
# Chiral Kinetic Theory and Anomalous Transport of Chiral Fluids

Di-Lun Yang

RIKEN

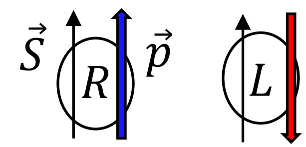
Yoshimasa Hidaka, Shi Pu, DY, arXiv:1612.04630, PRD 95 (2017) no.9, 091901
Yoshimasa Hidaka, Shi Pu, DY, arXiv:1710.00278, PRD 97 (2018) no.1, 016004
Yoshimasa Hidaka, DY, arXiv:1801.08253

# Anomalous transport in chiral matter

- Anomalous transport (for Weyl fermions):

chirality=helicity

Chiral magnetic effect (CME) : $\mathbf{J}_V = \boxed{\frac{\mu_5}{2\pi^2}}\mathbf{B}$




$$\mathbf{J}_V = \mathbf{J}_R + \mathbf{J}_L$$
$$\mathbf{J}_5 = \mathbf{J}_R - \mathbf{J}_L$$

$\mathcal{P}$-odd $\mathcal{T}$-even

A. Vilenkin, 80
D. E. Kharzeev, L. D. McLerran, H. J. Warringa, 08
K. Fukushima, D. E. Kharzeev, and H. J. Warringa, 08

- Chiral anomaly : $\partial_\mu J^\mu_{R/L} = \pm\frac{\mathbf{E}\cdot\mathbf{B}}{4\pi^2} \Rightarrow \partial_\mu J^\mu_5 = \frac{\mathbf{E}\cdot\mathbf{B}}{2\pi^2}$

S. Adler, J. Bell, R. Jackiw, 69
K. Fujikawa, 79

CME : $\mathbf{J}_V = \frac{1}{2\pi^2}\mu_5\mathbf{B}$

CSE : $\mathbf{J}_5 = \frac{1}{2\pi^2}\mu_V\mathbf{B}$

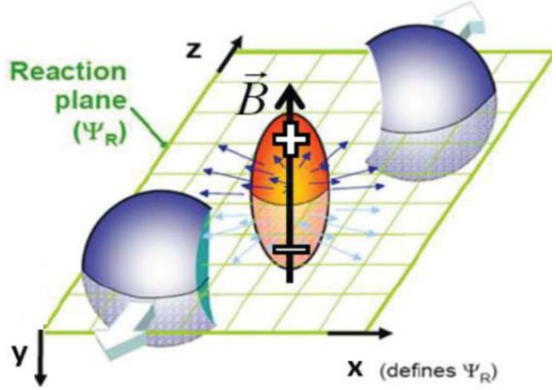
CVE : $\mathbf{J}_V = \frac{1}{\pi^2}\mu_5\mu_V\,\boldsymbol{\omega}$

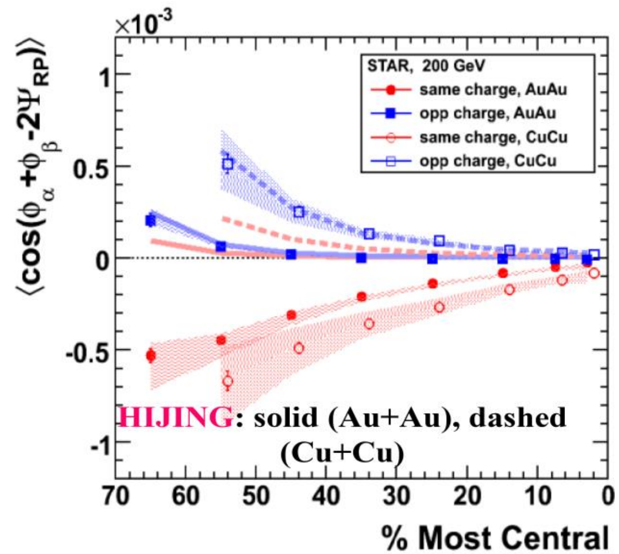
$$\mathbf{J}_5 = \left(\frac{\mu_V^2+\mu_5^2}{2\pi^2}+\frac{T^2}{6}\right)\boldsymbol{\omega}$$

A. Vilenkin, 79
K. Landsteiner, E. Megias, F. Pena-Benitez, 11

# Anomalous transport in the real world

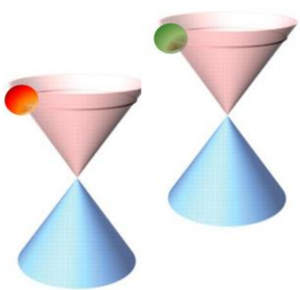
- Heavy ion collisions :



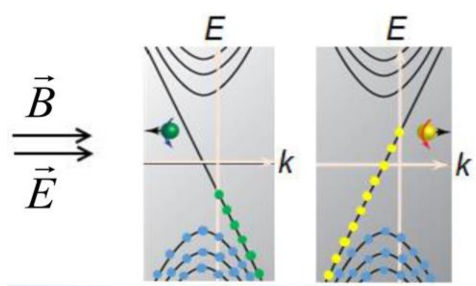




See H.Z. Huang's talk

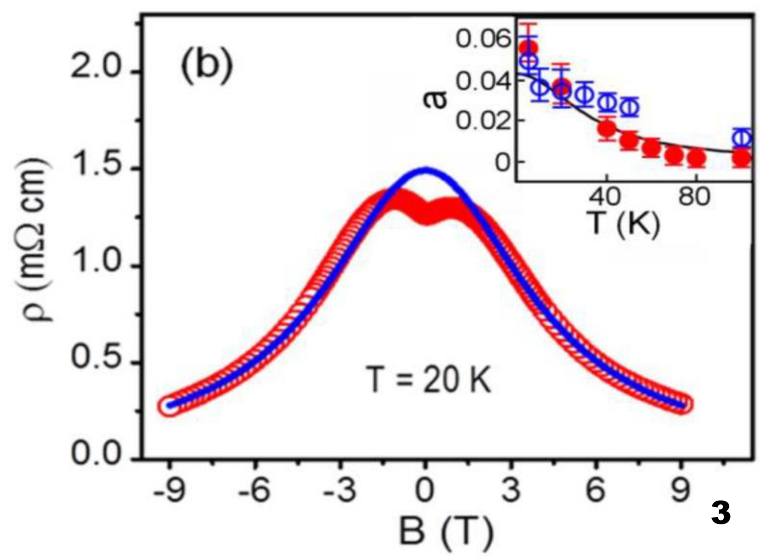
- Weyl semimetals :



TaAs
NbAs
NbP
TaP




charge pumping via parallel
E & B : generate $\mu_5 \sim n_5 \sim E \cdot B$




Qiang Li, et.al., Nature Phys. 12 (2016) 550-554

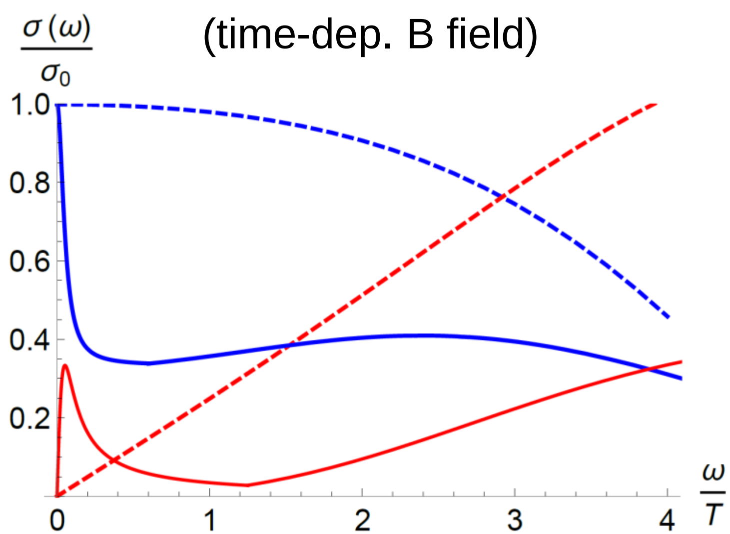
# Non-equilibrium transport

- CME in thermal equilibrium : independent of interactions
- CVE : coupling dependence when having dynamical gauge fields

  D.-F. Hou, H. Liu, and H.-c. Ren, 12. S. Golkar and D. T. Son, 12.

