Exact results in the Arcetri model of growing interfaces

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MH & **X. Durang**, J. Stat. Mech. P05022 (2015) [arxiv:1501.07745]

some words on geography/history

first mentioned $\sim$ 1050 (castle *Nanciacum*)

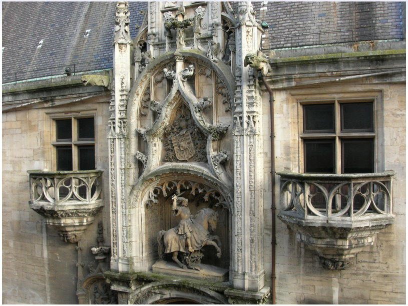
1265-1766 capital of dukedom of Lorraine

1572 foundation of the University

(at Pont-à-Mousson, since 1769 in Nancy)

1749 french translation of Newton's *Principia* — Voltaire & Marquise du Châtelet

1940s-1950s N. Bourbaki in Nancy ; theory of distributions — L. Schwartz

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L'art nouveau et l'École de Nancy ∼ 1895 - 1910






Overview :

0. Physical ageing : a reminder
1. Magnets and growing interfaces : analogies
2. Interface growth & KPZ universality class
3. Interface growth and Arcetri models : heuristics
4. First Arcetri model : simple ageing
5. Second Arcetri model : several marginally different length scales
6. Conclusions

0. Physical ageing : a reminder

known & practically used since prehistoric times (metals, glasses)
systematically studied in physics since the 1970s STRUIK '78

discovery : ageing effects **reproducible** & **universal** !

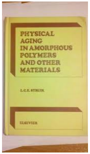
occur in widely different systems

(structural glasses, spin glasses, polymers, simple magnets, . . .)

Three defining properties of **ageing** :

1. slow relaxation (non-exponential !)
2. **no** time-translation-invariance (TTI)
3. dynamical scaling

without fine-tuning of parameters



Cooperative phenomenon, far from equilibrium

Two-time observables for simple magnets

time-dependent magnetisation = order-parameter = $\phi(t, \mathbf{r})$

two-time **correlator** $\quad C(t,s) := \langle \phi(t,\mathbf{r})\phi(s,\mathbf{r})\rangle - \langle \phi(t,\mathbf{r})\rangle \langle \phi(s,\mathbf{r})\rangle$

two-time **response** $\quad R(t,s) := \left. \frac{\delta \langle \phi(t,\mathbf{r})\rangle}{\delta h(s,\mathbf{r})}\right|_{h=0} = \left\langle \phi(t,\mathbf{r})\widetilde{\phi}(s,\mathbf{r})\right\rangle$

t : observation time, s : waiting time

a) system **at** equilibrium : **fluctuation-dissipation theorem** Kubo

$$R(t-s) = \frac{1}{T}\frac{\partial C(t-s)}{\partial s} \quad , \qquad T \text{ : temperature}$$

b) **far from equilibrium** : C and R **independent** !

The **fluctuation-dissipation ratio** (FDR) Cugliandolo, Kurchan, Parisi '94

$$X(t,s) := \frac{TR(t,s)}{\partial C(t,s)/\partial s}$$

measures the distance with respect to equilibrium : $\boxed{X_{\rm eq} = X(t-s) = 1}$

For quenches to $T \leq T_c$: $X \neq 1 \Longrightarrow$ system **never** reaches equilibrium

Scaling regime : $\boxed{t, s \gg \tau_{\rm micro} \text{ and } t - s \gg \tau_{\rm micro}}$

$$C(t,s) = s^{-b} f_C\left(\frac{t}{s}\right) \ , \ \ R(t,s) = s^{-1-a} f_R\left(\frac{t}{s}\right)$$

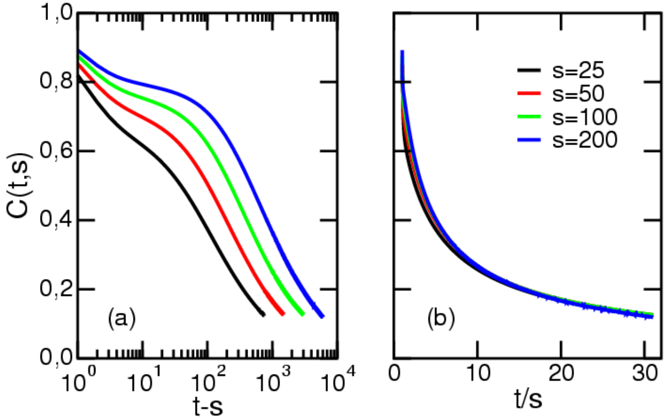
asymptotics : $\boxed{f_C(y) \sim y^{-\lambda_C/z},\ f_R(y) \sim y^{-\lambda_R/z}}$ for $y \gg 1$

λ_C : autocorrelation exponent, λ_R : autoresponse exponent,
z : dynamical exponent, a, b : ageing exponents

