Lattice QCD study for relation between quark-confinement and chiral symmetry breaking

Quantum Hadron Physics Laboratory, Nishina Center, RIKEN

Takahiro M. Doi (土居　孝寛)

In collaboration with

- Hideo Suganuma (Kyoto Univ.), Takumi Iritani (RIKEN)

  TMD, H. Suganuma, T. Iritani, Phys. Rev. D90, 094505 (2014)

  H. Suganuma, TMD, T. Iritani, PTEP 2016, 013B06 (2016)

- Krzysztof Redlich, Chihiro Sasaki (Univ. of Wroclaw)

  TMD, K. Redlich, C. Sasaki, H. Suganuma, Phys. Rev. D92, 094004 (2015)

NFQCD2018, Kyoto, YITP, 30, May 2018

Contents

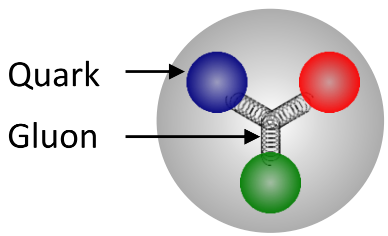
Introduction – confinement

Confinement : colored state cannot be observed (quark, gluon, ･･･)
only color-singlet states can be observed (meson, baryon, ･･･)

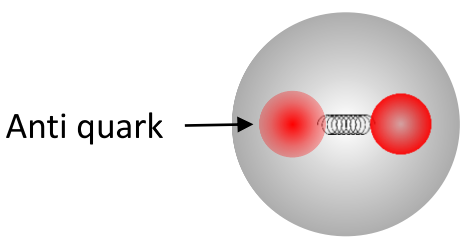
Observable	Not observable

Observable

Baryon

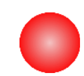


Meson

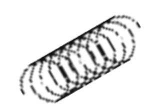


Not observable

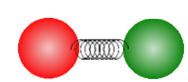
Single quark



Gluon



Two quarks(diquark)

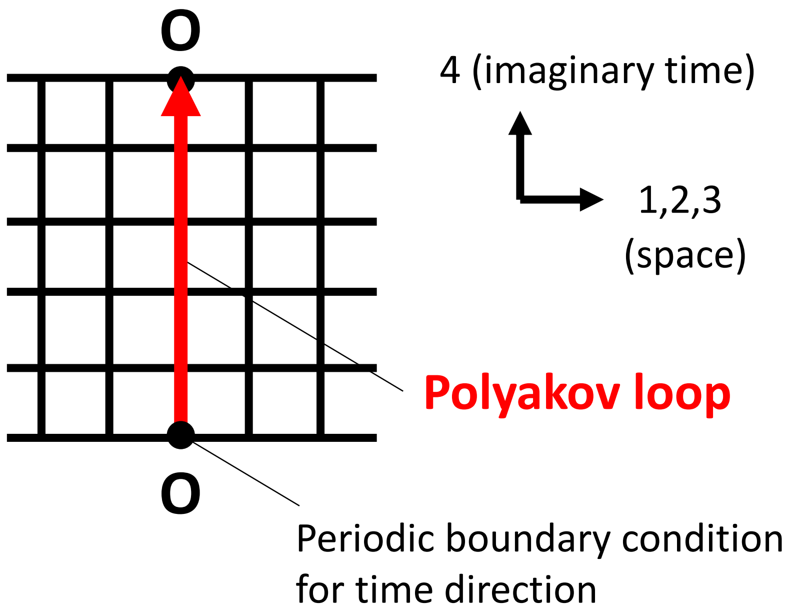


An order parameter for quark-confinement: Polyakov loop

L.D. McLerran and B. Svetitsky, Phys. Rev. D24, 450 (1981)

$$L(\boldsymbol{x}) = \mathrm{tr}\mathrm{T}e^{ig\int_0^\beta d\tau A_4(\boldsymbol{x},\tau)} \quad \text{(continuum theory)}$$

$$= \mathrm{tr}\prod_{s_4=1}^{N_\tau} U_4(\boldsymbol{s}, s_4) \quad \text{(lattice theory)}$$



$$\langle L\rangle = \frac{1}{N_c V}\sum_{\boldsymbol{x}}\langle L(\boldsymbol{x})\rangle$$: **Polyakov loop**

$$= e^{-\beta F_q}\begin{cases} = 0 & (F_q = \infty,\ \text{confinement phase}) \\ \neq 0 & (F_q : \text{finite},\ \text{deconfinement phase}) \end{cases}$$

F_q : free energy of the system with a single static quark

$\beta = 1/T$: inverse temperature

Introduction – Chiral Symmetry Breaking

- Chiral symmetry breaking : chiral symmetry is spontaneously broken

$$\mathrm{SU(N)_L} \times \mathrm{SU(N)_R} \underset{\text{CSB}}{\rightarrow} \mathrm{SU(N)_V}$$

for example $\mathrm{SU(2)}$

- u, d quarks get dynamical mass(constituent mass)
- Pions appear as NG bosons

- Chiral condensate : order parameter for chiral phase transition

$$\langle \bar{q}q \rangle \begin{cases} \neq 0 & \text{(chiral broken phase)} \\ = 0 & \text{(chiral restored phase)} \end{cases}$$

- Banks-Casher relation

$$\langle \bar{q}q \rangle = - \lim_{m \to 0} \lim_{V \to \infty} \pi \langle \rho(0) \rangle$$

$\hat{\not{D}}$:Dirac operator

$\hat{\not{D}}|n\rangle = i\lambda_n|n\rangle$:Dirac eigenvalue equation

$\rho(\lambda) = \frac{1}{V}\sum_n \delta(\lambda - \lambda_n)$:Dirac eigenvalue density

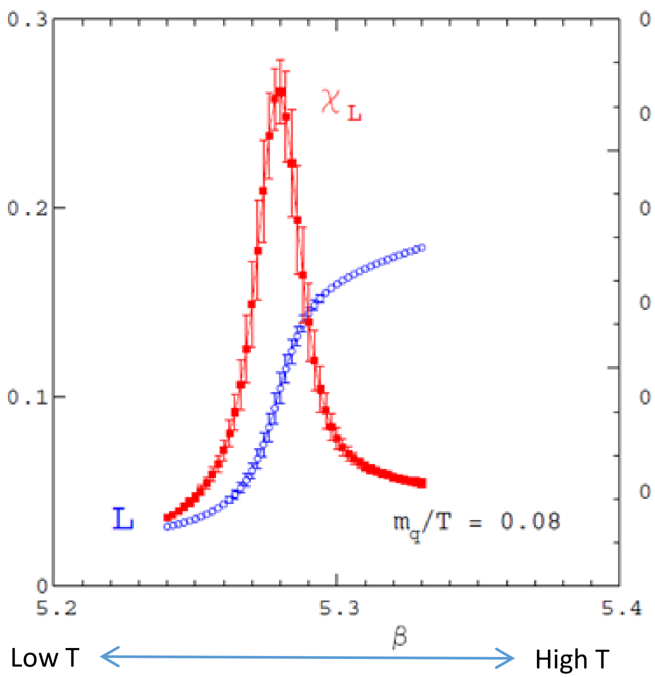
QCD phase transition at finite temperature

F. Karsch, Lect. Notes Phys. 583, 209 (2002)

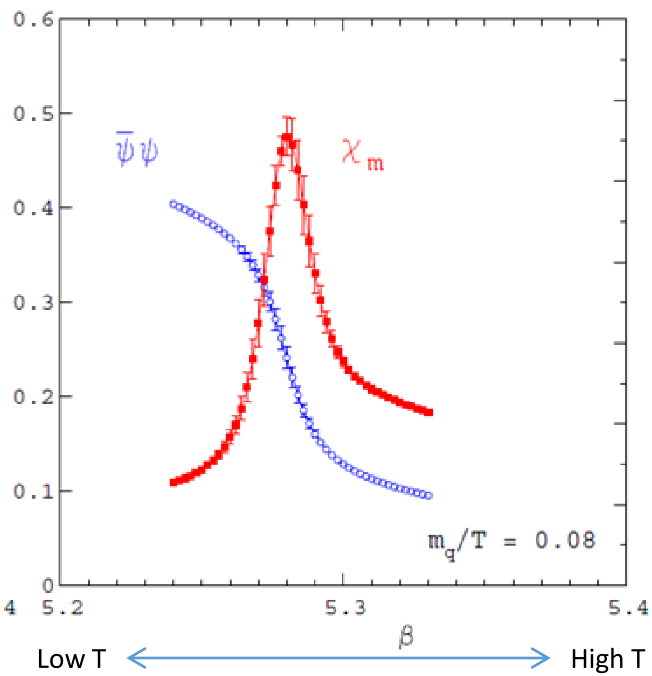
$\langle L \rangle, \chi_L$: Polyakov loop and its susceptibility