- Non-equilibrium cases : interactions could be involved
- E.g. AC conductivity of CME :

(time-dep. B field)




blue : real part
red : imaginary part
solid : kinetic theory
dashed : AdS/CFT

D. Kharzeev & H. Warringa, 09
D. Satow & H. U. Yee, 14.
D. Kharzeev, et.al., 17,

$$\sigma(\omega) = \sigma_0 \left(1 - \frac{2}{3}\frac{\omega}{\omega + i\tau_R^{-1}}\right)$$

$$\tau_R^{-1} \approx 1.3\,\alpha_s^2 \log(1/\alpha_s) T \quad (2-\text{flavor QCD})$$

"The magnetization current plays an important role."

- In practice, most of systems are non-equilibrium or just near equilibrium.
- Weakly coupled systems : chiral kinetic theory (CKT)

# Outline

- Recent development in chiral kinetic theory (CKT) : (QED with BFs)
  - Covariant CKT with background fields & collisions from QFT (Wigner-function approach)

    Hidaka, Pu, DY, 16

- Applications : non(near)-equilibrium transport for chiral fluids
  - Non-linear ($2^{nd}$-order) responses for anomalous transport

    Hidaka, Pu, DY, 17
    Hidaka, DY, 18

    ($2^{nd}$-order anomalous hydro)

  - Angular momenta of chiral fluids DY, preliminary results

(I will focus on R-handed fermions.)

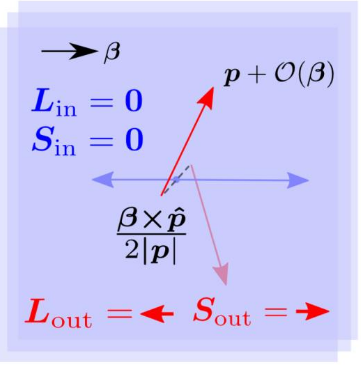
# Chiral kinetic theory

- Standard kinetic theory : $q^{\mu}\left(\partial_{\mu}+F_{\nu\mu}\frac{\partial}{\partial q_{\nu}}\right)f=\mathcal{C}[f] \Rightarrow \partial_{\mu}J^{\mu}=0$
- CKT : D. T. Son and N. Yamamoto,12 M. Stephanov and Y. Yin, 12 $\longrightarrow \partial_{\mu}J^{\mu}=\frac{\hbar}{4\pi^{2}}\mathbf{E}\cdot\mathbf{B}$ (for right-handed fermions)

❖ The semi-classical approach : classical action + Berry phase

$$\left[(1+\hbar\mathbf{B}\cdot\mathbf{\Omega_p})\,\partial_t+(\tilde{\mathbf{v}}+\hbar\mathbf{E}\times\mathbf{\Omega_p}+\hbar(\tilde{\mathbf{v}}\cdot\mathbf{\Omega_p})\mathbf{B})+\left(\tilde{\mathbf{E}}+\tilde{\mathbf{v}}\times\mathbf{B}+\hbar(\tilde{\mathbf{E}}\cdot\mathbf{B})\mathbf{\Omega_p}\right)\cdot\frac{\partial}{\partial\mathbf{p}}\right]f=0$$

Berry curvature : $\mathbf{\Omega_p}=\frac{\mathbf{p}}{2|\mathbf{p}|^{3}}$ $\quad \tilde{\mathbf{v}}=\partial\epsilon_{\mathbf{p}}/\partial\mathbf{p}$ $\quad \tilde{\mathbf{E}}=\mathbf{E}-\partial\epsilon_{\mathbf{p}}/\partial\mathbf{x}$




❖ Lorentz covariance : modified L.T. and side jumps

(responsible for the magnetization current & part of CVE)

J.-Y. Chen, et.al. 14 J.-Y. Chen, D. T. Son, and M. Stephanov, 15

❖ QFT derivation (WF) : limited conditions (steady state or high density)

J.-W. Chen, et.al. 13, D. T. Son & N. Yamamoto, 13

❖ A more general version : (non-)equilibrium + collisions Hidaka, Pu, DY, 16

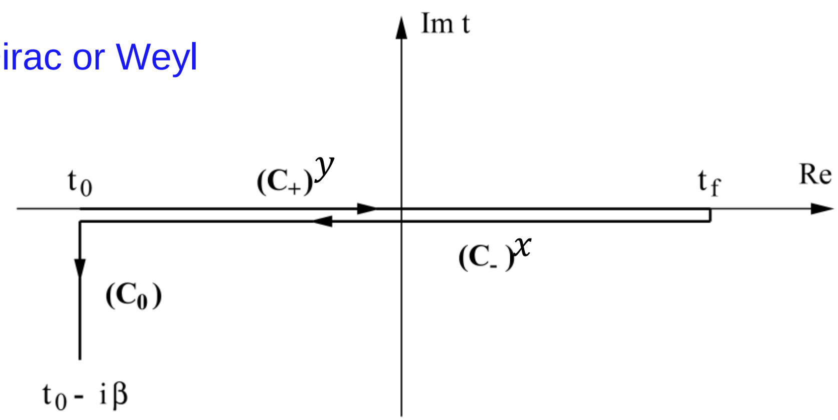
# Wigner functions (WF)

- less (greater) propagators :

Dirac or Weyl

$$S^{>}(x,y) = \langle \psi(x)\mathcal{P}\mathcal{U}^{\dagger}(A_\mu,x,y)\psi^{\dagger}(y)\rangle$$
$$S^{<}(x,y) = \langle \psi^{\dagger}(y)\boxed{\mathcal{P}\mathcal{U}(A_\mu,x,y)}\psi(x)\rangle$$

gauge link




$$X = \frac{x+y}{2},\ Y = x - y$$

review : J. Blaizot, E. Iancu, Phys.Rept. 359 (2002) 355-528

Wigner functions : $\grave{S}^{<(>)}(q,X) = \int d^4Y e^{\frac{iq\cdot Y}{\hbar}} S^{<(>)}\left(X+\frac{Y}{2}, X-\frac{Y}{2}\right)$

- Wigner functions are always covariant : $J^\mu = \int \frac{d^4q}{(2\pi)^4}\mathrm{tr}\left(\sigma^\mu \grave{S}^<\right)$

- Kadanoff-Baym-like equations up to $\mathcal{O}(\hbar)$ : ($q \gg \partial$ : weak fields)

$$\sigma^\mu\left(q_\mu + \frac{i\hbar}{2}\Delta_\mu\right)\grave{S}^< = \frac{i\hbar}{2}\left(\Sigma^<\grave{S}^> - \Sigma^>\grave{S}^<\right), \qquad \Delta_\mu = \partial_\mu + F_{\nu\mu}\partial/\partial q_\nu$$

systematically include collisions

$$\left(q_\mu - \frac{i\hbar}{2}\Delta_\mu\right)\grave{S}^<\sigma^\mu = -\frac{i\hbar}{2}\left(\grave{S}^>\Sigma^< - \grave{S}^<\Sigma^>\right).$$

(R-handed fermions)

# Quantum corrections for WF

- WF up to $\mathcal{O}(\hbar)$ : $\grave{S}^{<} = \bar{\sigma}_\mu \grave{S}^{<\mu}$, Hidaka, Pu, DY, 16,17

CME in equilibrium

$$\boxed{\grave{S}^{<\mu}(q,X) = 2\pi\bar{\epsilon}(q\cdot n)\left(q^\mu \delta(q^2) f_q^{(n)} + \boxed{\hbar\delta(q^2) S^{\mu\nu}_{(n)} \mathcal{D}_\nu f_q^{(n)}} + \hbar \epsilon^{\mu\nu\alpha\beta} q_\nu F_{\alpha\beta} \frac{\partial \delta(q^2)}{2\partial q^2} f_q^{(n)}\right)},$$

$\mathcal{D}_\beta f_q^{(n)} = \Delta_\beta f_q^{(n)} - \mathcal{C}_\beta, \quad C_\beta[f] = \Sigma^<_\beta \bar{f} - \Sigma^>_\beta f$

side-jump term : magnetization current

CVE or non-equilibrium effects

spin tensor : $S^{\mu\nu}_{(n)} = \dfrac{\epsilon^{\mu\nu\alpha\beta}}{2(q\cdot n)} q_\alpha n_\beta$

$S^{\mu\nu}_{(n)} \partial_\nu f_q^{(n)}$ : from inhomogeneity

- Frame vector $n^\mu$ : choice of the spin basis $\sigma^0 = I$ ($n^\mu = (1, \mathbf{0})$) $\Longrightarrow$ $n_\mu \sigma^\mu = I$
- The modified frame transformation :

