Spin Glass Approach to Restricted Isometry Constant

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• Background: Compressed sensing
• Problem setup: Restricted isometry constant (RIC)
• Spin glass approach
  – Replica symmetric analysis
  – Improvement by replica symmetry breaking
• Summary
Compressed sensing

Reconstruct original signal $X$ from its undersampled measurement $Y$.

$$M \{ Y \} = M \{ A \} X \{ N \}$$

Undersampled measurement

Original signal
Compressed sensing

Reconstruct original signal \( X \) from its undersampled measurement \( Y \).

Reconstruction is possible when \( X \) is sufficiently "sparse".
Practical relevance

• The problem is related to various technologies of modern signal processing

• Many application domains
  – Refraction seismic survey (mine examination)
  – Tomography (X-ray CT, MRI)
  – Single pixel camera
  – Noise removal of image
  – Data streaming computing
  – Group testing
  – etc.
Simulation of tomography

LT: Original (Logan-Shepp Phantom) #512x512

RT: Sampling 512 points of 2D FT from 22 directions.

LB: Recovery of pseudo-inverse (standard approach)

RB: Recovery utilizing the “sparseness” of spatial variations. “Original” is perfectly recovered.

Perfect recovery is realized by only 2% samples of what Nyquist-Shannon’s theory requires.

→ Breaking of the conventional limit!

Two major reconstruction methods

$l_0$ reconstruction

$$\hat{x} = \arg \min_x \|x\|_0, \text{subject to } Y = Ax$$

$l_1$ reconstruction

$$\hat{x} = \arg \min_x \|x\|_1, \text{subject to } Y = Ax$$

Sufficient conditions for the reconstruction are given by

**Restricted Isometriy Constant (RIC)**.
Restricted Isometry Constant (RIC)

Definition (Candès and Tao (2006)).

Let $A$ be a column-wisely normalized $M \times N$ matrix. Let $x \in \mathbb{R}^N$ be an arbitrary $S$-sparse vector, whose number of non-zero components is smaller than $S$.

Then, if there exists such a constant $d_S = \max \{d_S^{\text{min}}, d_S^{\text{max}}\}$ that satisfies

$$(1 - \delta_S^{\text{min}}) \|x\|_F^2 \leq \|Ax\|_F^2 \leq (1 + \delta_S^{\text{max}}) \|x\|_F^2,$$

$A$ is said to satisfy the $S$-restricted isometry property (RIP) with the restricted isometry constant (RIC) $d_S$.

Intuitively, $d_S$ quantifies how $A$ deviates from orthogonal transforms in terms of Frobenius ($L_2$) norm for $S$-sparse vectors.
Sufficient conditions for $l_0$, $l_1$ reconstruction

1. $l_0$ reconstruction gives the unique $S$-sparse solution when $d_{2S} < 1$.  

2. $l_1$ reconstruction gives the same unique $S$-sparse solution as $l_0$ reconstruction when $d_{2S} < 2^{1/2} - 1$.  
   

3. (Improvement of 2.)  

   $l_1$ reconstruction gives the same unique $S$-sparse solution as $l_0$ reconstruction when $(4\sqrt{2} - 3)\delta_{2S}^{\min} + \delta_{2S}^{\max} < 4(\sqrt{2} - 1)$.  

• RIC plays a key role in theoretical analysis and performance guarantee of compressed sensing.
• On top of this, it is purely of great interest as a fundamental problem of linear algebra.

• Unfortunately, “there are no known large (systematic) matrices with bounded RICs (and computing RICs is strongly NP-hard),…”.  
  – Wikipedia
• On the other hand, many random matrices have been shown to remain bounded. Therefore, much effort has been paid for improving the bounds.
  – Our study is along this line.
Why computationally difficult?

Suppose a situation where non-zero components are fixed.

Suppose a situation where non-zero components are fixed.

\( N = 5, \ V = \{1,2,3,4,5\}, \ S = 2, \ T = \{2,4\} \)

\[
\begin{align*}
\lambda_{\min}(A_T^T A_T) \|x_T\|_F^2 &\leq \|A_T x_T\|_F^2 \leq \lambda_{\max}(A_T^T A_T) \|x_T\|_F^2
\end{align*}
\]

The change of norm can be easily characterized by eigenvalues of “sub-matrix” \( A_T \).
In evaluation of RIC, the inequalities must hold for all possible combinations of the positions of non-zeros.

