Wilsonian Renormalization Approach to Nonlinear Sigma Models

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Nonlinear sigma models with $\mathcal{N} = 2$ supersymmetry is formulated in the framework of Wilson renormalization group in three dimensional space-time. Interesting conformal field theories are found as fixed points of the renormalization group equation. Any Einstein-Kähler manifold corresponds to a conformal field theory when the anomalous dimension is $\gamma = -1/2$.

§1. Introduction

Possible candidates for interacting field theories are severely restricted by the requirement of renormalizability in the perturbation theory, Interacting field theories cease to exist in the space-time whose dimension is larger than a critical value. The critical dimension of self-interacting scalar field theories or non-Abelian gauge theories is four, whereas the critical dimension is two in the gravity or non-linear sigma models. It is an important and interesting problem if these critical values remain unchanged even in non-perturbative treatment of field theories.

Although gauge and gravity theories are more important than non-linear sigma models, non-linear sigma models offer the theoretical playground for these models since there is deep similarity between gauge theories and sigma models. In this work, therefore, we study the non-perturbative renormalization property of three dimensional non-linear sigma models, non-renormalizable in the perturbation theory.

For that purpose, we use the Wilson's renormalization group.[1] In this method, field variables are divided into two parts, field ϕ_Λ and with wavelength shorter than $1/\Lambda$ and $\phi_>$ of higher frequency modes. Then, fields with frequencies higher than Λ are integrated out in the path integral formulation in order to obtain the effective action S_Λ describing the field dynamics with the ultraviolet cutoff Λ. Since it is difficult to integrate out all the field variables of higher frequencies, we integrate out field variables in an infinitesimal momentum range $\Lambda \cdot e^{-\delta t} < k \leq \Lambda$ to calculate the infinitesimal change of the effective action when the ultraviolet cutoff is changed infinitesimally from Λ to $\Lambda \cdot e^{-\delta t}$. The infinitesimal change of the effective action is described by the renormalization group equation(RGE).[1),2)]

The effective action, expanded in powers of space-time derivatives, contains all higher derivative terms, and the RGE is a set of coupled differential equations of infinite dimensions.[3] In order to solve these equations, we have to introduce some kind of truncations. In the simplest truncation method, one retains only terms without

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derivative, and is called the local potential approximation. In the next approximation, we retain all terms with two derivatives. We may call this approximation as the sigma model approximation, since the typical Lagrangian with two derivatives are written as

$$\mathcal{L} = \frac{1}{2} g_{ij} \partial_\mu \phi^i \partial^\mu \phi^j$$

which is nothing but the Lagrangian of non-linear sigma model, where $g_{ij}(\phi)$ describes the metric of the target manifold $\mathcal{M}$ where field variables ϕ reside. In this work, we use this sigma model approximation. Furthermore, we confine ourselves to theories with $\mathcal{N} = 2$ supersymmetry, in order to forbid the appearance of local potential terms.

The renormalization group equation for nonlinear sigma models describes the deformation of the metric of the target manifold $\mathcal{M}$ when the ultraviolet cutoff has been changed. The fixed points of the RGE correspond to conformal field theories which remain unchanged under the change of the mass scale. In this work, we study the fixed point theories of the RGE. It should be emphasized that RGE obtained in the perturbation theory has the similar form with the RGE obtained in the Wilson's renormalization method can only be used in the vicinity of the free field theory, whereas the Wilsonian RGE can be used to study even nontrivial conformal field theories located far away from the free field theory.

§2. Nonlinear sigma model with $\mathcal{N} = 2$ supersymmetry in three dimensions

Nonlinear sigma models with $\mathcal{N} = 2$ supersymmetry in three dimensions are defined by the so-called Kähler potential $K(\phi, \bar{\phi})$, which is a function of the chiral and anti-chiral superfields, ϕ^i and $\bar{\phi}^j$. A chiral superfield $\phi^i(x, \theta)$ consists of a complex scalar field $\varphi^i(x)$ and a complex fermion $\psi^i(x)$. The bosonic fields $\varphi^i(x)$ play the role of the coordinates of the target manifold $\mathcal{M}$. The metric, characterizing the target manifold $\mathcal{M}$, is obtained by the second derivative of this Kähler potential

$$g_{i\bar{j}} = \frac{\partial^2 K(\varphi, \bar{\varphi})}{\partial \varphi^i \partial \bar{\varphi}^j} \equiv K,_{i\bar{j}} \,.$$

