^{-4}\overline{\rho}^2 J_+(\rho^{-2}\overline{\rho}^{-2}\Delta^{-1}T_{\overline{m}n})],$$

$$B_2'^* = -\frac{1}{4}\rho^8\overline{\rho}\Delta^2 J_+[\rho^{-4}J_+(\rho^{-2}\overline{\rho}T_{\overline{mm}})]$$
$$-\frac{1}{2\sqrt{2}}\rho^8\overline{\rho}\Delta^2 J_+[\rho^{-4}\overline{\rho}^2\Delta^{-1}L_{-1}(\rho^{-2}\overline{\rho}^{-2}T_{\overline{m}n})], \tag{5.4}$$

with

$$\rho = (r - ia\cos\theta)^{-1},$$

$$L_j = \partial_\theta + \frac{m}{\sin\theta} - a\omega\sin\theta + j\cot\theta,$$

$$J_+ = \partial_r + \frac{iK}{\Delta}, \tag{5.5}$$

and $\overline{Q}$ denotes the complex conjugate of Q.

As we will see shortly, the terms proportional to $S_\parallel$ in the energy-momentum tensor do not contribute to the energy and angular momentum fluxes at linear order in S. In other words, the energy and angular momentum fluxes are the same for all orbits having the same $S_\perp$, irrespective of the value of $S_\parallel$. Thus, we ignore these terms in the following discussion. Further we recall that the particle can stay in the equatorial plane if $S_\parallel = 0$. Hence we fix $\theta = \pi/2$ in the following calculations.

Using the formula (2.5), we obtain the amplitude of gravitaional waves at infinity as

$$\tilde{Z}_{\ell m\omega} = \tilde{Z}_{\ell m\omega}^{nn} + \tilde{Z}_{\ell m\omega}^{\overline{m}n} + \tilde{Z}_{\ell m\omega}^{\overline{mm}}, \tag{5.6}$$

where

$$\tilde{Z}_{\ell m\omega}^{nn} = \frac{i\sqrt{2\pi}}{\omega B_{\ell m\omega}^{\rm in}}\delta(\omega - m\Omega)\left(\frac{dt}{d\tau}\right)^{-1}\left[A_{nn} - i\omega B_{nn}^t + imB_{nn}^\varphi - B_{nn}^r\frac{\partial}{\partial r}\right][L_1^\dagger\rho^{-4}(L_2^\dagger\rho^3{}_{-2}S_{\ell m}^{a\omega})]_{\theta=\pi/2}\frac{1}{r\Delta}R_{\ell m\omega}^{\rm in}\Big|_{r=r_0},$$

$$\tilde{Z}_{\ell m\omega}^{\overline{m}n} = \frac{i\sqrt{\pi}}{\omega B_{\ell m\omega}^{\rm in}}\delta(\omega - m\Omega)\left(\frac{dt}{d\tau}\right)^{-1}\left[A_{\overline{m}n} - i\omega B_{\overline{m}n}^t + imB_{\overline{m}n}^\varphi - B_{\overline{m}n}^r\frac{\partial}{\partial r}\right](L_2^\dagger{}_{-2}S_{\ell m}^{a\omega})_{\theta=\pi/2}\frac{1}{\sqrt{\Delta}}\left[2\frac{\partial}{\partial r} - \frac{2iK}{\Delta} - \frac{4}{r}\right]R_{\ell m\omega}^{\rm in}\Big|_{r=r_0}, \tag{5.7}$$

$$\tilde{Z}_{\ell m\omega}^{\overline{mm}} = \frac{i\sqrt{\pi}}{\omega B_{\ell m\omega}^{\rm in}}\delta(\omega - m\Omega)\left(\frac{dt}{d\tau}\right)^{-1}\left[A_{\overline{mm}} - i\omega B_{\overline{mm}}^t + imB_{\overline{mm}}^\varphi - B_{\overline{mm}}^r\frac{\partial}{\partial r}\right]$$
$$\times({}_{-2}S_{\ell m}^{a\omega})_{\theta=\pi/2}\left[\frac{\partial^2}{\partial r^2} - 2\left(\frac{1}{r} + \frac{iK}{\Delta}\right)\frac{\partial}{\partial r} - \left(\frac{iK}{\Delta}\right)_{,r} + \frac{2iK}{\Delta r} - \frac{K^2}{\Delta^2}\right]R_{\ell m\omega}^{\rm in}\Big|_{r=r_0},$$

and

$$A_{nn} = \frac{1}{4}\frac{1}{1-x^2}\{1 - S_\perp[(2\omega_1 + \omega_3)x + \omega_2]\},$$