This causes a combinatorial difficulty, yielding an exact expression of RIC as

$$\delta_S = \max\{1 - \min_{T:|T|=S, T \subseteq V} \lambda_{\min}(A_T^T A_T), \max_{T:|T|=S, T \subseteq V} \lambda_{\max}(A_T^T A_T) - 1\}.$$ 

Structures of the problem and the difficulty are analogous to those of spin glass problems.
Analogy to spin glasses

The choice of $T$ are represented by binary vector $\{c_i\} \in \{0, 1\}^N$.

$N = 5, S = 2, T = \{2, 4\} \rightarrow c = \{0, 1, 0, 1, 0\}$

$\lambda_{\min}(A^T_T A_T)$ and $\lambda_{\max}(A^T_T A_T)$ can be regarded as 'energy functions' of $c$ given $A$. 
Analogy to spin glasses

\[ \delta_S = \max \{1 - \min_{c: \sum_{i=1}^{N} c_i = S} \lambda_{\text{min}} (c \mid A), \max_{c: \sum_{i=1}^{N} c_i = S} \lambda_{\text{max}} (c \mid A) - 1\} \]

RIC is given by minimum and maximum ‘energy’ of \( c \).

- Energy of ‘0-1 spin’ \( c \)
  \[
  \Lambda_+ (c \mid A) = \lambda_{\text{min}} (c, A) \\
  \Lambda_- (c \mid A) = \lambda_{\text{max}} (c, A)
  \]

- ‘Canonical distribution’ of \( c \)
  \[
  P(c \mid A; \mu) \propto \exp \left( -\mu N \Lambda_{\text{sgn}(\mu)} (c \mid A) \right) \delta \left( \sum_{i=1}^{N} c_i - S \right)
  \]

- ‘Quenched randomness’
  \[
  P(A) = \left( 2\pi M^{-1} \right)^{-MN/2} \exp \left( -\frac{M}{2} \sum_{\mu, i} A_{\mu i}^2 \right)
  \]
Spin glass approach

- Free entropy (free energy)

\[
\phi(\mu \mid A) = \frac{1}{N} \log \left[ \sum_{c \in \{0,1\}^N} \exp \left( -\mu N \Lambda_{\text{sgn}(\mu)}(c \mid A) \right) \delta \left( \sum_{i=1}^{N} c_i - S \right) \right]
\]

- Typical properties can be assessed by the replica method.

\[
\phi(\mu) = \left[ \phi(\mu \mid A) \right]_A = \lim_{n \to 0} \frac{\partial}{\partial n} \log \left[ Z^n(\mu \mid A) \right]_A
\]

Energy (min/max eigenvalues)

\[
\lambda(\mu) = -\frac{\partial \phi(\mu)}{\partial \mu}
\]

Entropy (# of $T$ producing $\lambda$)

\[
\omega(\mu) = \phi(\mu) - \mu \frac{\partial \phi(\mu)}{\partial \mu}
\]
Possible minimum and maximum eigenvalues

\[ \omega_+ (\lambda_{\text{min}}) \]

Possible minimum eigenvalue

\[ \lambda^*_\text{min} \]

Possible maximum eigenvalue

\[ \lambda^*_\text{max} \]

\[ \delta_S = \max \{ 1 - \lambda^*_\text{min}, \lambda^*_\text{max} - 1 \} \]
Replica symmetric analysis

\[ \phi(\mu) = -\frac{\alpha}{2} \log[\alpha + \chi + \mu(1-q)] + \frac{\alpha}{2} \log(\alpha + \chi) \]

\[ - \frac{\alpha \mu q}{2\{\alpha + \chi + \mu(1-q)\}} + \frac{\hat{Q}}{2} - \frac{\hat{q}_1}{2} \left( 1 + \frac{\chi}{\mu} \right) + \frac{\hat{q}_0 q}{2} + K \rho \]

\[ + \int Dz \log \left( 1 + e^{-K} \int Dy \exp \left( \frac{\left( \sqrt{\hat{q}_1 - \hat{q}_0 y} + \sqrt{\hat{q}_0 z} \right)^2}{2\hat{Q}} \right) \right) \]

- \( \alpha = \frac{M}{N} \) (compression rate)
- \( \rho = \frac{S}{N} \) (fraction of non-zero components)
- \( \{q, \chi, \hat{Q}, \hat{q}_1, \hat{q}_0, K\} \) are determined by saddle point equations.
Replica symmetric entropy

