Avalanche contribution to nonlinear elasticity of granular materials

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Contents

1. Introduction : Jamming transition and shear modulus.

2. Critical behavior of shear modulus for frictionless granular materials under finite oscillatory shear.

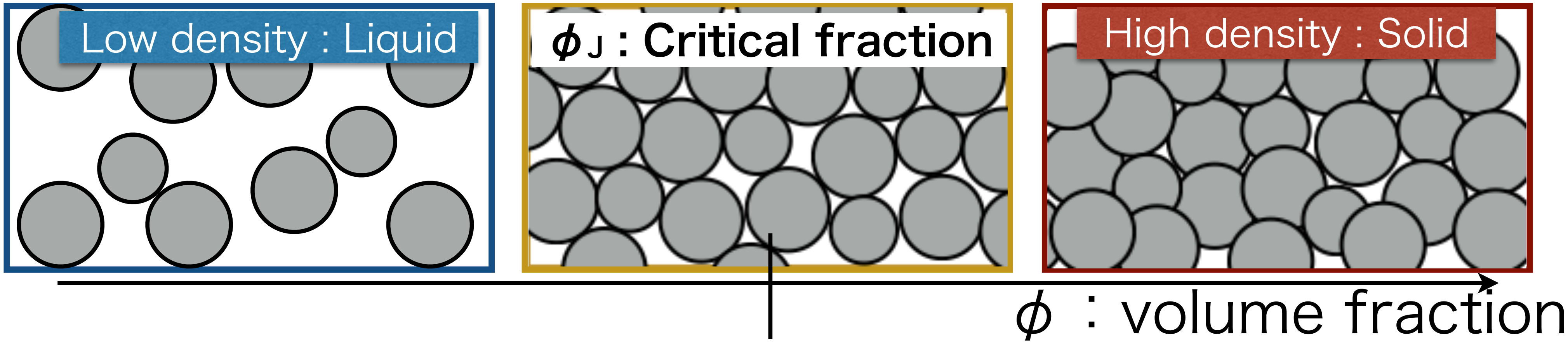
3. Theory for critical exponents.

MO and H. Hayakawa, PRE 90, 042202 (2014)

4. Effect of the friction between particles.

Jamming transition

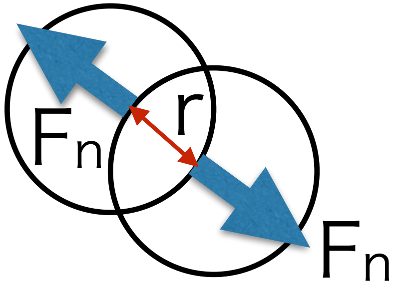
C. O'Hern, et al., Phys. Rev. Lett. 88, 075507 (2002)



- Granular materials can flow below a critical density ϕ_J.
- Above ϕ_J, the materials have rigidity and behave as solids.
- This transition is known as the jamming transition.

Scaling of shear modulus

M. Wyart, Ann. Phys. Fr. (2005)



Elastic interaction force

$$F_n \propto r^{\Delta}$$

contact length : r

$\Delta=1$ (linear force)

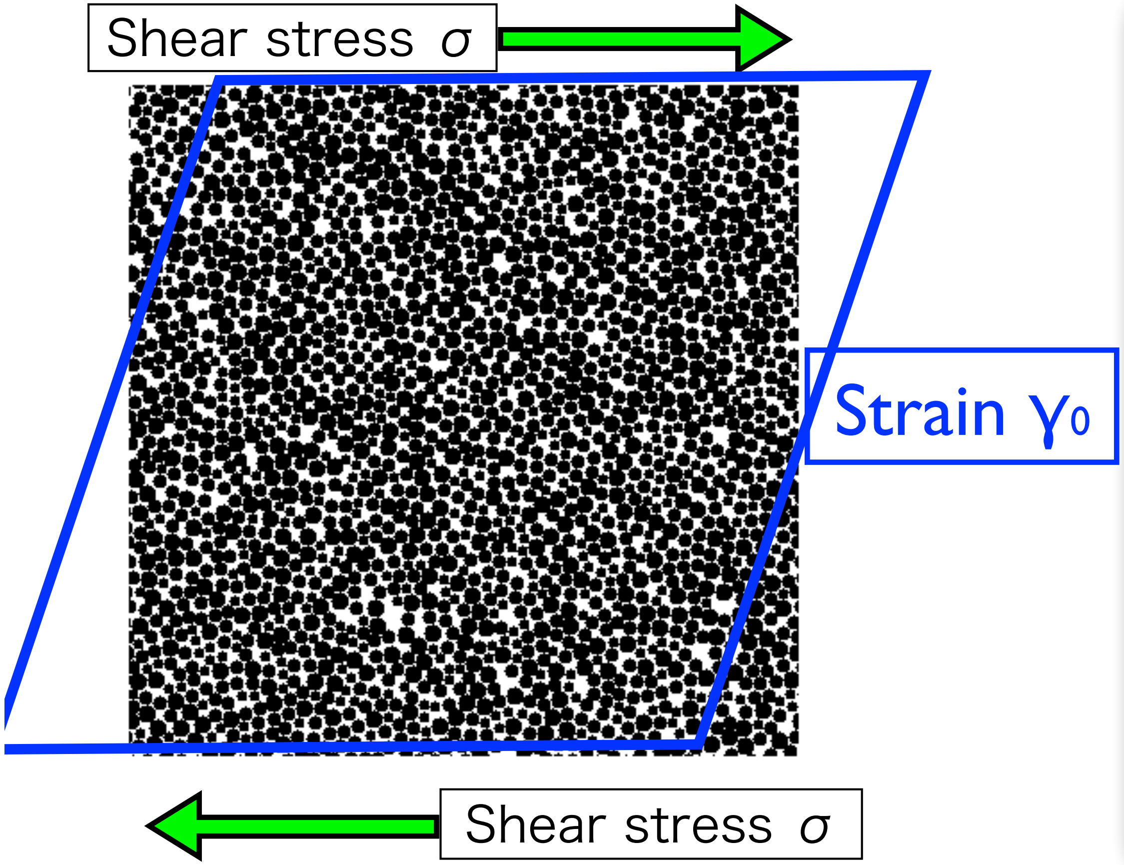
$\Delta = 3/2$ (Hertzian force)

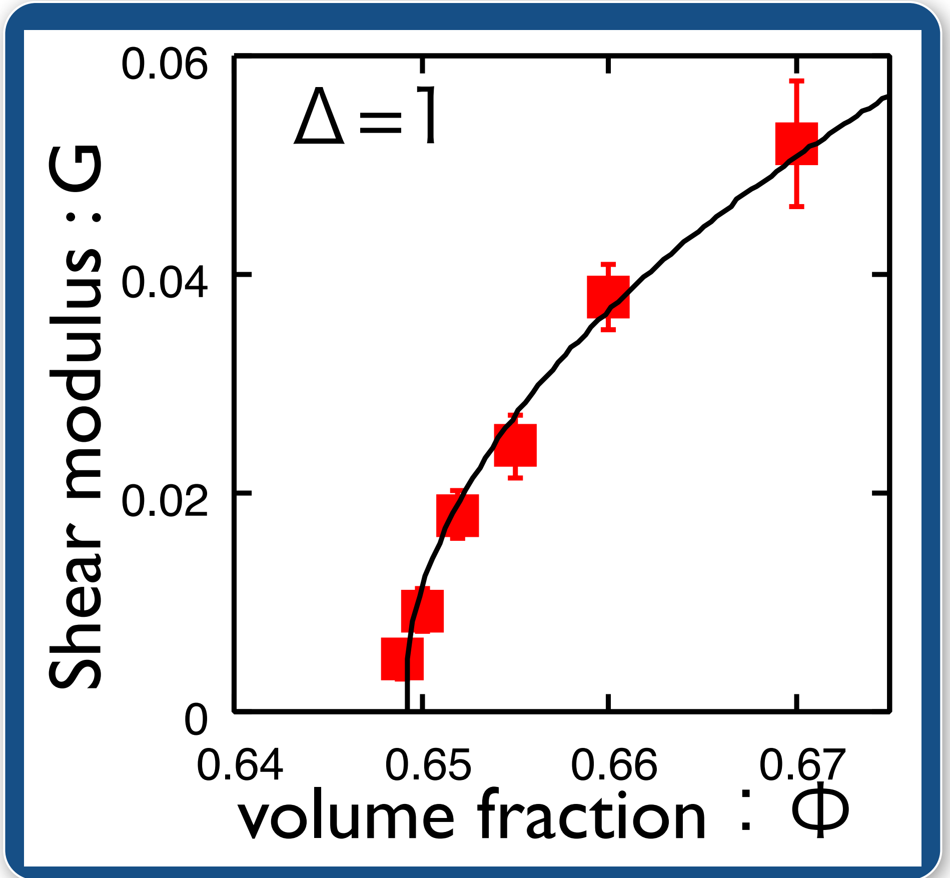
Shear modulus : $G = \sigma/\gamma_0$

Theory for elastic network :