Constat : exponents & scaling functions are universal,
i.e. independent of 'fine details'

may use simplified theoretical models to find their values

Dynamical scaling in the ageing $3D$ Ising model, $T < T_c$



no time-translation invariance

dynamical scaling

$C(t,s)$: autocorrelation function, quenched to $T < T_c$

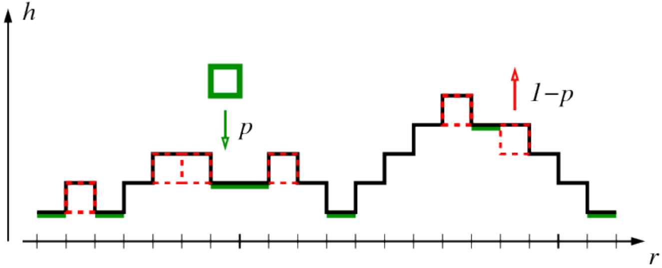
scaling regime : $t, s \gg \tau_{\rm micro}$ and $t - s \gg \tau_{\rm micro}$

data collapse evidence for **dynamical scale-invariance** MH & PLEIMLING 10

Interface growth

deposition (evaporation) of particles on a substrate

$\rightarrow$ height profile $h(t, \mathbf{r})$ slope profile $\mathbf{u}(t, \mathbf{r}) = \nabla h(t, \mathbf{r})$



$p =$ deposition prob.

$1 - p =$ evap. prob.

Questions :

* average properties of profiles & their fluctuations ?
* what about their relaxational properties ?
* are these also examples of physical ageing ?

? does dynamical scaling **always** exist ?

1. Magnets and growing interfaces : analogies

Common properties of critical and ageing phenomena :

* **collective** behaviour,
  very **large** number of interacting degrees of freedom
* **algebraic** large-distance and/or large-time behaviour
* described in terms of **universal** critical **exponents**
* very **few** relevant scaling operators
* justifies use of extremely **simplified** **mathematical models**
  with a remarkably rich and complex behaviour
* yet of **experimental significance**

Magnets

thermodynamic equilibrium state

order parameter $\phi(t,\mathbf{r})$

phase transition, at critical temperature T_c

variance :

$\left\langle (\phi(t,\mathbf{r}) - \langle\phi(t)\rangle)^2 \right\rangle \sim t^{-2\beta/(\nu z)}$

relaxation, after quench to $T \leq T_c$

autocorrelator

$C(t,s) = \langle \phi(t,\mathbf{r})\phi(s,\mathbf{r})\rangle_c$

Interfaces

growth continues forever

height profile $h(t,\mathbf{r})$

same generic behaviour throughout

roughness :

$w(t)^2 = \left\langle \left(h(t,\mathbf{r}) - \overline{h}(t) \right)^2 \right\rangle \sim t^{2\beta}$

relaxation, from initial substrate :

autocorrelator $C(t,s) =$

$\left\langle \left(h(t,\mathbf{r}) - \overline{h}(t) \right) \left(h(s,\mathbf{r}) - \overline{h}(s) \right) \right\rangle$

ageing scaling behaviour :

when $t,s\to\infty$, and $y := t/s > 1$ fixed, expect

$$C(t,s) = s^{-b} f_C\left(t/s\right) \quad \text{and} \quad f_C(y) \overset{y\to\infty}{\sim} y^{-\lambda_C/z}$$

b, β, ν and dynamical exponent z : **universal** & related to stationary state

autocorrelation exponent λ_C : **universal** & independent of stationary exponents

Magnets

exponent value $b = \begin{cases} 0 & ; \; T < T_c \\ 2\beta/\nu z & ; \; T = T_c \end{cases}$

Interfaces

exponent value $b = -2\beta$

models :

(a) gaussian field

$\mathcal{H}[\phi] = -\frac{1}{2}\int \mathrm{d}\mathbf{r}\,(\nabla\phi)^2$

(b) **Ising model**

$\mathcal{H}[\phi] = -\frac{1}{2}\int \mathrm{d}\mathbf{r}\,\left[(\nabla\phi)^2 + \tau\phi^2 + \frac{g}{2}\phi^4\right]$

such that $\tau = 0 \leftrightarrow T = T_c$

dynamical Langevin equation (Ising) :

$$\begin{aligned} \partial_t\phi &= -D\frac{\delta\mathcal{H}[\phi]}{\delta\phi} + \eta \\ &= D\nabla^2\phi + \tau\phi + g\phi^3 + \eta \end{aligned}$$

(a) Edwards-Wilkinson (EW) :

$\partial_t h = \nu\nabla^2 h + \eta$

(b) **Kardar-Parisi-Zhang** (KPZ) :

$$\partial_t h = \nu\nabla^2 h + \frac{\mu}{2}(\nabla h)^2 + \eta$$

$\eta(t,\mathbf{r})$ is the usual white noise, $\langle\eta(t,\mathbf{r})\eta(t',\mathbf{r}')\rangle = 2T\delta(t-t')\delta(\mathbf{r}-\mathbf{r}')$

phase transition exactly solved $d = 2$

relaxation exactly solved $d = 1$

Onsager '44, Glauber '63, ...

growth exactly solved $d = 1$

Calabrese & Le Doussal '11

Sasamoto & Spohn '10

Question : obtain qualitative understanding by approximate treatment of the non-linearity ?

Ising model : **yes, certainly !** $\boxed{\Rightarrow \text{ **spherical model** !}}$

BERLIN & KAC 52
LEWIS & WANNIER 52

(a) for a lattice model : replace Ising spins $\sigma_i = \pm 1 \mapsto S_i \in \mathbb{R}$, with (mean) spherical constraint $\sum_i \langle S_i^2 \rangle = \mathcal{N}$
(b) for continuum field : replace $\phi^3 \mapsto \phi\langle\phi^2\rangle$ and spherical constraint $\int \mathrm{d}\mathbf{r}\, \langle\phi^2\rangle \sim 1$.

