A Lattice Formulation of Super Yang-Mills Theories with Exact Supersymmetry*)

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We construct SU(N) super Yang-Mills theories with extended supersymmetry on hypercubic lattices of various dimensions keeping one or two supercharges exactly.

It is based on topological field theory formulation for the super Yang-Mills theories. Gauge fields are represented by compact unitary link variables, and the exact supercharges on the lattice are nilpotent up to gauge transformations. In particular, the lattice models are free from the vacuum degeneracy problem, which was encountered in earlier approaches. Thus, we do not need to introduce any supersymmetry breaking terms, and the exact supersymmetry is preserved wholly in the process of taking the continuum limit.

Among the models, we show that the desired continuum theories are obtained without any fine tuning of parameters for the cases $\mathcal{N} = 2, 4, 8$ in two-dimensions. Also, the cases $\mathcal{N} = 4, 8$ in three-dimensions are investigated, and a problem arising in four-dimensional models is discussed.

§1. Introduction

Nonperturbative aspects in supersymmetric gauge theory are quite interesting not only from the field-theoretical point of view beyond the standard model, but also from the AdS/CFT duality between gauge theory and gravity in string theory.

A conventional approach to the nonperturbative study is lattice formulation. However, there has been difficulty on the lattice approach to supersymmetry, because of lack of infinitesimal translational invariance on the lattice and breakdown of the Leibniz rule. As we will discuss here, in spite of the difficulty, it is possible to construct lattice models, which do not have manifest full-fledged supersymmetry but flow to the desired supersymmetric theories in the continuum limit.

Supersymmetric theories with extended supersymmetry have some supercharges, which are not related to the infinitesimal translations and can be seen as fermionic internal symmetries. It is possible to realize a part of such supercharges as exact symmetry on lattice, and the exact supersymmetry is expected to play a key role to restore the full supersymmetry in the continuum limit with fine tuning of less or hopefully no parameters.

In sections 2 and 3, we construct lattice models for two-dimensional $\mathcal{N} = 2, 4$ SYM theories based on (balanced) topological field theory formulation, and discuss on renormalization near the continuum limit. In sections 4 and 5, starting naive lattice actions for four-dimensional $\mathcal{N} = 2, 4$ SYM theories, we construct lattice models for $\mathcal{N} = 4, 8$ in two-dimensions and for $\mathcal{N} = 8$ in three-dimensions. Section 6 is devoted to the summary and discussion on the results obtained here.

Throughout this paper, we focus on the gauge group $G = \mathrm{SU}(N)$. At the points

*) This presentation is based on the works 1)–3).

discussing continuum theories, notations of repeated indices in formulas are assumed to be summed. On the other hand, when treating lattice theories, we explicitly write the summation over the indices except the cases of no possible confusion.

§2. 2D $\mathcal{N}=2$ SYM

2.1. *Continuum Action*

The action of $\mathcal{N}=2$ SYM in two-dimensions can be written as the 'topological field theory (TFT) form':[5)]

$$S_{2D\mathcal{N}=2} = Q\frac{1}{2g^2}\int d^2x\, \mathrm{tr}\left[\frac{1}{4}\eta\,[\phi,\,\bar{\phi}] - i\chi\Phi + \chi H - i\psi_\mu D_\mu\bar{\phi}\right], \qquad (2\cdot 1)$$

where μ is the index for two-dimensional space-time. Bosonic fields are gauge fields A_μ, complex scalars ϕ, $\bar{\phi}$, and auxiliary field H. The other fields ψ_μ, χ, η are fermionic, and $\Phi = 2F_{12}$. Q is one of the supercharges of $\mathcal{N}=2$ supersymmetry, and its transformation rule is given as

$$\begin{aligned} QA_\mu &= \psi_\mu, \quad Q\psi_\mu = iD_\mu\phi, \\ Q\phi &= 0, \\ Q\chi &= H, \quad QH = [\phi,\,\chi], \\ Q\bar{\phi} &= \eta, \quad Q\eta = [\phi,\,\bar{\phi}]. \end{aligned} \qquad (2\cdot 2)$$

Q is nilpotent up to infinitesimal gauge transformations with the parameter ϕ. Note that the action has $\mathrm{U}(1)_R$ symmetry whose charge assignment is $+2$ for ϕ, -2 for $\bar{\phi}$, $+1$ for ψ_μ, -1 for χ and η, 0 for A_μ and H.