$$G \propto (\phi - \phi_J)^{\Delta - 1/2}$$

$$P \propto (\phi - \phi_J)^{\Delta}$$





Different scaling relations

G : shear modulus, P : pressure

$$G \propto P$$

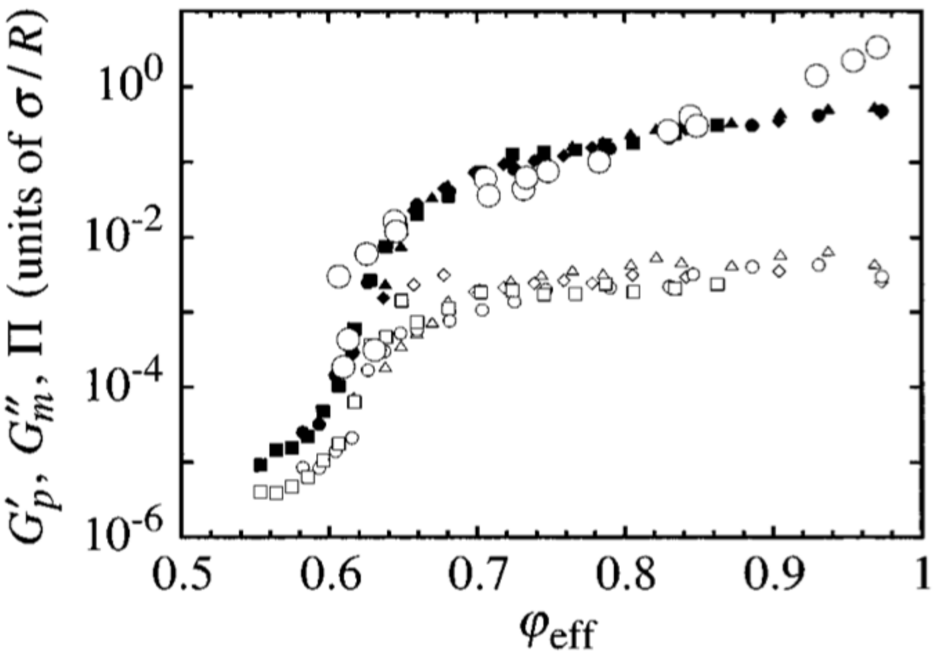
Mason et al., PRE 1996, (experiments of emulsions)

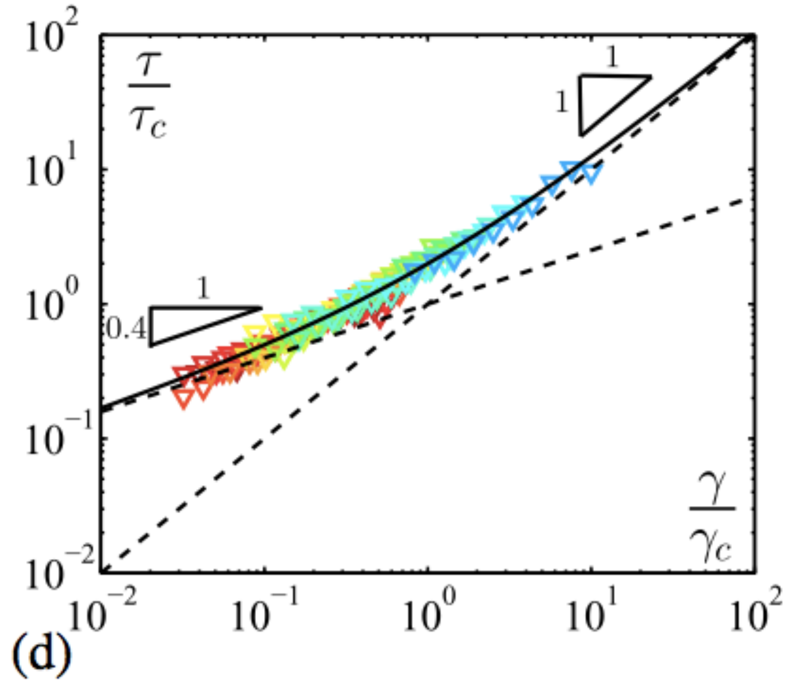
$$G \propto (\phi - \phi_J)^{\Delta}$$

Okamura & Yoshino 2013, (Replica theory with small temperature)

$$G \propto \gamma_0^{-c}(\phi - \phi_J)$$

Coulais, Seguin, and Dauchot, PRL 2014





Purpose

G : shear modulus, P : pressure

$$G \propto (\phi - \phi_J)^{\Delta - 1/2}$$

$$P \propto (\phi - \phi_J)^{\Delta}$$

C. O'Hern, et al., Phys. Rev. Lett. 88, 075507 (2002)

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Okamura & Yoshino 2013, (Replica theory with small temperature)

$$G \propto \gamma_0^{-c} (\phi - \phi_J)$$

Coulais, Seguin, and Dauchot, PRL 2014

Purpose :

- We would like to clarify the relationship between different scaling relations.
- For this purpose, we perform a simulation of granular materials under oscillatory shear.

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MO and H. Hayakawa, PRE 90, 042202 (2014)

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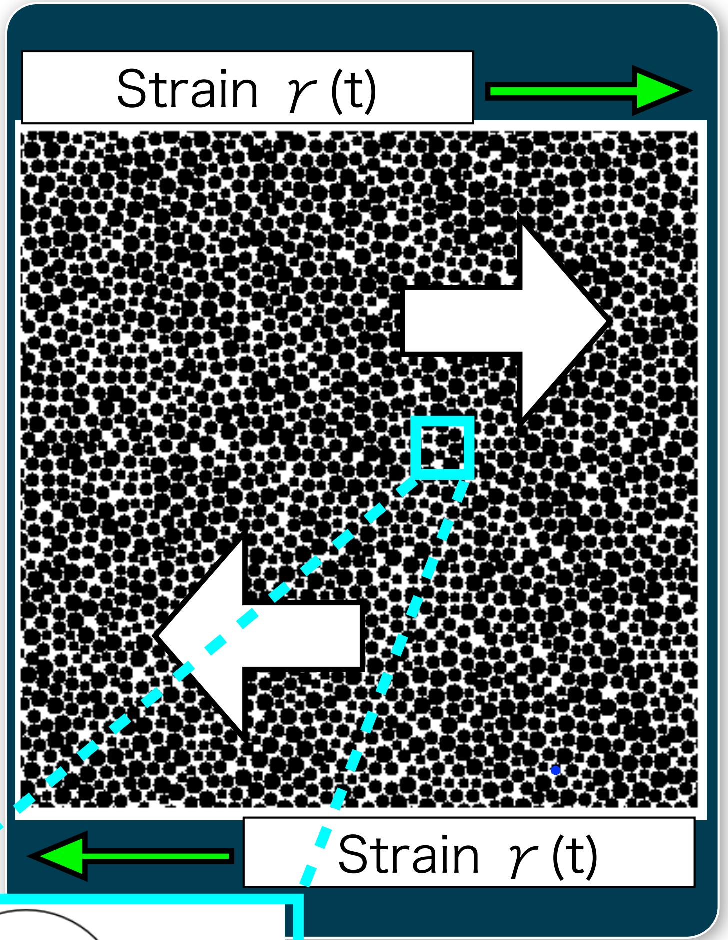
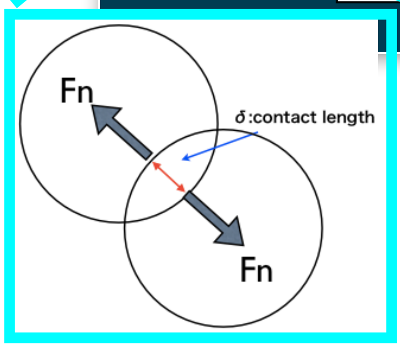
Model of frictionless particles





Normal interaction force

$$F_n = kr^{\Delta} + \eta\dot{r}$$

- Oscillatory shear strain

$$\boxed{\gamma(t) = \gamma_0 \cos(\omega t)}$$

- Frequency : ω **Quasi-static limit : $\omega \to 0$**
- **Strain amplitude : γ_0**
- Shear stress : σ(t)

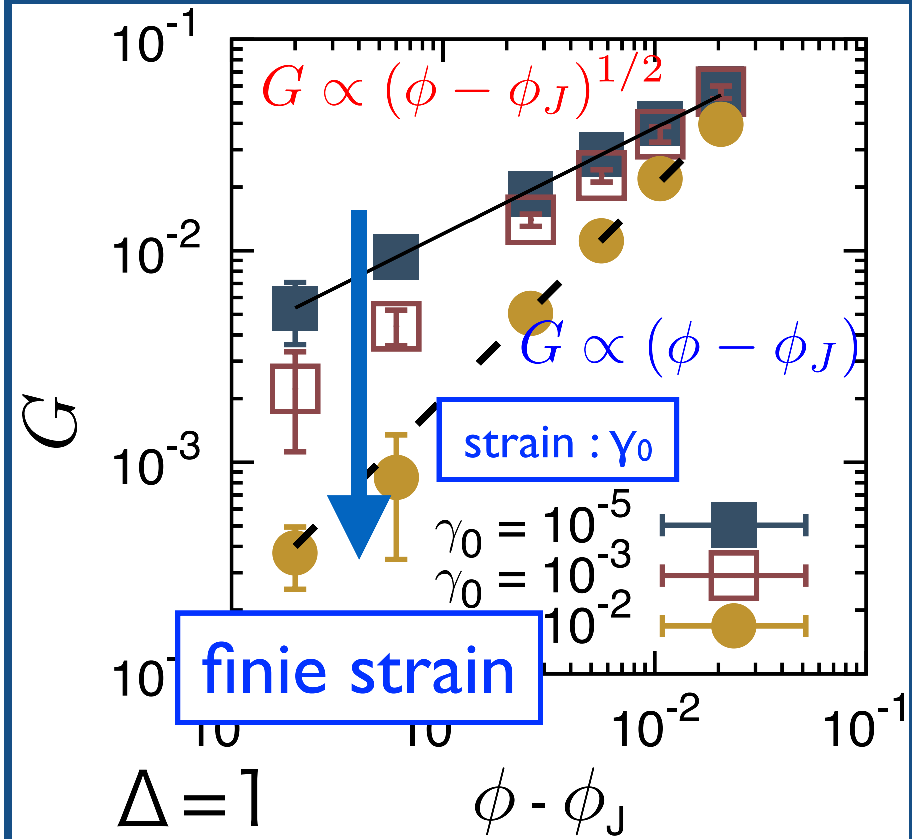
Shear modulus (storage modulus)

$$G(\gamma_0, \phi) = \frac{\omega}{\pi}\int_0^{2\pi/\omega} dt \frac{\sigma(t)\cos(\omega t)}{\gamma_0}$$