$$B_{nn}^\mu = \frac{1}{4r}S_\perp\frac{1}{1-x^2}\left(\frac{r^2+a^2}{\sqrt{\Delta}}x + a, -\sqrt{\Delta}x, 0, \frac{a}{\sqrt{\Delta}}x + 1\right), \tag{5.8}$$

$$A_{\overline{mn}}=\frac{i}{4\sqrt{2}}\frac{1}{1-x^2}\{2x-S_\perp(\omega_1x^2-4\omega_2x-\omega_3)\},$$

$$B^\mu_{\overline{mn}}=\frac{i}{4\sqrt{2}r}S_\perp\frac{1}{1-x^2}\times\left(\frac{r^2+a^2}{\sqrt{\Delta}}x^2+ax,-\sqrt{\Delta}(1+x^2),0,\frac{a}{\sqrt{\Delta}}x^2+x\right),$$

$$A_{\overline{mm}}=-\frac{1}{2}\frac{1}{1-x^2}\{x^2+S_\perp(\omega_2(1+2x^2)+\omega_3x)\},$$

$$B^\mu_{\overline{mm}}=\frac{1}{2r}S_\perp\frac{1}{1-x^2}(0,\sqrt{\Delta}x,0,0),$$

and

$$L_j^\dagger=\partial_\theta-\frac{m}{\sin\theta}+a\omega\,\sin\theta+j\,\cot\theta. \tag{5.9}$$

The Lorentz factor $dt/d\tau$ which appears in Eqs. (5.7) can be calculated from Eqs. (4.25) as

$$\frac{dt}{d\tau}=\frac{1}{r\sqrt{1-x^2}}\left(ax+\frac{r^2+a^2}{\sqrt{\Delta}}\right). \tag{5.10}$$

When the orbit is quasiperiodic, the Fourier component of gravitational waves does not have a continuous spectrum but takes the form

$$\tilde{Z}_{\ell m\omega}=\sum_n\ \delta(\omega-\omega_n)Z_{\ell m\omega_n}. \tag{5.11}$$

Then the time-averaged energy flux is given by the formula [19]

$$\left\langle\frac{dE}{dt}\right\rangle_{\rm GW}=\sum_{\ell,m,n}\frac{|Z_{\ell m\omega_n}|^2}{4\pi\omega_n^2}=:\sum_{\ell,m,n}\left(\frac{dE}{dt}\right)_{\ell mn}. \tag{5.12}$$

The z component of the angular momentum flux is also given by a similar formula

$$\left\langle\frac{dJ_z}{dt}\right\rangle_{\rm GW}=\sum_{\ell,m,n}\frac{m|Z_{\ell m\omega_n}|^2}{4\pi\omega_n^3}=:\sum_{\ell,m,n}\left(\frac{dJ_z}{dt}\right)_{\ell mn}. \tag{5.13}$$

In the present case of circular orbits in the equatorial plane, the index n degenerates to the angular index m and ω_n is simply given by $m\Omega$ $(n=m)$. Hence we eliminate the index n in the following discussion.

Here we mention the effect of nonzero $S_\parallel$. If we recall that all the terms which are proportional to $S_\parallel$ have the time dependence of $e^{\pm i\Omega_p\tau}$, we see that they give a contribution to the sidebands. That is to say, their contributions in $\tilde{Z}_{\ell m\omega}$ are all proportional to $\delta(\omega-m\Omega\pm\Omega_p)$. Then, since the energy and angular momentum fluxes are quadratic in $Z_{\ell m\omega_n}$, they are not affected by the presence of $S_\parallel$ as long as we are working only up to linear order in S.

In order to express the post-Newtonian corrections to the energy flux, we define $\eta_{\ell m\omega}$ as

$$\left(\frac{dE}{dt}\right)_{\ell m}=:\frac{1}{2}\left(\frac{dE}{dt}\right)_N\eta_{\ell m}, \tag{5.14}$$

where $(dE/dt)_N$ is the Newtonian quadrupole formula:

$$\left(\frac{dE}{dt}\right)_N=\frac{32\mu^2M^3}{5r^5}=:\frac{32}{5}\left(\frac{\mu}{M}\right)^2v^{10}. \tag{5.15}$$