$\langle \bar{\psi}\psi \rangle, \chi_m$: chiral condensate and its susceptibility

deconfinement transition



chiral transition



- $\mu = 0$
- two flavor QCD with light quarks

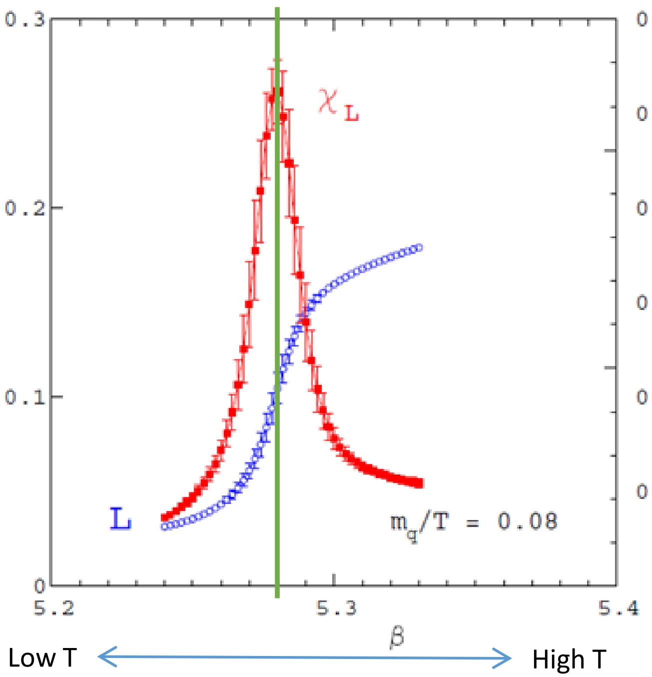
QCD phase transition at finite temperature

F. Karsch, Lect. Notes Phys. 583, 209 (2002)

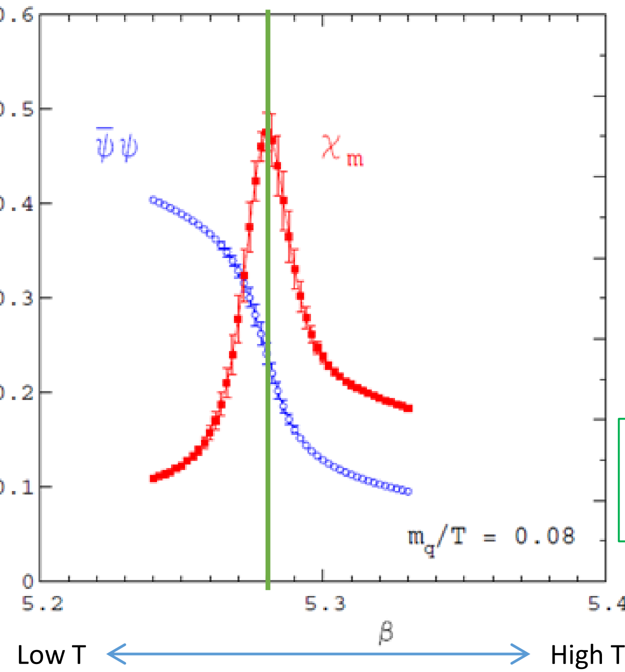
$\langle L \rangle, \chi_L$: Polyakov loop and its susceptibility

$\langle \bar{\psi}\psi \rangle, \chi_m$: chiral condensate and its susceptibility

deconfinement transition



chiral transition



- $\mu = 0$
- two flavor QCD with light quarks

We define critical temperature as the peak of susceptibility

$\Longrightarrow$ These two phenomena are strongly correlated(?)

Our strategy

Our strategy to study relation between confinement and chiral symmetry breaking :

anatomy of Polyakov loop in terms of Dirac mode

Our strategy

Our strategy to study relation between confinement and chiral symmetry breaking :

anatomy of Polyakov loop in terms of Dirac mode

Polyakov loop L **:**

an order parameter of **deconfinement** transition.

Our strategy

Our strategy to study relation between confinement and chiral symmetry breaking :

anatomy of Polyakov loop in terms of Dirac mode

Polyakov loop L **:**

an order parameter of **deconfinement** transition.

Dirac eigenmode $\hat{\not{D}}|n\rangle = i\lambda_n|n\rangle$ **:**

low-lying Dirac modes (with small eigenvalue $|\lambda_n| \sim 0$)
are essential modes for **chiral symmetry breaking**.

recall Banks-Casher relation: $\langle \bar{q}q \rangle = -\lim_{m\to 0}\lim_{V\to\infty} \pi\langle\rho(0)\rangle$

Our strategy

Our strategy to study relation between confinement and chiral symmetry breaking :

anatomy of Polyakov loop in terms of Dirac mode

Polyakov loop L **:**

an order parameter of **deconfinement** transition.

Dirac eigenmode $\hat{\not{D}}|n\rangle = i\lambda_n|n\rangle$ **:**

low-lying Dirac modes (with small eigenvalue $|\lambda_n| \sim 0$)
are essential modes for **chiral symmetry breaking**.

recall Banks-Casher relation: $\langle \bar{q}q \rangle = -\lim_{m\to 0}\lim_{V\to\infty} \pi\langle\rho(0)\rangle$

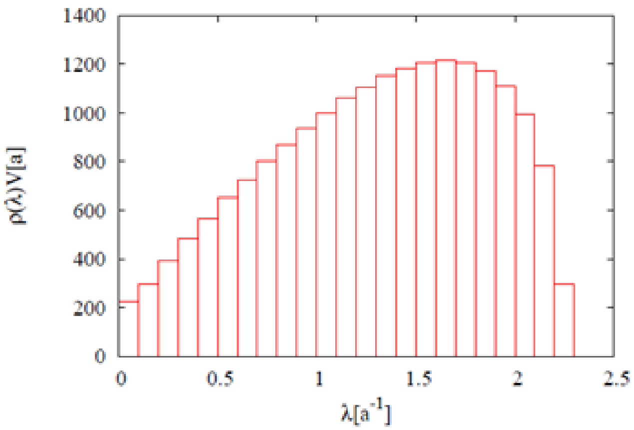
What we want to show:

If the contribution to the Polyakov loop from the low-lying Dirac modes is very small,
the important modes for chiral symmetry breaking are not important for confinement.

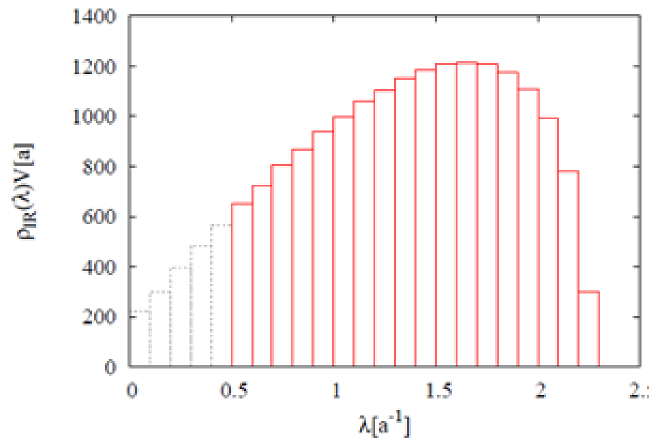
Polyakov loop with Dirac-infrared(IR) cut