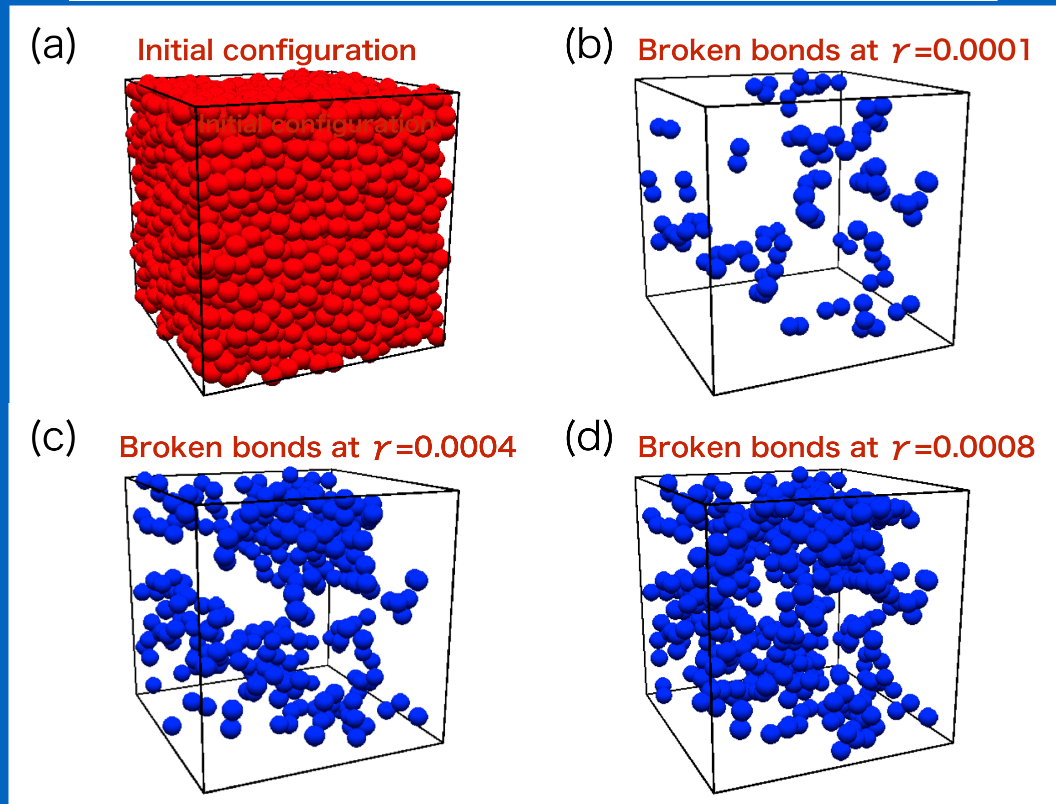
We investigate the dependence of G on γ_0 and Φ.

Shear modulus under finite strain ($\Delta=1$)

MO and H. Hayakawa, PRE (2014)



Avalanche-like bond breakage



Infinitesimal strain : $G \propto (\phi - \phi_J)^{1/2}$

Finite strain : $G \propto \gamma_0^{-c}(\phi - \phi_J)$

Origin : Avalanche (correlated bond breakage)

Critical scaling of G

MO and H. Hayakawa, PRE (2014)

$$G(\gamma_0,\phi) = (\phi-\phi_J)^a \mathcal{G}(\gamma_0(\phi-\phi_J)^{-b})$$

$$\lim_{x\to 0}\mathcal{G}(x) = \text{const.} \qquad \lim_{x\to\infty}\mathcal{G}(x) \propto x^{-c}$$

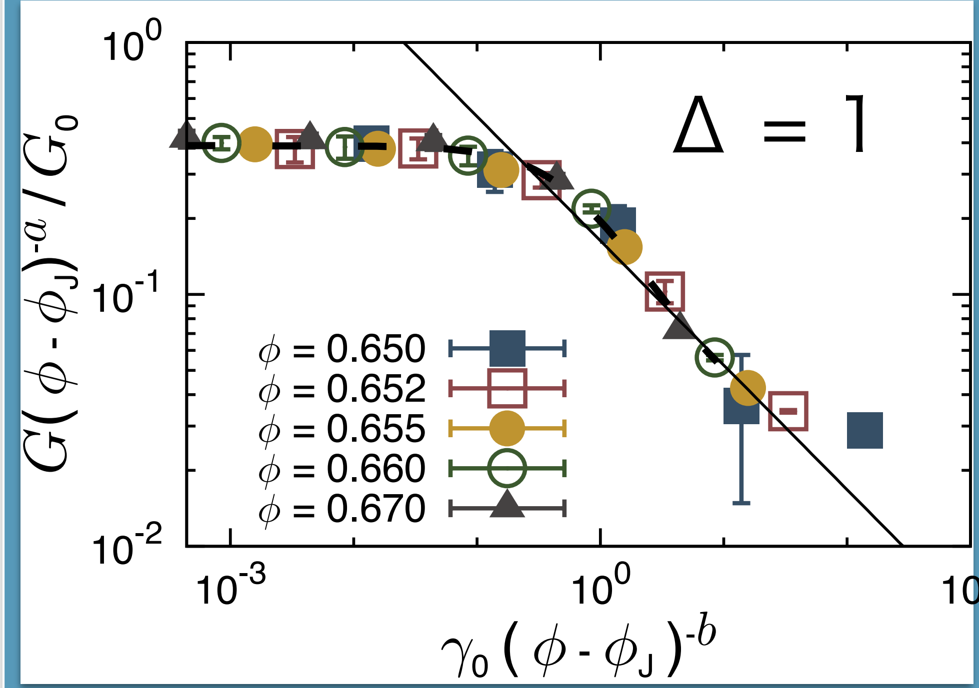
G : shear modulus,
ϕ : volume fraction,
γ_0 : strain amplitude

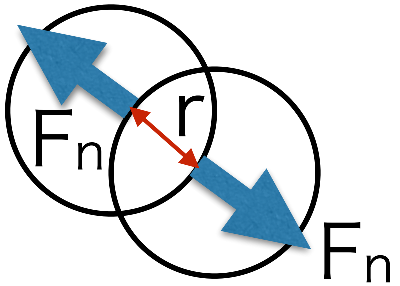
critical exponents **a = Δ - 1/2, b = 1, c = ?**

$$\lim_{\gamma_0\to 0} G(\gamma_0,\phi) \propto (\phi-\phi_J)^a$$

$$G \propto (\phi-\phi_J)^{\Delta-1/2}$$

M. Wyart, Ann. Phys. Fr. (2005)





Elastic interaction force

$$F_n \propto r^{\Delta}$$

contact length : r

Δ=1 (linear force)

Δ= 3/2 (Hertzian force)

Exponent c

MO and H. Hayakawa, PRE (2014)

$$G(\gamma_0, \phi) = (\phi - \phi_J)^a \mathcal{G}(\gamma_0(\phi - \phi_J)^{-b})$$

G : shear modulus,
ϕ : volume fraction,
γ_0 : strain amplitude

$$\lim_{x\to 0} \mathcal{G}(x) = \text{const.} \qquad \lim_{x\to\infty} \mathcal{G}(x) \propto x^{-c}$$

critical exponents **a = Δ - 1/2, b = 1, c = ?**

Finite strain :

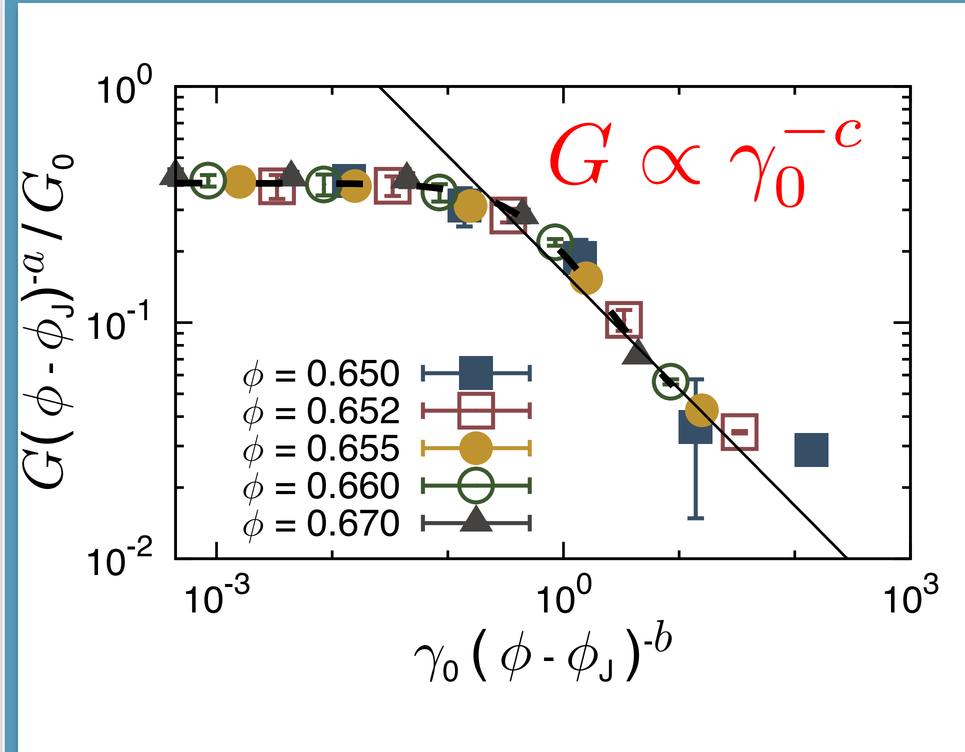
$$G \propto \gamma_0^{-c}(\phi - \phi_J)^{a+bc}$$

Numerical estimation :

C = 1/2 ?