We calculate $\eta_{\ell m}$ up to 2.5PN order, i.e., to $O(v^5)$. The result is

$$\begin{aligned}
\eta_{2\pm2}&=1-\frac{107}{21}v^2+\left(4\pi-6q-\frac{19}{3}\hat{s}\right)v^3+\left(\frac{4784}{1323}+2q^2+9q\hat{s}\right)v^4+\left(-\frac{428}{21}\pi+\frac{4216}{189}q+\frac{2134}{63}\hat{s}\right)v^5,\\
\eta_{2\pm1}&=\frac{1}{36}v^2+\left(-\frac{1}{12}q+\frac{1}{12}\hat{s}\right)v^3+\left(-\frac{17}{504}+\frac{1}{16}q^2-\frac{1}{8}q\hat{s}\right)v^4+\left(\frac{1}{18}\pi-\frac{793}{9072}q-\frac{535}{1008}\hat{s}\right)v^5,\\
\eta_{3\pm3}&=\frac{1215}{896}v^2-\frac{1215}{112}v^4+\left(\frac{3645}{448}\pi-\frac{1215}{112}q-\frac{10\,935}{896}\hat{s}\right)v^5,\\
\eta_{3\pm2}&=\frac{5}{63}v^4+\left(-\frac{40}{189}q+\frac{20}{63}\hat{s}\right)v^5,\\
\eta_{3\pm1}&=\frac{1}{8064}v^2-\frac{1}{1512}v^4+\left(\frac{1}{4032}\pi-\frac{17}{9072}q-\frac{1}{8064}\hat{s}\right)v^5,\\
\eta_{4\pm4}&=\frac{1280}{567}v^4,\\
\eta_{4\pm2}&=\frac{5}{3969}v^4,
\end{aligned} \tag{5.16}$$

where $q:=a/M$ and $\hat{s}:=S_\perp/M$. The rest of $\eta_{\ell m}$ are all of higher order. We should mention that if we regard the spinning particle as a model of a black hole or neutron star, S is of order μ. Therefore the corrections due to S are generally small compared with the S-independent terms in the test particle limit $\mu/M\ll1$.

Putting all together, we obtain, to 2.5PN order,

$$\left\langle \frac{dE}{dt} \right\rangle_{\rm GW} = \left(\frac{dE}{dt} \right)_N \left[1 - \frac{1247}{336} v^2 + \left(4\pi - \frac{73}{12} q - \frac{25}{4} \hat{s} \right) v^3 + \left(-\frac{44\,711}{9072} + \frac{33}{16} q^2 + \frac{71}{8} q\hat{s} \right) v^4 + \left(-\frac{8191}{672}\pi + \frac{3749}{336} q + \frac{2403}{112} \hat{s} \right) v^5 \right]. \quad (5.17)$$

Since v is defined in terms of the coordinate radius of the orbit, the expansion with respect to v does not have a clear gauge-invariant meaning. In particular, for the purpose of the comparison with the standard post-Newtonian calculations, it is better to write the result by means of the angular velocity observed at infinity. Using the post-Newtonian expansion of Eq. (4.26),

$$M\Omega = v^3 \left[1 - \left(\frac{3}{2}\hat{s} + q \right) v^3 + \frac{3}{2} q\hat{s} v^4 + O(v^6) \right], \quad (5.18)$$

the above result can be rewritten as

$$\left\langle \frac{dE}{dt} \right\rangle_{\rm GW} = \left(\frac{\widetilde{dE}}{dt} \right)_N \left[1 - \frac{1247}{336} (M\Omega)^{2/3} + \left(4\pi - \frac{11}{4} q - \frac{5}{4}\hat{s} \right) \times (M\Omega) + \left(-\frac{44\,711}{9072} + \frac{33}{16} q^2 + \frac{31}{8} q\hat{s} \right) (M\Omega)^{4/3} + \left(-\frac{8191}{672}\pi + \frac{59}{16} q - \frac{13}{16}\hat{s} \right) (M\Omega)^{5/3} \right], \quad (5.19)$$

where

$$\left(\frac{\widetilde{dE}}{dt} \right)_N := \frac{32}{5} \left(\frac{\mu}{M} \right)^2 (M\Omega)^{10/3}. \quad (5.20)$$

Since there is no sideband contribution in the present case, the angular momentum flux is simply given by $\langle dJ_z/dt \rangle_{\rm GW} = \Omega^{-1} \langle dE/dt \rangle_{\rm GW}$. The result (5.19) is consistent with the one obtained by the standard post-Newtonian approach [11,12] to 2PN order in the limit $\mu/M \to 0$. The $\hat{s}$-dependent term of order $(M\Omega)^{5/3}$ is the one which is newly obtained here.

VI. RADIATION REACTION

In this section, we consider the effect of radiation reaction on the orbit by equating the gravitational energy and angular momentum fluxes with their loss rates of the system.