S. Gongyo, T. Iritani, H. Suganuma, PRD86, 034510 (2012)
T. Iritani and H. Suganuma, PTEP 2014, 033B03 (2014)

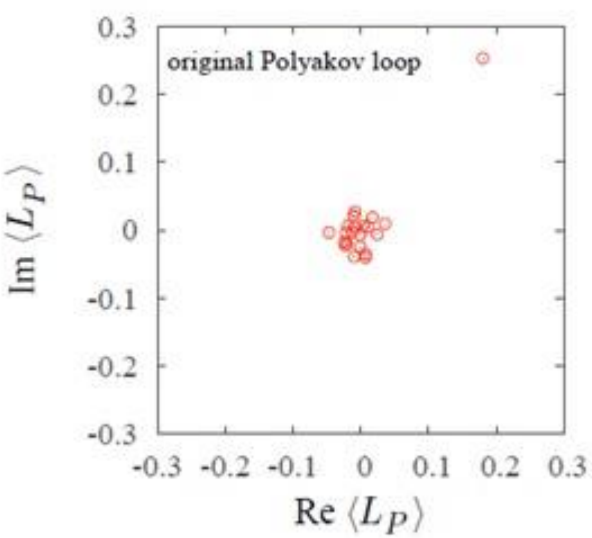
$\{U\}_n$

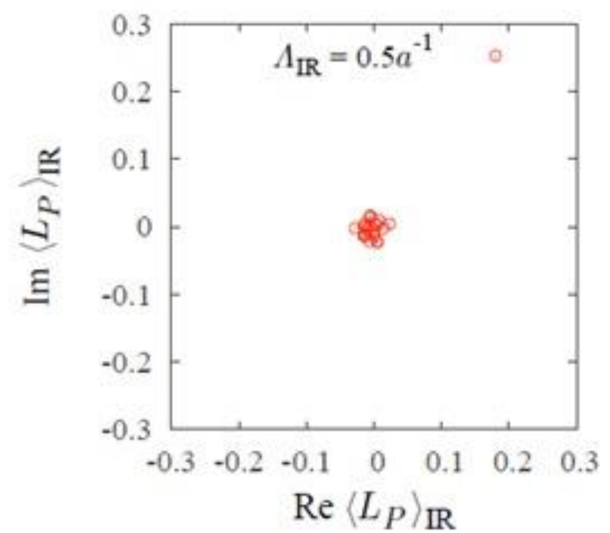


$\{U_{\mathrm{IR-cut}}\}_n$



remove low-lying modes
(Dirac-IR cut)





Polyakov loop is almost unchanged by Dirac-IR cut

Analytical relation between Polyakov loop and Dirac modes with twisted boundary condition

C. Gattringer, Phys. Rev. Lett. 97 (2006) 032003.

$$L = \frac{1}{8V}\left(2\sum_{\lambda}\lambda^{N_4} - (1+i)\sum_{\lambda_+}\lambda_+^{N_4} - (1-i)\sum_{\lambda_-}\lambda_-^{N_4}\right)$$

twisted boundary condition:

$$U_4(\mathbf{x}, N_4) \to \pm i U_4(\mathbf{x}, N_4), \qquad \forall \mathbf{x}$$

$$D(x,y) \to D_{\pm}(x,y)$$

λ : Eigenvalue of $D(x|y)$

$\lambda_{\pm}$: Eigenvalue of $D_{\pm}(x|y)$

$$D(x|y) = (4+m)\delta_{x,y} - \frac{1}{2}\sum_{\mu=\pm 1}^{\pm 4}[1 \mp \gamma_\mu]U_\mu(x)\delta_{x+\mu,y}$$

: Wilson Dirac operator

The twisted boundary condition is not the periodic boundary condition.

However,
the temporal periodic boundary condition is physically important
for the imaginary-time formalism at finite temperature.

(The b.c. for link-variables is p.b.c., but the b.c. for Dirac operator is twisted b.c.)

Our study

The relation btw quark-confinement and chiral symmetry breaking in QCD
is investigated by deriving analytical formulae
connecting the order parameters for confinement and Dirac eigenmodes
with proper (anti)periodic boundary condition for temporal direction.

- Polyakov loop L

  TMD, H. Suganuma, T. Iritani, Phys. Rev. D90, 094505 (2014)

- Polyakov loop fluctuations

  TMD, K. Redlich, C. Sasaki, H. Suganuma, Phys. Rev. D92, 094004 (2015)

- Wilson loop

  H. Suganuma, TMD, T. Iritani, PTEP 2016, 013B06 (2016)

Our study

The relation btw quark-confinement and chiral symmetry breaking in QCD is investigated by deriving analytical formulae connecting the order parameters for confinement and Dirac eigenmodes with proper (anti)periodic boundary condition for temporal direction.

- Polyakov loop L

  TMD, H. Suganuma, T. Iritani, Phys. Rev. D90, 094505 (2014)

- Polyakov loop fluctuations

  TMD, K. Redlich, C. Sasaki, H. Suganuma, Phys. Rev. D92, 094004 (2015)

- Wilson loop

  H. Suganuma, TMD, T. Iritani, PTEP 2016, 013B06 (2016)

An analytical relation between Polyakov loop and Dirac mode on temporally odd-number lattice

TMD, H. Suganuma, T. Iritani, Phys. Rev. D 90, 094505 (2014).
H. Suganuma, TMD, T. Iritani, Prog. Theor. Exp. Phys. 2016, 013B06 (2016).

$$L = -\frac{(2ai)^{N_4-1}}{12V}\sum_n \lambda_n^{N_4-1}\langle n|\hat{U}_4|n\rangle$$ on temporally odd number lattice: N_4 is odd

notation:

- Polyakov loop : L
- link variable operator : $\langle s|\hat{U}_\mu|s'\rangle = U_\mu(s)\delta_{s+\hat{\mu},s'}$

  with anti p.b.c. for time direction: $\langle N_4, \mathbf{x}|\hat{U}_4|1,\mathbf{x}\rangle = -U_4(N_4,\mathbf{x})$
- Dirac eigenmode : $\hat{\not{D}}|n\rangle = i\lambda_n|n\rangle$

  Dirac operator : $\hat{\not{D}} = \frac{1}{2a}\sum_\mu \gamma_\mu(\hat{U}_\mu - \hat{U}_{-\mu})$ $\quad \sum_n |n\rangle\langle n| = 1$
- The Dirac-matrix element of the link variable operator: $\langle n|\hat{U}_4|n\rangle$

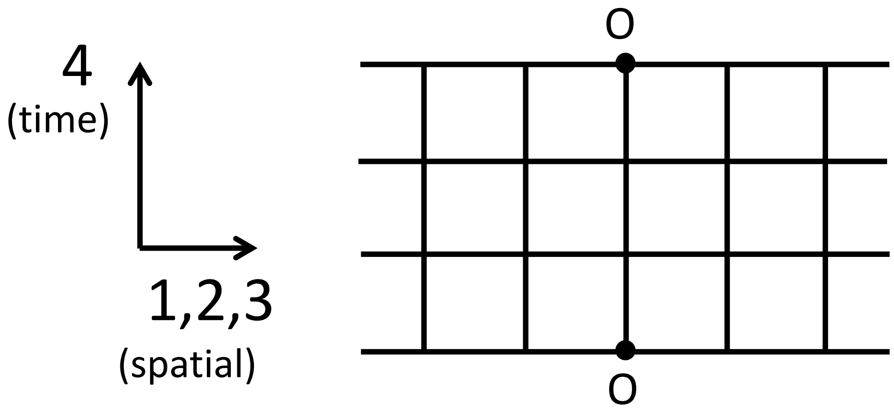
  $|\langle n|\hat{U}_4|n\rangle| < 1$

properties :

- This formula is valid in full QCD and at the quenched level.
- This formula exactly holds for each gauge-configuration $\{U\}_n$ and for arbitrary fermionic kernel K[U]

$$Z = \int \mathcal{D}\bar{q}\mathcal{D}q\mathcal{D}U e^{-S_G[U]+\bar{q}K[U]q} = \int \mathcal{D}U e^{-S_G[U]}\det K[U]$$

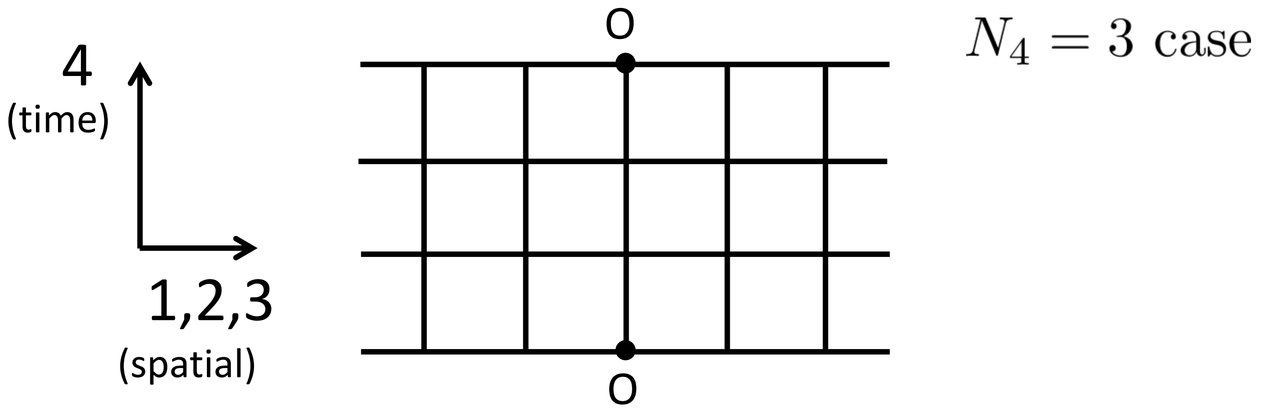
An analytical relation between Polyakov loop and Dirac mode on temporally odd-number lattice