Δ-dependence?

We theoretically estimate the value of c from the strain dependence of G.



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MO and H. Hayakawa, PRE 90, 042202 (2014)

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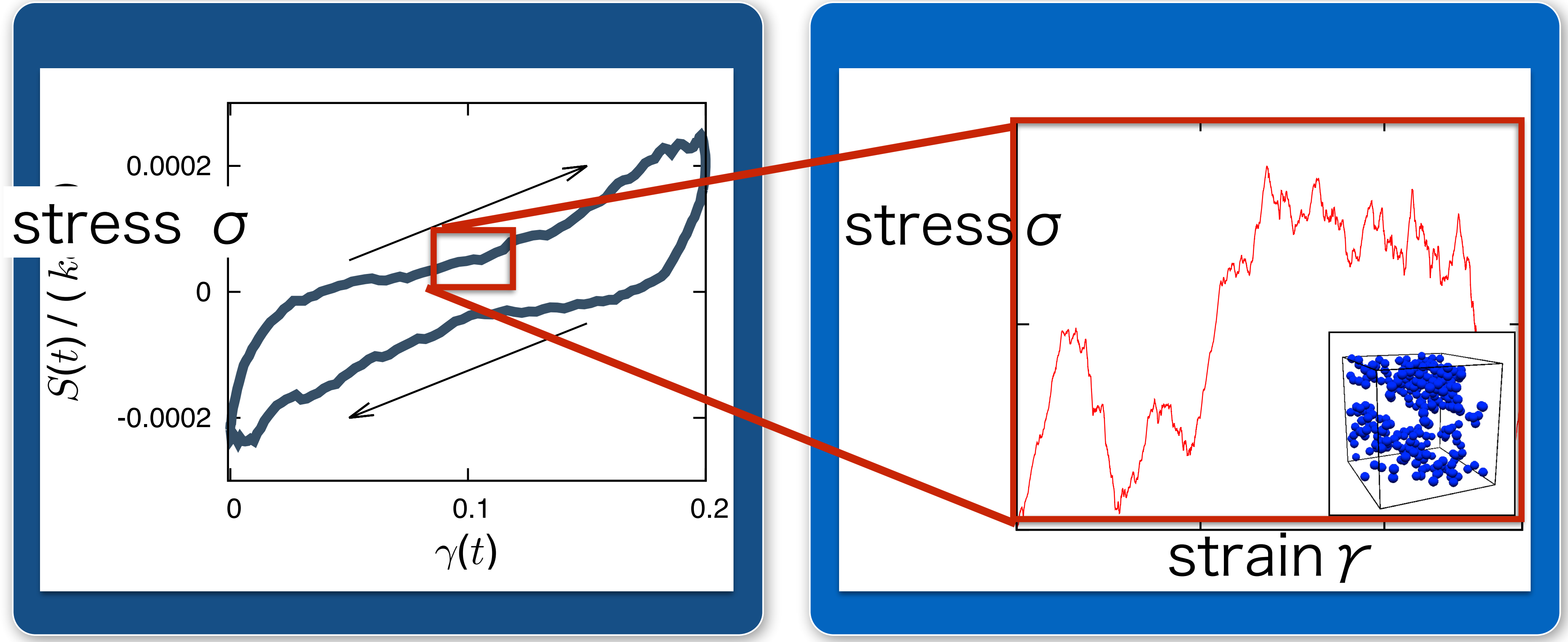
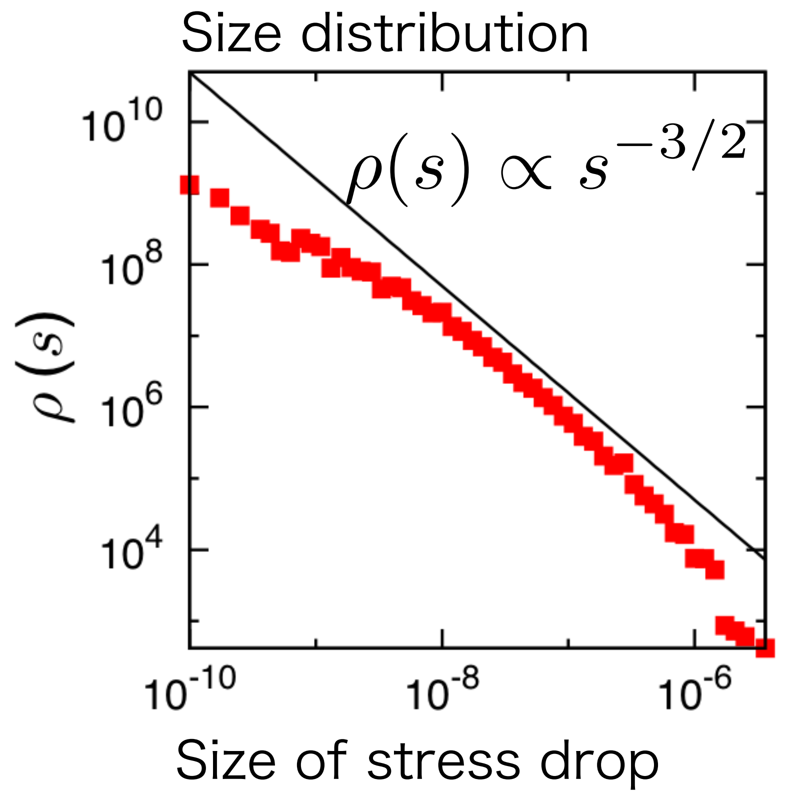
Effect of avalanches



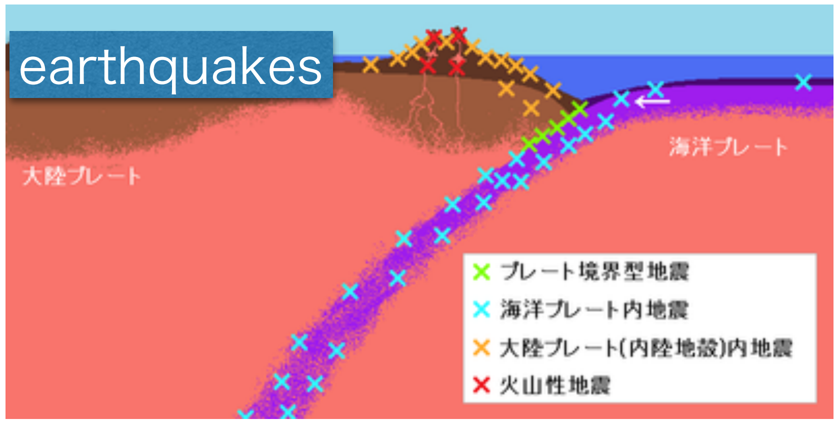
Non-linear and hysteric relation.

There exist stress drops due to avalanches appear.