A. Conserved quantities

Here we consider the conserved quantities which are the first integrals of the equations of motion. First we give two conserved quantities which follow from the Killing vectors of the Kerr spacetime. The timelike Killing vector is given by

$$\xi_\mu = \sqrt{\frac{\Delta}{\Sigma}}\, e^0_{\ \mu} + \frac{a\sin\theta}{\sqrt{\Sigma}} e^3_{\ \mu}, \quad (6.1)$$

and its derivative is

$$\xi_{\mu;\nu} = -\frac{2M(r^2 - a^2\cos^2\theta)}{\Sigma^2} e^1_{\ [\mu} e^0_{\ \nu]} + \frac{4Mar\cos\theta}{\Sigma^2} e^3_{\ [\mu} e^2_{\ \nu]}. \quad (6.2)$$

The rotational Killing vector is given by

$$\chi_\mu = a\sin^2\theta \sqrt{\frac{\Delta}{\Sigma}}\, e^0_{\ \mu} + \frac{(r^2+a^2)\sin\theta}{\sqrt{\Sigma}} e^3_{\ \mu}, \quad (6.3)$$

and its derivative is

$$\chi_{\mu;\nu} = -\frac{2a\sin^2\theta}{\Sigma^2}[(r-M)\Sigma + 2Mr^2] e^1_{\ [\mu} e^0_{\ \nu]} - \frac{2a\sqrt{\Delta}\sin\theta\cos\theta}{\Sigma} e^2_{\ [\mu} e^0_{\ \nu]} - \frac{2r\sin\theta\sqrt{\Delta}}{\Sigma} e^1_{\ [\mu} e^3_{\ \nu]} + \frac{2\cos\theta}{\Sigma^2}[a^2\sin^2\theta\Delta - (r^2+a^2)^2] e^2_{\ [\mu} e^3_{\ \nu]}. \quad (6.4)$$

Then following the discussion around Eq. (3.18), we can construct the conserved quantities describing the energy and the z component of the angular momentum from these Killing vectors. They are given by

$$\frac{E}{\mu} := u^\mu \xi_\mu - \frac{1}{2\mu} S^{\mu\nu} \xi_{\mu;\nu} = \sqrt{\frac{\Delta}{\Sigma}}\, u^0 + \frac{a\sin\theta}{\sqrt{\Sigma}} u^3 + \frac{M(r^2 - a^2\cos^2\theta)}{\Sigma^2} \frac{S^{10}}{\mu} + \frac{2Mar\cos\theta}{\Sigma^2} \frac{S^{23}}{\mu},$$

$$\frac{J_z}{\mu} := u^\mu \chi_\mu - \frac{1}{2\mu} S^{\mu\nu} \chi_{\mu;\nu} = a\sin^2\theta \sqrt{\frac{\Delta}{\Sigma}}\, u^0 + \frac{(r^2+a^2)\sin\theta}{\sqrt{\Sigma}} u^3 + \frac{a\sin^2\theta}{\Sigma^2}[(r-M)\Sigma + 2Mr^2] \frac{S^{10}}{\mu} + \frac{a\sqrt{\Delta}\sin\theta\cos\theta}{\Sigma} \frac{S^{20}}{\mu} + \frac{r\sqrt{\Delta}\sin\theta}{\Sigma} \frac{S^{13}}{\mu} - \frac{\cos\theta}{\Sigma^2}[a^2\sin^2\theta\Delta - (r^2+a^2)^2] \frac{S^{23}}{\mu}, \quad (6.5)$$

Since terms such as $\cos\theta S^{23}$ and $\cos\theta S^{20}$ are higher order in $\widetilde{\theta}$, we neglect them in the following discussion.

For a spinless particle, there exists one more conserved quantity on the Kerr spacetime, known as the Carter constant [28]. It is associated with the Killing tensor $K_{\mu\nu}$ which satisfies $K_{(\mu\nu;\sigma)}=0$. However, for a spinning particle, no such conserved quantity has been known. Nevertheless, one can show that there exists an approximately conserved quantity which corresponds to the Carter constant for a spinless particle. It is constructed as follows.

In addition to the Killing vectors, the Kerr spacetime has an antisymmetric Killing Yano tensor

$$f_{\mu\nu}=2a\cos\theta\, e^1_{[\mu}e^0_{\nu]}+2re^2_{[\mu}e^3_{\nu]}, \tag{6.6}$$

which satisfies

$$f_{\mu(\nu;\sigma)}=0. \tag{6.7}$$

Note that from $f_{\mu\nu}$, we can construct the Killing tensor $K_{\mu\nu}$ as

$$K_{\mu\nu}=f_{\mu\sigma}f_\nu{}^\sigma=r^2g_{\mu\nu}+2\Sigma l_{(\mu}n_{\nu)}. \tag{6.8}$$