$N_4 = 3$ case

In this study, we use
- standard square lattice
- with ordinary periodic boundary condition for gluons,
- with the odd temporal length N_4
  (temporally odd-number lattice)

An analytical relation between Polyakov loop and Dirac mode on temporally odd-number lattice

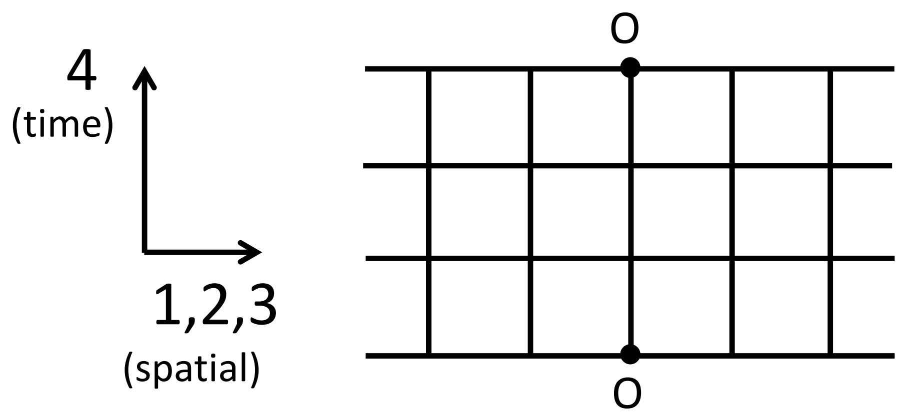


In this study, we use
- standard square lattice
- with ordinary periodic boundary condition for gluons,
- with the odd temporal length N_4

  (temporally odd-number lattice)

Note: in the continuum limit of $a \to 0$, $N_\tau \to \infty$,
any number of large N_4 gives the same result.
Then, it is no problem to use the odd-number lattice.

An analytical relation between Polyakov loop and Dirac mode on temporally odd-number lattice



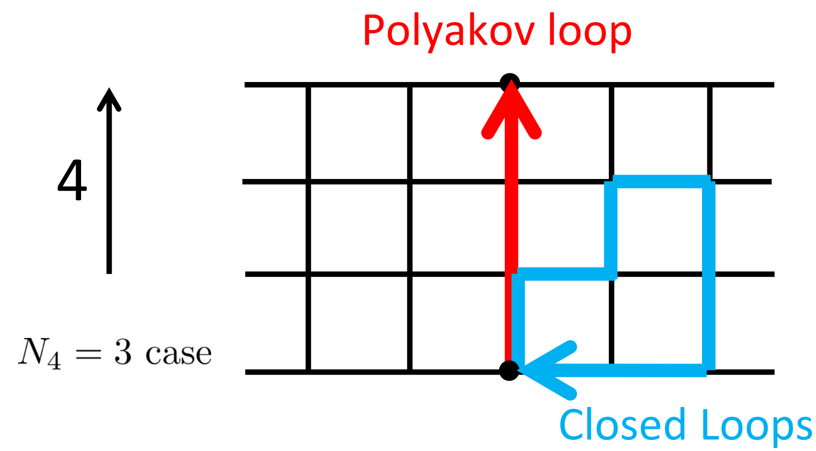
$N_4 = 3$ case

In this study, we use
- standard square lattice
- with ordinary periodic boundary condition for gluons,
- with the odd temporal length N_4
  (temporally odd-number lattice)

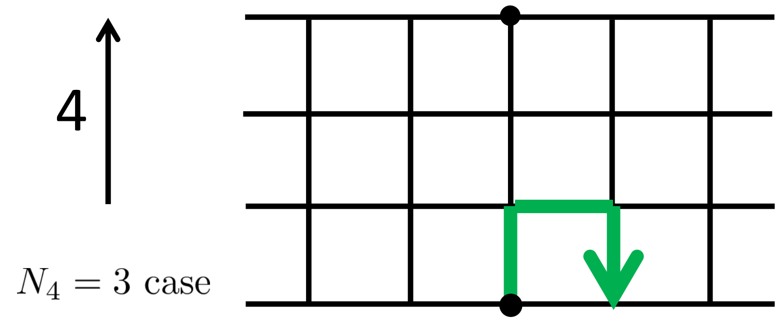
For the simple notation,
we take the lattice unit a=1 hereafter.

An analytical relation between Polyakov loop and Dirac mode on temporally odd-number lattice

In general, only gauge-invariant quantities such as Closed Loops and the Polyakov loop survive in QCD. (Elitzur's Theorem)



All the non-closed lines are gauge-variant and their expectation values are zero.

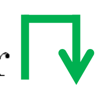


Nonclosed Lines

e.g.

$$\mathrm{Tr}\hat{U}_4\hat{U}_1\hat{U}_{-4} = \sum_s \mathrm{tr}\{U_4(s)U_1(s+\hat{4})U_4^\dagger(s+\hat{1})\}\underline{\delta_{s,s+\hat{1}}} = 0$$

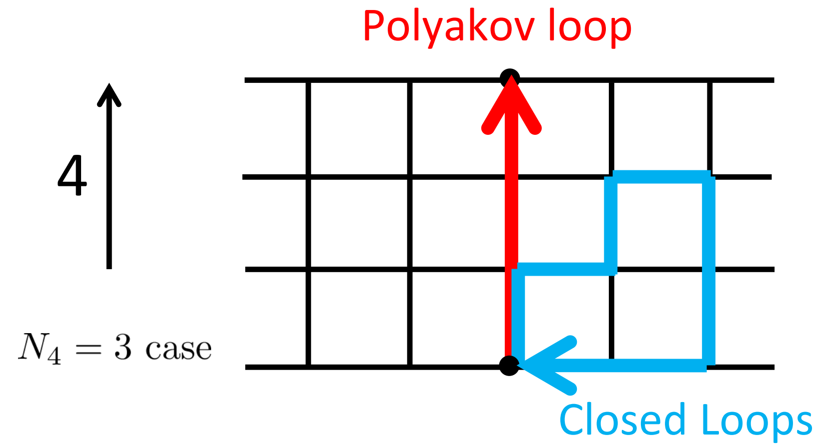
gauge-variant

(Tr  =0)

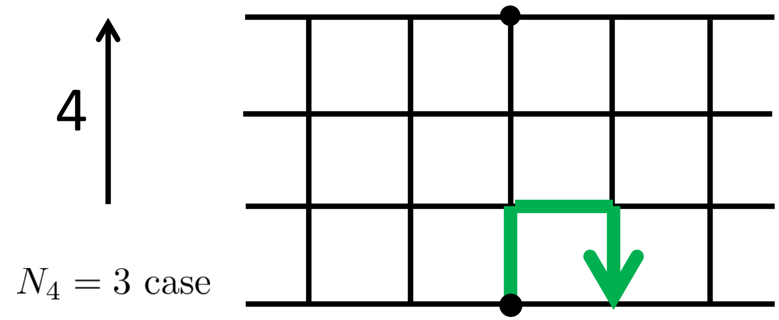
$$\langle s|\hat{U}_\mu|s'\rangle = U_\mu(s)\delta_{s+\hat{\mu},s'}$$

An analytical relation between Polyakov loop and Dirac mode on temporally odd-number lattice

In general, only gauge-invariant quantities such as Closed Loops and the Polyakov loop survive in QCD. (Elitzur's Theorem)

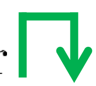


All the non-closed lines are gauge-variant and their expectation values are zero.



e.g.

$$\underbrace{\mathrm{Tr}\hat{U}_4\hat{U}_1\hat{U}_{-4}}_{\text{gauge-variant}} = \sum_s \mathrm{tr}\{U_4(s)U_1(s+\hat{4})U_4^\dagger(s+\hat{1})\}\underline{\delta_{s,s+\hat{1}}} = 0$$

(Tr  =0)

Key point

Note: any closed loop needs even-number link-variables on the square lattice.