- $\alpha = 0.5, \rho = 0.1$

$$\omega_+ (\lambda_{\text{min}})$$

- $\alpha = \frac{M}{N}, \rho = \frac{S}{N}$

$$\omega_- (\lambda_{\text{max}})$$

$$\delta_S = \max \{1 - \lambda^*_{\text{min}}, \lambda^*_{\text{max}} - 1\}$$
Comparison with prior works

- $d_S$ at $\alpha = 0.5$

\[ \alpha = \frac{M}{N}, \rho = \frac{S}{N} \]

\[
\begin{align*}
\alpha & = M/N, \\
\rho & = S/N
\end{align*}
\]

Replica symmetric RIC
$l_0, l_1$ reconstruction limit

※ $\alpha = \frac{M}{N}, \rho = \frac{S}{N}$

※ Dashed lines are Bah – Tanner (2010)
The RS RIC estimate is lower than *any existing upper bounds*, being consistent with a known lower bound.

On the other hand, there are non-negligible deviations from the experimental data.

In fact, detailed analysis shows that the replica symmetry is broken for the left and right edges of the entropy curves.

However, physical interpretation of RSB indicates that the RS estimates still serve as *upper bounds* of RIC, and the bounds are improved as the higher RSBs are taken into account.
• $\alpha = 0.5, \rho = 0.1$

$\omega_+ (\lambda_{\text{min}}) \quad \star \quad \omega_- (\lambda_{\text{max}})$

\[ \delta_S = \max \{ 1 - \lambda_{\text{min}}^*, \lambda_{\text{max}}^* - 1 \} \]
Physical interpretation of RSB

State space

RS | $|\mu_d|$ | RS, 1RSB degenerated | $|\mu_c|$ | 1RSB

Complexity = entropy of pure states

$$\Sigma(\lambda,s) = \frac{1}{N} \log(\#\text{pure states specified by } (\lambda,s))$$

Non-negativity constraint: Must be non-negative.

Total entropy

$$\omega(\lambda) = \frac{1}{N} \log\left(\int ds \exp\left(N\left(s + \Sigma(\lambda,s)\right)\right)\right) = \max_s \left\{s + \Sigma(\lambda,s)\right\}$$

Physical interpretation of RSB

RS evaluation: the complexity constraint is ignored.

\[ \omega_{RS}(\lambda) = \max_s \left\{ s + \sum (\lambda, s) \right\} \]

1RSB evaluation: the complexity constraint is taken into account.

\[ \omega_{1RSB}(\lambda) = \max_s \left\{ s + \sum (\lambda, s) \mid \sum (\lambda, s) \geq 0 \right\} \]

The non-negativity constraint of the 1RSB evaluation means

\[ \omega_{RS}(\lambda) \geq \omega_{1RSB}(\lambda). \]
Physical interpretation of RSB

State space

Single dominant cluster

\[
\begin{align*}
\omega_{\text{RS}} (\lambda) & \geq \omega_{1\text{RSB}} (\lambda) \geq \omega_{2\text{RSB}} (\lambda) \\
\text{Applying the similar argument repeatedly concludes}
\end{align*}
\]

\[
\omega_{\text{RS}} (\lambda) \geq \omega_{1\text{RSB}} (\lambda) \geq \omega_{2\text{RSB}} (\lambda) \geq \ldots
\]
The series of inequalities

$$\omega_{RS}(\lambda) \geq \omega_{1RSB}(\lambda) \geq \omega_{2RSB}(\lambda) \geq \ldots.$$  

indicates that bounds are monotonically improved by incorporating the higher RSB.
1RSB results

- The estimates are actually improved by the 1RSB solution!
- Two scenarios for the RSB transition
  - $\lambda_{\text{min}}$: Random first order transition (RFOT)
  - $\lambda_{\text{max}}$: de Almeida-Thouless instability (full RSB)

$\alpha = 0.5, \rho = 0.1$
Summary

• Evaluation of the restricted isometry constant (RIC) can be formulated as a spin glass problem.
• We provided a replica based-framework for accurate evaluation of RICs.
  – Replica evaluation provides the current best accuracy.
  – Although the RS solution is not thermodynamically stable, the physical interpretation of RSB implies that the RS estimates still have the meaning of “upper-bounds”.
  – The bounds are monotonically improved by taking the higher RSB into account.

• Future work
  – Application to other matrix ensembles
  – Mathematical justification
Thank you for your attention.

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