When there is a Killing Yano tensor, the system possesses a quantity whose τ derivative is of $O(S^2)$ and hence conserved to linear order in S. Introducing the totally antisymmetric tensor

$$f_{\mu\nu\sigma}:=f_{\mu\nu;\sigma}=6\left(\frac{a\sin\theta}{\sqrt{\Sigma}}e^0_{[\mu}e^1_\nu e^2_{\sigma]}+\sqrt{\frac{\Delta}{\Sigma}}\,e^1_{[\mu}e^2_\nu e^3_{\sigma]}\right), \tag{6.9}$$

the approximate conserved quantity is expressed by [27]

$$\begin{aligned}\frac{Q}{\mu^2}:=&\frac{1}{2}f_{\mu\sigma}f_\nu{}^\sigma u^\mu u^\nu-u^\mu\frac{S^{\rho\sigma}}{\mu}(f^\nu{}_\sigma f_{\mu\rho\nu}-f_\mu{}^\nu f_{\rho\sigma\nu})\\ =&\frac{1}{2}\{\Sigma[(u^0)^2-(u^1)^2]-r^2\}-\frac{a\sin\theta}{\mu\sqrt{\Sigma}}\{r(u^0S^{13}+u^1S^{30}-2u^3S^{10})+a\cos\theta u^3S^{23}\}\\ &-\frac{1}{\mu}\sqrt{\frac{\Delta}{\Sigma}}\{a\cos\theta(u^2S^{30}-u^3S^{20}+2u^0S^{23})-ru^0S^{10}\}.\end{aligned} \tag{6.10}$$

The quantity corresponding to the Carter constant is

$$C:=2Q-(J_z-aE)^2. \tag{6.11}$$

For the case of circular orbits under consideration, we find

$$\frac{C}{\mu^2}=-2aS_\perp. \tag{6.12}$$

B. Frequency shift due to a radiation reaction

The orbit of a nonspinning particle is completely determined by the three constants of motion, E, J_z, and C. As mentioned above the counterpart of C for a spinning particle also exists in the linear order of S. This means that we need to calculate the radiation reaction to these quantities to obtain the orbital evolution of a spinning particle. In both nonspinning and spinning cases, the radiation reaction to E and J_z can be evaluated by equating their loss rates with the corresponding fluxes emitted by gravitational waves. However, we do not know how to determine the reaction to C from the asymptotic behavior of emitted gravitational waves at infinity. Furthermore, there are also spin degrees of freedom in the present case. It is not clear at all how to evaluate the back reaction to the spin. In order to fully understand the radiation reaction, it will be necessary to derive some regularized radiation reaction force which acts on the particle. However, this is beyond the scope of the present paper.

Although the rigorous evaluation of the change rate of spin seems formidable, there exists an order-of-magnitude estimate by Apostolatos *et al.* [29], in which they calculate a torque acting inside a spinning star due to the radiation reaction force. According to their estimate, the rate of change of the spin is $O(v^{10})$ smaller than that of the orbital angular momentum. This means we can safely ignore the time variation of the spin if $v\ll 1$. Since we are interested in the case $v\ll 1$ in this paper, we may then assume that the circular orbit obtained in Sec. IV remains circular with smaller r but with the same values of $S_\perp$ and $S_\parallel$ under the radiation reaction. Note that this implies that the approximate Carter constant, Eq. (6.12), is conserved under the radiation reaction.

If this assumption is correct, we can obtain an evolution sequence of the orbits under the radiation reaction. Namely, the orbit is quasicircular and slowly spiraling in with constant $S_\perp$ and $S_\parallel$. Although not sufficient, we can consider the necessary condition for this assumption to be true by examining the consistency with the energy and angular momentum loss rates evaluated in terms of their fluxes emitted by gravitational waves. Here the consistency means that the relation

$$\left\langle\frac{dE}{dt}\right\rangle_{\rm GW}=\frac{\delta E}{\delta J_z}\left\langle\frac{dJ_z}{dt}\right\rangle_{\rm GW}, \tag{6.13}$$

holds, where the variations δE and δJ_z are taken keeping the orbit circular with fixed $S_\perp$ and $S_\parallel$. As noted before, since there are no sideband contributions to the energy and angular momentum fluxes in the present case, we have $\langle dE/dt\rangle_{\rm GW}=\Omega\langle dJ_z/dt\rangle_{\rm GW}$. Therefore the condition (6.13) is equivalent to

$$\frac{\delta E}{\delta J_z}=\Omega. \tag{6.14}$$

In the following, we prove that Eq. (6.14) indeed holds. Since the $S_\parallel$ does not appear in the expressions for E, J_z, $\langle dE/dt\rangle_{\rm GW}$ and $\langle dJ_z/dt\rangle_{\rm GW}$ to linear order in S, we only have to examine the situation with $S_\parallel=0$.