The manifold defined by a Kähler potential is called the Kähler manifold. The Lagrangian of nonlinear sigma model with $\mathcal{N} = 2$ supersymmetry reads

$$\mathcal{L} = g_{i\bar{j}} \partial_\mu \varphi^i \partial^\mu \bar{\varphi}^j + i g_{i\bar{j}} \bar{\psi}^j (\not{D} \psi)^i + \frac{1}{4} R_{i\bar{j}k\bar{l}} \psi^i \psi^k \bar{\psi}^j \bar{\psi}^l,$$

where the covariant derivative for the fermion fields is given by

$$(D_\mu \psi)^i = \partial_\mu \psi^i + \partial_\mu \varphi^j \Gamma^i{}_{jk} \psi^k.$$

The connection and the Riemann curvature are also written in terms of the Kähler potential K as follws

$$\Gamma^k{}_{ij} = g^{k\bar{l}} g_{j\bar{l},i} = g^{k\bar{l}} K,_{ij\bar{l}} \,,$$
$$R_{i\bar{j}k\bar{l}} \equiv g_{i\bar{m}} R^{\bar{m}}{}_{\bar{j}k\bar{l}} = K,_{i\bar{j}k\bar{l}} - g^{m\bar{n}} K,_{m\bar{j}\bar{l}} K,_{\bar{n}ik} \,.$$

§3. Renormalization group equation

Renormalization group equation for the metric of the target manifold $\mathcal{M}$ in three dimensional sigma models has been derived in[4)–6)]

$$-\frac{d}{dt}g_{i\bar{j}} = \frac{1}{2\pi^2}R_{i\bar{j}} - g_{i\bar{j}} + \nabla_i\xi_{\bar{j}} + \nabla_{\bar{j}}\xi_i \tag{3·1}$$

where t parametrize the change of the cutoff $\Lambda \to \mathrm{e}^{-t}\Lambda$. The vector field ξ^i can be written

$$\xi^i = \left(\frac{1}{2} + \gamma\right)\varphi^i \tag{3·2}$$

in the Kähler normal coordinate.[7)] The anomalous dimension γ of the field ϕ is introduced to normalize the field at the origin

$$g_{i\bar{j}}\,|_{\varphi=0} = \delta_{i\bar{j}}. \tag{3·3}$$

The renormalization group equation (3·1), called the Ricci flow in mathematical literature,[8)] describes the deformation of the target manifold of the effective theory when the renormalization mass scale is changed from Λ to $\mathrm{e}^{-t}\Lambda$.

The fixed point, invariant under the change of the mass scale, is obtained by solving an equation

$$\frac{1}{2\pi^2}R_{i\bar{j}} - g_{i\bar{j}} + \nabla_i\xi_{\bar{j}} + \nabla_{\bar{j}}\xi_i = 0. \tag{3·4}$$

The metric $g_{i\bar{j}}$ satisfying this equation defines a conformal field theory.

In the rest of this section, let us discuss the change of the volume of the target manifold along the flow of the RGE. The change of the volume of the manifold along the renormalization group flow (3·1) is given by

$$\begin{aligned}\frac{d}{dt}\int\sqrt{\det(g)}d\varphi d\varphi^* &= \frac{1}{2}\int\sqrt{\det(g)}\mathrm{Tr}\left(g^{i\bar{j}}\frac{dg_{i\bar{j}}}{dt}\right)d\varphi d\varphi^*\\ &= -\frac{1}{4\pi^2}\int\sqrt{\det(g)}Rd\varphi d\varphi^* + \frac{dim_{\boldsymbol{C}}\mathcal{M}}{2}\int\sqrt{\det(g)}d\varphi d\varphi^*\end{aligned}$$

where use has been made of the covariant constancy of the metric $\nabla g = 0$ and the scalar curvature R is defined by

$$R = g^{i\bar{j}}R_{i\bar{j}}.$$

Terms depending on the anomalous dimension γ has disappeared in this equation, therefore the volume of the target manifold is independent of the anomalous dimension along the flow of RGE. When $dim_{\boldsymbol{C}}\mathcal{M} = 1$, the change of the volume is determined by the Euler number $\chi(\mathcal{M})$ of the manifold

$$\chi(\mathcal{M}) = \frac{1}{4\pi}\int\sqrt{\det(g)}Rd\varphi d\varphi^* = 2(1-g)$$

where the genus is 0,1 for S^2 and T^2. Especially, the volume at the fixed point is given by

$$V(\mathcal{M}) = \int \sqrt{\det(g)} d\varphi d\varphi^* = \frac{2}{\pi}\chi(\mathcal{M})$$

If the manifold is a round sphere S^2 with radius a,

$$V(S^2) = 4\pi a^2 = \frac{2}{\pi}$$

implies the radius at the fixed point is

$$a^2 = \frac{1}{2\pi^2}.$$

§4. Fixed point theory for $\gamma = -\frac{1}{2}$

When the anomalous dimension of the field takes a specific value $-\frac{1}{2}$, the fixed point of the renormalization group equation has an extremely simple form

$$\frac{1}{2\pi^2} R_{i\bar{j}} - g_{i\bar{j}} = 0. \tag{4·1}$$

By comparing with the equation for the Einstein-Kähler manifolds *)

$$R_{i\bar{j}} - h\lambda^2 g_{i\bar{j}} = 0, \tag{4·2}$$

with a positive cosmological constant $h\lambda^2 > 0$, we find the coupling constant λ (inverse radius of the Einstein-Kähler manifold) of the fixed point theory is given by

$$\lambda^2 = \frac{2\pi^2}{h}. \tag{4·3}$$

We found that any Einstein-Kähler manifold corresponds to the conformally invariant field theory, when the radius, the inverse coupling constant, takes a specific value (4·3).