2.2. *Lattice Supersymmetry Q*

We formulate the theory (2·1) on the two-dimensional square lattice keeping the supersymmetry Q. In the lattice theory, gauge fields $A_\mu(x)$ are promoted to the compact unitary variables

$$U_\mu(x) = e^{iaA_\mu(x)} \qquad (2\cdot 3)$$

defined on the link $(x, x+\hat{\mu})$. 'a' stands for the lattice spacing, and $x \in \mathbf{Z}^2$ the lattice site. All other variables are distributed at sites. (See Fig.1.)

Interestingly, the Q-transformation (2·2) is extendible to the lattice variables preserving the property

$$Q^2 = (\text{infinitesimal gauge transformation with the parameter } \phi) \qquad (2\cdot 4)$$

as follows:

$$\begin{aligned} &QU_\mu(x) = i\psi_\mu(x)U_\mu(x), \\ &Q\psi_\mu(x) = i\psi_\mu(x)\psi_\mu(x) - i\left(\phi(x) - U_\mu(x)\phi(x+\hat{\mu})U_\mu(x)^\dagger\right), \\ &Q\phi(x) = 0, \\ &Q\chi(x) = H(x), \quad QH(x) = [\phi(x),\,\chi(x)], \\ &Q\bar{\phi}(x) = \eta(x), \quad Q\eta(x) = [\phi(x),\,\bar{\phi}(x)]. \end{aligned} \qquad (2\cdot 5)$$

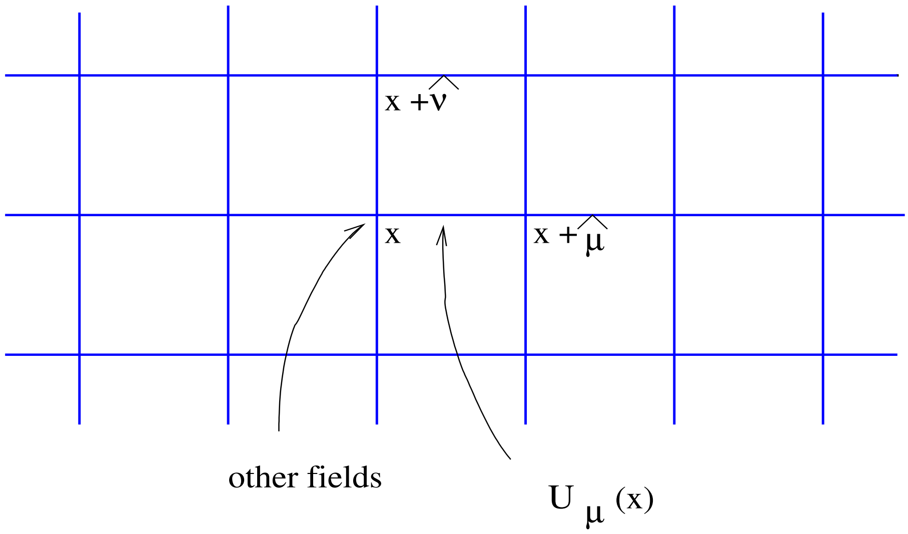

Fig. 1. On the lattice, the unitary variable $U_\mu(x)$ is defined on the link $(x, x+\hat{\mu})$. Other fields are put on sites.

Note that we use the dimensionless variables here, and that various quantities are of the following orders:

$$\psi_\mu(x), \chi(x), \eta(x) = O(a^{3/2}), \quad \phi(x), \bar{\phi}(x) = O(a), \quad H(x) = O(a^2),$$
$$Q = O(a^{1/2}). \tag{2·6}$$

The first term in the RHS of "$Q\psi_\mu(x) = \cdots$" in (2·5) is of subleading order $O(a^3)$ and irrelevant in the continuum limit.