Interest : analytically solvable for any d and in more general contexts than Ising model, all exponents **. . .** known exactly, non-trivial for $2 < d < 4$. Very useful to illustrate general principles in a specific setting. New universality class, distinct from the Ising model ($\mathrm{O}(N)$ model with $N \to \infty$).

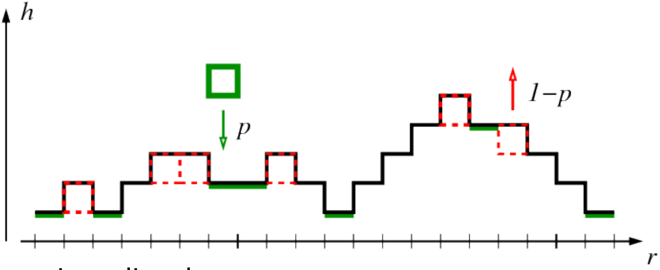
STANLEY 68

Question : can one find a similar procedure, based on the KPZ equation ?

Are there new universality class(es) for interface growth ?
Behaviour different from the rather trivial EW-equation ?

2. Interface growth & KPZ class

deposition (evaporation) of particles on a substrate $\to$ height profile $h(t, \mathbf{r})$
generic situation : RSOS (**r**estricted **s**olid-**o**n-**s**olid) model KIM & KOSTERLITZ 89



$p =$ deposition prob.
$1 - p =$ evap. prob.

here $p = 0.98$

some universality classes :

(a) KPZ $\partial_t h = \nu \nabla^2 h + \frac{\mu}{2} (\nabla h)^2 + \eta$ KARDAR, PARISI, ZHANG 86

(b) EW $\partial_t h = \nu \nabla^2 h + \eta$ EDWARDS, WILKINSON 82

η is a gaussian white noise with $\langle \eta(t, \mathbf{r}) \eta(t', \mathbf{r}') \rangle = 2\nu T \delta(t - t') \delta(\mathbf{r} - \mathbf{r}')$

Family-Viscek scaling on a spatial lattice of extent L^d : $\overline{h}(t) = L^{-d} \sum_j h_j(t)$

Family & Viscek 85

$$w^2(t;L) = \frac{1}{L^d} \sum_{j=1}^{L^d} \left\langle \left(h_j(t) - \overline{h}(t) \right)^2 \right\rangle = L^{2\alpha} f\left(tL^{-z} \right) \sim \begin{cases} L^{2\alpha} & ; \text{ if } tL^{-z} \gg 1 \\ t^{2\beta} & ; \text{ if } tL^{-z} \ll 1 \end{cases}$$

β : growth exponent (≥ 0), α : roughness exponent, $\boxed{\alpha = \beta z}$

two-time correlator :

limit $L \to \infty$

$$C(t,s;\mathbf{r}) = \left\langle \left(h(t,\mathbf{r}) - \langle \overline{h}(t) \rangle \right) \left(h(s,\mathbf{0}) - \langle \overline{h}(s) \rangle \right) \right\rangle = s^{-b} F_C\left(\frac{t}{s}, \frac{\mathbf{r}}{s^{1/z}} \right)$$

with ageing exponent : $\boxed{b = -2\beta}$

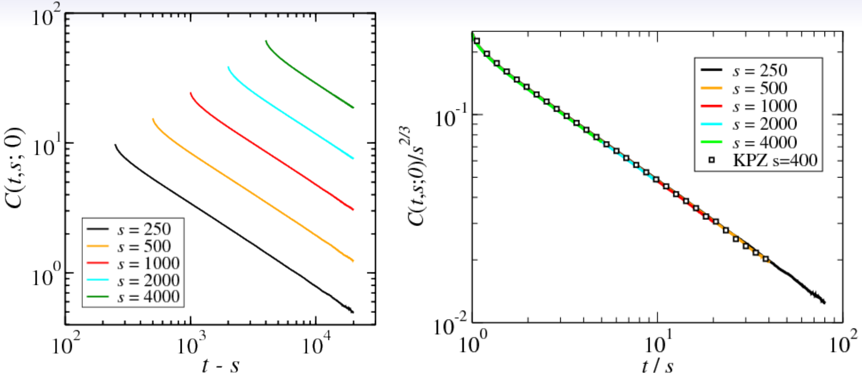
Kallabis & Krug 96

expect for $y = t/s \gg 1$: $F_C(y,\mathbf{0}) \sim y^{-\lambda_C/z}$ autocorrelation exponent

rigorous bound : $\boxed{\lambda_C \geq (d + zb)/2}$

Yeung, Rao, Desai 96 ; mh & Durang 15

$1D$ relaxation dynamics, starting from an initially flat interface



observe all **3** properties of **ageing** : $\begin{cases} \text{slow dynamics} \\ \text{no TTI} \\ \text{dynamical scaling} \end{cases}$

confirm **simple ageing** for the $1D$ KPZ universality class

confirm expected exponents $b = -2/3$, $\lambda_C/z = 2/3$ *pars pro toto*

Kallabis & Krug 96 ; Krech 97 ; Bustingorry *et al.* 07-10 ; Chou & Pleimling 10 ;
D'Aquila & Täuber 11/12 ; mh, Noh, Pleimling 12 ...