(agrees with J.-Y. Chen, et.al. 14)

$$\boxed{f_q^{(n')} = f_q^{(n)} + \frac{\hbar \epsilon^{\nu\mu\alpha\beta} q_\alpha n'_\beta n_\mu}{2(q\cdot n)(q\cdot n')} \mathcal{D}_\nu f_q^{(n)}}$$

- CKT (for $n^\mu = n^\mu(X)$) :

$$\boxed{\delta\left(q^2 - \hbar \frac{B\cdot q}{q\cdot n}\right)\left[q\cdot\tilde{\mathcal{D}} + \frac{\hbar S^{\mu\nu}_{(n)} E_\mu}{q\cdot n}\mathcal{D}_\nu + \hbar S^{\mu\nu}_{(n)}(\partial_\mu F_{\rho\nu})\partial_q^\rho + \hbar(\partial_\mu S^{\mu\nu}_{(n)})\mathcal{D}_\nu\right] f_q^{(n)} = 0}$$

energy shift

$\tilde{\mathcal{D}}_\mu f_q^{(n)} = \Delta_\mu f_q^{(n)} - \tilde{\mathcal{C}}_\mu \qquad \tilde{\mathcal{C}}^\mu = \mathcal{C}^\mu + \hbar \dfrac{\epsilon^{\mu\nu\alpha\beta} n_\nu}{2q\cdot n}\left(\bar{f}_q^{(n)} \Delta^>_\alpha \Sigma^<_\beta - f_q^{(n)} \Delta^<_\alpha \Sigma^>_\beta\right)$

# Local equilibrium

- Local equilibrium $(T, \mu \neq \text{const.})$ :
  $f$ can be nontrivially defined in $n^\mu = u^\mu$(fluid velocity) such that $C[f] = 0$ :

$$\Rightarrow \quad f_q^{\mathrm{eq}(u)} = \left(\exp\left[\beta(q\cdot u - \mu) + \boxed{\frac{\hbar q\cdot\omega}{2q\cdot u}}\right] + 1\right)^{-1}, \quad \omega^\mu = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}u_\nu(\partial_\alpha u_\beta).$$

spin-vorticity coupling

J.-Y. Chen, D. T. Son, and M. A. Stephanov, 15

Hidaka, Pu, DY, 17

- Near local equilibrium : $T^{\mu\nu} = \int \frac{d^4q}{(2\pi)^4}\left[q^\mu \grave{S}^{<\nu} + q^\nu \grave{S}^{<\mu}\right], \quad J^\mu = 2\int \frac{d^4q}{(2\pi)^4}\grave{S}^{<\mu}.$

$$T^{\mu\nu} = \underbrace{u^\mu u^\nu \epsilon - p\Theta^{\mu\nu}}_{\mathcal{O}(1)} + \underbrace{\Pi^{\mu\nu}_{\mathrm{non}}}_{\mathcal{O}(\hbar)} + \underbrace{\Pi^{\mu\nu}_{\mathrm{dis}}}_{\mathcal{O}(1)+\mathcal{O}(\hbar)}, \quad J^\mu = \underbrace{N_0 u^\mu}_{\mathcal{O}(1)} + \underbrace{v^\mu_{\mathrm{non}}}_{\mathcal{O}(\hbar)} + \underbrace{v^\mu_{\mathrm{dis}}}_{\mathcal{O}(1)+\boxed{\mathcal{O}(\hbar)}}, \quad \Theta^{\mu\nu} = \eta^{\mu\nu} - u^\mu u^\nu$$

our focus

- Equilibrium anomalous transport : $F_{\alpha\beta} = -\epsilon_{\mu\nu\alpha\beta}B^\mu u^\nu + u_\beta E_\alpha - u_\alpha E_\beta$

$$v^\mu_{\mathrm{non}} = \hbar\sigma_B B^\mu + \hbar\sigma_\omega\omega^\mu \qquad \Pi^{\mu\nu}_{\mathrm{non}} = \hbar\xi_\omega\left(\omega^\mu u^\nu + \omega^\nu u^\mu\right) + \hbar\xi_B\left(B^\mu u^\nu + B^\nu u^\mu\right)$$

$$\sigma_\omega = \frac{T^2}{12}\left(1 + \frac{3\bar{\mu}^2}{\pi^2}\right), \quad \sigma_B = \frac{\mu}{4\pi^2}, \quad \xi_\omega = \frac{T^3}{6}\left(\bar{\mu} + \frac{\bar{\mu}^3}{\pi^2}\right), \quad \xi_B = \frac{T^2}{24}\left(1 + \frac{3\bar{\mu}^2}{\pi^2}\right).$$

(agree with different approaches, e.g. Son & Surowka, 09. K. Landsteiner, et.al. Lect. Notes, 13.)

CVE CME

Hidaka, Pu, DY, 17

# RT approximation & matching conditions

- 2$^{nd}$-order anomalous responses :

Hidaka, Pu, DY, 17  Hidaka, DY, 18

- Solve CKT for $f_q^{(u)} - f_q^{\rm eq} = \delta f_q = \delta f_q^{(c)} + \hbar \delta f_q^{(Q)}$
- Relaxation time approximation (RTA) : $\square(q,X) f_q^{(u)} = \mathcal{C}_{\rm full},$

  (neglect R-L int.)

$$\mathcal{C}_{\rm full} = -\tau_R^{-1}(q\cdot u)\delta f_q$$

- Constrains from hydrodynamic EOM : $\partial_\mu T^{\mu\nu} = F^{\nu\rho} J_\rho, \quad \partial_\mu J^\mu = \frac{\hbar}{4\pi^2}(\mathbf{E}\cdot\mathbf{B})$

  (to satisfy *charge & energy conservation) $\Longrightarrow$ $DT, D\mu, Du^\mu. \qquad D = u\cdot\partial$

- Matching conditions for RTA :

$$\partial_\mu J^\mu = -\frac{\hbar}{4\pi^2} E_\mu B^\mu - \frac{u_\mu \delta J^\mu}{\tau_R}, \quad \partial_\mu T^{\mu\nu} = F^{\nu\mu} J_\mu - \frac{u_\mu \delta T^{\mu\nu}}{\tau_R}.$$

*charge & energy conservation $\Longrightarrow$ $\boxed{u_\mu \delta J^\mu = u_\mu \delta T^{\mu\nu} = 0}$

No non-equilibrium corrections on charge density
& energy-density current (same as standard KT)

# Non-equilibrium charged currents

- $2^{nd}$-order corrections on currents : $V^\mu_\perp \equiv P^\mu_\nu V^\nu \quad P^{\mu\nu} \equiv \eta^{\mu\nu} - u^\mu u^\nu$

| $J^\mu_{Q\perp}$ | $\mathcal{O}(\partial)$ | $\mathcal{O}(\partial^2)$ |
|---|---|---|
| $\mathcal{O}(\hbar)$ | $\sigma_B B^\mu + \sigma_\omega \omega^\mu$ | $\tau_R\big[\epsilon^{\mu\nu\alpha\beta}u_\nu\big(\hat{\gamma}_E\partial_\alpha E_\beta + \hat{\gamma}_\mu E_\alpha\partial_\beta\mu + \hat{\gamma}_T E_\alpha\partial_\beta T + \hat{\gamma}_{T\mu}(\partial_\alpha T)(\partial_\beta\mu)\big) + \delta\hat{\sigma}_{BL}\theta B^\mu + \delta\hat{\sigma}_{BH}\pi^{\mu\nu}B_\nu + \delta\hat{\sigma}_{\omega L}\theta\omega^\mu + \delta\hat{\sigma}_{\omega H}\pi^{\mu\nu}\omega_\nu\big]$ |

$\theta \equiv \partial\cdot u$

$\pi^{\mu\nu} \equiv P^\mu_\rho P^\nu_\sigma(\partial^\rho u^\sigma + \partial^\sigma u^\rho - 2\eta^{\rho\sigma}\theta/3)/2$

Hidaka, Pu, DY, 17
Hidaka, DY, 18

Viscous corrections for CME/CVE

- $2^{nd}$-order quantum transport coefficients : $\mathcal{P}, \mathcal{T} - odd$
- The origin of viscous corrections : time-dep. B

$\partial_\nu \tilde{F}^{\mu\nu} = 0$

$$\Rightarrow \boxed{u\cdot\partial B^\rho + B^\rho\partial\cdot u - B\cdot\partial u^\rho + u^\rho B^\mu u\cdot\partial u_\mu} + \epsilon^{\rho\mu\alpha\beta}\big(u_\beta\partial_\mu E_\alpha + u_\mu E_\alpha u\cdot\partial u_\beta\big) = 0$$

- Recall that AC conductivity of CME is int.-dep.