2.3. *Lattice Action*

Once we obtain the Q-transformation rule closed among lattice variables, it is almost straightforward to construct the lattice action with the exact supersymmetry Q:

$$S^{\rm LAT}_{2D\mathcal{N}=2} = Q\frac{1}{2g_0^2}\sum_x \operatorname{tr}\left[\frac{1}{4}\eta(x)\left[\phi(x),\, \bar{\phi}(x)\right] - i\chi(x)(\Phi(x) + \Delta\Phi(x)) + \chi(x)H(x)\right.$$
$$\left. + i\sum_{\mu=1}^{2}\psi_\mu(x)\left(\bar{\phi}(x) - U_\mu(x)\bar{\phi}(x+\hat{\mu})U_\mu(x)^\dagger\right)\right], \tag{2·7}$$

where

$$\Phi(x) = -i\left[U_{12}(x) - U_{21}(x)\right], \tag{2·8}$$
$$\Delta\Phi(x) = -r(2 - U_{12}(x) - U_{21}(x)). \tag{2·9}$$

$U_{\mu\nu}$ are plaquette variables written as

$$U_{\mu\nu}(x) \equiv U_\mu(x)U_\nu(x+\hat{\mu})U_\mu(x+\hat{\nu})^\dagger U_\nu(x)^\dagger. \tag{2·10}$$

The action (2·7) is clearly Q-invariant from its Q-exactness, and is $\mathrm{U}(1)_R$ symmetric. It is an almost straightforward latticization of the continuum action (2·1) except $\Delta\Phi(x)$ introduced. We will explain a role of $\Delta\Phi(x)$.

After acting Q in the RHS, the action takes the form

$$\begin{aligned} S^{\rm LAT}_{2D\mathcal{N}=2} = \frac{1}{2g_0^2}\sum_x \mathrm{tr}\, \Bigg[& \frac{1}{4}[\phi(x),\,\bar{\phi}(x)]^2 + H(x)^2 - iH(x)(\Phi(x)+\Delta\Phi(x)) \\ & +\sum_{\mu=1}^{2}\left(\phi(x)-U_\mu(x)\phi(x+\hat{\mu})U_\mu(x)^\dagger\right)\left(\bar{\phi}(x)-U_\mu(x)\bar{\phi}(x+\hat{\mu})U_\mu(x)^\dagger\right) \\ & -\frac{1}{4}\eta(x)[\phi(x),\,\eta(x)] - \chi(x)[\phi(x),\,\chi(x)] \\ & -\sum_{\mu=1}^{2}\psi_\mu(x)\psi_\mu(x)\left(\bar{\phi}(x)+U_\mu(x)\bar{\phi}(x+\hat{\mu})U_\mu(x)^\dagger\right) + i\chi(x)Q(\Phi(x)+\Delta\Phi(x)) \\ & - i\sum_{\mu=1}^{2}\psi_\mu(x)\left(\eta(x)-U_\mu(x)\eta(x+\hat{\mu})U_\mu(x)^\dagger\right)\Bigg]. \end{aligned} \tag{2·11}$$

In order to see the relevance of $\Delta\Phi(x)$, let us consider the case without $\Delta\Phi(x)$ in the action. After integrating out $H(x)$, induced $\Phi(x)^2$ term yields the gauge kinetic term as the form

$$\frac{1}{2g_0^2}\sum_x\sum_{\mu<\nu}\mathrm{tr}\,\left[-(U_{\mu\nu}(x)-U_{\nu\mu}(x))^2\right], \tag{2·12}$$

which is different from the standard Wilson action

$$\frac{1}{2g_0^2}\sum_x\sum_{\mu<\nu}\mathrm{tr}\,\left[2-U_{\mu\nu}(x)-U_{\nu\mu}(x)\right]. \tag{2·13}$$