An analytical relation between Polyakov loop and Dirac mode on temporally odd-number lattice

We consider the functional trace I on the temporally odd-number lattice:

$$I \equiv \mathrm{Tr}_{c,\gamma}(\hat{U}_4 \hat{\not{D}}^{N_4-1}) \quad (N_4 : \mathrm{odd})$$

Dirac operator : $\hat{\not{D}} = \frac{1}{2}\sum_{\mu} \gamma_\mu (\hat{U}_\mu - \hat{U}_{-\mu})$

definition:

$\langle s|\hat{U}_\mu|s'\rangle = U_\mu(s)\delta_{s+\hat{\mu},s'}$

$\mathrm{Tr}_{c,\gamma} \equiv \Sigma_s \mathrm{tr}_c \mathrm{tr}_\gamma$

site & color & spinor

An analytical relation between Polyakov loop and Dirac mode on temporally odd-number lattice

We consider the functional trace I on the temporally odd-number lattice:

$$I \equiv \mathrm{Tr}_{c,\gamma}(\hat{U}_4 \hat{\not{D}}^{N_4-1}) \quad (N_4 : \mathrm{odd})$$

definition:

$$\langle s|\hat{U}_\mu|s'\rangle = U_\mu(s)\delta_{s+\hat{\mu},s'}$$

$$\mathrm{Tr}_{c,\gamma} \equiv \Sigma_s \mathrm{tr}_c \mathrm{tr}_\gamma$$

site & color & spinor

Dirac operator : $\hat{\not{D}} = \frac{1}{2}\sum_\mu \gamma_\mu(\hat{U}_\mu - \hat{U}_{-\mu})$

$\hat{U}_4 \hat{\not{D}}^{N_4-1}$ is expressed as a sum of products of N_4 link-variable operators because the Dirac operator $\hat{\not{D}}$ includes one link-variable operator in each direction $\hat{\mu}$.

An analytical relation between Polyakov loop and Dirac mode on temporally odd-number lattice

We consider the functional trace I on the temporally odd-number lattice:

$$I \equiv \mathrm{Tr}_{c,\gamma}(\hat{U}_4 \hat{\not{D}}^{N_4-1}) \quad (N_4 : \mathrm{odd})$$

definition:

$$\langle s|\hat{U}_\mu|s'\rangle = U_\mu(s)\delta_{s+\hat{\mu},s'}$$

$$\mathrm{Tr}_{c,\gamma} \equiv \Sigma_s \mathrm{tr}_c \mathrm{tr}_\gamma$$

site & color & spinor

Dirac operator : $\hat{\not{D}} = \frac{1}{2}\sum_\mu \gamma_\mu(\hat{U}_\mu - \hat{U}_{-\mu})$

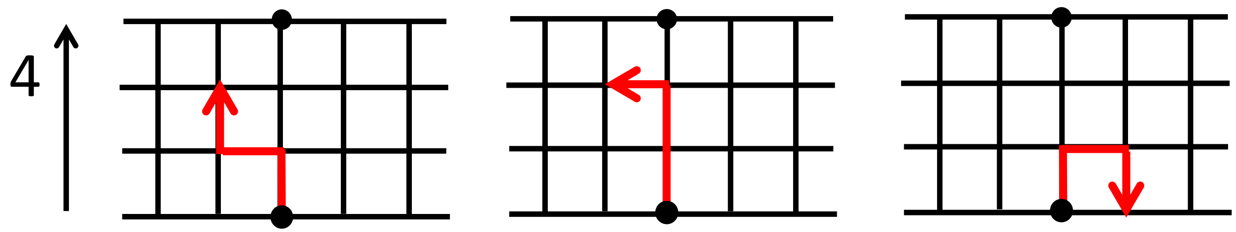
$\hat{U}_4 \hat{\not{D}}^{N_4-1}$ is expressed as a sum of products of N_4 link-variable operators because the Dirac operator $\hat{\not{D}}$ includes one link-variable operator in each direction $\hat{\mu}$.

$I \equiv \mathrm{Tr}_{c,\gamma}(\hat{U}_4 \hat{\not{D}}^{N_4-1})$ includes many trajectories on the square lattice.

$N_4 = 3$ case

length of trajectories: $N_4 = 3$

odd



$\hat{U}_4\hat{U}_{-1}\hat{U}_4$ $\quad$ $\hat{U}_4^2\hat{U}_{-1}$ $\quad$ $\hat{U}_4\hat{U}_1\hat{U}_{-4}$

An analytical relation between Polyakov loop and Dirac mode on temporally odd-number lattice

We consider the functional trace I on the temporally odd-number lattice:

$$I \equiv \mathrm{Tr}_{c,\gamma}(\hat{U}_4 \hat{\not{D}}^{N_4-1}) \quad (N_4 : \mathrm{odd})$$

definition:

$$\langle s|\hat{U}_\mu|s'\rangle = U_\mu(s)\delta_{s+\hat{\mu},s'}$$

$$\mathrm{Tr}_{c,\gamma} \equiv \Sigma_s \mathrm{tr}_c \mathrm{tr}_\gamma$$

site & color & spinor

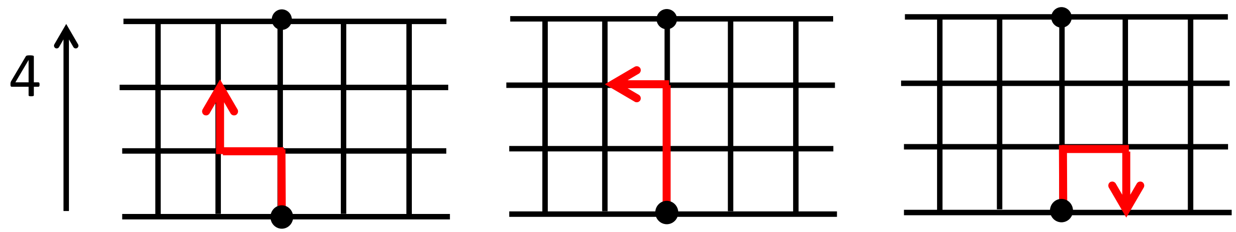
Dirac operator : $\hat{\not{D}} = \frac{1}{2}\sum_\mu \gamma_\mu(\hat{U}_\mu - \hat{U}_{-\mu})$

$\hat{U}_4 \hat{\not{D}}^{N_4-1}$ is expressed as a sum of products of N_4 link-variable operators because the Dirac operator $\hat{\not{D}}$ includes one link-variable operator in each direction $\hat{\mu}$.

$I \equiv \mathrm{Tr}_{c,\gamma}(\hat{U}_4 \hat{\not{D}}^{N_4-1})$ includes many trajectories on the square lattice.

$N_4 = 3$ case

length of trajectories: $N_4 = 3$ odd



Key point

Note: any closed loop needs even-number link-variables on the square lattice.

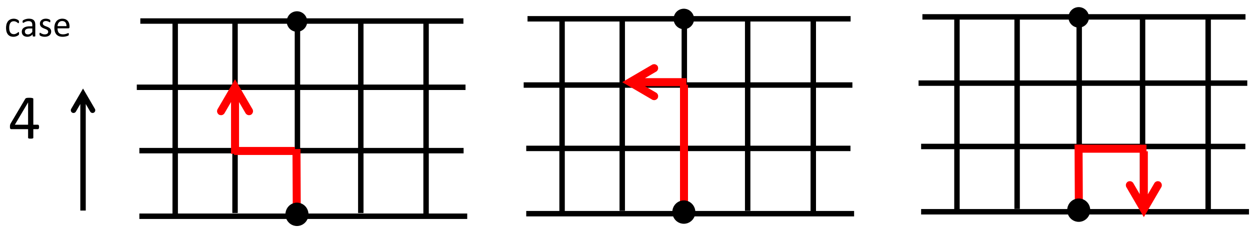
An analytical relation between Polyakov loop and Dirac mode on temporally odd-number lattice

$$I \equiv \mathrm{Tr}_{c,\gamma}(\hat{U}_4 \, \hat{\not{D}}^{N_4-1}) \qquad (N_4 : \mathrm{odd})$$

In this functional trace $I \equiv \mathrm{Tr}_{c,\gamma}(\hat{U}_4 \, \hat{\not{D}}^{N_4-1})$,
it is impossible to form a closed loop on the square lattice,
because the length of the trajectories, N_4, is odd.

Almost all trajectories are gauge-variant & give no contribution.

