Understanding the Mechanism for the Spontaneous Breakdown of Rotational Symmetry in the IIB Matrix Model *

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Recently, the Gaussian expansion method has been applied to investigate the dynamical generation of 4d space-time in the IIB matrix model, which is a conjectured nonperturbative definition of type IIB superstring theory in 10 dimensions. Evidence for such a phenomenon, which is associated with the spontaneous breaking of the $SO(10)$ symmetry down to $SO(4)$, has been obtained up to 7-th order calculations. Here we discuss an application of the same method to a simplified model, which is considered to exhibit an analogous spontaneous symmetry breaking via the same mechanism as conjectured for the IIB matrix model. The results up to 9-th order demonstrate a clear convergence, which allows us to unambiguously identify the actual symmetry breaking pattern by comparing the free energy of possible vacua and to calculate the extent of "space-time" in each direction.

§1. Introduction

It has long been believed that matrix models may be useful as a nonperturbative formulation of string theory, and hence play an important role similar to the lattice formulation of quantum field theory. Indeed, matrix models have been quite successful in formulating non-critical string theory, and after the development of such notions as string duality and D-branes, the idea has been extended also to critical strings. The IIB matrix model[2)] is one such proposal, which is conjectured to be a nonperturbative definition of type IIB superstring theory in 10 dimensions. It is a supersymmetric matrix model, which can be formally obtained as the zero-volume limit of 10d $SU(N)$ super Yang-Mills theory.

In this model space-time is represented by the eigenvalue distribution of ten bosonic matrices.[3)] If the distribution collapses dynamically to a four-dimensional hypersurface, which, in particular, requires the $SO(10)$ symmetry of the model to be spontaneously broken, we may naturally understand the dimensionality of our space-time as a result of the nonperturbative dynamics of superstring theory. In Ref. 4) the first evidence for the above scenario was obtained by calculating the free energy of space-times with various dimensionalities using the Gaussian expansion method up to 3rd order. Higher-order calculations,[5),6)] as well as tests of the method itself in simpler models,[7),8)] have strengthened the conclusion considerably. Further evidence for the emergence of a four-dimensional space-time based on perturbative calculations around fuzzy-sphere like solutions is provided in Ref. 9)

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While these results are certainly encouraging, it is desirable to understand the mechanism responsible for the spontaneous symmetry breaking (SSB) of rotational symmetry. In Ref. 10) it is pointed out that the phase of the fermion determinant favors lower-dimensional configurations, since the phase becomes stationary around such configurations. [1] Indeed, Monte Carlo simulations show that SSB does *not* occur in various models *without* such a phase factor.[13)–15)] Unfortunately, including the effects of the phase in Monte Carlo simulations is technically difficult, due to the so-called sign problem, but a new method,[16)] which has been tested in random matrix theory,[17)] is capable of producing some preliminary results, which appear to be promising. In Ref. 18) a simple matrix model which realizes the above mechanism is proposed. The model contains $N_{\rm f}$ flavors of Weyl fermions in the fundamental representation of $SU(N)$, which yields a complex fermion determinant, and the large N limit is taken with the ratio $r = N_{\rm f}/N$ fixed. The model can be solved exactly for infinitesimal r, and it has been shown that the $SO(4)$ symmetry breaks down to $SO(3)$.

In this talk we present a study[1)] of this model at *finite* r using the Gaussian expansion method. Because this model is much simpler than the IIB matrix model, we can perform calculations up to 9-th order with reasonable effort. The results demonstrate a clear convergence for $r \lesssim 2$, which allows us to unambiguously identify the symmetry breaking pattern and to calculate the extent of "space-time" in each direction.

In fact, it turns out that the $SO(4)$ symmetry is broken down to $SO(2)$ at finite r. However, at small r we obtain the same free energy and extent of "space-time" in each direction as in Ref. 18), which implies that the $SO(3)$ symmetry is realized asymptotically as r approaches zero. In the large r region, on the other hand, the extent of "space-time" in two directions, in which the $SO(2)$ symmetry is realized, becomes much larger than in the remaining two directions. This behavior can be understood from the viewpoint of Ref. 10), because the phase of the fermion determinant becomes stationary for two-dimensional configurations, and increasing r tends to amplify the effect of the phase. Thus our results nicely demonstrate the proposed mechanism for the dynamical generation of space-time in the IIB matrix model.