3. Interface growth & Arcetri models : heuristics

? KPZ $\longrightarrow$ $\boxed{\text{intermediate model}}$ $\longrightarrow$ EW ?

preferentially exactly solvable, and this in $d \geq 1$ dimensions

inspiration : mean **spherical model** of a ferromagnet — Berlin & Kac 52, Lewis & Wannier 52

Ising spins $\sigma_i = \pm 1$ obey $\sum_i \sigma_i^2 = \mathcal{N} = \#$ sites

spherical spins $S_i \in \mathbb{R}$ spherical constraint $\left\langle \sum_i S_i^2 \right\rangle = \mathcal{N}$

hamiltonian $\mathcal{H} = -J \sum_{(i,j)} S_i S_j - \lambda \sum_i S_i^2$ Lagrange multiplier λ

exponents **non**-mean-field for $2 < d < 4$ and $T_c > 0$ for $d > 2$

kinetics from Langevin equation $\boxed{\partial_t \phi = -D \frac{\delta \mathcal{H}[\phi]}{\delta \phi} + \mathfrak{z}(t)\phi + \eta}$

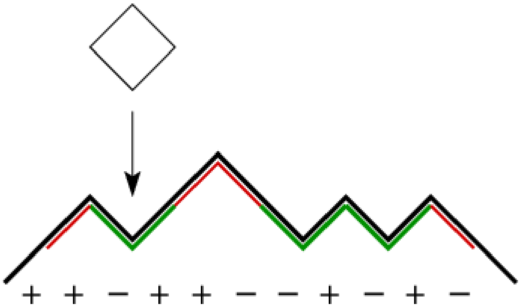
time-dependent Lagrange multiplier $\mathfrak{z}(t)$ fixed from spherical constraint

all equilibrium and ageing exponents exactly known, for $T < T_c$ and $T = T_c$

Ronca 78, Coniglio & Zannetti 89, Cugliandolo, Kurchan, Parisi 94, Godrèche & Luck '00, Corberi, Lippiello, Fusco, Gonnella & Zannetti 02-14 ...

consider **RSOS**-adsorption process :

rigorous : continuum limit gives KPZ   BERTINI & GIACOMIN 97



use **not** the heights $h_n(t) \in \mathbb{N}$ on a discrete lattice,
but rather the **slopes** $u_n(t) = \frac{1}{2}\left(h_{n+1}(t) - h_{n-1}(t)\right) = \pm 1$ **RSOS**

? let $u_n(t) \in \mathbb{R}$, & impose a spherical constraint $\boxed{\sum_n \langle u_n(t)^2 \rangle \stackrel{!}{=} \mathcal{N}}$?

? consequences of the 'hardening' of a soft EW-interface by a 'spherical constraint' on the u_n ?

KPZ equation for height $h(t,r)$: $$\partial_t h = \nu \partial_r^2 h + \frac{\mu}{2} (\partial_r h)^2 + \eta$$

Burger's equation for slope $u(t,r) = \partial_r h(t,r)$:

$$\partial_t u = \nu \partial_r^2 u + \mu u \partial_r u + \partial_r \eta$$

model $\mathrm{A}I$: $\boxed{\partial_t u = \nu \partial_r^2 u + \mathfrak{z}(t) u + \partial_r \eta, \qquad \int \mathrm{d}r \, \langle u^2 \rangle \sim 1}$

$\mathfrak{z}(t) \sim \langle\!\langle \partial_r u \rangle\!\rangle \sim$ curvature

model $\mathrm{A}II$: $\boxed{\partial_t u = \nu \partial_r^2 u + \mathfrak{z}(t) \partial_r u + \partial_r \eta, \qquad \int \mathrm{d}r \, \langle u^2 \rangle \sim 1}$

$\mathfrak{z}(t) \sim \langle\!\langle u \rangle\!\rangle \sim$ slope

model $\mathrm{A}III$: $\boxed{\partial_t h = \nu \partial_r^2 h + \mathfrak{z}(t) \partial_r h + \eta, \qquad \int \mathrm{d}r \, \langle (\partial_r h)^2 \rangle \sim 1}$

$\mathfrak{z}(t) \sim \langle\!\langle \partial_r h \rangle\!\rangle \sim$ slope

? interface rough or smooth ?

? long-time properties and ageing behaviour ?

? does dynamical scaling resp. simple ageing always hold ?

4. First Arcetri model $\mathrm{A}I$: simple ageing

slope $u(t,x) = \partial_x h(t,x)$ obeys Burgers' equation,

replace its non-linearity by a mean spherical condition $\Longrightarrow$

$$\begin{aligned} \partial_t u_n(t) &= \nu\left(u_{n+1}(t) + u_{n-1}(t) - 2u_n(t)\right) + \mathfrak{z}(t) u_n(t) \\ &\quad + \frac{1}{2}\left(\eta_{n+1}(t) - \eta_{n-1}(t)\right) \\ \sum_n \left\langle u_n(t)^2 \right\rangle &= N \end{aligned}$$

$\langle \eta_n(t)\eta_m(s)\rangle = 2T\nu\delta(t-s)\delta_{n,m}$

Extension to $d \geq 1$ dimensions :