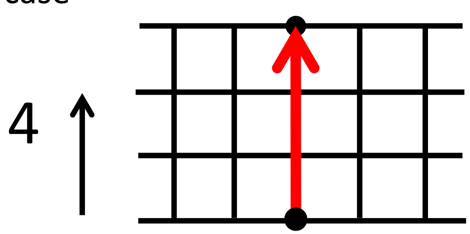
$N_4 = 3$ case



gauge variant
(no contribution)

Only the exception is the Polyakov loop.

$N_4 = 3$ case



gauge invariant

I is proportional to the Polyakov loop.

$$I \propto L$$

L : Polyakov loop

An analytical relation between Polyakov loop and Dirac mode on temporally odd-number lattice

On the one hand,

$$I = \frac{12V}{2^{N_4-1}} L \qquad \cdots ①$$

On the other hand, take the Dirac modes as the basis for functional trace

$$\begin{aligned} I &= \mathrm{Tr}_{c,\gamma}(\hat{U}_4 \, \hat{\not{D}}^{N_4-1}) \\ &= \sum_n \langle n | \hat{U}_4 \, \hat{\not{D}}^{N_4-1} | n \rangle \\ &= i^{N_4-1} \sum_n \lambda_n^{N_4-1} \langle n | \hat{U}_4 | n \rangle \qquad \cdots ② \end{aligned}$$

Dirac eigenmode

$$\hat{\not{D}} | n \rangle = i \lambda_n | n \rangle$$

$$\sum_n | n \rangle \langle n | = 1$$

from ①、②

$$L = \frac{(2i)^{N_4-1}}{12V} \sum_n \lambda_n^{N_4-1} \langle n | \hat{U}_4 | n \rangle$$

An analytical relation between Polyakov loop and Dirac mode on temporally odd-number lattice

TMD, H. Suganuma, T. Iritani, Phys. Rev. D 90, 094505 (2014).
H. Suganuma, TMD, T. Iritani, Prog. Theor. Exp. Phys. 2016, 013B06 (2016).

$$L = -\frac{(2ai)^{N_4-1}}{12V}\sum_n \lambda_n^{N_4-1}\langle n|\hat{U}_4|n\rangle$$

on temporally odd number lattice: N_4 is odd

notation:

- Polyakov loop : L
- link variable operator : $\langle s|\hat{U}_\mu|s'\rangle = U_\mu(s)\delta_{s+\hat{\mu},s'}$

  with anti p.b.c. for time direction: $\langle N_4, \mathbf{x}|\hat{U}_4|1,\mathbf{x}\rangle = -U_4(N_4,\mathbf{x})$
- Dirac eigenmode : $\hat{\not{D}}|n\rangle = i\lambda_n|n\rangle$

  Dirac operator : $\hat{\not{D}} = \frac{1}{2a}\sum_\mu \gamma_\mu(\hat{U}_\mu - \hat{U}_{-\mu})$ $\qquad \sum_n |n\rangle\langle n| = 1$

properties :

- This formula is valid in full QCD and at the quenched level.
- This formula exactly holds for each gauge-configuration {U} and for arbitrary fermionic kernel K[U]

$$Z = \int \mathcal{D}\bar{q}\mathcal{D}q\mathcal{D}U \mathrm{e}^{-S_{\mathrm{G}}[U]+\bar{q}K[U]q} = \int \mathcal{D}U\mathrm{e}^{-S_{\mathrm{G}}[U]}\mathrm{det}K[U]$$

The relation also can be obtained on the even lattice. (See our paper ↑)

An analytical relation between Polyakov loop and Dirac mode on temporally odd-number lattice

TMD, H. Suganuma, T. Iritani, Phys. Rev. D 90, 094505 (2014).
H. Suganuma, TMD, T. Iritani, Prog. Theor. Exp. Phys. 2016, 013B06 (2016).

$$L = -\frac{(2ai)^{N_4-1}}{12V} \sum_n \lambda_n^{N_4-1} \langle n|\hat{U}_4|n\rangle$$ on temporally odd number lattice: N_4 is odd

notation:

- Polyakov loop : L
- link variable operator : $\langle s|\hat{U}_\mu|s'\rangle = U_\mu(s)\delta_{s+\hat{\mu},s'}$

  with anti p.b.c. for time direction: $\langle N_4, \mathbf{x}|\hat{U}_4|1, \mathbf{x}\rangle = -U_4(N_4, \mathbf{x})$
- Dirac eigenmode : $\hat{\not{D}}|n\rangle = i\lambda_n|n\rangle$

  Dirac operator : $\hat{\not{D}} = \frac{1}{2a}\sum_\mu \gamma_\mu(\hat{U}_\mu - \hat{U}_{-\mu})$ $\quad \sum_n |n\rangle\langle n| = 1$

Discussion:

- Low-lying Dirac-modes $(|\lambda_n| \sim 0)$ are important for CSB (Banks-Casher relation)
- Low-lying Dirac-modes have little contribution to the Polyakov loop because of damping factor: $\lambda_n^{N_4-1}$

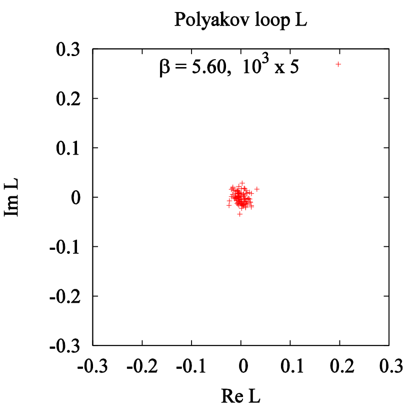
⇒ Contribution from the important modes of chiral symmetry breaking is negligible to the Polyakov loop, namely an order parameter of the deconfinement.

⇒ The important modes of chiral symmetry breaking are not important for confinement.

λ_n V.S. $(n|\hat{U}_4|n)$, $\lambda_n^{N_4-1}(n|\hat{U}_4|n)$

$$L = \frac{(2i)^{N_4-1}}{3V} \sum_n \lambda_n^{N_4-1}(n|\hat{U}_4|n)$$

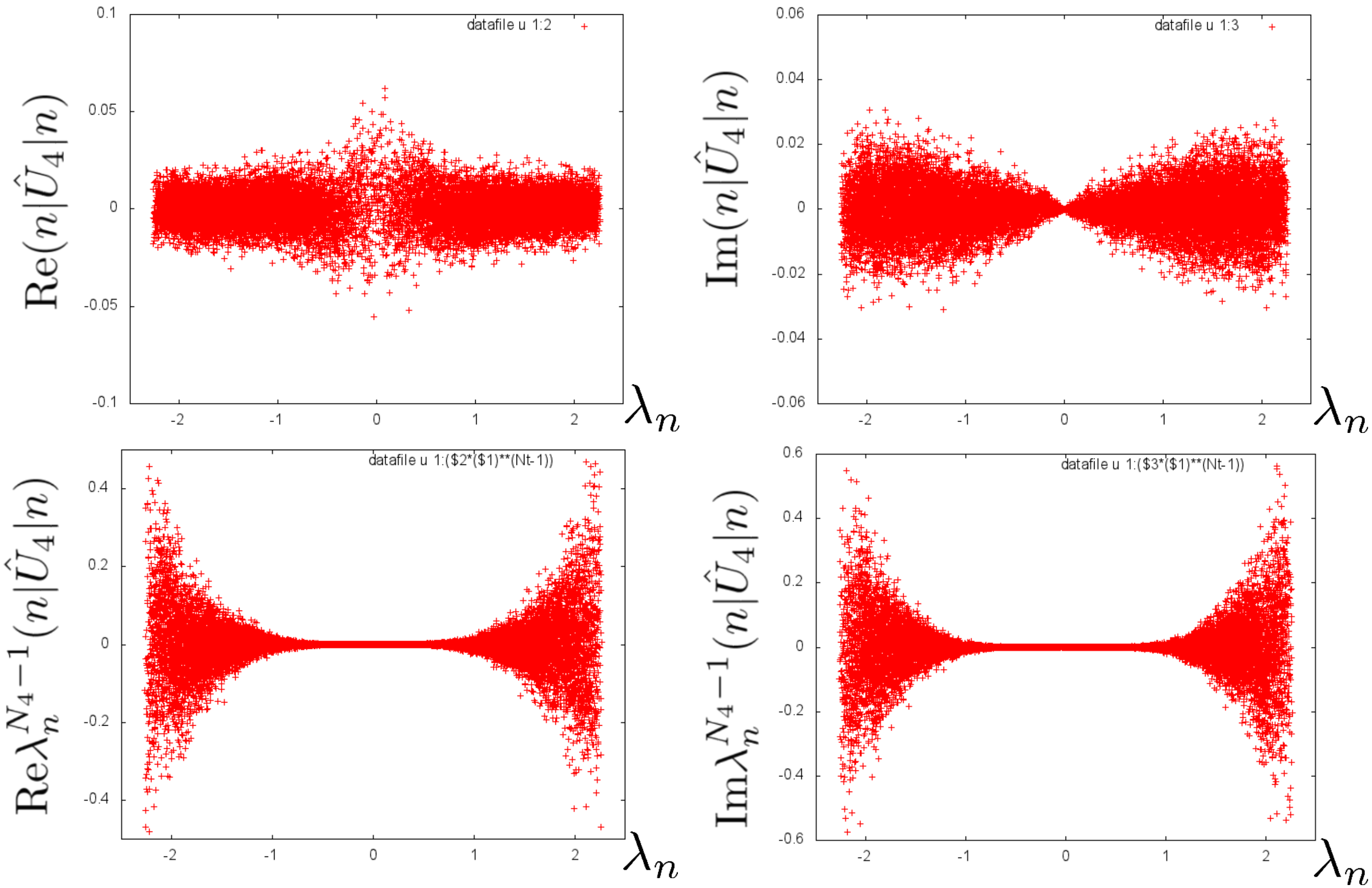
$\beta = 5.6$
lattice size : $10^3 \times 5$