In contrast with (2·13) giving the unique minimum $U_{\mu\nu}(x)=1$, the action (2·12) has many classical vacua

$$U_{\mu\nu}(x) = \mathrm{diag}\,(\pm 1,\cdots,\pm 1) \tag{2·14}$$

up to gauge transformations, where any combinations of ± 1 with '-1' appearing even times are allowed in the diagonal entries. Since the configurations (2·14) can be taken freely for each plaquette, it leads a huge degeneracy of vacua with the number growing as exponential of the number of the plaquettes. In order to see the dynamics of the model, we need to sum up contributions from all of the minima, and the ordinary weak field expansion around a single vacuum $U_{\mu\nu}(x)=1$ can not be justified.*) Thus, we can not say anything about the continuum limit of the lattice model (2·11) without its nonperturbative investigations. In order to resolve the difficulty without affecting the Q-supersymmetry, we introduce the $\Delta\Phi(x)$ terms with an appropriate choice of the parameter $r=\cot\theta$:**)

$$e^{i2\ell\theta}\neq 1 \quad \text{for } \forall\ell=1,\cdots,N. \tag{2·15}$$

*) This kind of difficulty already appeared in Ref. 4).

**) For a discussion about how the degeneracy is removed, see Ref. 2).

2.4. *Renormalization*

At the classical level, the lattice action (2·7) leads to the continuum action (2·1) in the limit $a \to 0$ with $g^{-2} \equiv a^2 g_0^{-2}$ kept fixed, and thus the $\mathcal{N} = 2$ supersymmetry and rotational symmetry in two-dimensions are restored. We will check whether the symmetry restoration persists against quantum corrections, i.e. whether symmetries of the lattice action forbid any relevant or marginal operators induced which possibly obstruct the symmetry restoration.

$p = a + b + 3c$	$\varphi^a \partial^b \psi^{2c}$				
0			1		
1			φ		
2			φ^2		
3		φ^3,	$\psi\psi$,	$\varphi\partial\varphi$	
4	φ^4,	$\varphi^2\partial\varphi$,	$(\partial\varphi)^2$,	$\psi\partial\psi$,	$\varphi\psi\psi$

Table I. List of operators with $p \leq 4$.

Assuming that the model has the critical point $g_0 = 0$ from the asymptotic freedom, we shall consider the renormalization effect perturbatively. The mass dimension of the coupling g^2 is two. After a rescaling of fields as indicated in (2·6), for generic boson field φ (other than the auxiliary fields) and fermion field ψ, the dimensions are 1 and 3/2 respectively. Thus, operators of the type $\varphi^a \partial^b \psi^{2c}$ have the dimension $p \equiv a + b + 3c$, where '∂' means a derivative with respect to a coordinate. From the dimensional analysis, we can see that the operators receive the following radiative corrections up to some powers of possible logarithmic factors:

$$\left(\frac{a^{p-4}}{g^2} + c_1 a^{p-2} + c_2 a^p g^2 + \cdots \right) \int \mathrm{d}^2 x \, \varphi^a \partial^b \psi^{2c}, \tag{2·16}$$

where $c_1, c_2, \cdots$ are constants dependent on N. The first, second and third terms in the parentheses represent the contributions at tree, one-loop and two-loop levels. It is easily seen from the fact that g^2 appears as an overall factor in front of the action and plays the same role as the Planck constant $\hbar$. Due to the super-renormalizable property of two-dimensional theory, the relevant corrections terminate at the two-loop. From the above formula, it is seen that the following operators can be relevant or marginal in the $a \to 0$ limit: operators with $p \leq 2$ induced at the one-loop level and with $p = 0$ at the two-loop level. Operators with $p \leq 4$ are listed in Table I.

Since the identity operator does not affect the spectrum, we have to check operators of the types φ and φ^2 only. Gauge symmetry and $\mathrm{U}(1)_R$ invariance*) allow the operator $\mathrm{tr}\,\phi\bar{\phi}$, however it is forbidden by the supersymmetry Q. Hence, no relevant or marginal operators except the identity are generated by radiative corrections, which means that in the continuum limit full supersymmetry and rotational symmetry are considered to be restored without any fine tuning.

*) Note that the $\mathrm{U}(1)_R$ symmetry is not anomalous for $G = \mathrm{SU}(N)$ in the two-dimensions.