Size distribution of avalanches



Avalanches



Analysis of a mean field lattice model:

Dahmen et al., (2010)

$$\rho(s) \propto s^{-\tau}$$

$$\tau = 3/2$$

Elastic-plastic model

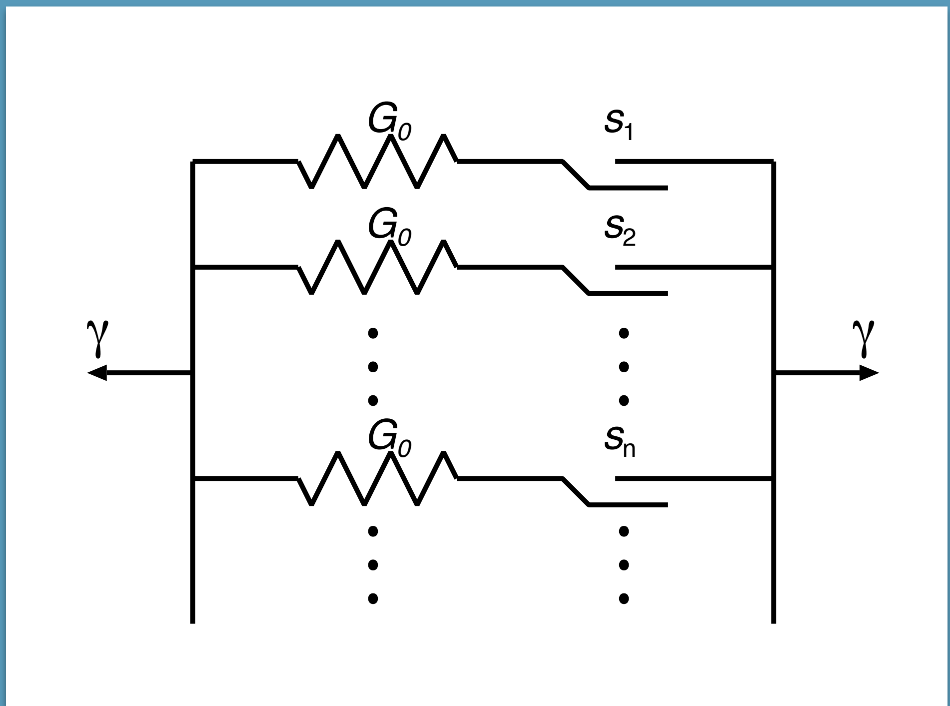
MO and H. Hayakawa, PRE (2014)

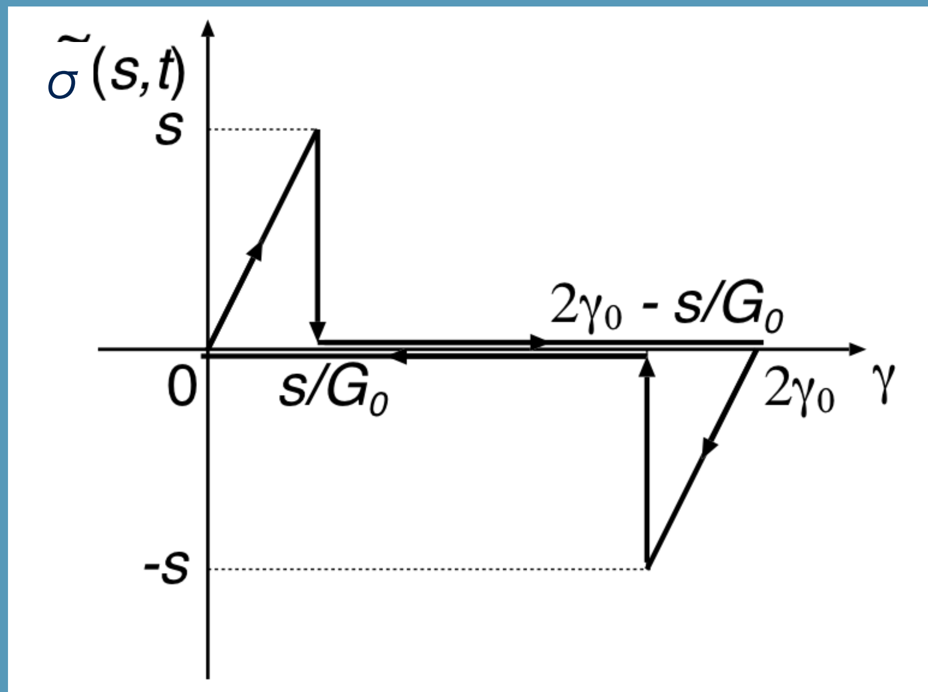
$$\sigma(t) = \int_0^\infty ds\ \rho(s)\tilde{\sigma}(s,t)$$

$\tilde{\sigma}$: stress of element

s : yield stress = stress drop

$\rho(s)$: size distribution





each elements have different yield stress

Phenomenological result

MO and H. Hayakawa, PRE (2014)

$$G(\gamma_0, \phi) = \frac{\omega}{\pi}\int_0^{2\pi/\omega} dt \frac{\sigma(t)\cos(\omega t)}{\gamma_0}$$

$$\sigma(t) = \int_0^\infty ds\ \rho(s)\tilde{\sigma}(s,t)$$

$$\rho(s) \propto s^{-\tau}$$

$$G \propto \gamma_0^{-(\tau-1)} \qquad \tau = 3/2$$

Dahmen et al., (2010)

$$G \propto \gamma_0^{-c}$$ critical scaling

$$\boxed{c = \tau - 1 = 1/2}$$

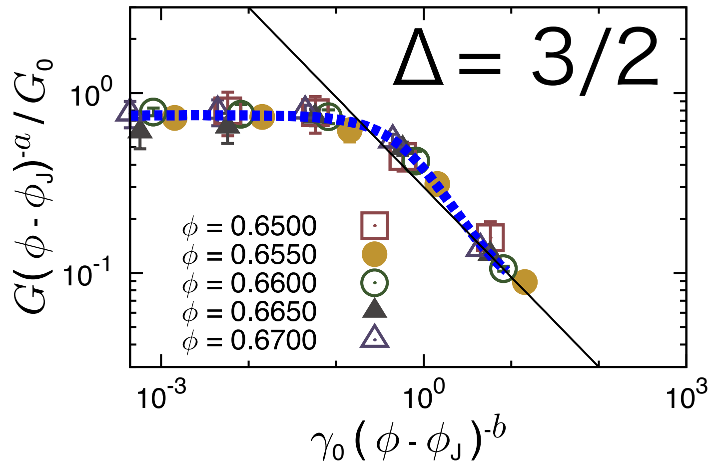
no Δ-dependence

$$G(\gamma_0, \phi) = (\phi - \phi_S)^a \mathcal{G}(\gamma_0(\phi - \phi_S)^{-b})$$

a = Δ - 1/2, b = 1, c = 1/2

$$G \propto (\phi - \phi_J)^{\Delta - 1/2}$$ small γ_0

$$G \propto \gamma_0^{-1/2}(\phi - \phi_J)^{\Delta}$$ large γ_0



This is consistent with the previous talk.
C . Coulais, et al., PRL. 113, 198001 (2014)

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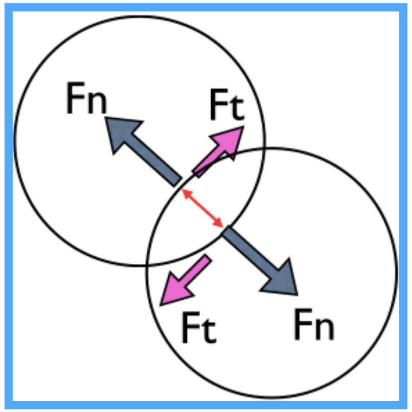
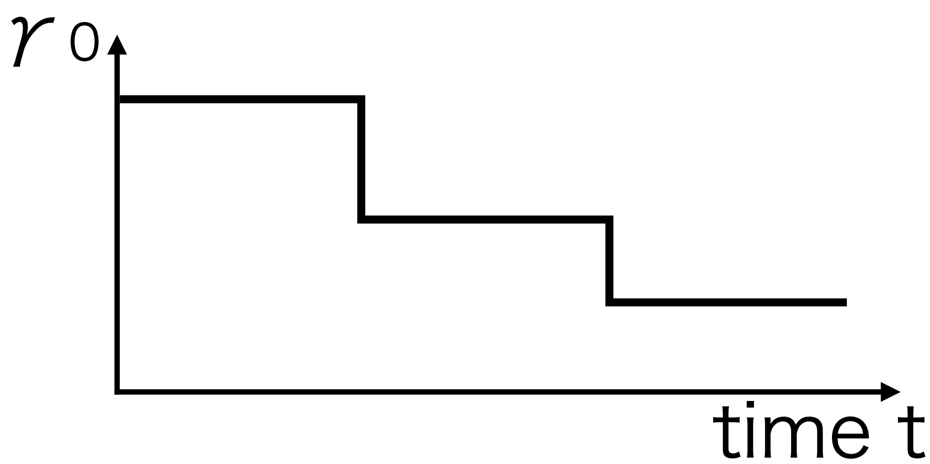
Effect of friction