$\langle L \rangle = 0$
(confined phase)

$$\not{D}|n\rangle = i\lambda_n|n\rangle$$

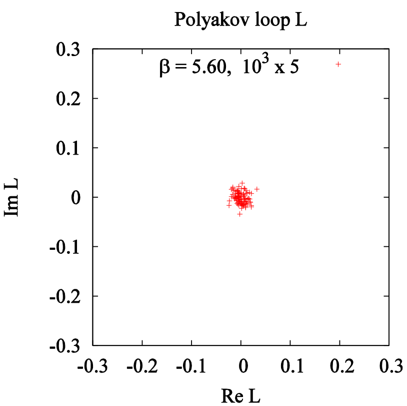
Dirac eigenvalue: $i\lambda_n$



λ_n V.S. $(n|\hat{U}_4|n)$, $\lambda_n^{N_4-1}(n|\hat{U}_4|n)$

$$L = \frac{(2i)^{N_4-1}}{3V} \sum_n \lambda_n^{N_4-1}(n|\hat{U}_4|n)$$

$\beta = 5.6$
lattice size : $10^3 \times 5$

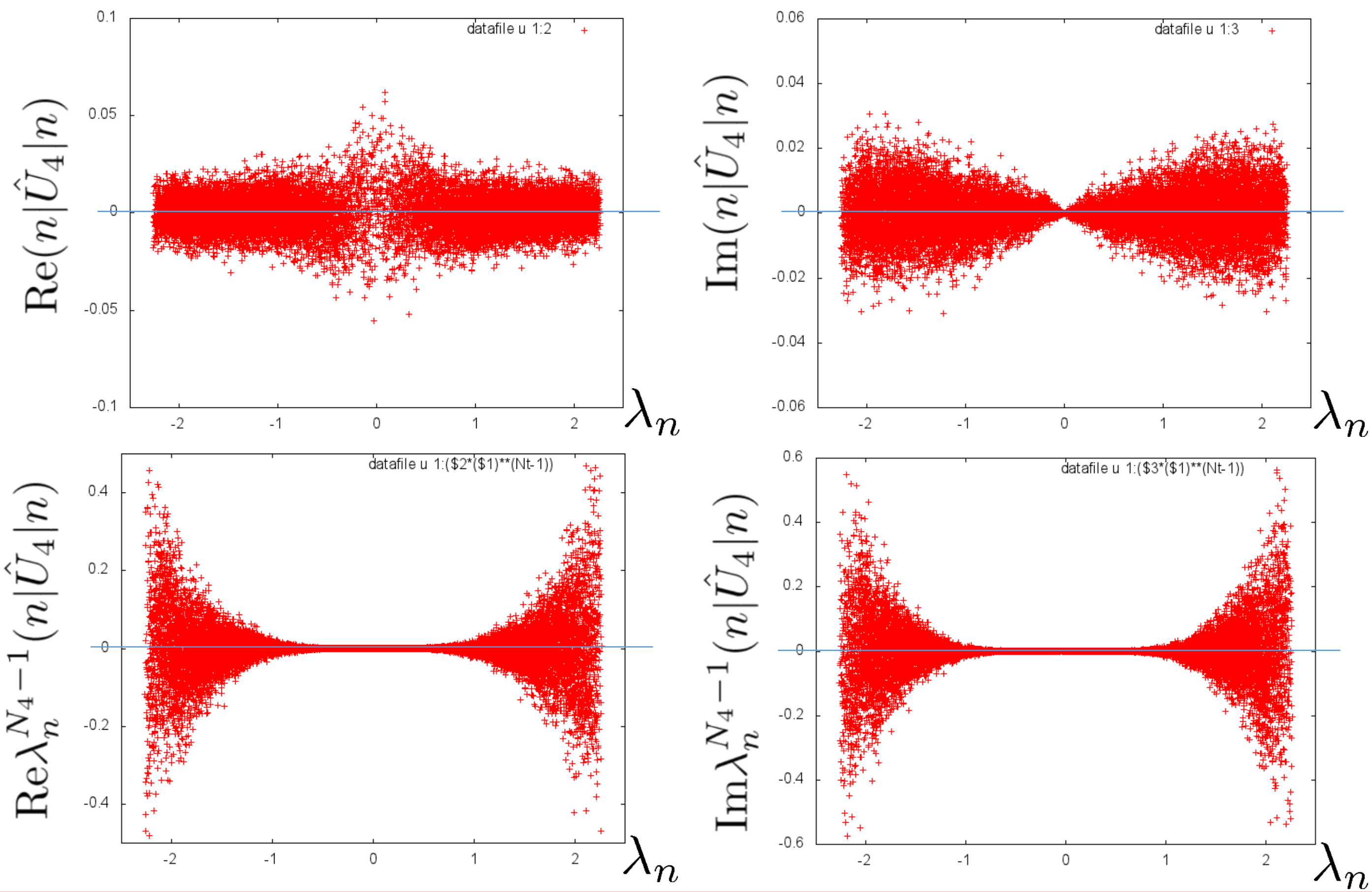


$\langle L \rangle = 0$
(confined phase)

$$\not{D}|n\rangle = i\lambda_n|n\rangle$$

Dirac eigenvalue: $i\lambda_n$

confined phase



$\langle L \rangle = 0$ is due to the symmetric distribution of positive/negative value of $(n|\hat{U}_4|n)$, $\lambda_n^{N_4-1}(n|\hat{U}_4|n)$

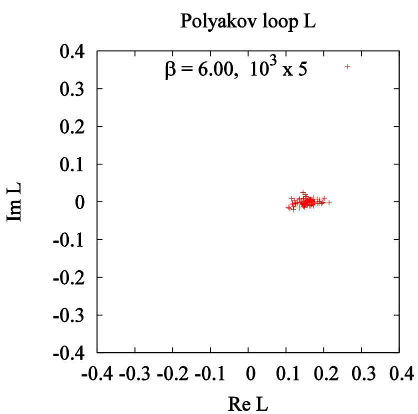
Low-lying Dirac modes have little contribution to Polyakov loop.