We introduce the function

$$V(E,J_z,r,S_\perp):=-g^{tt}u_tu_t-2g^{t\varphi}u_tu_\varphi-g^{\varphi\varphi}u_\varphi u_\varphi-1, \tag{6.15}$$

which is guaranteed to be non-negative and it becomes zero when the orbit has no radial or vertical motion, i.e., $v^1=v^2=0$. If we set E and J_z to the values for a circular orbit with $r=r_0$, we have $V=0$ at $r=r_0$ and $V>0$ at any other points near $r=r_0$. Hence $\partial V/\partial r|_{r=r_0}=0$.

The momentum components $-\mu u_t$ and μu_φ are different from the conserved energy and angular momenta due to the presence of spin. From Eqs. (6.5), we see

$$-\mu u_t=E-\mu S_\perp\varepsilon(u^0(E,J_z),u^3(E,J_z),r),$$
$$\mu u_\varphi=J_z-\mu S_\perp j(u^0(E,J_z),u^3(E,J_z),r), \tag{6.16}$$

where

$$\varepsilon(u^0,u^3,r)=\frac{M}{r^2}u^3,$$
$$j(u^0,u^3,r)=\frac{a}{r^2}(M+r)u^3+\frac{\sqrt{\Delta}}{r}u^0. \tag{6.17}$$

Inserting these expressions into Eq. (6.15), taking the variation of V while keeping the orbit circular with fixed spin, and recalling $\partial V/\partial r|_{r=r_0}=0$, we obtain the relation between δE and δJ_z as

$$\mu\,\delta V=2u^t\left[\delta E-\mu S_\perp\left(\frac{\partial\varepsilon}{\partial E}\delta E+\frac{\partial\varepsilon}{\partial J_z}\delta J_z\right)\right]-2u^\varphi\left[\delta J_z-\mu S_\perp\left(\frac{\partial j}{\partial E}\delta E+\frac{\partial j}{\partial J_z}\delta J_z\right)\right]=0. \tag{6.18}$$

This gives

$$\frac{\delta E}{\delta J_z}=\frac{u^\varphi}{u^t}\left\{1+\mu S_\perp\left[\frac{1}{u^\varphi}\left(-\frac{\partial\varepsilon}{\partial J_z}u^t+\frac{\partial j}{\partial J_z}u^\varphi\right)+\frac{1}{u^t}\left(-\frac{\partial\varepsilon}{\partial E}u^t+\frac{\partial j}{\partial E}u^\varphi\right)\right]+O(S^2)\right\}. \tag{6.19}$$

By using the relations

$$u^0=\frac{1}{r\sqrt{\Delta}}[(r^2+a^2)E-aJ_z],$$
$$u^3=\frac{1}{r}(J_z-aE),$$
$$u^t=\frac{1}{\sqrt{\Delta}}\frac{r^2+a^2}{r}u^0+\frac{a}{r}u^3,$$
$$u^\varphi=\frac{1}{\sqrt{\Delta}}\frac{a}{r}u^0+\frac{1}{r}u^3, \tag{6.20}$$

which hold in the lowest order in S, it is easy to verify that the terms in the square parentheses on the right-hand side of Eq. (6.19) become of $O(S^2)$ or higher at $r=r_0$. Thus Eq. (6.14) is shown to hold to linear order in S and our assumption of the stability of the quasicircular orbit is found to be consistent.

Under the assumption that the orbit remains quasicircular with fixed spin, we can evaluate the frequency shift due to the radiation reaction by

$$\frac{d\Omega}{dt}=-\left(\frac{dE}{d\Omega}\right)^{-1}\left\langle\frac{dE}{dt}\right\rangle_{\rm GW}. \tag{6.21}$$

The post-Newtonian expansion of this quantity is calculated to become

$$\frac{d\Omega}{\Omega^2dt}=\frac{96}{5}\frac{\mu}{M}(M\Omega)^{5/3}\left[1-\frac{743}{336}(M\Omega)^{2/3}+\left(4\pi-\frac{113}{12}q-\frac{25}{4}\hat{s}\right)(M\Omega)+\left(\frac{34\,103}{18\,144}+\frac{81}{16}q^2+\frac{79}{8}q\hat{s}\right)(M\Omega)^{4/3}+\left(-\frac{4159}{672}\pi-\frac{31\,319}{1008}q-\frac{809}{84}\hat{s}\right)(M\Omega)^{5/3}\right]. \tag{6.22}$$

VII. SUMMARY AND DISCUSSION

In this paper we have investigated the gravitational waves emitted by a spinning particle in circular orbits around a rotating black hole. First we have solved the equations of motion of a spinning particle in Kerr spacetime, assuming the spin of the particle is small and the orbit is close to the equatorial plain. Applying the Teukolsky formalism of the black hole perturbation, we have then calculated the first order corrections due to spin to the energy flux up to 2.5PN order. The effect of spin is always small $[=O(\mu/M)]$ compared with the spin-independent contributions if we take the limit $\mu/M\to0$. However, the result will be a useful guideline to the standard post-Newtonian calculations because our approach is totally different from the standard post-Newtonian approach and gives the leading spin-dependent terms in the μ/M expansion. Up to 2PN order, our results are in complete agreement with the previous ones obtained by the standard post-Newtonian method. The spin-dependent term at 2.5PN order, which we have newly obtained, will be verified by the standard post-Newtonian approach in the future.