The rest of this article is organized as follows. In §2 we define the model and review existing results. In §3 we explain how to apply the Gaussian expansion method to the model. In §4 we present our results. §5 is devoted to a summary and discussions.

§2. The model

The model we are going to discuss is defined by the partition function[18)]

$$Z = \int dA\, d\psi\, d\bar{\psi}\, e^{-(S_{\rm b}+S_{\rm f})} \ , \tag{2·1}$$

[1] See Refs. 3),11) and 12) for discussions of other possible mechanisms.

$$S_{\rm b} = \frac{1}{2} N \, {\rm tr}\, (A_\mu)^2 \ , \tag{2·2}$$

$$S_{\rm f} = - N \bar{\psi}^f_\alpha (\Gamma_\mu)_{\alpha\beta} A_\mu \psi^f_\beta \ , \tag{2·3}$$

where A_μ ($\mu = 1, \cdots, 4$) are $N \times N$ traceless [2] hermitian matrices and $\bar{\psi}^f_\alpha$ and ψ^f_α ($\alpha = 1, 2$; $f = 1, \cdots, N_{\rm f}$) are N-dimensional row and column vectors, respectively, making the system $SU(N)$ invariant. The integration measure for A_μ is given by

$$dA = \prod_{a=1}^{N^2-1} \prod_{\mu=1}^{4} \frac{dA^a_\mu}{\sqrt{2\pi}} \ , \tag{2·4}$$

where A^a_μ is the coefficient in the expansion $A_\mu = \sum_{a=1}^{N^2-1} A^a_\mu T^a$ with respect to the $SU(N)$ generators T^a ($a = 1, \cdots, (N^2 - 1)$) normalized as ${\rm tr}\,(T^a T^b) = \frac{1}{2}\delta^{ab}$. The integration measure for the fermions is given by

$$d\psi \, d\bar{\psi} = \prod_{f=1}^{N_{\rm f}} \prod_{i=1}^{N} \prod_{\alpha=1}^{2} d\psi^{fi}_\alpha \, d\bar{\psi}^{fi}_\alpha \ . \tag{2·5}$$

The system possesses $SO(4)$ symmetry, under which A_μ transforms as a vector, and ψ^f_α and $\bar{\psi}^f_\alpha$ transform as Weyl spinors. The 2×2 matrices Γ_μ are the gamma matrices after the Weyl projection. As one example of their explicit forms we have,

$$\Gamma_1 = i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \Gamma_2 = i \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \Gamma_3 = i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \Gamma_4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{2·6}$$

The fermionic part of the model can be thought of as the zero-volume limit of the system of Weyl fermions in four dimensions interacting with a background gauge field via a fundamental coupling.

We take the large N limit, keeping the ratio $r \equiv N_{\rm f}/N$ fixed (the so-called Veneziano limit). In order to study the SSB of the $SO(4)$ symmetry in this limit, we consider the "moment of inertia tensor,"[3),13)]

$$T_{\mu\nu} = \frac{1}{N} {\rm tr}\,(A_\mu A_\nu) \ , \tag{2·7}$$

which is a 4×4 real symmetric tensor, and denote its eigenvalues as $\{\lambda_i \, ; \, i = 1, \cdots, 4\}$, with the specified order

$$\lambda_1 \ge \lambda_2 \ge \lambda_3 \ge \lambda_4 \ . \tag{2·8}$$

If the vacuum expectation values (VEVs) $\langle \lambda_i \rangle$ ($i = 1, \cdots, 4$) do not agree in the large N limit, we can conclude that SSB occurs. Thus $\langle \lambda_i \rangle$ plays the role of an order parameter. In the present model, the sum of the VEVs is given by

$$\sum_{i=1}^{4} \langle \lambda_i \rangle = \sum_{\mu=1}^{4} \left\langle \frac{1}{N} {\rm tr}\,(A_\mu)^2 \right\rangle = 4 \left(1 - \frac{1}{N^2} \right) + 2\, r \tag{2·9}$$

[2] The tracelessness condition was not imposed in the original paper.[18)] While this condition does not affect the large N limit of the model, it simplifies our calculation drastically (see Footnote in page 4.).