D. Kharzeev & H. Warringa, 09
D. Satow & H. U. Yee, 14.
D. Kharzeev, et.al., 17,

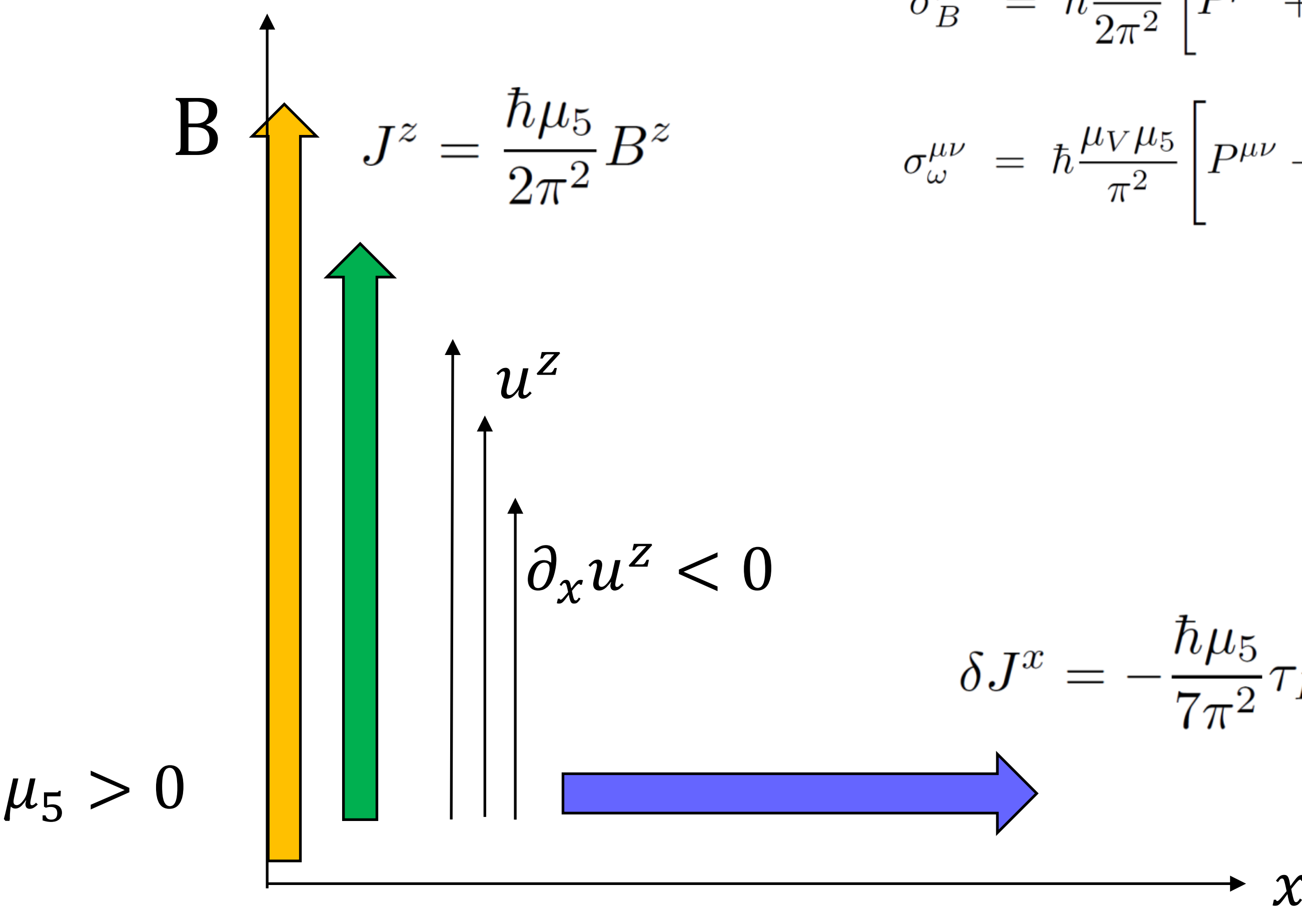
# High-temperature limit

- Viscous corrections on CME/CVE ($\bar{\mu} \ll 1$) : $J^{\mu}_{Q\perp} = \sigma^{\mu\nu}_{B} B_{\nu} + \sigma^{\mu\nu}_{\omega}\omega_{\nu},$

Hidaka, DY, 18

$$\sigma^{\mu\nu}_{B} = \hbar\frac{\mu_5}{2\pi^2}\left[P^{\mu\nu} + \tau_R\left(\frac{10P^{\mu\nu}}{9}\theta - \frac{2\pi^{\mu\nu}}{7}\right)\right],$$

$$\sigma^{\mu\nu}_{\omega} = \hbar\frac{\mu_V\mu_5}{\pi^2}\left[P^{\mu\nu} - \tau_R\left(\frac{37P^{\mu\nu}}{264}\theta + \frac{10\pi^{\mu\nu}}{63}\right)\right].$$

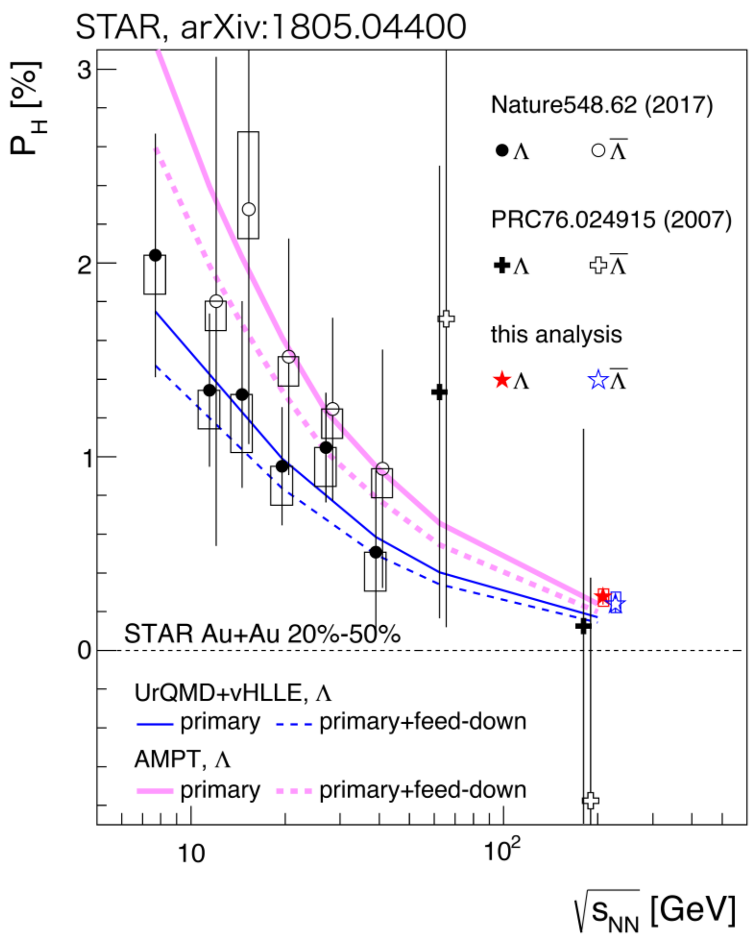


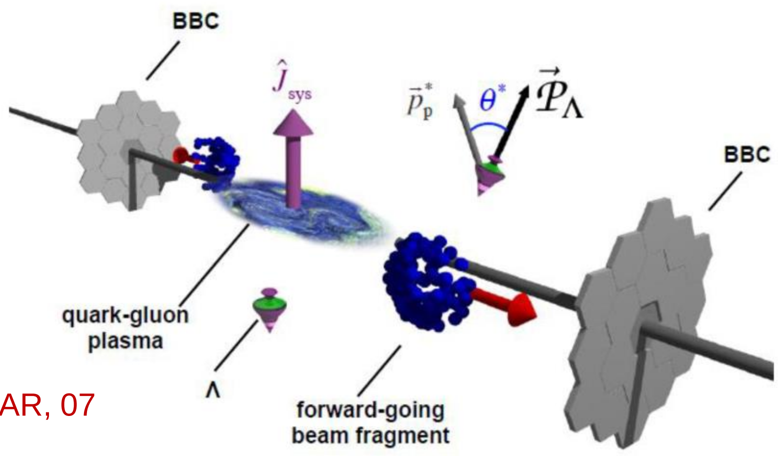

$$J^z = \frac{\hbar\mu_5}{2\pi^2}B^z$$

$$\delta J^x = -\frac{\hbar\mu_5}{7\pi^2}\tau_R B^z \partial_x u^z$$

# Rotating fluids with spins

- Global polarization of Λ hyperons :