In this paper we have restricted our analysis to a class of orbits which are circular and stays in the equatorial plane if the spin vector is orthogonal to it. In this case, the assumption that the orbit remains in this class under a radiation reaction has been found to be consistent with the energy and the angular momentum loss rates evaluated from the gravitational wave flux at infinity. Although we have not considered it here, it seems possible to incorporate the orbital inclination. However, if the present restriction is relaxed, the orbit may become too complicated and the same technique used here may not work well. As long as one considers a spinless particle, the orbit is parametrized by the three conserved quantities, i.e., the energy, the z component of the angular momentum, and the Carter constant. In that case, the particle will sweep through a certain region of phase space restricted by these conserved quantities sufficiently fast compared with the time scale of the radiation reaction. Hence the quasiperiodicity will be a good approximation. On the other hand, for a spinning particle, there is not a sufficient number of conserved quantities to confine the orbit to a restricted region of phase space. This means that the orbit is not guaranteed to be quasiperiodic. When we try to make a better template to be used for interferometric gravitational wave detectors in the future by taking account of the effect of spin, this point may cause a difficult problem.

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APPENDIX

In this appendix, we present an exact solution of the equations of motion in the case when the spin vector S^a is parallel or antiparallel to the rotation axis of the Kerr black hole. We assume that the orbit is circular and lies on the equatrial plane. Thus the only nonvanishing component of the spin vector is $S\zeta^2=-S_\perp$ and we set $v^1=v^2=0$ exactly. Further we assume $u^1=u^2=0$, which will be found to be consistent.

With these assumptions, the only nonvanishing component of R^a is

$$SR^1=-S_\perp \frac{M}{r^3}[2v^0u^3+v^3u^0)]. \tag{A1}$$

Then all but one of the equations of motion are trivially satisfied and the remaining nontrivial equation is

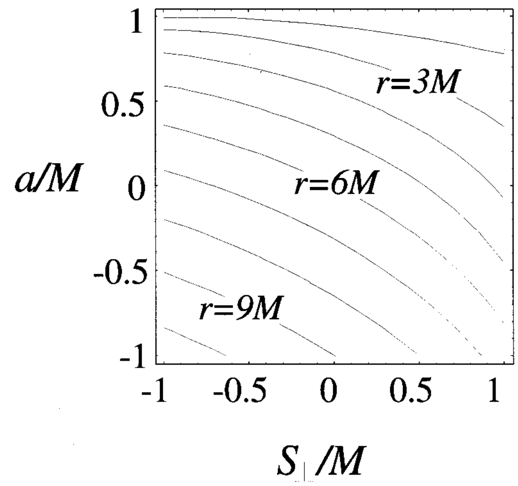

FIG. 2. The contours of radii of the innermost stable circular orbits on the $(a,S_\perp)$ plane. The conserved angular momentum J_z is assumed to be positive so that the orbits with $a>0$ are corotating with the black hole and those with $a<0$ are counterrotating.

$$\dot{u}^1=\omega_1 v^0u^0+\omega_2(v^0u^3+v^3u^0)+\omega_3 v^3u^3 + S_\perp \frac{M}{r^3}(2v^0u^3+v^3u^0)=0. \tag{A2}$$

In addition to the equations of motion there are constraint equations to be satisfied. We list them below.

(1) From the time derivative of the center-of-mass condition $S^{ab}u_b=0$, we have the relation between u^a and v^a as

$$u^a=v^a-\frac{S^{ab}}{\mu}SR_b, \tag{A3}$$

which is equivalent to Eq. (3.5). This gives

$$u^0=v^0-\frac{MS_\perp^2}{r^3}(2v^0u^3+v^3u^0)u^3, \tag{A4}$$

$$u^3=v^3-\frac{MS_\perp^2}{r^3}(2v^0u^3+v^3u^0)u^0. \tag{A5}$$

(2) The mass conservation $u_au^a=-1$ gives

$$(u^0)^2-(u^3)^2=1. \tag{A6}$$

(3) The normalization condition $u_av^a=-1$ of v^a is

$$u^0v^0-u^3v^3=1. \tag{A7}$$

We may consider $S_\perp$ and r as freely specifiable variables. Then the variables to be determined are v^0, v^3, u^0, and u^3.