for arbitrary N and r, as found from a "virial theorem".[18)]

For infinitesimal r, the VEVs can be obtained in the large N limit as[18)]

$$\begin{aligned} \langle\lambda_1\rangle = \langle\lambda_2\rangle = \langle\lambda_3\rangle &= 1 + r + o(r) \ , \\ \langle\lambda_4\rangle &= 1 - r + o(r) \ , \end{aligned} \tag{2·10}$$

which means that the $SO(4)$ symmetry is spontaneously broken down to $SO(3)$. The SSB is associated with the formation of a condensate $\langle\bar{\psi}^f_\alpha \psi^f_\alpha\rangle$, which is invariant under $SO(3)$, but not under the full $SO(4)$ transformation.

An important feature of the model that is relevant to the SSB is that the fermion determinant $\det \mathcal{D}$, where $\mathcal{D}$ is a $2N \times 2N$ matrix given by $\mathcal{D} = \Gamma_\mu A_\mu$, is complex in general. If one replaces the fermion determinant by its absolute value, [3] applying the same analysis to the case of infinitesimal r leads to an $SO(4)$ symmetric result.[18)] Thus the SSB of the original model is due to the phase of the fermion determinant.

For large r, the effect of the phase is amplified, and we may expect that the configurations for which the phase is stationary dominate the path integral. Analogously to the situation in the IIB matrix model,[10)] the phase becomes stationary for 2-dimensional configurations in the present model. Therefore, we anticipate the emergence of a 2-dimensional "space-time" (λ_1, $\lambda_2 \gg \lambda_3$, λ_4) as r increases. [4]

§3. The Gaussian expansion method

In this section, we derive results for the model (2·1) in the case of finite r using the Gaussian expansion method. This method has a long history. The original idea appeared around 1980 in the context of solving quantum mechanical systems.[19),20)] In those works, it was shown that the expansion converges in some specific examples.[21)] The method also proved useful in field theories[22)] in various contexts. Applications to superstring/M theories using their matrix model formulations have been advocated by Kabat and Lifschytz,[23)] and a subsequent series of papers[24)] revealed interesting black hole thermodynamics. Applications to simplified versions of the IIB matrix model were initiated in Ref. 25). An earlier application to random matrix models can be found in Ref. 26).

In analogy to the case of the IIB matrix model,[4)–6)] let us introduce the $SU(N)$-invariant Gaussian action [5]

$$S_0 = \frac{1}{2} N \sum_{\mu=1}^{4} t_\mu \, \mathrm{tr}\,(A_\mu)^2 + N \sum_{f=1}^{N_\mathrm{f}} \sum_{\alpha,\beta=1}^{2} \mathcal{A}_{\alpha\beta} \, \bar{\psi}^f_\alpha \, \psi^f_\beta \ , \tag{3·1}$$

[3] This may be achieved by introducing $N_\mathrm{f}/2$ left-handed Weyl fermions and $N_\mathrm{f}/2$ right-handed Weyl fermions, or in other words, by introducing $N_\mathrm{f}/2$ Dirac fermions. In this case, the theory becomes vector-like.

[4] Note that the phase of the fermion determinant is invariant under the scale transformation $A_\mu \mapsto \alpha A_\mu$. Therefore it is only the *ratio* of the eigenvalues that matters for the stationarity.

[5] If we do not impose the tracelessness condition on A_μ, we would need to consider a linear term, such as $S_\mathrm{lin} = N \sum_\mu h_\mu \mathrm{tr}\, A_\mu$, in the Gaussian action (3·1). The Gaussian expansion method can be extended to such a case,[8)] but the calculation would be more involved.