§3. 2D $\mathcal{N}=4$ SYM

3.1. *Continuum Action*

The action of $\mathcal{N}=4$ SYM in two-dimensions can be written as the following 'Balanced Topological Field Theory (BTFT) form':[6),7)]

$$
\begin{aligned}
S_{2D\mathcal{N}=4} &= Q_+Q_-\mathcal{F}_{2D\mathcal{N}=4},\\
\mathcal{F}_{2D\mathcal{N}=4} &= \frac{1}{2g^2}\int \mathrm{d}^2x\,\mathrm{tr}\left[-iB\Phi-\psi_{+\mu}\psi_{-\mu}-\chi_+\chi_--\frac{1}{4}\eta_+\eta_-\right],
\end{aligned}
\tag{3·1}
$$

where $Q_\pm$ are two of supercharges of the $\mathcal{N}=4$ theory, and $\Phi\equiv 2F_{12}$. Bosons are gauge fields A_μ ($\mu=1,2$) and scalar fields B, C, ϕ, $\bar\phi$. Also, there are auxiliary fields $\tilde H_\mu$, H. Other fields $\psi_{\pm\mu}$, $\chi_\pm$, $\eta_\pm$ are fermions. Transformation rule of the supersymmetry $Q_\pm$ is given by

$$
\begin{aligned}
&Q_+A_\mu=\psi_{+\mu},\quad Q_+\psi_{+\mu}=iD_\mu\phi,\quad Q_-\psi_{+\mu}=\frac{i}{2}D_\mu C-\tilde H_\mu,\\
&Q_-A_\mu=\psi_{-\mu},\quad Q_-\psi_{-\mu}=-iD_\mu\bar\phi,\quad Q_+\psi_{-\mu}=\frac{i}{2}D_\mu C+\tilde H_\mu,\\
&Q_+\tilde H_\mu=[\phi,\,\psi_{-\mu}]-\frac{1}{2}[C,\,\psi_{+\mu}]-\frac{i}{2}D_\mu\eta_+,\\
&Q_-\tilde H_\mu=[\bar\phi,\,\psi_{+\mu}]+\frac{1}{2}[C,\,\psi_{-\mu}]+\frac{i}{2}D_\mu\eta_-,
\end{aligned}
\tag{3·2}
$$

$$
\begin{aligned}
&Q_+B=\chi_+,\quad Q_+\chi_+=[\phi,\,B],\quad Q_-\chi_+=\frac{1}{2}[C,\,B]-H,\\
&Q_-B=\chi_-,\quad Q_-\chi_-=-[\bar\phi,\,B],\quad Q_+\chi_-=\frac{1}{2}[C,\,B]+H,\\
&Q_+H=[\phi,\,\chi_-]+\frac{1}{2}[B,\,\eta_+]-\frac{1}{2}[C,\,\chi_+],\\
&Q_-H=[\bar\phi,\,\chi_+]-\frac{1}{2}[B,\,\eta_-]+\frac{1}{2}[C,\,\chi_-],
\end{aligned}
\tag{3·3}
$$

$$
\begin{aligned}
&Q_+C=\eta_+,\quad Q_+\eta_+=[\phi,\,C],\quad Q_-\eta_+=-[\phi,\,\bar\phi],\\
&Q_-C=\eta_-,\quad Q_-\eta_-=-[\bar\phi,\,C],\quad Q_+\eta_-=[\phi,\,\bar\phi],\\
&Q_+\phi=0,\quad Q_-\phi=-\eta_+,\quad Q_+\bar\phi=\eta_-,\quad Q_-\bar\phi=0.
\end{aligned}
\tag{3·4}
$$

This transformation leads the following nilpotency of $Q_\pm$ (up to gauge transformations):

$$
\begin{aligned}
Q_+^2 &= (\text{infinitesimal gauge transformation with the parameter } \phi),\\
Q_-^2 &= (\text{infinitesimal gauge transformation with the parameter } -\bar\phi),\\
\{Q_+,Q_-\} &= (\text{infinitesimal gauge transformation with the parameter } C).
\end{aligned}
\tag{3·5}
$$