However, there are five equations to be satisfied, i.e., Eqs. (A2) and (A4)–(A7). Hence one might think the system is overdetermined and inconsistent. Fortunately this is not the case because Eq. (A7) is guaranteed to hold by Eqs. (A4), (A5), and (A6), as can be seen by contracting Eq. (A3) with u_a. Thus our assumptions, in particular $u^1=u^2=0$, turn out to be consistent.

In order to solve the above set of equations, we introduce the new variables

$$x_v=\frac{v^3}{v^0} \quad \text{and} \quad x_u=\frac{u^3}{u^0}. \tag{A8}$$

In terms of x_v and x_u, Eq. (A2) is rewritten as

$$a^2-Mr+a\sqrt{\Delta}(x_v+x_u)+\Delta x_v x_u+\sigma M\sqrt{\Delta}(2x_u+x_v)=0, \tag{A9}$$

where

$$\sigma:=\frac{S_\perp}{r}=\frac{M}{r}\hat{s}. \tag{A10}$$

Multiplying Eq. (A4) by u^3 and Eq. (A5) by u^0, equating the right-hand sides of them, and dividing it by $u^0 v^0$, we obtain

$$x_v-\frac{M}{r}\sigma^2(2x_u+x_v)(u^0)^2=x_u-\frac{M}{r}\sigma^2(2x_u+x_v)(u^3)^2. \tag{A11}$$

Using Eq. (A6), one readily sees this reduces to

$$x_v-x_u-\frac{M}{r}\sigma^2(2x_u+x_v)=0. \tag{A12}$$

Thus we have obtained the coupled equations (A9) and (A12) for x_v and x_u. They are solved to give

$$x_v=\frac{-(2ra+3Mr\sigma+aM\sigma^2)\pm\sqrt{4Mr^3+12aMr^2\sigma+13M^2r^2\sigma^2+6aM^2r\sigma^3-8M^3r\sigma^4+9a^2M^2\sigma^4}}{2\sqrt{\Delta}(r-M\sigma^2)},$$

$$x_u=\frac{r-M\sigma^2}{r+2M\sigma^2}x_v. \tag{A13}$$

Then from Eqs. (A6) and (A7), u^0, u^3, v^0, and v^3 are found as

$$u^0=\frac{1}{\sqrt{1-x_u^2}}, \quad u^3=\frac{x_u}{\sqrt{1-x_u^2}},$$

$$v^0=\frac{\sqrt{1-x_u^2}}{1-x_v x_u}, \quad v^3=\frac{x_v\sqrt{1-x_u^2}}{1-x_v x_u}. \tag{A14}$$

We note that the terms inside the square root of the expression for x_v in Eq. (A13) are not positive definite. Thus circular orbits do not exist for very large values of σ $(=S_\perp/r)$. However, they always exist for physically reasonable values of σ, i.e., for $\sigma<M/r<1$. We also note that the $+$ $(-)$ sign in front of the square root corresponds to a corotating (counterrotating) orbit if we restrict the range of a to be non-negative, i.e., $0\leqslant a<M$. On the other hand, if we extend the range of a to $-M<a<M$, the $\pm$ signs become redundant. Here we take the latter option and take the $+$ sign.

Then a matter of interest is the stability of these circular orbits. In Fig. 2, we show the contours of radii of the innermost stable circular orbits on the $(a,S_\perp)$ plane. The contour of $r=6M$ passes through the $(a,S_\perp)=(0,0)$, which is the well-known minimum radius for a spinless particle in Schwarzchild spacetime. One readily notices that the minimum radius decreases as a increases for a fixed $S_\perp$ and it is smaller for larger $S_\perp$. Another interesting feature is that the minimum radius approaches $r=a$ in the limit $a\to M$ irrespective of the values of $S_\perp$. Although this latter feature can be explained only in a fully relativistic context, the main feature of the contours can be understood as a consequence of the spin-orbit coupling, which is the dominant effect in a mildly relativistic situation. It is repulsive when the spin and orbital angular momentum vectors are parallel and attractive when they are antiparallel. Now, if the contribution of the particle's spin to the spin-orbit interaction could be neglected, the contours of the minimum radii would be parallel to the $S_\perp$ axis, with decreasing minimum radii for larger a. On the other hand, if the particle were another Kerr black hole with the same mass M, spins of the black hole and the particle would contribute to the spin-orbit interaction in an exactly symmetric way, and the contours would be straight lines at 45° downward in the right direction. In reality, neither can the spin of the particle be neglected nor is its contribution as large as that of the black hole. This approximately explains the feature of the contours.

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