$\mathfrak{z}(t)$ Lagrange multiplier

define gradient fields $u_a(t,\mathbf{r}) := \nabla_a h(t,\mathbf{r})$, $a = 1, \ldots, d$:

$$\begin{aligned} \partial_t u_a(t,\mathbf{r}) &= \nu \nabla_{\mathbf{r}} \cdot \nabla_{\mathbf{r}} u_a(t,\mathbf{r}) + \mathfrak{z}(t) u_a(t,\mathbf{r}) + \nabla_a \eta(t,\mathbf{r}) \\ \sum_{\mathbf{r}} \sum_{a=1}^{d} \left\langle u_a(t,\mathbf{r})^2 \right\rangle &= dN^d \end{aligned}$$

interface height : $\boxed{\widehat{u}_a(t,\mathbf{q}) = \mathrm{i}\sin q_a\, \widehat{h}(t,\mathbf{q})}$; $\mathbf{q} \neq \mathbf{0}$ in Fourier space exact solution :

$\omega(\mathbf{q}) = \sum_{a=1}^{d}(1-\cos q_a)$, $\mathbf{q} \neq \mathbf{0}$

$$\widehat{h}(t,\mathbf{q}) = \widehat{h}(0,\mathbf{q})e^{-2t\omega(\mathbf{q})}\sqrt{\frac{1}{g(t)}} + \int_0^t \mathrm{d}\tau\ \widehat{\eta}(\tau,\mathbf{q})\sqrt{\frac{g(\tau)}{g(t)}}\, e^{-2(t-\tau)\omega(\mathbf{q})}$$

in terms of the auxiliary function $g(t) = \exp\left(-2\int_0^t \mathrm{d}\tau\, \mathfrak{z}(\tau)\right)$,
which satisfies Volterra equation

$$g(t) = f(t) + 2T\int_0^t \mathrm{d}\tau\, g(\tau)f(t-\tau) \ \ , \ \ f(t) := d\,\frac{e^{-4t}I_1(4t)}{4t}\left(e^{-4t}I_0(4t)\right)^{d-1}$$

* for $d=1$, identical to 'spherical spin glass', with $T = 2T_{\rm SG}$:
hamiltonian $\mathcal{H} = -\frac{1}{2}\sum_{i,j} J_{ij}S_iS_j$; J_{ij} random matrix, its eigenvalues distributed according to Wigner's semi-circle law CUGLIANDOLO & DEAN 95

* also related to distribution of first gap of random matrices PERRET & SCHEHR 15/16

a further auxiliary function : $F_{\mathbf{r}}(t) := \prod_{a=1}^{d} e^{-2t}I_{r_a}(2t)$ I_n : modified Bessel function
for initially uncorrelated heights and initially flat interface

height autocorrelator :

$$C(t,s) = \langle h(t,\mathbf{r})h(s,\mathbf{r})\rangle_c = \frac{2F_0(t+s)}{\sqrt{g(t)g(s)}} + \frac{2T}{\sqrt{g(t)g(s)}} \int_0^s \mathrm{d}\tau\, g(\tau) F_0(t+s-2\tau)$$

interface width : $w^2(t) = C(t,t) = \frac{2F_0(2t)}{g(t)} + \frac{2T}{g(t)} \int_0^t \mathrm{d}\tau\, g(\tau) F_0(2t-2\tau)$

slope autocorrelator :

$$A(t,s) = \sum_{a=1}^d \langle u_a(t,\mathbf{r})u_a(s,\mathbf{r})\rangle_c = \frac{2f((t+s)/2)}{\sqrt{g(t)g(s)}} + \int_0^s \mathrm{d}\tau\, \frac{2Tg(\tau)}{\sqrt{g(t)g(s)}} f((t+s)/2-\tau)$$

height response : $R(t,s;\mathbf{r}) = \left.\frac{\delta\langle h(t,\mathbf{r})\rangle}{\delta j(s,\mathbf{0})}\right|_{j=0} = \Theta(t-s)\sqrt{\frac{g(s)}{g(t)}}\, F_{\mathbf{r}}(t-s)$

slope autoresponse : $Q(t,s;\mathbf{0}) = \Theta(t-s)\sqrt{\frac{g(s)}{g(t)}}\, f((t-s)/2)$

* **correspondence of $1D$ $\mathrm{A}I$ model with spherical spin glass** : $\boxed{\text{spins } S_i \leftrightarrow \text{slopes } u_n}$

spin glass autocorrelator $\boxed{C_{\mathrm{SG}}(t,s) = \frac{1}{\mathcal{N}}\sum_{i=1}^{\mathcal{N}} \overline{\langle S_i(t)S_i(s)\rangle} = A(t,s)}$

spin glass response $\boxed{R_{\mathrm{SG}}(t,s) = \sum_{i=1}^{\mathcal{N}} \left.\frac{\delta\overline{\langle S_i(t)\rangle}}{\delta h_i(s)}\right|_{h=0} = 2Q(t,s)}$

* kinetics of heights $h_n(t)$ in model $\mathrm{A}I$ driven by phase-ordering of the spherical spin glass $\equiv 3D$ kinetic spherical model

phase transition : long-range correlated surface growth for $T \leq T_c$

$$\frac{1}{T_c(d)} = \frac{1}{2}\int_0^\infty \mathrm{d}t\; e^{-dt} t^{-1} I_1(t) I_0(t)^{d-1} \quad ; \quad T_c(1) = 2,\; T_c(2) = \frac{2\pi}{\pi - 2}$$

Some results : $\boxed{\text{always simple ageing}}$ upper critical dimension $d^* = 2$

1. $T = T_c,\; d < 2$:

   rough interface, width $w(t) = t^{(2-d)/4} \Longrightarrow \beta = \frac{2-d}{4} > 0$

   ageing exponents $a = b = \frac{d}{2} - 1,\; \lambda_R = \lambda_C = \frac{3d}{2} - 1\,;\; z = 2$