which breaks the $SO(4)$ symmetry for general t_μ and $\mathcal{A}_{\alpha\beta}$. [We do not include terms proportional to $(\psi^f_\alpha)^\top \psi^f_\beta$ and $\bar{\psi}^f_\alpha (\bar{\psi}^f_\beta)^\top$, since they are not $SU(N)$ invariant.] The 2×2 complex matrix $\mathcal{A}$ can be expanded in terms of gamma matrices as

$$\mathcal{A} = \sum_{\mu=1}^{4} u_\mu \Gamma_\mu \ , \tag{3·2}$$

using four complex parameters u_μ. Then we consider the action

$$S_{\mathrm{GEM}}(t,u;\lambda) = \frac{1}{\lambda}\left[\left\{ S_0 + \lambda(S_{\mathrm{b}} - S_0)\right\} + S_{\mathrm{f}}\right] \ , \tag{3·3}$$

which reduces to the original action for $\lambda = 1$. The Gaussian expansion amounts to calculating various quantities in an expansion with respect to λ up to some finite order and then setting $\lambda = 1$ at the end. As we discuss below, the free parameters t_μ and u_μ in the Gaussian action S_0 play a crucial role in the method.

In fact, the Gaussian expansion can be interpreted as a loop expansion with the "classical action" $(S_0 + S_{\mathrm{f}})$ and the "one-loop counterterms" $(S_{\mathrm{b}} - S_0)$. This becomes clear upon rescaling A_μ and ψ as $A_\mu \mapsto \lambda A_\mu$ $\psi^f_\alpha \mapsto \sqrt{\lambda}\,\psi^f_\alpha$, so that the partition function takes the form

$$Z = \int dA\, d\psi\, d\bar{\psi}\, e^{-(S_{\mathrm{cl}} + S_{\mathrm{c.t.}})} \ , \tag{3·4}$$

$$S_{\mathrm{cl}}(t,u) = S_0 + \sqrt{\lambda}\, S_{\mathrm{f}} \quad , \quad S_{\mathrm{c.t.}}(t,u) = \lambda\,(S_{\mathrm{b}} - S_0) \ . \tag{3·5}$$

In actual calculations, the "one-loop counter terms" can be incorporated easily by exploiting the relation

$$S_{\mathrm{cl}}(t,u) + S_{\mathrm{c.t.}}(t,u) = S_{\mathrm{cl}}(t + \lambda(1-t), u - \lambda u) \ . \tag{3·6}$$

As an example, let us consider evaluating the free energy $F = -\frac{1}{N^2}\ln Z$ using the Gaussian expansion method. We first calculate the free energy $\mathcal{F}(t,u)$ for the "classical action" $S_{\mathrm{cl}}(t,u)$ defined by

$$\exp[-N^2 \mathcal{F}(t,u)] = \int dA\, d\psi\, d\bar{\psi}\, e^{-S_{\mathrm{cl}}(t,u)} \ . \tag{3·7}$$

This can be done with ordinary Feynman diagrammatic calculations, where use of Schwinger-Dyson (SD) equations reduces the number of diagrams considerably.[5] Suppose we obtain the result up to K-th order as

$$\mathcal{F}_K(t,u) = \sum_{k=0}^{K} \widetilde{\mathcal{F}}_k(t,u)\,\lambda^k \ . \tag{3·8}$$

We shift the arguments, and obtain the new coefficients $\tilde{F}_k(t,u)$ in the expansion

$$\mathcal{F}_K(t + \lambda(1-t), u - \lambda u) = \sum_{k=0}^{K} \tilde{F}_k(t,u)\,\lambda^k + O(\lambda^{K+1}) \ . \tag{3·9}$$

Then the free energy for the original model can be evaluated as

$$F_K(t,u) = \sum_{k=0}^{K} \tilde{F}_k(t,u) \ . \tag{3·10}$$

The result of such calculations depends on the free parameters in the Gaussian action. However, in various models[5),7),8)] it was found that a "plateau" region, in which the result becomes almost constant, develops in the parameter space as one goes to higher orders of the expansion. Moreover, it turns out that the height of the plateau agrees very accurately with the correct value obtained with another method. Therefore it is reasonable to conjencture that the method is valid in general if one can identify a plateau in the parameter space. In early works, the free parameters were determined in such a way that the result is most insensitive to changes in the parameter values.[20)] However, it is actually the formation of the plateau that ensures the validity of the method, as first recognized in Ref. 5).