STAR, 07

Theoretical studies :

❖ Spin-orbit coupling model

Z.-T. Liang & X.-N. Wang, 05

❖ Wigner-function method

Q. Wang, et.al. 16

❖ Statistical-hydro model

F. Becattini, et.al. 13

See e.g. QM 17 review by Qun Wang

➢ Relativistic hydrodynamics with spins : charge/energy-momentum + angular-momentum conservation Florkowski, et.al. 17

# Angular momenta for relativistic fermions

- Considering just fermions : $$\mathcal{L} = \bar{\psi}\left(\frac{i\hbar}{2}\gamma^\mu \overleftrightarrow{D}_\mu - m\right)\psi$$
- Canonical EM tensor (from Noether's theorem + EOM) :

$$\bar{T}^{\mu\nu} = T^{\mu\nu} + T_A^{\mu\nu}, \quad T^{\mu\nu} = \frac{i\hbar}{4}\bar{\psi}\gamma^{\{\mu}\overleftrightarrow{D}^{\nu\}}\psi, \quad T_A^{\mu\nu} = \frac{i\hbar}{4}\bar{\psi}\gamma^{[\mu}\overleftrightarrow{D}^{\nu]}\psi$$

- Canonical AM tensor :

review : E. Leader & C. Lorce, 13

$$M_C^{\lambda\mu\nu} = M_{\text{spin}}^{\lambda\mu\nu} + M_{\text{orbit}}^{\lambda\mu\nu},$$

$$M_{\text{spin}}^{\lambda\mu\nu} = \frac{\hbar}{2}\bar{\psi}\{\gamma^\lambda, \Sigma^{\mu\nu}\}\psi = -\frac{\hbar}{2}\epsilon^{\lambda\mu\nu\rho}\bar{\psi}\gamma_5\gamma_\rho\psi,$$

$$M_{\text{orbit}}^{\lambda\mu\nu} = \frac{i\hbar}{2}\bar{\psi}\gamma^\lambda\left(x^\mu\overleftrightarrow{D}^\nu - x^\nu\overleftrightarrow{D}^\mu\right)\psi = x^\mu T^{\lambda\nu} - x^\nu T^{\lambda\mu} + x^\mu T_A^{\lambda\nu} - x^\nu T_A^{\lambda\mu}$$

- Conservation (in global equilibrium) : $\partial_\mu \bar{T}^{\mu\nu} = 0, \quad \partial_\lambda M_C^{\lambda\mu\nu} = 0.$
- In relation to the Belinfante one :

(using EOM)

$$M_C^{\lambda\mu\nu} = M_B^{\lambda\mu\nu} + \partial_\beta V^{[\beta\lambda][\mu\nu]},$$

$$M_B^{\lambda\mu\nu} = x^\mu T^{\lambda\nu} - x^\nu T^{\lambda\mu}$$

# Angular momenta in Wigner functions

- Phase-space distributions in terms of WFs :

$$T^{\mu\nu}(q,X) = \Pi^{\{\nu}\grave{S}_V^{<\mu\}}(q,X), \quad T_A^{\mu\nu}(q,X) = \Pi^{[\nu}\grave{S}_V^{<\mu]}(q,X),$$

$$M_{\rm spin}^{\lambda\mu\nu}(q,X) = -\hbar\epsilon^{\lambda\mu\nu\rho}\grave{S}_{5\rho}^{<}(q,X),$$

$$M_{\rm orbit}^{\lambda\mu\nu}(q,X) = X^\mu\bar{T}^{\lambda\nu}(q,X) - X^\nu\bar{T}^{\lambda\mu}(q,X) + \boxed{\hbar\big(\partial_q^\mu\nabla^\nu - \partial_q^\nu\nabla^\mu\big)\grave{S}_V^{<\lambda}(q,X),}$$

quantum corrections on OAM

$$\nabla_\mu = \partial_\mu + F_{\nu\mu}\partial_q^\nu - \frac{\hbar^2}{24}(\partial_\rho\partial_q^\rho)^2F_{\nu\mu}\partial_q^\nu + \mathcal{O}(\hbar^4),$$

$$\Pi_\mu = q_\mu + \frac{\hbar^2}{12}\partial_\rho\partial_q^\rho F_{\nu\mu}\partial_q^\nu + \mathcal{O}(\hbar^4).$$

- AM-tensor density :

$$M_C^{\lambda\mu\nu}(X) = -\frac{\hbar}{2}\epsilon^{\lambda\mu\nu\rho}J_{5\rho}(X) + \Big(X^\mu\bar{T}^{\lambda\nu}(X) - X^\nu\bar{T}^{\lambda\mu}(X)\Big)$$

- Relativistic angular momentum (or polarization) : (depends on the choice of a temporal direction)

$$M_C^{\mu\nu}(q,X) = \bar{n}_\lambda M^{\lambda\mu\nu}(q,X)$$

Pauli-Lubanski pseudo vector :

$$W_C^\mu(q,X) = -\frac{1}{2}\epsilon^{\mu\nu\alpha\beta}\Pi_\nu(M_C)_{\alpha\beta}(q,X)$$

# AM conservation in global equilibrium

- Global equilibrium (no collisions) : $\partial_\mu J^\mu_{V/5} = \partial_\mu T^{\mu\nu} = 0,$
  $(\omega^\mu \neq 0, \ \ B^\mu \neq 0.)$
- Conservation of canonical EM & AM tensors :

$$\partial_\mu \bar{T}^{\mu\nu} = 0, \qquad \partial_\lambda M_C^{\lambda\mu\nu} = 0. \quad \Rightarrow \quad \underbrace{\boxed{-\frac{\hbar}{2}\epsilon^{\lambda\mu\nu\rho}\partial_\lambda J_{5\rho}}}_{\text{spin}} + \underbrace{\boxed{2T_A^{\mu\nu}}}_{\text{orbit}} = 0$$

- Massive fermions : $T_A^{\mu\nu} = 0$ $\Rightarrow$ spin alone is conserved Florkowski, et.al. 17
- Weyl fermions : (quantum corrections from just $f$)

$$T_A^{\mu\nu} = \underbrace{\boxed{\frac{\hbar}{2}N_A\left(\omega^\mu u^\nu - \omega^\nu u^\mu\right)}}_{\text{from side-jumps}} \quad \sim \omega^{[\mu}S^{\nu]} : \boldsymbol{\tau} \sim \mathbf{S} \times \boldsymbol{\omega}$$

$$M^{\lambda\mu\nu}_{\text{spin}}(X) = \frac{\hbar}{2}\epsilon^{\lambda\mu\nu\rho}\Big(N_A u_\rho + \underbrace{\boxed{\hbar\sigma_{BA}B_\rho + \hbar\sigma_{\omega A}\omega_\rho}}_{\text{CSE \& CVE}}\Big)$$

- $\mathcal{O}(\hbar)$ : spin-orbit cancellation
- Higher orders : we need higher-order WFs.

# Near local equilibrium

- Local equilibrium : (collisions are involved) (R-handed fermions)

$$J^\mu_R = N_R u^\mu + \tau_R\left(\frac{2}{3}\sigma_{\omega R} - \frac{N_R^2}{4p_R}\right)\mathcal{E}^\mu_{R\perp} + \mathcal{O}(\hbar) \qquad \mathcal{E}_{R\mu} = E_\mu + T\partial_\mu\bar{\mu}_R$$

$$T^{\mu\nu}_{AR} = -\frac{\hbar}{2}\xi_{\omega R}(u^\mu\omega^\nu - u^\nu\omega^\mu) - \frac{\hbar\epsilon^{\mu\nu\alpha\beta}u_\alpha}{2}\left(\sigma_{\omega R}T\partial_\beta\bar{\mu}_R + N_R\frac{\partial_\beta T}{T} - \frac{\sigma_{\omega R}\mathcal{E}_{R\beta}}{3}\right)$$

$$\Rightarrow \quad T^{\mu\nu}_{AR} \neq \frac{\hbar}{4}\epsilon^{\lambda\mu\nu\rho}\partial_\lambda J_{R\rho} \quad \Rightarrow \quad \partial_\mu\bar{T}^{\mu\nu}_R \neq 0,\ \partial_\lambda M^{\lambda\mu\nu}_{CR} \neq 0.$$

even for E,B=0

- Suggested by KB equations :

$$\boxed{T^{\mu\nu}_{AR} = \frac{\hbar}{4}\epsilon^{\mu\nu\alpha\beta}\left(\partial_\alpha J_{R\beta} + 2\int_q\left((q\cdot n)\mathcal{C}_{R\perp\mu} - n\cdot\mathcal{C}_R q_{\perp\mu}\right)\right) = \frac{\hbar}{4}\epsilon^{\mu\nu\alpha\beta}\left(\partial_\alpha J_{R\beta} + \frac{u_\alpha\delta J_{R\perp\beta}}{\tau_R}\right)}$$

$$\Rightarrow \quad \partial_\lambda M^{\lambda\mu\nu}_{CR} = X^{[\mu}F^{\nu]\rho}J_{R\rho} - \frac{u_\rho}{\tau_R}\cancelto{0}{X^{[\mu}\delta T^{\rho\nu]}_R} - \boxed{\frac{\hbar}{4}\partial_\lambda\left(X^{[\mu}\epsilon^{\nu]\lambda\alpha\beta}\frac{u_\alpha\delta(J_R)_{\perp\beta}}{\tau_R}\right)}$$

~spin-orbit int.