$\boxed{\textbf{exponents } z, \beta, a, b \textbf{ same as } \mathrm{EW}\textbf{, but exponent } \lambda_C = \lambda_R \textbf{ different}}$

2. $T = T_c,\; d > 2$:

   smooth interface, width $w(t) = \mathrm{cste.} \Longrightarrow \beta = 0$

   ageing exponents $a = b = \frac{d}{2} - 1,\; \lambda_R = \lambda_C = d\,;\; z = 2$

$\boxed{\textbf{same asymptotic exponents as } \mathrm{EW}\textbf{, but scaling functions are distinct}}$

3. $T < T_c$:

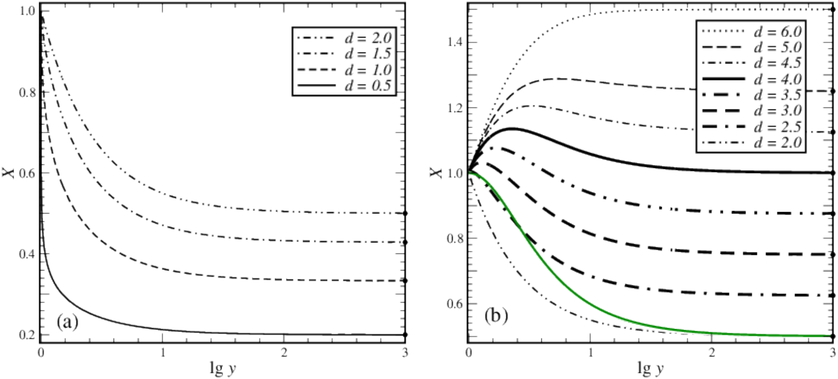
   rough interface, width $w^2(t) = (1 - T/T_c)t \Longrightarrow \beta = \frac{1}{2}$

   ageing exponents $a = \frac{d}{2} - 1,\; b = -1,\; \lambda_R = \lambda_C = \frac{d-2}{2}\,;\; z = 2$

Illustration : Shape of the height fluctuation-dissipation ratio, $T = T_c$

$$X(t,s) \quad := \quad TR(t,s)\left(\frac{\partial C(t,s)}{\partial s}\right)^{-1} = X\left(\frac{t}{s}\right)$$

$$\stackrel{t/s\to\infty}{\longrightarrow} \quad X_\infty = \begin{cases} d/(d+2) & ;\ 0<d<2 \\ d/4 & ;\ 2<d \end{cases}$$

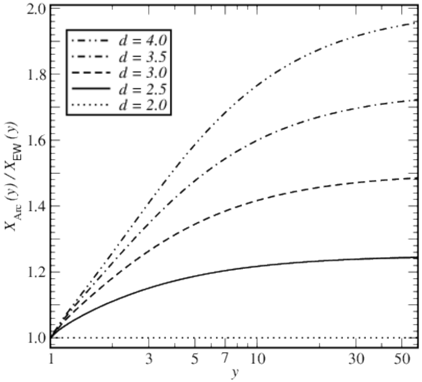


distinct from $X_{\mathrm{EW},\infty} = 1/2$ for all $d > 0$ green line : X_{EW} for $d = 4$

$\Longrightarrow$ although for $d > 2$ the non-equilibrium exponent $\lambda_C = \lambda_R = d$ is the same for the Arcetri and EW models, the scaling functions are different

in simple magnets : $\boxed{X_\infty \text{ is an } \textbf{universal} \text{ constant}}$

Godrèche & Luck 00



use universal value of X_∞ as **diagnostic tool**,

(provided that $a = b$,
valid in the Arcetri model at $T = T_c$)

N.B. : for $d < 2$, the slope FDR $X_\infty^{(\text{slope})} = d/(d+2) = X_\infty^{\text{SM}}\big|_{d+2 \text{ dim.}}$,
same as X_∞ in the spherical ferromagnet in $d+2$ dimensions

Relationship with the critical diffusive bosonic pair-contact process (BPCPD)

Howard & Täuber 97 ; Houchmandzadeh 02 ; Paessens & Schütz 04 ; Baumann, mh, Pleimling, Richert 05

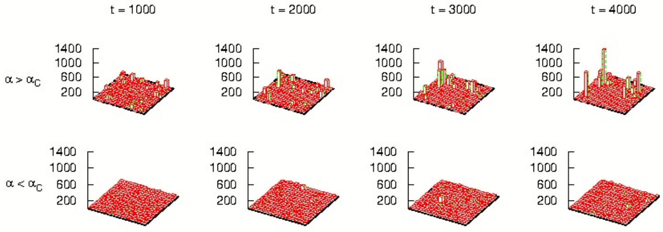
* each site of a hypercubic lattice is occupied by $n_i \in \mathbb{N}_0$ particles
* single particles hop to a nearest-neighbour site with diffusion rate D
* on-site reactions, with rates $\Gamma[2A \to (2+k)A] = \Gamma[2A \to (2-k)A] = \mu$

  k is either 1 or 2
* control parameter $\alpha := k^2\mu/D$

$\Longrightarrow$ $\boxed{\text{for } d>2\text{, particles cluster on a few sites only, if } \alpha > \alpha_C}$ BHPR 05