Identification of a plateau becomes non-trivial when there are many parameters in the Gaussian action. The histogram prescription,[7),8)] which is effective when there are only one or two real parameters, apparently becomes inapplicable when the number exceeds three. (Note that we have four real and four complex parameters in the present case.) We have also attempted a Monte Carlo simulation *in parameter space* to search for a plateau, but we have had little success. We therefore use the prescription adopted for the IIB matrix model. First we solve the "self-consistency equations"

$$\begin{aligned} \frac{\partial}{\partial t_\mu} F_K(t,u) &= 0 \ , \\ \frac{\partial}{\partial u_\mu} F_K(t,u) &= 0 \ . \end{aligned} \tag{3·11}$$

Typically we obtain many solutions as we go to higher orders. If we observe that solutions are concentrated in some region of the parameter space, we consider this to be an indication of plateau formation.

Although the number of parameters in the present case is much less than that in the IIB matrix model (where we have 10 real and 120 complex parameters), it is still technically difficult to obtain all the solutions of the self-consistency equations at high orders. As is done in the case of the IIB matrix model,[4)–6)] we search for solutions assuming that some subgroup of the full rotational symmetry is preserved. Here we consider the following Ansätze. In each case, there are two real and one complex independent parameters.

***SO*(3) Ansatz :** We assume $SO(3)$ symmetry in the x_2, x_3 and x_4 directions. Then the parameters are restricted as

$$t_2 = t_3 = t_4 (\equiv \tilde{t}), \qquad u_2 = u_3 = u_4 = 0 \ . \tag{3·12}$$

***SO*(2) Ansatz :** We assume $SO(2)$ symmetry in the x_3 and x_4 directions. Furthermore we impose discrete symmetry under $x_1 \to x_2$, $x_2 \to x_1$, $x_4 \to -x_4$. Then the parameters are restricted as

$$t_1 = t_2, \quad t_3 = t_4 (\equiv \tilde{t}), \qquad u_1 = u_2, \quad u_3 = u_4 = 0 \ . \tag{3·13}$$

Note that the free energy calculated at the zero-th order involves a term $-r \ln(\det \mathcal{A})$. Therefore, in order to obtain a real value for the free energy, we require $\mathit{\Delta} \equiv \det \mathcal{A} = \sum_{i=1}^{4} (u_i)^2$ to be real and positive. With the above Ansätze, it turns out that everything can be written in terms of t_1, $\tilde{t}$ and $\mathit{\Delta}$. We solve the self-consistency equation (3·11) with the Newton method, using initial values chosen in the region $0.2 \le t_1 \le 4$, $0.2 \le \tilde{t} \le 4$, $0.5 \le \mathit{\Delta} \le 10$. This parameter region is discretized into a $20 \times 20 \times 20$ lattice, and we use each lattice point as a set of initial values. In Figs. 1 and 2 we show all the solutions obtained in this way.

Although we cannot think of a natural "$SO(4)$ Ansatz", the Gaussian expansion method with the above two Ansätze can, in principle, lead to the conclusion that the $SO(4)$ symmetry is not spontaneously broken, because a plateau may form around $t_1 = \cdots = t_4$, $u_1 = \cdots = u_4 \sim 0$ as we increase the order. The results in the following, however, show that this is not the case.