A special class of the Kähler-Einstein manifolds is provided by the hermitian symmetric space(HSS) of the form G/H. The compact HSS is completely classified and listed in the following table, where h denotes the dual coxeter number of the group G. **)

G/H	$D = dim_{\mathbf{C}}(G/H)$	h
$SU(N)/SU(N-1) \times U(1)$	$N-1$	N
$U(N)/U(N-M) \times U(M)$	$M(N-M)$	N
$SO(N)/SO(N-2) \times U(1)$	$N-2$	$N-2$
$Sp(N)/U(N)$	$\frac{1}{2}N(N+1)$	$N+1$
$SO(2N)/U(N)$	$\frac{1}{2}N(N-1)$	$N-1$
$E_6/SO(10) \times U(1)$	16	12
$E_7/E_6 \times U(1)$	27	18

*) A parameter λ has been introduced to satisfy the renormalization condition (3·3) at the origin.

**) For S^2, λ^2 is related to the radius a^2 of the sphere by $\lambda^2 = 1/2a^2$ and $h = 2$.

The metric of HSS is explicitly constructed by using the gauge theory technique in ref.,[9] therefore it is possible to write down the Lagrangian of conformal field theories explicitly.

§5. Two-dimensional manifold

Although it is difficult to solve eq.(3·4) explicitly for $\gamma \neq -\frac{1}{2}$, it can be solved for two-dimensional target space $\mathcal{M}$ by using a graphical method. In this section, we use real variables to describe the target manifold $\mathcal{M}$, and choose a special gauge where the line element of $\mathcal{M}$ takes the following form

$$ds^2 = dr^2 + e^2(r) d\phi^2 \tag{5·1}$$

where we have assumed the rotational symmetry in the ϕ direction. Here $e(r)$ denotes the radius of a circle for a fixed value of r. In this coordinate system, the Ricci tensor and the vector field ξ^i takes the following form

$$\begin{aligned} R_{rr} = R^{\phi}_{r\phi r} = -\frac{e''}{e}, \quad R_{\phi\phi} = R^{r}_{\phi r\phi} = -ee'', \\ \xi^r = ce(r), \quad \xi^{\phi} = 0, \qquad (c = \frac{1}{2} + \gamma). \end{aligned} \tag{5·2}$$

Corresponding to the renormalization condition (3·3), we impose a boundary condition for $e(r)$

$$lim_{r\to 0} \frac{e(r)}{r} = 1. \tag{5·3}$$

The RG equation reads in this gauge

$$-a^2 e'' - e + 2cee' = 0. \tag{5·4}$$

When $c \neq 0$, it is convenient to rewrite the second order differential equation to a set of the first order differential equations

$$\begin{aligned} e' &= p \\ p' &= -\frac{1}{a^2} e(1 - 2cp) \end{aligned} \tag{5·5}$$

with the boundary condition

$$e(0) = 0, \quad p(0) = 1 \tag{5·6}$$

The vector field of the flow (5·5) is shown in fig.1. When $0 \leq 2c < 1$, this equation defines a compact manifold since the trajectory starting from the initial point (5·6) comes back to $e = 0$ at a finite r implying the circumference of the circle at that r vanishes. On the other hand, the solution corresponds to a noncompact manifold for $2c \geq 1$.

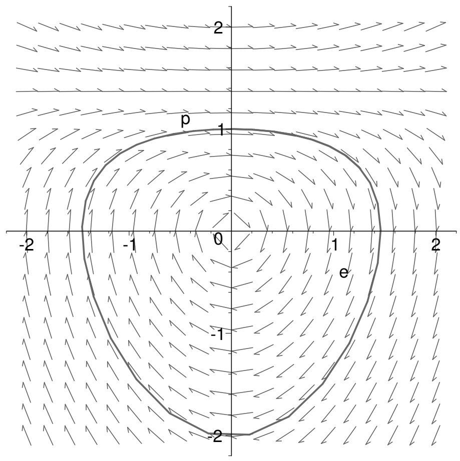

Fig. 1. Flow of the first order differential equations (5·5) for $2c < 1$ in the "phase space" $(e(r), p(r))$. The solid line represents the solution specified by the boundary condition.

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