Figure : $2D$ section of BPCPD in $d=3$; height of columns $\sim$ particle number Baumann 07

$\Longrightarrow$ fluctuations grow with t when $\alpha > \alpha_C$ & are bounded for $\alpha < \alpha_C$



bosonic creation operator $a^\dagger(t,\mathbf{r})$, commutator $[a(t,\mathbf{r}),a^\dagger(t,\mathbf{r}')]=\delta(\mathbf{r}-\mathbf{r}')$
$\Longrightarrow$ average particle number is constant !

$$n(t,\mathbf{r})=\langle a^\dagger(t,\mathbf{r})a(t,\mathbf{r})\rangle=\langle a(t,\mathbf{r})\rangle=\rho_0=\mathrm{cste.}$$

clustering transition at $\alpha=\alpha_C$, caracterised by changes in the variance.

$$\bar{C}(t,s) \;:=\; \left\langle a^\dagger(t,\mathbf{r})a(s,\mathbf{r})\right\rangle-\rho_0^2 \stackrel{t,s\to\infty}{\simeq} \langle n(t,\mathbf{r})n(s,\mathbf{r})\rangle-\rho_0^2=s^{-b}f_C(t/s)$$

$$\bar{R}(t,s) \;:=\; \left.\frac{\delta\langle a(t,\mathbf{r})\rangle}{\delta j(s,\mathbf{r})}\right|_{j=0}=s^{1-a}f_R(t/s)$$

obey simple ageing for $\alpha\leq\alpha_C$. Precisely **at** the clustering transition $\alpha=\alpha_C$, for $2<d<4$, the scaling functions are **identical** :

$$\boxed{\mathrm{BPCPD}\;:\quad b+1=a=d/2-1} \qquad \boxed{\mathrm{Arcetri}\;:\quad b=a=d/2-1}$$

$$f_{R,\mathrm{BPCPD}}(y) \;=\; (y-1)^{d-2} \;=\; f_{R,\mathrm{Arc}}(y)$$

$$f_{C,\mathrm{BPCPD}}(y) \;=\; (y+1)^{-d/2}{}_2F_1\left(\frac{d}{2},\frac{d}{2};\frac{d}{2}+1;\frac{2}{1+y}\right) \;=\; f_{C,\mathrm{Arc}}(y)$$

N.B. : for $d>4$, Arcetri $\neq$ BPCPD $\neq$ EW, although all exponents, up to b, agree.

Summary of results in the $\mathrm{A}l$ model :

Captures at least some qualitative properites of growing interfaces.

* phenomenology of relaxation analogous to domain growth in simple magnets $\boxed{\Longrightarrow \text{dynamical scaling form of simple ageing}}$

* existence of a critical point $T_c(d) > 0$ for all $d > 0$ **as a magnet**

* at $T = T_c$, rough interface for $d < 2$, smooth interface for $d > 2$; upper critical dimension $d^* = 2$

* at $T = T_c$, $d < 2$, the stationary exponents (β, z) are those of EW, but the non-stationary ageing exponents are different

explicit example for expectation from field-theory renormalisation group in domain growth of independent exponents $\lambda_{C,R}$

different from EW and KPZ classes, where $\lambda_C = d$ for all $d < 2$ KRECH 97

* at $T = T_c$, $d > 2$, **distinct from** EW, although all exponents agree

* for $d = 1$, equivalent to $p = 2$ spherical spin glass

* at $T = T_c$ and $2 < d < 4$, same ageing behaviour as at the multicritical point of the bosonic pair-contact process with diffusion (BPCPD)

* distinct universality class for $T < T_c$

5. Second Arcetri model $\mathrm{A}II$: several length scales

$d = 1$ only ; work in progress

$$\boxed{\partial_t u = \nu \partial_r^2 u + \mathfrak{z}(t)\partial_r u + \partial_r \eta, \qquad \int \mathrm{d}r \, \langle u^2 \rangle \sim 1}$$

requirement : stationary solution should remain roughly flat

but find $\nu u'' + \mathfrak{z} u' = 0 \Longrightarrow u = u^{(0)} + u^{(1)} e^{-(\mathfrak{z}/\nu)r}$ exponential growth **?**

N.B. : equation of motion couples even and odd contributions to slope profile

decompose $\boxed{u(t,r) = a(t,r) + b(t,r)}$

with $a(t,r) = a(t,-r)$ **even** and $b(t,r) = -b(t,-r)$ **odd**

gives $\nu a'' + \mathfrak{z} b' = 0$, $\nu b'' + \mathfrak{z} a' = 0 \Longrightarrow$ exponential growth as $r \to \pm\infty$ **?**

$\boxed{u(t,r) = a(t,r) + b(t,r)}$ with a even and b odd

construct pair of equations of motion, with an **important modification**

$$\begin{aligned}\partial_t a(t,r) &= \nu\partial_r^2 a(t,r) + \mathfrak{z}(t)\partial_r b(t,r) + \partial_r \eta^-(t,r) \\ \partial_t b(t,r) &= \nu\partial_r^2 b(t,r) - \mathfrak{z}(t)\partial_r a(t,r) - \partial_r \eta^+(t,r) \\ & \langle \sum_r \left(a(t,r) + b(t,r)\right)^2 \rangle = \mathcal{N}\end{aligned}$$

with symmetrised noise $\eta^\pm(t,r) = \frac{1}{2}\left(\eta(t,r) \pm \eta(t,-r)\right)$