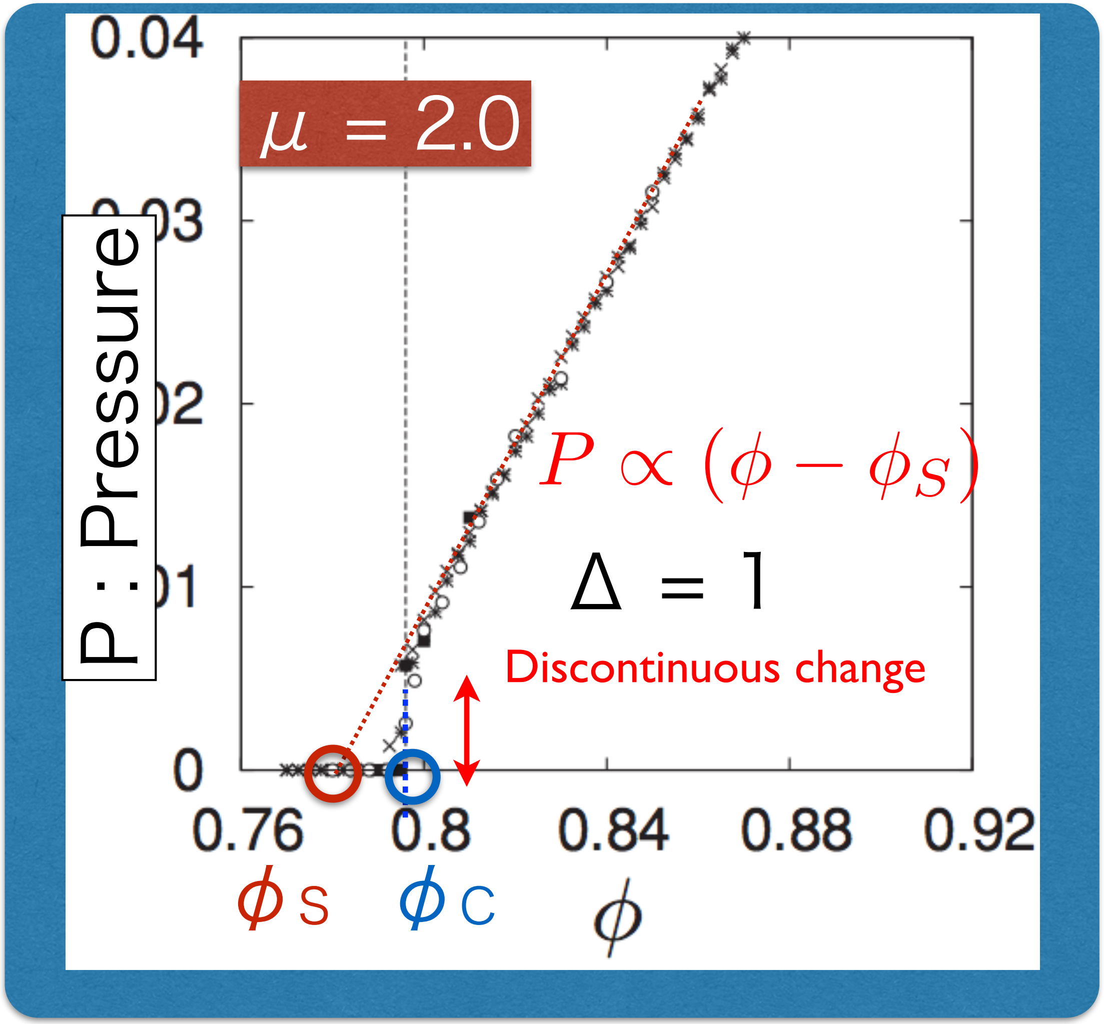


F_n : Normal force
F_t : Tangential force

μ : friction coefficient

Protocol :





Discontinuous transition

Two critical densities :

ϕ_C : True critical density

ϕ_S : (fictitious) critical density for scaling $P \propto (\phi - \phi_S)$

Frictional grains (μ=0.1)

$$G(\gamma_0, \phi) = (\phi - \phi_S)^a \mathcal{G}(\gamma_0(\phi - \phi_S)^{-b})$$

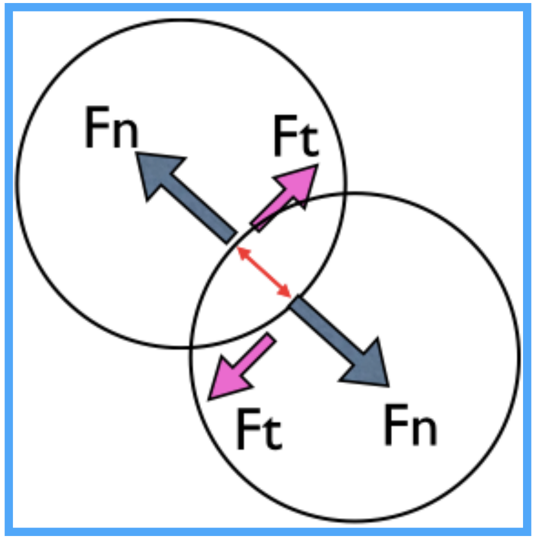
$\Delta = 1$

$$\lim_{x \to \infty} \mathcal{G}(x) \quad \propto \quad x^{-c}$$

$\mu = 0$: $\boxed{a = 1/2, \quad b = 1, \quad c = 1/2}$

This scaling law may be superficial.