# Conclusions & outlook

- We have presented a covariant CKT with background fields and collisions for Weyl fermions from the WF formalism.
- Novel 2nd –order non-equilibrium anomalous transport including the viscous corrections on CME/CVE is found.
- Nontrivial spin-orbit int. in chiral fluids.

Outlook :

- Phenomenological applications : HIC, Weyl semimetals, etc.
  - Realistic collisions beyond RTA
  - Inclusion of dynamical gluons/photons

- Theoretical issues : how to go beyond the weak-field limit?
  - Landau-level WF : subjected to constant B (no Lorentz sym.).
  - Perturbative sol. up to $\mathcal{O}(\hbar^2)$ : new phenomena expected, interpolation with L.L. (IR singularity could be more severe), OAM of chiral fluids

Thank you!

# Solving WF perturbatively

- To derive anomalous(quantum) corrections : Hidaka, Pu, DY, 16
  - work in Weyl bases ( $\grave{S}^{<} = \bar{\sigma}^{\mu}\grave{S}^{<}_{\mu}$, for R-handed fermions).
  - solve for the perturbative solution : $\grave{S}^{<}_{\mu} = 2\pi q_{\mu}\delta(q^2)f + \hbar\delta S^{<}_{\mu}$
  - usually choosing a basis for $\sigma^0 = I$.
  - a more general choice : $n_{\mu}\sigma^{\mu} = I$ with $n^{\mu}$ being a frame vector.
- The frame vector can be regarded as the zeroth component of a vierbein.

Local coordinates : $\sigma^a = (I, \sigma^i)$ $\longrightarrow$ Global spacetime : $\sigma^{\mu} = e^{\mu}_{a}\sigma^{a}$

$\bar{\sigma}^a = (I, -\sigma^i)$

(local spin direction)

$\boxed{n^{\mu} \equiv e^{\mu}_{0}}$

$n^2 = 1$

flat : $e^{\mu}_{a}(X) = \delta^{\mu}_{a}$

$n^{\mu} = (1, \mathbf{0})$

- Global spacetime coordinate transf. $\longleftrightarrow$ Frame transf. $n^{\mu} \to n'^{\mu}$
- KB eq. :

$$\mathcal{D}\cdot\grave{S}^{<} = 0, \quad q\cdot\grave{S}^{<} = 0,$$

$$\mathcal{D}_{\beta}f^{(n)}_{q} = \Delta_{\beta}f^{(n)}_{q} - \mathcal{C}_{\beta},$$

$$\Delta_{\mu} = \partial_{\mu} + F_{\nu\mu}\partial/\partial q_{\nu}$$

$$C_{\beta}[f] = \Sigma^{<}_{\beta}\bar{f} - \Sigma^{>}_{\beta}f$$

give WF

$$\left\{\begin{aligned} &2\pi\delta(q^2)\left(q\cdot n\mathcal{D}_{\mu} - q_{\mu}n\cdot\mathcal{D}\right)f = -2\epsilon_{\alpha\mu\nu\beta}n^{\alpha}q^{\nu}\delta\grave{S}^{<\beta},\\ &2\pi\epsilon_{\alpha\mu\nu\beta}\delta(q^2)n^{\alpha}q^{\nu}\mathcal{D}^{\beta}f = 2\left(q\cdot n\delta\grave{S}^{<}_{\mu} - q_{\mu}n\cdot\delta\grave{S}^{<}\right).\end{aligned}\right.$$

# Origin of side-jumps

- An alternative way to derive WF (no background fields & no collisions) with a "scalar $f$". Hidaka, Pu, DY, 16
- Nontrivial phase for massless particles with helicity:

Helicity dep. phase

S. Weinberg, QFT, Vol. I

$$\text{L. T. :}\ v_+(\Lambda p) = e^{i\boxed{\Phi(p,\Lambda)}}U(\Lambda)v_+(p)$$

- Second quantization : $\psi(x) = \int \frac{d^3p}{(2\pi)^3\sqrt{2|\mathbf{p}|}} e^{-ip\cdot x} v_+(p) a_{\mathbf{p}}$ (neglect anti-fermions)

$$\grave{S}^<(x,y) = \langle \psi^\dagger(y)\psi(x)\rangle = \int \frac{d^3p}{(2\pi)^3\sqrt{2|\mathbf{p}|}} \int \frac{d^3p'}{(2\pi)^3\sqrt{2|\mathbf{p}'|}} v_+(p)v_+^\dagger(p')\langle a_{\mathbf{p}'}^\dagger a_{\mathbf{p}}\rangle e^{i(p'-p)\cdot X - \frac{i}{2}(p'+p)\cdot Y}$$

Helicity dep. phase

- Under the L.T. : $N(p',p) \equiv \langle a_{\mathbf{p}'}^\dagger a_{\mathbf{p}}\rangle \to e^{-i\left(\boxed{\Phi(\Lambda,p)-\Phi(\Lambda,p')}\right)}\langle a_{\mathbf{p}'}^\dagger a_{\mathbf{p}}\rangle$
- Could we define a scalar distribution function?
  - Introduce a phase field : $\phi(p) \to \phi'(\Lambda p) = \phi(p) - \Phi(p,\Lambda)$
  - Reparametrize the wave function and annihilation operator :

$$v_+(p) \to e^{i\phi(p)}v_+(p)$$
$$a_{\mathbf{p}} \to e^{-i\phi(p)}a_{\mathbf{p}}$$

# Manifestation of Lorentz symmetry

- From $a_{\mathbf{p}} \to e^{-i\phi(p)} a_{\mathbf{p}}$ , we may define a scalar distribution function :

scalar　　non-scalar　　Hidaka, Pu, DY, 16

$$\boxed{\check{N}(p',p)} \equiv e^{-i(\phi(p)-\phi(p'))} \boxed{N(p',p)} \quad \Rightarrow \quad \check{f}(q,X) \equiv \int \frac{d^3\bar{p}}{(2\pi)^3} \check{N}\left(q-\frac{\bar{p}}{2}, q+\frac{\bar{p}}{2}\right) e^{-i\bar{p}\cdot X}$$

$$N(p',p) \equiv \langle a^\dagger_{\mathbf{p}'} a_{\mathbf{p}} \rangle$$

- The derivation of WF implicitly involves the contribution from anti-fermions.