These are the defining equations of the model $\mathrm{A}II$

gives $\nu a'' + \mathfrak{z} b' = 0$, $\nu b'' - \mathfrak{z} a' = 0 \Longrightarrow \nu^2 a''' = -\mathfrak{z}^2 a'$, $\nu^2 b''' = -\mathfrak{z}^2 b'$

$\Longrightarrow$ $\boxed{\text{profiles remain bounded as } r \to \pm\infty \ \ !}$

analogous procedure for third Arcetri model $\mathrm{A}III$

initial condition :

interface flat on average, initial slopes uncorrelated,
spherical constraint respected

work out spherical constraint : let $Z(t) := \int_0^t \mathrm{d}\tau\, \mathfrak{z}(\tau)$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{d}k\; \cosh(2\sin k Z(t))e^{-4\nu\omega(k)t}$$
$$+\quad \frac{\nu T}{\pi}\int_{-\pi}^{\pi}\mathrm{d}k\; \sin^2 k\int_0^t \mathrm{d}\tau\; \cosh(2\sin k(Z(t)-Z(\tau)))e^{-4\nu\omega(k)(t-\tau)} = 1$$

concentrate on case $T = 0$: dynamics driven by initial fluctuations

much as in phase-ordering kinetics in simple magnets

spherical constraint : $\boxed{e^{4\nu t} = I_0\left(\sqrt{(4\nu t)^2 + (2Z(t))^2}\,\right)}$

asymptotic solution for $t \gg 1$: $Z(t) \simeq (\nu t \ln(\pi\nu t))^{1/2}$

slope response

choose units such that $\nu = 1$

$$R_{x,y}(t,s) = \left\langle \left. \frac{\partial a(t,x)}{\partial j^{+}(s,y)} \right|_{j=0} \right\rangle + \left\langle \left. \frac{\partial b(t,x)}{\partial j^{-}(s,y)} \right|_{j=0} \right\rangle$$

$$= \quad \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{d}k \; \sin k e^{-2\omega(k)(t-s)} \sinh(\sin k(Z(t) - Z(s))) \cos k(x-y)$$

slope correlator

$$C_{x,y}(t,s) = \langle a(t,x)a(s,y) + b(t,x)b(s,y) \rangle$$

$$= \quad \frac{1}{2\pi} \int_{\pi}^{\pi} \mathrm{d}k \; e^{-2\omega(k)(t+s)} \cosh(\sin k(Z(t) + Z(s))) \cos k(x-y)$$

both can be evaluated as sums of modified Bessel functions

analysis of the long-time scaling behaviour, $T = 0$

it turns out that simple ageing is not obeyed !

rather, consider as a scaling variable $\boxed{\tau := t - s = ys \ln^{-\varsigma} \pi s}$

$\boxed{\text{scaling limit } t, s \to \infty \text{ with } y \text{ fixed and } \varsigma > 0}$ *'logarithmic sub-ageing'*

use $Z(t) \simeq \sqrt{t \ln \pi t}$ for $t \to \infty$:

slope autocorrelator $C(t,s) = C_{0,0}(t,s)$

$$
\begin{aligned}
C(t,s) &= \frac{I_0\left(2(t+s)\sqrt{(1+(Z(t)+Z(s))^2/(2(t+s))^2}\right)}{I_0\left(2(t+s)\sqrt{1+Z^2((t+s)/2)}\right)} \\
&\simeq \exp\left(-\frac{y^2}{32}\ln^{1-2\varsigma}\pi s\right)
\end{aligned}
$$

* try simple ageing $\varsigma = 0$: $\Longrightarrow$ no data collapse & multiscaling !

* only find dynamical scaling if $\boxed{\varsigma = \frac{1}{2} > 0}$

* same sub-ageing behaviour as in the $2D$ spherical magnet with conserved order parameter (model B) Berthier 00

slope autoresponse $R(t,s) = R_{0,0}(t,s)$

$$R(t,s) \simeq \sqrt{\frac{2}{\pi}}\, s^{-1} y^{-3/2} \ln^{1+3\varsigma/2} \pi s$$

* looks very similar to simple ageing
* but additional logarithmic factor breaks dynamical scale-invariance

spatial equal-time correlator $C_n(t) = C_{n,0}(t,t)$

$$\begin{aligned} C_n(t) &= \frac{I_n\left(4t\sqrt{1+Z^2(t)/4t^2}\,\right)\cos\left(n \arctan Z(t)/2t\right)}{I_0\left(4t\sqrt{1+Z^2(t)/4t^2}\,\right)} \\ &\simeq \exp\left(-\left(\frac{n}{\sqrt{8t}}\right)^2\right)\cos\left(\frac{n}{\sqrt{2t/\ln \pi t}}\right) \end{aligned}$$

* find two marginally different length scales
* simple scaling ansatz leads to **multiscaling**
* analogue : spherical magnet at $T=0$, conserved order-parameter CONIGLIO & ZANNETTI 89
  but the $\mathrm{A}\mathit{II}$ model does not have a macroscopic conservation law !

6. Conclusions

* long-time dynamics of growing interfaces naturally evolves towards
  dynamical scaling & ageing
* phenomenology very similar to ageing phenomena in simple magnets
* subtleties in the precise scaling forms
* exactly solvable model with proven sub-ageing, although the $\mathrm{A}II$ does
  not have a macroscopic conservation law !

proving dynamical symmetries can remain a delicate affair !