In this formulation, among the SU(4) internal symmetry of the $\mathcal{N}=4$ theory, its subgroup $SU(2)_R$ is manifest which rotates (Q_+,Q_-). Under the $SU(2)_R$, each of $(\psi^a_{+\mu}, \psi^a_{-\mu})$, (χ^a_+, χ^a_-), $(\eta^a_+, -\eta^a_-)$ and (Q_+, Q_-) transforms as a doublet, and $(\phi^a, C^a, -\bar{\phi}^a)$ as a triplet. Also, let us note a symmetry of the action (3·1) under exchanging the two supercharges $Q_+ \leftrightarrow Q_-$ with

$$\begin{aligned}
&\phi \to -\bar{\phi}, \quad \bar{\phi} \to -\phi, \quad B \to -B, \\
&\chi_+ \to -\chi_-, \quad \chi_- \to -\chi_+, \quad \tilde{H}_\mu \to -\tilde{H}_\mu, \\
&\psi_{\pm\mu} \to \psi_{\mp\mu}, \quad \eta_\pm \to \eta_\mp.
\end{aligned} \tag{3·6}$$

3.2. *Lattice Supersymmetry* $Q_\pm$

Similarly to the $\mathcal{N} = 2$ cases, it is possible to define the theory (3·1) on the square lattice preserving the two supercharges $Q_\pm$. The transformation rule (3·2) is modified as

$$\begin{aligned}
Q_+ U_\mu(x) &= i\psi_{+\mu}(x) U_\mu(x), \\
Q_- U_\mu(x) &= i\psi_{-\mu}(x) U_\mu(x), \\
Q_+ \psi_{+\mu}(x) &= i\psi_{+\mu}\psi_{+\mu}(x) - i\left(\phi(x) - U_\mu(x)\phi(x+\hat{\mu})U_\mu(x)^\dagger\right), \\
Q_- \psi_{-\mu}(x) &= i\psi_{-\mu}\psi_{-\mu}(x) + i\left(\bar{\phi}(x) - U_\mu(x)\bar{\phi}(x+\hat{\mu})U_\mu(x)^\dagger\right), \\
Q_- \psi_{+\mu}(x) &= \frac{i}{2}\left\{\psi_{+\mu}(x),\, \psi_{-\mu}(x)\right\} - \frac{i}{2}\left(C(x) - U_\mu(x)C(x+\hat{\mu})U_\mu(x)^\dagger\right) - \tilde{H}_\mu(x), \\
Q_+ \psi_{-\mu}(x) &= \frac{i}{2}\left\{\psi_{+\mu}(x),\, \psi_{-\mu}(x)\right\} - \frac{i}{2}\left(C(x) - U_\mu(x)C(x+\hat{\mu})U_\mu(x)^\dagger\right) + \tilde{H}_\mu(x), \\
Q_+ \tilde{H}_\mu(x) &= -\frac{1}{2}\left[\psi_{-\mu}(x),\, \phi(x) + U_\mu(x)\phi(x+\hat{\mu})U_\mu(x)^\dagger\right] \\
&\quad +\frac{1}{4}\left[\psi_{+\mu}(x),\, C(x) + U_\mu(x)C(x+\hat{\mu})U_\mu(x)^\dagger\right] \\
&\quad +\frac{i}{2}\left(\eta_+(x) - U_\mu(x)\eta_+(x+\hat{\mu})U_\mu(x)^\dagger\right) \\
&\quad +\frac{i}{2}\left[\psi_{+\mu}(x),\, \tilde{H}_\mu(x)\right] + \frac{1}{4}\left[\psi_{+\mu}(x)\psi_{+\mu}(x),\, \psi_{-\mu}(x)\right], \\
Q_- \tilde{H}_\mu(x) &= -\frac{1}{2}\left[\psi_{+\mu}(x),\, \bar{\phi}(x) + U_\mu(x)\bar{\phi}(x+\hat{\mu})U_\mu(x)^\dagger\right] \\
&\quad -\frac{1}{4}\left[\psi_{-\mu}(x),\, C(x) + U_\mu(x)C(x+\hat{\mu})U_\mu(x)^\dagger\right] \\
&\quad -\frac{i}{2}\left(\eta_-(x) - U_\mu(x)\eta_-(x+\hat{\mu})U_\mu(x)^\dagger\right) \\
&\quad +\frac{i}{2}\left[\psi_{-\mu}(x),\, \tilde{H}_\mu(x)\right] - \frac{1}{4}\left[\psi_{-\mu}(x)\psi_{-\mu}(x),\, \psi_{+\mu}(x)\right].
\end{aligned} \tag{3·7}$$