Key eq. : $\mathrm{Im}\left[c^\dagger_\pm(q)\sigma^k \frac{\partial}{\partial q_\beta} c_\pm(q)\right] = \mp a^\beta_\pm v^k - \frac{1}{2|\mathbf{q}|}\epsilon^{kj\beta} v_j$ from $c_+(p)c^\dagger_+(p) + c_-(p)c^\dagger_-(p) = I$

$$\Rightarrow \grave{S}^<_\mu(q,X) = (2\pi)\theta(q^0)\delta(q^2)\boxed{\left(q_\mu\left(1-\hbar(\partial^\nu_q\phi - a^\nu)\partial_\nu\right) + \hbar\delta_{\mu i}\epsilon_{ijk}\frac{q_j}{2|\mathbf{q}|}\partial_k\right)}\check{f}(q,X) \, ,$$

covariant　　scalar

- Compare to the previous expression :

$$\grave{S}^{<\mu} = 2\pi\theta(q^0)\delta(q^2)\left(q^\mu + \hbar\delta^{\mu i}\epsilon^{ijk}\frac{q_j}{2|\mathbf{q}|}\partial_k\right) f(q,X)$$

"the origin of side-jumps"

$$\Rightarrow f(q,X) = \check{f}\left(q_\mu, X^\mu \boxed{-\hbar\partial^\mu_q\phi(q) + \hbar a^\mu}\right)$$

non-scalar

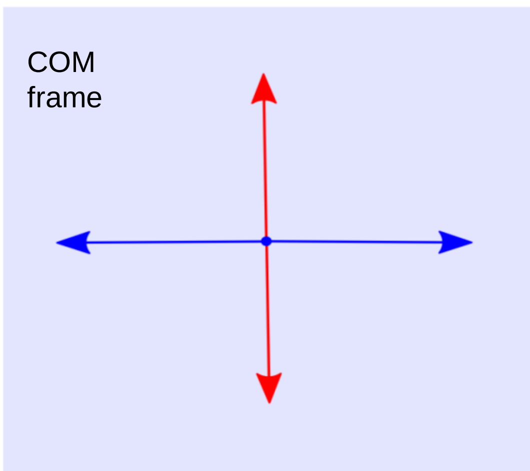
- Choices of phase field corresponds to the gauge degrees of freedom for the Berry connection.
- The perturbative solution could be uniquely determined by Lorentz symmetry.

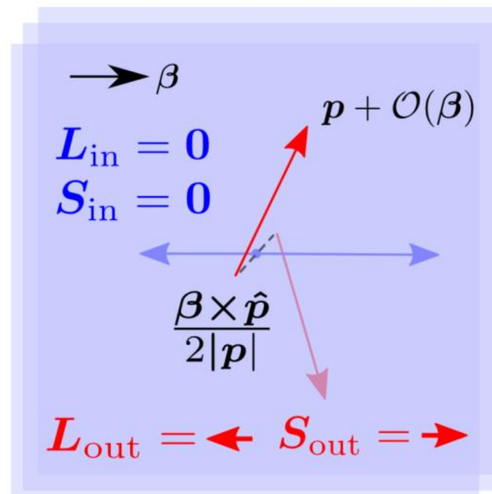
# A no-jump frame

- Conservation of the angular momentum : COM frame = no-jump frame

  J.-Y. Chen, D. T. Son, and M. A. Stephanov, 15

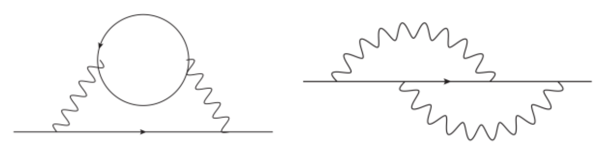
- A phenomenological argument for side jumps in collisions :




side-jump corrections for $C$

E.g. 2-2 Coulomb scattering ( $n_c^\mu = (q^\mu + q'^\mu)/\sqrt{s}$ ) : $\Sigma =$



$$q^\mu C_\mu[f^{(n_c)}] = \frac{1}{4}\int_{\mathbf{q}',\mathbf{k},\mathbf{k}'} |\mathcal{M}|^2 \Big[\bar{f}^{(n_c)}(q)\bar{f}^{(n_c)}(q')f^{(n_c)}(k)f^{(n_c)}(k') - f^{(n_c)}(q)f^{(n_c)}(q')\bar{f}^{(n_c)}(k)\bar{f}^{(n_c)}(k')\Big]$$

Hidaka, Pu, DY, 16

"no side-jump corrections"

# The no-jump frame in 2-2 scattering

- Introducing a frame : $\tilde{S}^{<}_{\mu} = 2\pi\delta(q^2)\Big[q_\mu f - \frac{\hbar}{2q\cdot u}\epsilon_{\mu\nu\alpha\beta}u^\nu q^\alpha\Big(\partial^\beta f + \Sigma^{>\beta} f - \Sigma^{<\beta}\bar{f}\Big)\Big]$

- Conservation of the angular momentum : COM frame = no-jump frame

  J.-Y. Chen, D. T. Son, and M. A. Stephanov, Phys. Rev. Lett. 115, 021601 (2015)

- Choosing the COM frame : $u_c^\mu = (q+q')^\mu/\sqrt{s}$

$$\mathcal{P}(q',k,k') = 4e^4\left(\frac{1}{(q-k)^2} + \frac{1}{(q-k')^2}\right)^2,$$

$$\Sigma^{<}_{\mu} = \int_{q',k,k'} \mathcal{P}(q',k,k')\tilde{S}^{>}_{\mu}(q')\big(\tilde{S}^{<}(k)\cdot\tilde{S}^{<}(k')\big),$$

$$\int_{\mathbf{q}',\mathbf{k},\mathbf{k}'} = \int \frac{d^3\mathbf{q}'d^3\mathbf{k}d^3\mathbf{k}'}{(2\pi)^5}\frac{\delta^{(4)}(q+q'-k-k')}{8E_{q'}E_kE_{k'}}.$$

$$\Rightarrow \quad \tilde{S}^{>\mu}\Sigma^{<}_{\mu} = 2\pi\delta(q^2)\int_{\mathbf{q}',\mathbf{k},\mathbf{k}'}\mathcal{P}(q',k,k')(k\cdot k')(q\cdot q') \times \bar{f}^{(u_c)}(q)\bar{f}^{(u_c)}(q')f^{(u_c)}(k)f^{(u_c)}(k'),$$

- No "explicit" $\mathcal{O}(\hbar)$ corrections in $C_\mu$ : $\partial_\mu\tilde{S}^{\mu<} = 2\pi\delta(q^2)q^\mu C_\mu\big[f^{(u_c)}\big],$

$$q^\mu C_\mu\big[f^{(u_c)}\big] = \frac{1}{4}\int_{\mathbf{q}',\mathbf{k},\mathbf{k}'}\boxed{|\mathcal{M}|^2\times\Big[\bar{f}^{(u_c)}(q)\bar{f}^{(u_c)}(q')f^{(u_c)}(k)f^{(u_c)}(k') - f^{(u_c)}(q)f^{(u_c)}(q')\bar{f}^{(u_c)}(k)\bar{f}^{(u_c)}(k')\Big]}$$

no side-jumps

- The final expression is concise but not pragmatic.
- $u_c$ is momentum-dependent : hard to write down $f^{(u_c)}$ with different $q'$

# Anomalous Hydrodynamics

- Anomalous hydro : $(T, \bar{\mu} = \frac{\mu}{T}, u^\mu$: free parameters)

E & B : $n^\nu F_{\mu\nu} = E_\mu,$

$$\frac{1}{2}\epsilon^{\mu\nu\alpha\beta} n_\nu F_{\alpha\beta} = B^\mu$$

constrains : $\partial_\mu T^{\mu\nu} = F^{\nu\rho} J_\rho, \quad \partial_\mu J^\mu = \frac{\hbar}{4\pi^2}(\mathbf{E}\cdot\mathbf{B})$

in the local rest frame

$u^\mu \approx (1, \mathbf{0})$

$$\frac{\partial_0 T}{T} = \hbar \boldsymbol{\mathcal{E}} \cdot \left(\tilde{T}_B \mathbf{B} + \tilde{T}_\omega \boldsymbol{\omega} T\right), \quad \partial_0 \bar{\mu} = \hbar \boldsymbol{\mathcal{E}} \cdot \left(\tilde{\mu}_B \mathbf{B} + \tilde{\mu}_\omega \boldsymbol{\omega} T\right), \quad \mathcal{E}_\nu = E_\nu + T\partial_\nu \bar{\mu}$$

$$\partial_0 \mathbf{u} = \partial_0 \mathbf{u}^{(0)} + \hbar \partial_0 \delta \mathbf{u},$$

Hidaka, Pu, DY, 17

$$\partial_0 \mathbf{u}^{(0)} = -\frac{\nabla T}{T} + \frac{N_0 \boldsymbol{\mathcal{E}}}{4p}, \quad \hbar \partial_0 \delta \mathbf{u} = \hbar \left(\tilde{U}_E \nabla \times \mathbf{E} + \tilde{U}_T \frac{\mathbf{E} \times \nabla T}{T} + \tilde{U}_{\bar{\mu}} \mathbf{E} \times \nabla \bar{\mu}\right)$$