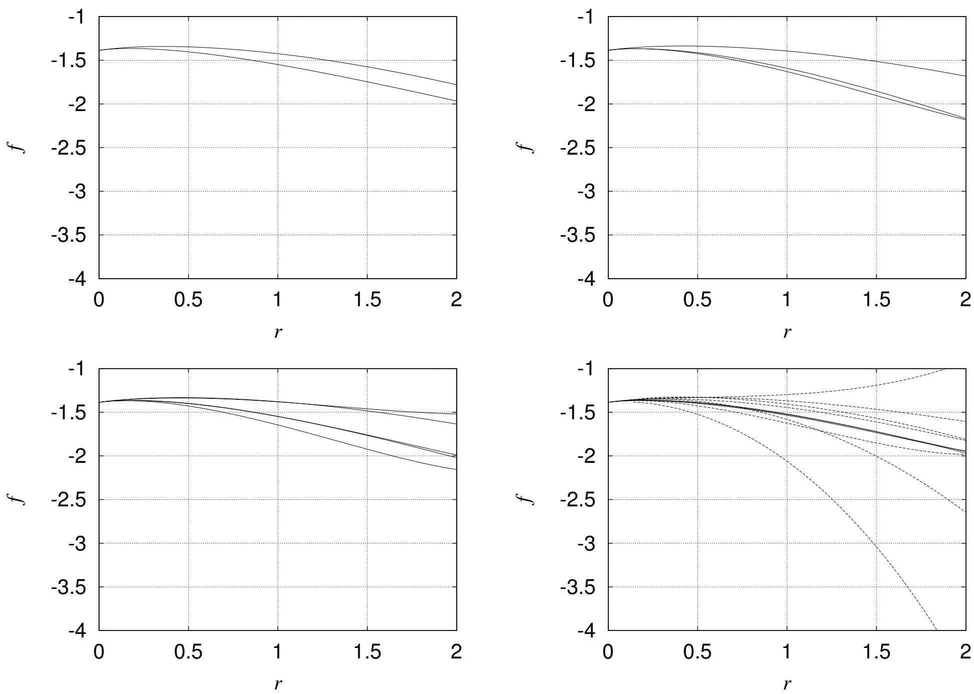

Fig. 1. The free energy obtained for the $SO(3)$ Ansatz at orders 3 (left top), 5 (right top), 7 (left bottom) and 9 (right bottom) is plotted as a function of r.

§4. Results

For each solution of the self-consistency equations obtained with the symmetry Ansatz, we calculate the free energy. The free energy we plot below is actually defined by

$$f = \lim_{N\to\infty} \{F - 2(1-r)\ln N\} \ , \tag{4·1}$$

where the subtraction is necessary to make the quantity finite. The exact result at infinitesimal r is given by[18)]

$$f = -2\ln 2 + (1 - \ln 2)r + \mathrm{o}(r) \ . \tag{4·2}$$

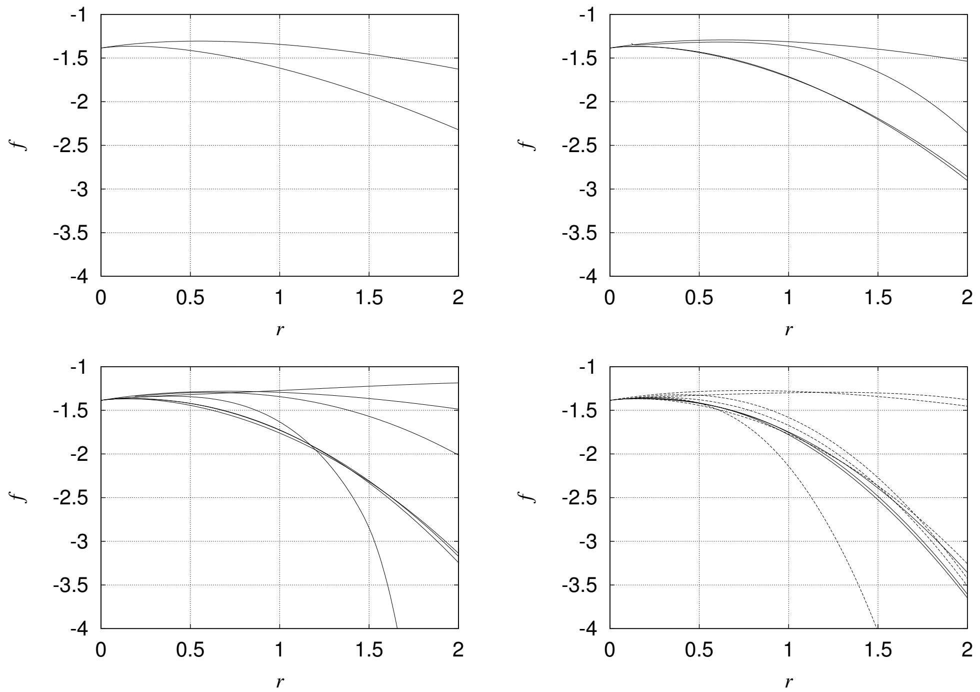

Fig. 2. The free energy obtained for the $SO(2)$ Ansatz at orders 3 (left top), 5 (right top), 7 (left bottom) and 9 (right bottom) is plotted as a function of r.