$\boxed{a = 0.13, \quad b = 1, \quad c = 0.87}$

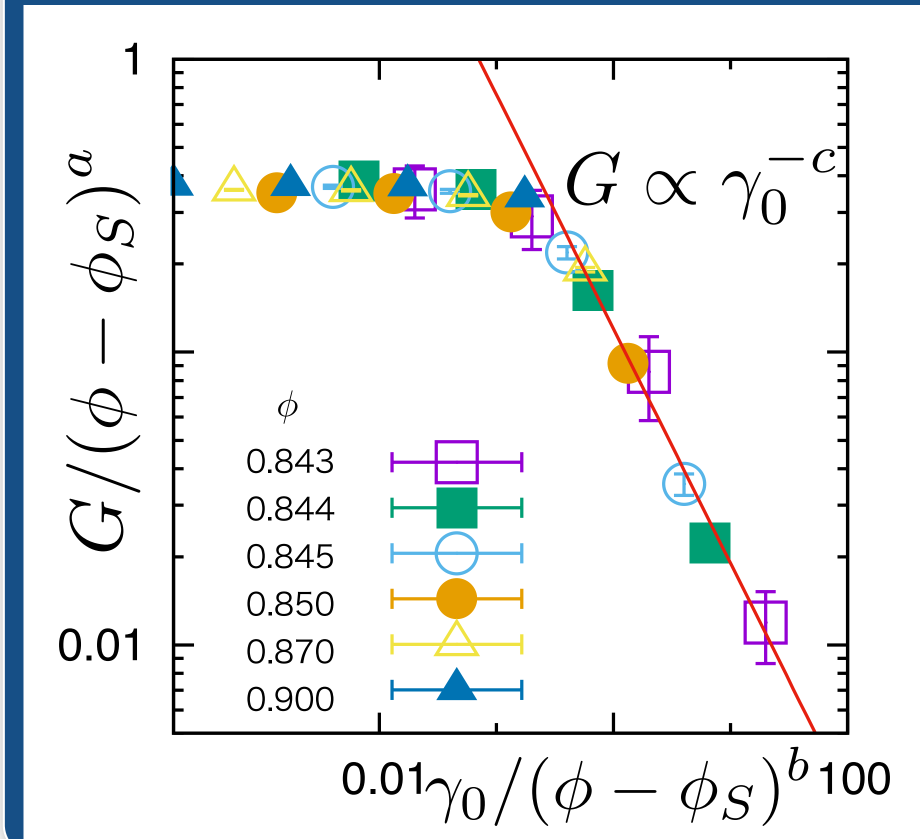


F_n : Normal force

F_t : Tangential force

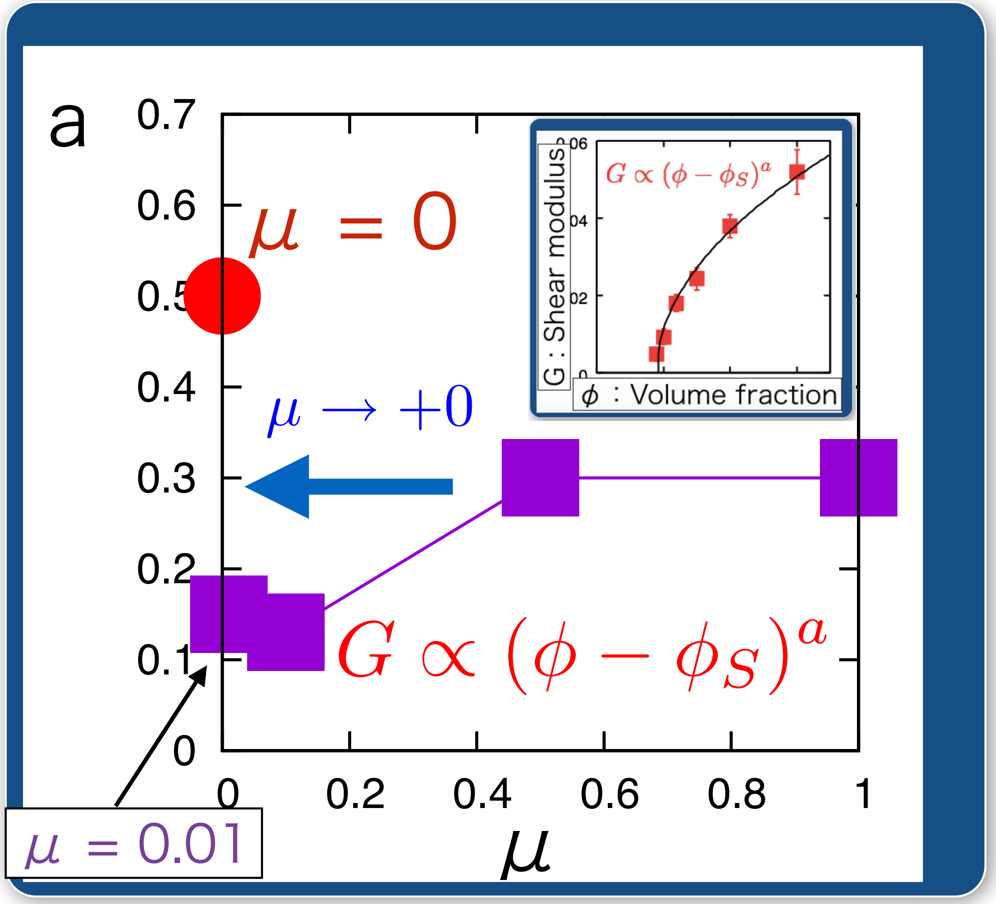
μ : friction coefficient

elastic network with friction

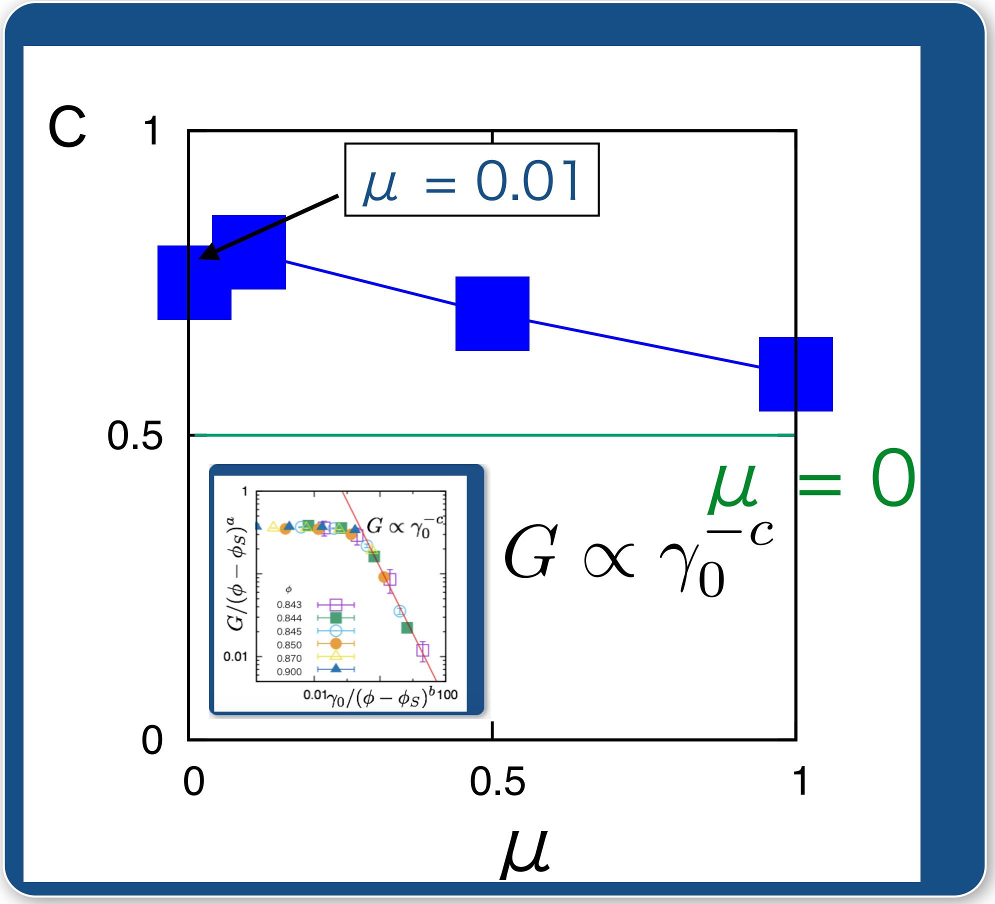


μ-dependence of exponents

Exponent for linear elasticity



Exponent for strain dependence



The exponents do not converge to that of the frictionless particles.

Discussion (Linear elasticity)

Our estimation : $G \propto (\phi - \phi_J)^a$ infinitesimal strain

Somfai, et al., PRE (2007) : $G = G_0(\mu) + A(\phi - \phi_J)^{1/2}$ $\Delta = 1$

elastic network with friction

$$\lim_{\mu \to +0} G_0(\mu) = \mathrm{const.}$$

$$G \propto Z - Z_{\mathrm{iso}}$$

$$Z = Z_J(\mu) + \beta(\phi - \phi_J)^{1/2}$$

Z : coordination number,

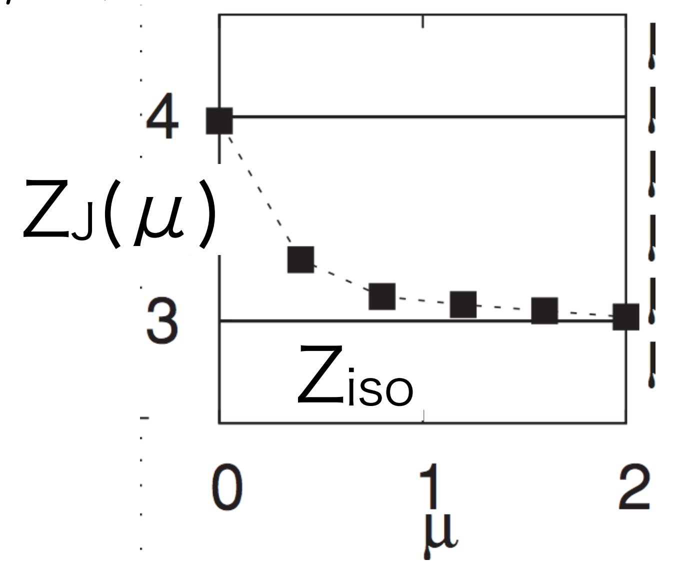
Z_{iso} : coordination number at isostatic state

Z_J : coordination number at jamming

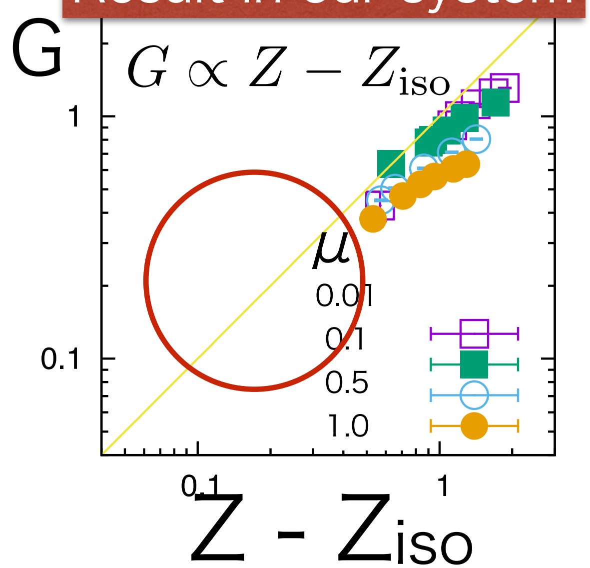
$G \propto Z - Z_{\mathrm{iso}}$ cannot be verified in our system

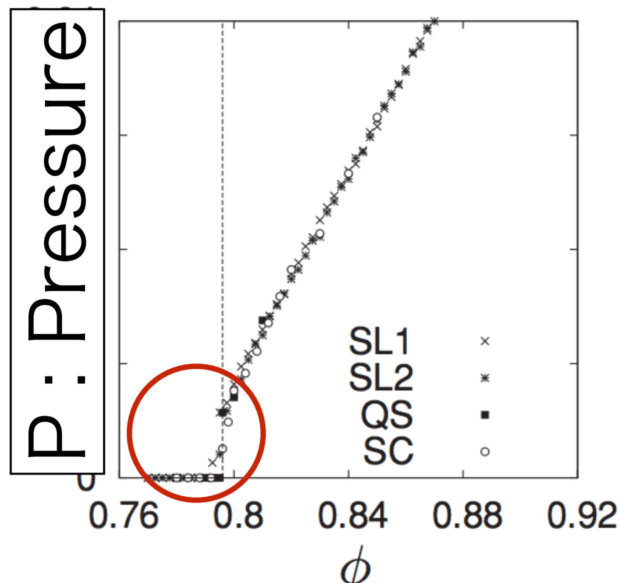
Origin : discontinuous transition

$$\lim_{\mu \to +0} \{Z_J(\mu) - Z_{\mathrm{iso}}\} > 0$$



Result in our system



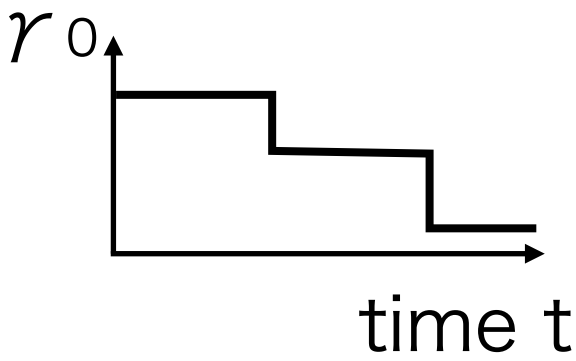


Discussion (Protocol dependence)

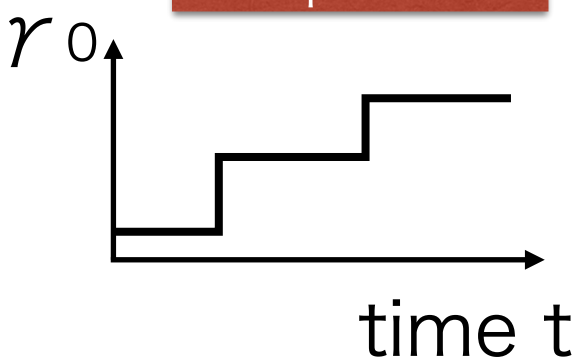
Somfai, et al., PRE (2007) : $G = G_0 + A(\phi - \phi_J)^{1/2}$