- Free to choose a different frame in hydrodynamics :

e.g. Landau frame :

$$\tilde{u}^\mu = u^\mu + \hbar \frac{\xi_\omega \omega^\mu + \xi_B B^\mu}{\epsilon + p},$$

$$T^{\mu\nu}_{\text{leq}} = \tilde{u}^\mu \tilde{u}^\nu \epsilon - p\tilde{\Theta}^{\mu\nu},$$

$$J^\mu_{\text{leq}} = N_0 \tilde{u}^\mu + \hbar \tilde{\sigma}_\omega \omega^\mu + \hbar \tilde{\sigma}_B B^\mu,$$

$$\partial_0 \tilde{\mathbf{u}} = -\frac{\nabla T}{T} + \frac{N_0 \boldsymbol{\mathcal{E}}}{4p} - \frac{\hbar \sigma_\omega}{8p} \boldsymbol{\omega} \times \mathbf{B}$$

(yields Chiral Alfven waves)

N. Yamamoto, Phys. Rev. Lett. 115, 141601 (2015)

# Non-equilibrium distribution functions

- Solving CKT for non-equilibrium fluctuations of $f$ : $f_q^{(u)} - f_q^{\text{eq}} = \delta f_q = \delta f_q^{(c)} + \hbar\delta f_q^{(Q)}$

classical quantum

$$\left[q\cdot\Delta + \hbar\frac{S^{\mu\nu}_{(u)}E_\mu}{(q\cdot u)}\Delta_\nu + \hbar S^{\mu\nu}_{(u)}(\partial_\mu F_{\rho\nu})\partial_q^\rho + \hbar\big(\hat{\Pi}^\mu_1 + \hat{\Pi}^\mu_2\big)\Delta_\mu\right] f_q^{(u)} = \mathcal{C}_{\text{full}},\ \ \partial_\mu S^{\mu\nu}_{(u)} = \hat{\Pi}^\nu_1 + \hat{\Pi}^\nu_2.$$

- $\mathcal{C}_{full}$ : $\mathcal{C}_{\text{full}} = q^\mu\mathcal{C}_\mu + \hbar\frac{S^{\mu\nu}_{(u)}E_\mu}{(q\cdot u)}\mathcal{C}_\nu + \hbar\big(\partial_\rho S^{\rho\mu}_{(u)}\big)\mathcal{C}_\mu$ $\xrightarrow{\text{RTA}}$ $\mathcal{C}_{\text{full}} = -\frac{1}{\tau_R}\left(q\cdot u + \hbar\frac{q^\mu\mathcal{A}_\mu}{(q\cdot u)^2}\right)\delta f_q$

$$\partial_\rho S^{\rho\mu}_{(u)} = \frac{1}{2}\left[\omega^\mu - \frac{(q\cdot\omega)u^\mu}{q\cdot u} - \frac{(q\cdot\omega)q^\mu}{(q\cdot u)^2}\right] + \frac{\epsilon^{\mu\nu\alpha\beta}}{2q\cdot u}\left(q_\alpha\kappa_{\beta\nu} - \frac{u_\nu q_\alpha q^\rho}{q\cdot u}(\sigma_{\beta\rho} + \kappa_{\beta\rho})\right)$$

- The perturbative solution :

Hidaka, Pu, DY, 17

$$\delta f_q = -\frac{\tau_R}{q\cdot u}\left(1 - \frac{\hbar q\cdot\mathcal{A}}{q\cdot u}\right)\left[q\cdot\Delta + \hbar\frac{S^{\mu\nu}_{(u)}E_\mu}{(q\cdot u)}\Delta_\nu + \hbar S^{\mu\nu}_{(u)}(\partial_\mu F_{\rho\nu})\partial_q^\rho + \hbar\big(\hat{\Pi}^\mu_1 + \hat{\Pi}^\mu_2\big)\Delta_\mu\right] f_q^{\text{eq}}$$

- Different contributions for $\hbar\delta f_q^{(Q)} = \delta f_q^{\mathcal{K}} + \delta f_q^{\mathcal{H}} + \delta f_q^{\mathcal{C}}$

from CKT from hydro EOM from collisions

(neglect nonlinear classical responses)

- The non-equilibrium four current :

$$\delta J^\mu_Q = 2\hbar\int\frac{d^4q}{(2\pi)^3}\bar{\epsilon}(q\cdot u)\delta(q^2)\left[q^\mu\delta f_q^{(Q)} - \frac{1}{2}\left(\epsilon^{\mu\nu\alpha\beta}\frac{u_\nu q_\alpha}{q\cdot u}\Delta_\beta + \epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}\frac{\partial_{q\nu}}{2}\right)\delta f_q^{(c)}\right]$$

# Anomalous Hall currents

- Non-equilibrium (quantum) charge currents : Hidaka, Pu, DY, 17

$$\delta \boldsymbol{J}_{QR/L} = \mp\frac{\hbar\tau_R}{84\pi^2}\left[\mu_{R/L}\nabla\times\boldsymbol{E} + \frac{7\mu_{R/L}}{2T}\boxed{\boldsymbol{E}\times\nabla T} - \frac{1}{2}\boldsymbol{E}\times\nabla\mu_{R/L} + \frac{12\mu_{R/L}}{T}\boxed{(\nabla\mu_{R/L})\times(\nabla T)}\right], \quad \bar{\mu}_{R/L} \ll 1$$

emerge from hydro.

$$\delta \boldsymbol{J}_{QR/L} = \pm\frac{\hbar\tau_R}{4\pi^2}\left[\frac{\pi^2 T^2}{3\mu_{R/L}}\nabla\times\boldsymbol{E} + \frac{2\mu_{R/L}}{3T}\boxed{\boldsymbol{E}\times\nabla T} - \frac{2}{3}\boldsymbol{E}\times\nabla\mu_{R/L} - \frac{2\mu_{R/L}}{3T}\boxed{(\nabla\mu_{R/L})\times(\nabla T)}\right], \quad \bar{\mu}_{R/L} \gg 1$$

- Some observations :
  - $B$ and $\omega$ do not contribute to nonlinear corrections in the inviscid case.
  - The transport coefficients are parity-odd (charged currents from chiral imbalance).
- A caveat : part of anomalous Hall currents vanishes without hydro in the “naïve” RT ($\tau_R$ = const.) approximation.

exist without hydro.
(could exist in Weyl semimetals?)

$$\tau_R(T,\mu): \ \delta\boldsymbol{J}_{\tau_R} = \frac{\hbar}{12\pi^2}\Big((\mu\partial_T\tau_R - I_1\partial_{\bar{\mu}}\tau_R)\boxed{(\nabla\mu)\times(\nabla T)} + \bar{\mu}(\bar{\mu}\partial_{\bar{\mu}}\tau_R - T\partial_T\tau_R)\boxed{\mathbf{E}\times\nabla T} - \bar{\mu}(\partial_{\bar{\mu}}\tau_R)\mathbf{E}\times\nabla\mu\Big)$$

# Matching Conditions & Entropy Production

- From CKT :

$$\partial_\mu J^\mu = \frac{\hbar}{4\pi^2}(\mathbf{E}\cdot\mathbf{B}) + 2\int_q \Big[\delta(q^2)q^\mu + \hbar\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}\frac{\partial_{q\nu}\delta(q^2)}{4}\Big]\mathcal{C}_\mu$$

$$\partial_\mu T^{\mu\nu} = F^{\nu\mu}J_\mu + 2\int_q \delta(q^2)\Big[q^\nu q^\mu + \frac{\hbar\epsilon^{\sigma\mu\alpha\beta}}{4}\Big(\eta^\nu_\sigma\left(q_\beta\partial_\alpha + F_{\alpha\beta}\right) + q^\nu F_{\alpha\beta}\partial_{q\sigma}\Big)\Big]\mathcal{C}_\mu$$

- Matching conditions for the classical RT approximation : $u_\mu \delta J^\mu = 0,$ $u_\mu \delta T^{\mu\nu} = 0$

- Entropy currents : $s^\mu = \frac{1}{T}\Big(pu^\mu + T^{\mu\nu}u_\nu - \mu J^\mu\Big) + \hbar(D_B B^\mu + D_\omega \omega^\mu)$

$$T^{\mu\nu} = u^\mu u^\nu \epsilon - pP^{\mu\nu} + \Pi^{\mu\nu}_{\rm dis} + \Pi^{\mu\nu}_{\rm non}$$

- 2nd law is protected by classical parts :

$$\partial\cdot s = \frac{1}{T}\Big[\Pi^{\mu\nu}_{\rm dis}\partial_\mu u_\nu - (E_\mu + T\partial_\mu\bar{\mu})\delta J^\mu\Big]$$

Hidaka, DY, 18