In Fig. 1 the free energy calculated for the $SO(3)$ Ansatz at orders 3, 5, 7 and 9 is plotted as a function of r. We find that at orders 5 and 7 there are two solutions which almost coincide over almost the entire range [6] $0 \le r \le 2$. At 9-th order there are actually three solutions lying on top of each other, which are represented by the solid curves in order to distinguish them from the other solutions. We consider this as an indication of plateau formation. Similar behavior is observed for the $SO(2)$ Ansatz, as can be seen from Fig. 2. Again the three solid curves in the right bottom panel represent the solutions that we consider to be concentrating.

We consider the three solutions which concentrate at 9-th order for the two Ansätze and plot them in Fig. 3 for comparison. Throughout the entire range of r

[6] For $r \gtrsim 2$ the solutions that concentrate at 9-th order start to separate, and we cannot obtain reliable results in that regime. Note in this regard that the number of fermion loops gives the power of r in the result of the Gaussian expansion. Therefore it is reasonable that the convergence becomes slower as r increases.

considered, the free energy for the $SO(2)$ Ansatz is smaller than that for the $SO(3)$ Ansatz. Thus we conclude that the true vacuum is described by the $SO(2)$ Ansatz. However, the results for the two Ansätze converge to each other as r approaches zero. The meaning of this behavior is clarified below.

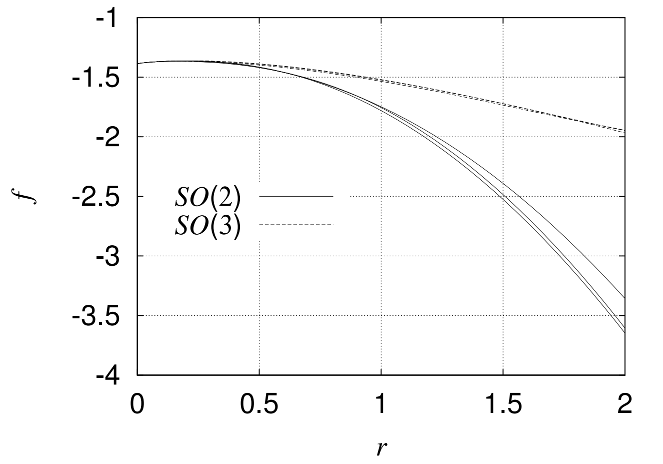

Fig. 3. Comparison of the free energy obtained for the $SO(2)$ and $SO(3)$ Ansätze (solid curves and dashed curves, respectively). The solutions that concentrate at 9-th order are extracted from Figs. 1 and 2.

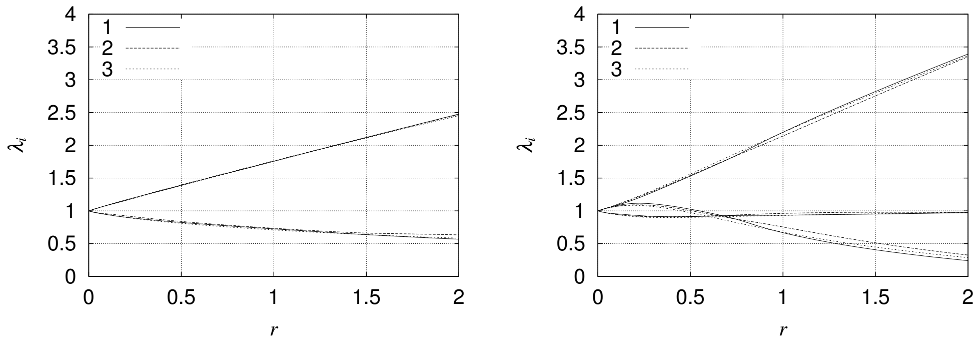

Fig. 4. The four eigenvalues of the "moment of inertia tensor" (2·7) obtained for the $SO(3)$ Ansatz (left) and the $SO(2)$ Ansatz (right) at 9-th order are shown as functions of r. Note that the largest eigenvalue has 3-fold (2-fold) degeneracy for the $SO(3)$ Ansatz and the $SO(2)$ Ansatz, respectively. The three types of curves correspond to the three solutions of the self-consistency equations (3·11) that concentrate at 9-th order.