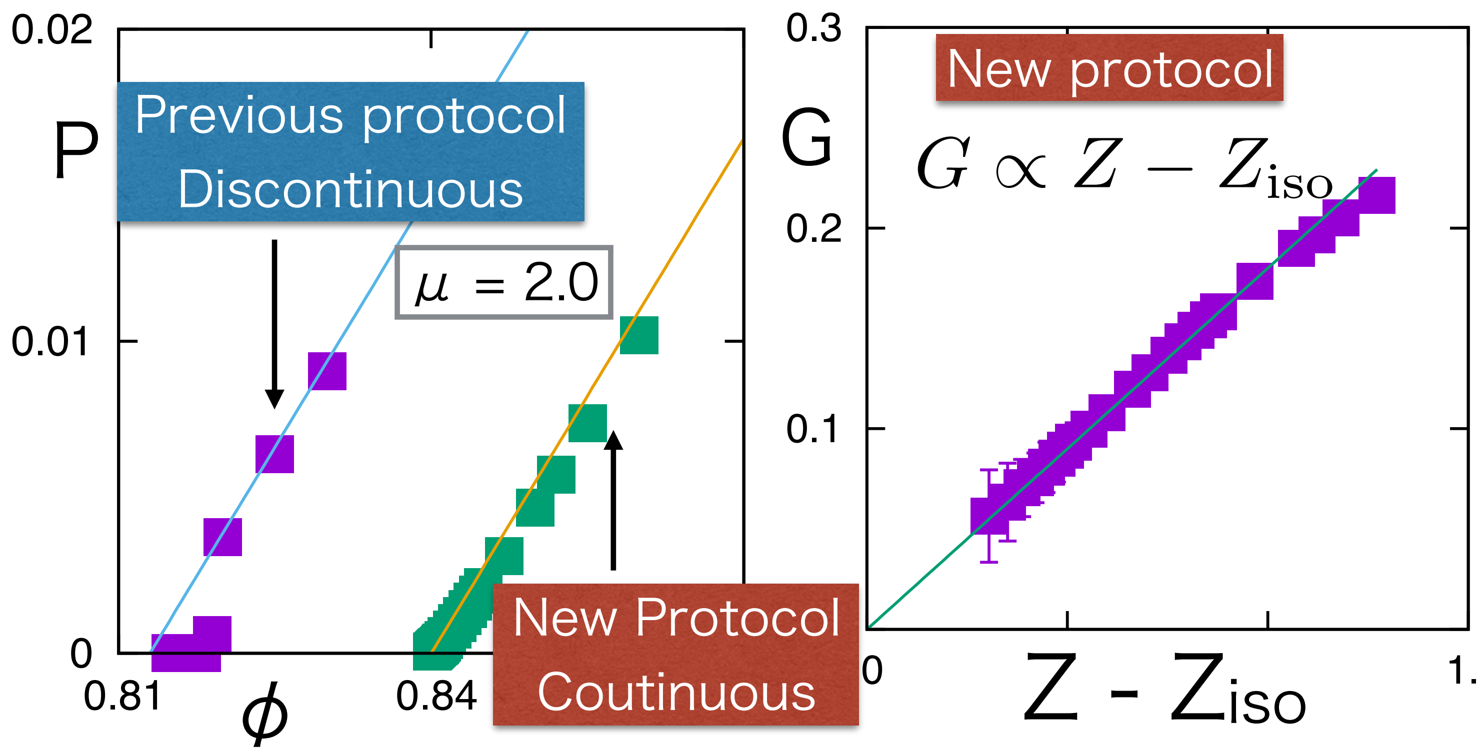
elastic network with friction $G \propto Z - Z_{iso}$

Previous protocol :



New protocol :





$G = G_0 + A(\phi - \phi_J)^{1/2}$ is plausible in new protocol.

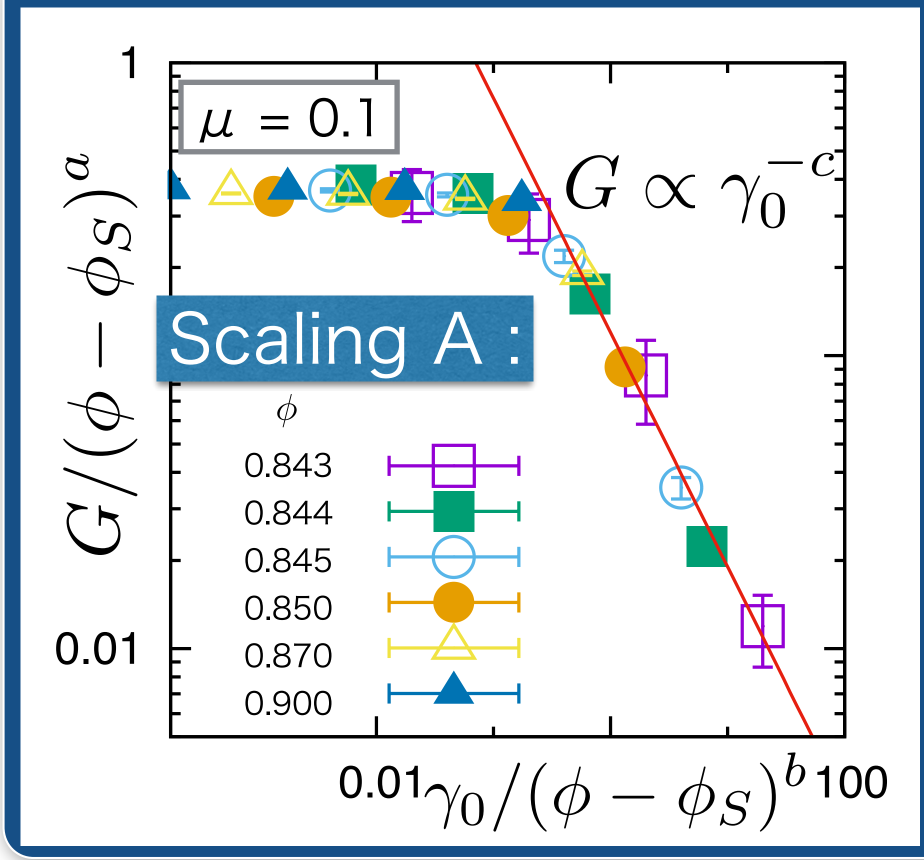
Discussion (Other critical scaling law)

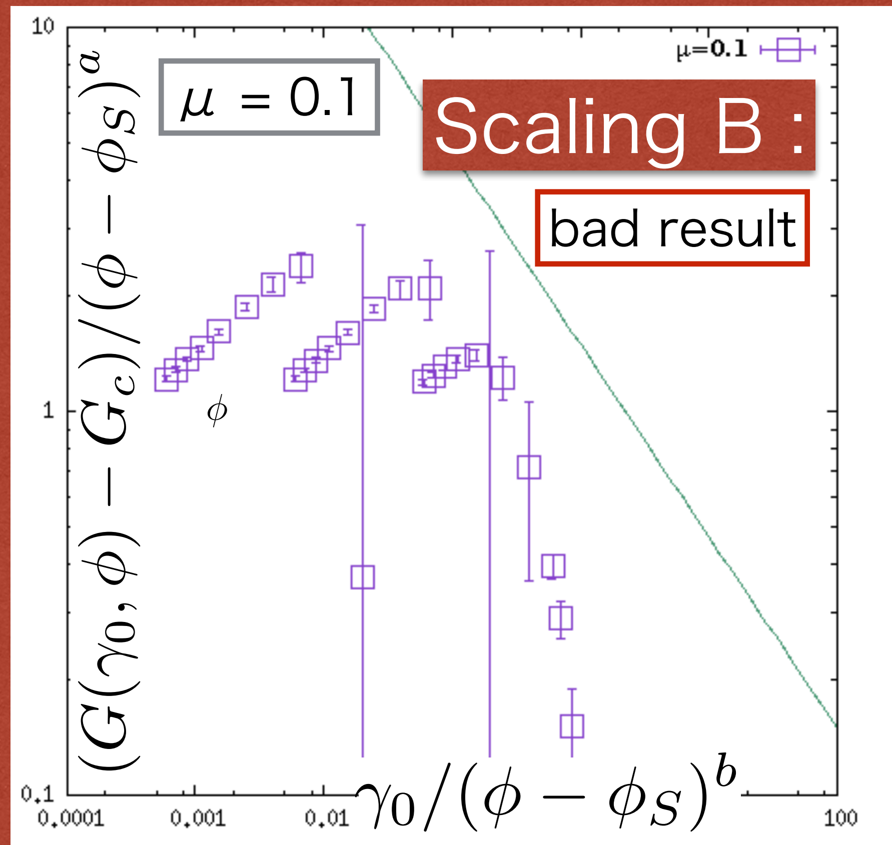
Scaling A : $$G(\gamma_0, \phi) = (\phi - \phi_S)^a \mathcal{G}(\gamma_0(\phi - \phi_S)^{-b})$$

The critical exponent a depend on μ.

Scaling B : $$G(\gamma_0, \phi, \mu) = \underline{G_0(\mu)} + (\phi - \phi_J)^a \mathcal{G}(\gamma_0(\phi - \phi_J)^{-b})$$

G_0 depends on μ. a = 1/2





We don't have a clear understanding yet.

Summary

MO and H. Hayakawa, PRE 90, 042202 (2014) [Frictionless case]

- We perform simulations for frictionless grains under oscillatory shear.
- We found a crossover from the known exponent for the jamming to the non-trivial behavior.
- Non-trivial exponent can be understood by the mean field analysis for avalanches.
- We discuss the effect of the friction coefficient.