The other transformations (3·3, 3·4) do not change the form under the latticization. Note that this modification keeps the nilpotency (3·5).

Making use of the $Q_\pm$-transformation rule in terms of lattice variables, we construct lattice actions with the exact supercharges $Q_\pm$ as

$$S^{\rm LAT}_{2D\mathcal{N}=4} = Q_+ Q_- \frac{1}{2g_0^2} \sum_x \mathrm{tr}\, \Bigg[-iB(x)(\Phi(x) + \Delta\Phi(x)) - \sum_{\mu=1}^{2} \psi_{+\mu}(x)\psi_{-\mu}(x)$$

$$-\chi_+(x)\chi_-(x) - \frac{1}{4}\eta_+(x)\eta_-(x)\Bigg], \qquad (3\cdot8)$$

where $\Phi(x)$ and $\Delta\Phi(x)$ are given by (2·8) and (2·9), respectively. Note that the lattice formulation retains the symmetries under $\mathrm{SU}(2)_R$ as well as the $Q_+ \leftrightarrow Q_-$ exchange.

Similarly to the $\mathcal{N}=2$ case, $\Delta\Phi(x)$ ensures to remove the vacuum degeneracy. With respect to the renormalization argument, symmetries of the lattice action are sufficient to restore full supersymmetry and rotational invariance in the continuum limit. For instance, gauge invariance and $\mathrm{SU}(2)_R$ symmetry allow the operators $\mathrm{tr}\,(4\phi\bar{\phi}+C^2)$ and $\mathrm{tr}\,B^2$, but they are not admissible from the supersymmetry $Q_\pm$. Thus, radiative corrections are not allowed to generate any relevant or marginal operators except the identity, which means the restoration of full supersymmetry and rotational invariance in the continuum limit.

§4. 3D $\mathcal{N}=4$

Also for $\mathcal{N}=2$ theory in four-dimensions, we can write the action in the 'TFT form', and construct a *naive* lattice action as

$$S^{\mathrm{LAT}}_{4D\mathcal{N}=2} = Q\frac{1}{2g_0^2}\sum_x \mathrm{tr}\Bigg[\frac{1}{4}\eta(x)\,[\phi(x),\,\bar{\phi}(x)] - i\vec{\chi}(x)\cdot(\vec{\Phi}(x)+\Delta\vec{\Phi}(x)) + \vec{\chi}(x)\cdot\vec{H}(x)$$
$$+ i\sum_{\mu=1}^{4}\psi_\mu(x)\left(\bar{\phi}(x) - U_\mu(x)\bar{\phi}(x+\hat{\mu})U_\mu(x)^\dagger\right)\Bigg], \qquad (4\cdot1)$$

where $\vec{H}(x)$, $\vec{\chi}(x)$, $\vec{\Phi}(x)$ and $\Delta\vec{\Phi}(x)$ are three-component vectors, and

$$\Phi_{\mathtt{A}}(x) = -i\left[U_{4,-\mathtt{A}}(x) - U_{-\mathtt{A},4}(x) + \frac{1}{2}\sum_{\mathtt{B},\mathtt{C}=1}^{3}\varepsilon_{\mathtt{ABC}}\left(U_{\mathtt{BC}}(x) - U_{\mathtt{CB}}(x)\right)\right], \qquad (4\cdot2)$$

$$\Delta\Phi_1(x) = -r\left[W_{4,-1}(x) + W_{23}(x)\right],$$
$$\Delta\Phi_2(x) = -r\left[W_{4,-2}(x) + W_{31}(x)\right],$$
$$\Delta\Phi_3(x) = -r\left[W_{4,-3}(x) + W_{12}(x)\right]. \qquad (4\cdot3)$$