Let us move on to the calculation of observables. Similarly to the free energy, we can calculate an observable as an expansion with respect to λ using the action (3·3). In our model, the observables of primary interest are the eigenvalues λ_i of the "moment of inertia tensor" (2·7). We calculate them for the $SO(3)$ and $SO(2)$ Ansätze at 9-th order as functions of the free parameters in the Gaussian action, and we plug in the three solutions that were found to concentrate in the study of the free energy. Figure 4 displays the results. (We have made analogous plots for orders 5 and 7, and the results turned out to converge to that obtained for 9-th order as we move from order 5 to order 7.) Note that we are ultimately interested in the results for the $SO(2)$ Ansatz, since it gives the smaller free energy.

For the $SO(3)$ ($SO(2)$) Ansatz, the curves that increase almost linearly actually represent three (two) eigenvalues, which are degenerate due to the assumed symmetry. For the $SO(2)$ Ansatz, it turns out that the third largest eigenvalue comes closer to the two degenerate largest ones as $r = 0$ is approached, thus realizing the $SO(3)$ symmetry asymptotically. In fact, the results obtained for the $SO(2)$ Ansatz are indistinguishable from those obtained for the $SO(3)$ Ansatz at small r. This is consistent with our observation in Fig. 3 that the free energy for the $SO(2)$ Ansatz has the same asymptotic behavior as the $SO(3)$ Ansatz for $r \to 0$. Actually, we find that both the free energy and the observable agree asymptotically with the exact results given in (4·2) and (2·10) at infinitesimal r.

For large r, by contrast, the results for the $SO(2)$ Ansatz show a clear tendency for the two degenerate eigenvalues to become much larger than the other two, which implies the emergence of a two-dimensional "space-time". This is consistent with the argument based on the phase stationarity given at the end of §2. Thus the Gaussian expansion method enables us to obtain results at finite r, which naturally interpolate between the $SO(3)$ symmetric exact result at infinitesimal r and the two-dimensional behavior expected at large r.

§5. Summary and discussion

In this talk we have discussed an application of the Gaussian expansion method to a matrix model that is considered to exhibit SSB of rotational symmetry due to the phase of the fermion determinant. The free energy calculated with the Gaussian expansion method depends on the free parameters in the Gaussian action, and the formation of a plateau in the parameter space is necessary for the validity of the method. For each Ansatz considered with regard to the possible breaking pattern of the $SO(4)$ symmetry, we obtained clear evidence for plateau formation. By comparing the free energies obtained for all Ansätze, we conclude that the true vacuum is described by the $SO(2)$ Ansatz. Our results for the extent of "space-time" in each direction are consistent with the exact result for infinitesimal r and also with the behavior expected at large r.

The mechanism for the SSB of rotational symmetry demonstrated in the present model is expected to exist also in the IIB matrix model. However, we should note the difference between the two models. In the present model, the fermionic degrees of freedom match the bosonic ones at $r = 1$, but there, the "space-time" is not really two dimensional. Rather, it takes a form similar to a four-dimensional rugby ball, with two directions more extended than the other two. Moreover, if we consider the ten-dimensional version of the present model, we expect that there will be eight directions more extended than the other two. In the IIB matrix model, contrastingly, the result of the Gaussian expansion method shows that the ratio of the extent of space-time in four directions to that in the remaining six directions increases with order, up to 7-th order.[4)–6)] This suggests that *four* directions are *much more* (possibly, infinitely more) extended than the remaining six directions. It is conceivable that supersymmetry plays an important role here. Let us recall that the effective theory for the eigenvalues of the bosonic matrices in the IIB matrix model is a weakly bound system, like a branched polymer, due to a cancellation between the bosonic and fermionic contributions.[3)] This makes it easy for the space-time to collapse. Note also that the convergence of the Gaussian expansion is not as clear in the IIB matrix model as in the present model.

While it is certainly worthwhile to proceed to the 8-th or 9-th order in the IIB matrix model, we believe that Monte Carlo simulations similar to those of Ref. 16) are necessary to definitely confirm the emergence of a four-dimensional space-time. That approach is also expected to provide an intuitive understanding of why the space-time should be four-dimensional, rather than three-dimensional or five-dimensional. Note in this regard that the present simplified model, like the IIB matrix model, has a technical difficulty involving the implementation of the phase of the fermion determinant in the Monte Carlo simulation. The results obtained here therefore provide a nice testing ground for the new method of simulating the IIB matrix model.

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