$W_{\mu\nu}(x)$ are defined by

$$W_{\mu\nu}(x) \equiv 2 - U_{\mu\nu}(x) - U_{\nu\mu}(x), \quad U_{-\mu}(x) = U_\mu(x-\hat{\mu})^\dagger. \qquad (4\cdot4)$$

The vacuum degeneracy is removed with the choice $r=\cot\theta$: $0<\theta\le\frac{\pi}{2N}$.*) It turns out, however, that the quadratic terms in A_μ in $\mathrm{tr}\,(\vec{\Phi}(x)+\Delta\vec{\Phi}(x))^2$ have surplus zero-modes (other than gauge degrees of freedom) carrying the nonzero momentum in the fourth direction. Fermion kinetic terms also have zero-modes at the same momenta, which is consistent to the exact supersymmetry Q because the surplus modes are

*) For a detailed discussion, see Ref. 3).

not exact zero-modes of the full action (only of the quadratic terms) and a fermionic partner necessarily exist for each bosonic surplus modes.

Here, we do not resolve the problem, but consider the dimensional reduction with respect to the fourth direction. Then, the four-dimensional $\mathcal{N} = 2$ theory reduces to three-dimensional $\mathcal{N} = 4$ theory where the surplus modes are all killed. Thus, the dimensionally reduced lattice model reproduces desired three-dimensional $\mathcal{N} = 4$ theory in the classical continuum limit. The renormalization argument tells that necessary is fine-tuning of three parameters for the counter terms with the mass dimension three:

$$Q\sum_{\mu=1}^{3} \mathrm{tr}\,(\psi\bar{\phi}), \quad Q\,\mathrm{tr}\,(\psi_4\bar{\phi}), \quad Q\sum_{\mathtt{A}=1}^{3} \mathrm{tr}\,(\chi_{\mathtt{A}} A_4)$$

in order to arrive at the desired continuum theory at the quantum level.

§5. 3D $\mathcal{N} = 8$ and 2D $\mathcal{N} = 8$

We can similarly construct a *naive* lattice action for four-dimensional $\mathcal{N} = 4$ SYM, where however the same problem of the surplus modes occurs. Considering the dimensional reduction with respect to the fourth direction, we obtain a lattice model for three-dimensional $\mathcal{N} = 8$ SYM which reproduces the desired theory in the classical continuum limit. Also, further reduction with respect to the third direction leads two-dimensional $\mathcal{N} = 8$ theory. For the three-dimensional $\mathcal{N} = 8$ model, one parameter fine-tuning for an operator of the mass dimension three is required, while the two-dimensional model of $\mathcal{N} = 8$ needs no fine-tuning.[3)]

§6. Summary and Discussion

We have constructed varioius lattice models for SYM theories of $\mathcal{N} = 2, 4, 8$ in two-dimensions and of $\mathcal{N} = 4, 8$ in three-dimensions, based on '(balanced) topological field theory form' of the theories. The formulation exactly realizes a part of the supersymmetry and employes compact link variables for the gauge fields on hypercubic lattice. From the renormalization argument, we have shown that the desired continuum theories are obtained by fine-tuning three and one parameters for the three-dimensional $\mathcal{N} = 4$ and 8 theories respectively, while the two-dimensional theories require no tunings.

We have also seen that there exist surplus modes in four-dimensional naive lattice models for $\mathcal{N} = 2, 4$. It may be related to exact realization of the topological term $\mathrm{tr}\, F \wedge F$ on the lattice which needs a nonabelian extension of the solution for the U(1) case.[8)*)]

*) Catterall has proposed lattice models for $\mathcal{N} = 2, 4$ SYM theories in two- and four-dimensions respectively, both of which are free from the problem of the surplus modes.[9)] However, all the fields appearing in the models are complexified and we have to pick up the real parts in order to get the correct theories. Unfortunately, it has been unclear how to do it with keeping the exact lattice supersymmetry.

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