λ_n V.S. $(n|\hat{U}_4|n)$, $\lambda_n^{N_4-1}(n|\hat{U}_4|n)$

$$L = \frac{(2i)^{N_4-1}}{3V}\sum_n \lambda_n^{N_4-1}(n|\hat{U}_4|n)$$

$\beta = 6.0$

lattice size : $10^3 \times 5$

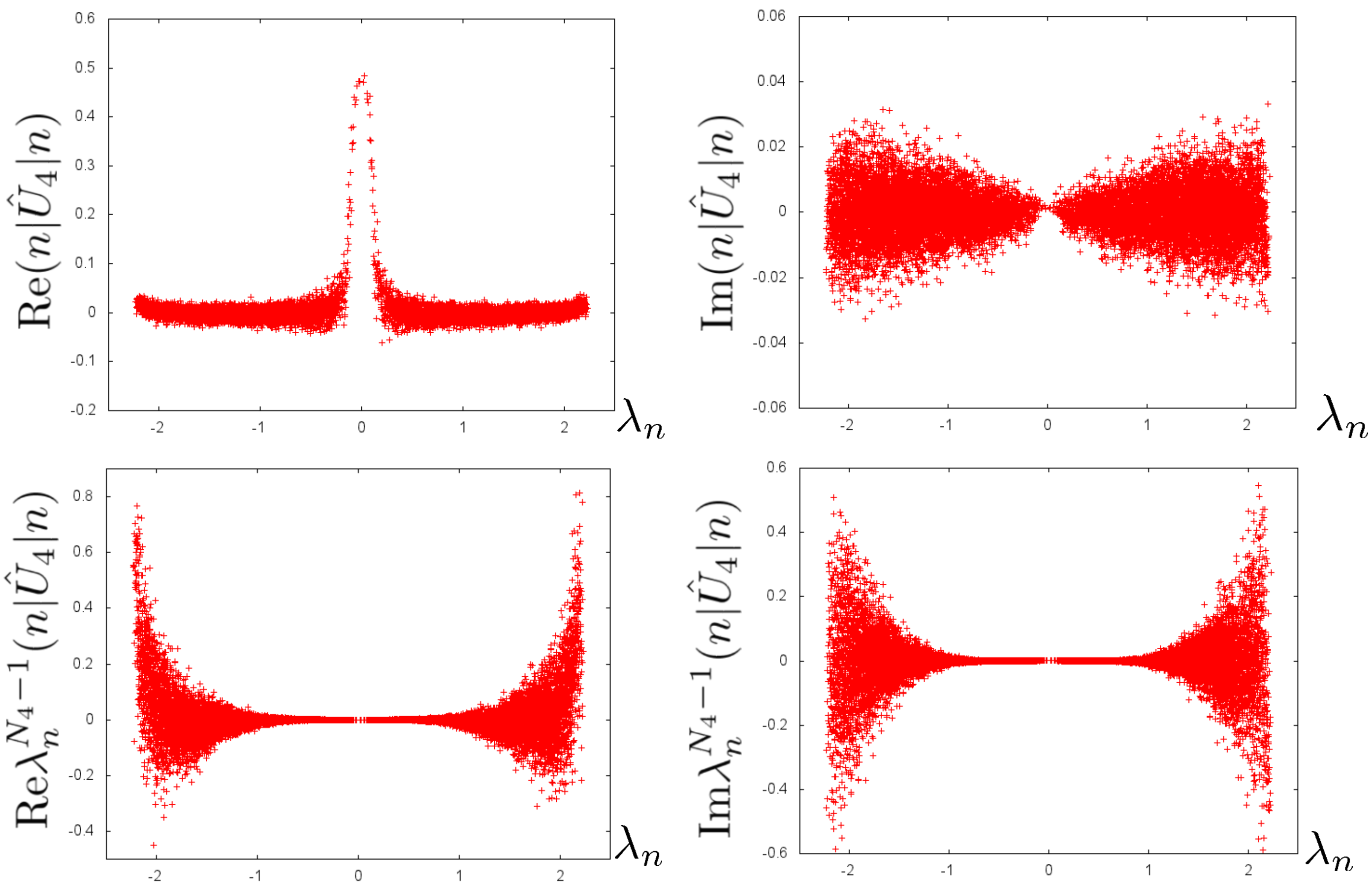


$\langle L \rangle \neq 0$

(deconfined phase)

$$\not{D}|n\rangle = i\lambda_n|n\rangle$$

Dirac eigenvalue: $i\lambda_n$

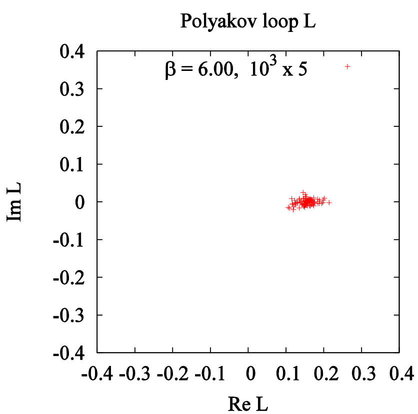


We mainly investigate the real Polyakov-loop vacuum, where the Polyakov loop is real, so only real part is different from it in confined phase.

λ_n V.S. $(n|\hat{U}_4|n)$, $\lambda_n^{N_4-1}(n|\hat{U}_4|n)$

$$L = \frac{(2i)^{N_4-1}}{3V}\sum_n \lambda_n^{N_4-1}(n|\hat{U}_4|n)$$

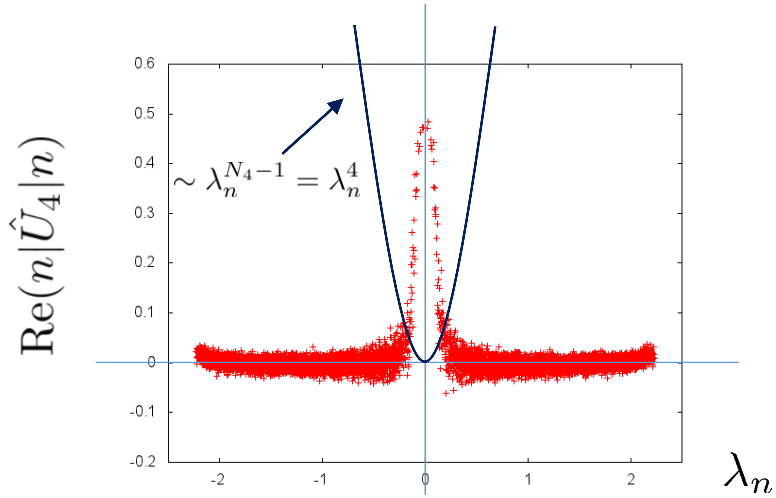
$\beta = 6.0$
lattice size : $10^3 \times 5$

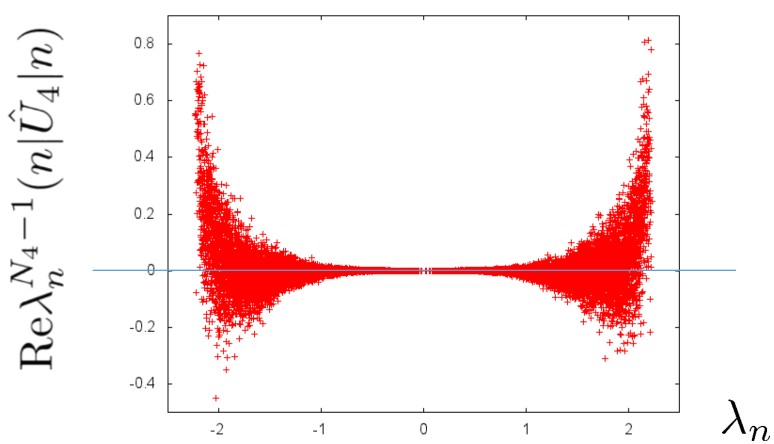


$\langle L \rangle \neq 0$
(deconfined phase)

$$\not{D}|n\rangle = i\lambda_n|n\rangle$$

Dirac eigenvalue: $i\lambda_n$





In low-lying Dirac modes region, $\mathrm{Re}(n|\hat{U}_4|n)$ has a large value,
but contribution of low-lying (IR) Dirac modes to Polyakov loop is very small
because of dumping factor $\lambda_n^{N_4-1}$

Our study

The relation btw quark-confinement and chiral symmetry breaking in QCD is investigated by deriving analytical formulae connecting the order parameters for confinement and Dirac eigenmodes with proper (anti)periodic boundary condition for temporal direction.

- Polyakov loop L

  TMD, H. Suganuma, T. Iritani, Phys. Rev. D90, 094505 (2014)

- Polyakov loop fluctuations

  TMD, K. Redlich, C. Sasaki, H. Suganuma, Phys. Rev. D92, 094004 (2015)

- Wilson loop

  H. Suganuma, TMD, T. Iritani, PTEP 2016, 013B06 (2016)

Polyakov loop fluctuations

P.M. Lo, B. Friman, O. Kaczmarek, K. Redlich and C. Sasaki,
Phys. Rev. D88, 014506 (2013); Phys. Rev. D88, 074502 (2013)

- Polyakov loop: $L \equiv \frac{1}{N_c V} \sum_s \mathrm{tr}_c \{ \prod_{i=0}^{N_\tau - 1} U_4(s + i\hat{4}) \}$
- Z3 rotated Polyakov loop: $\tilde{L} = L e^{2\pi k i/3}$
- longitudinal Polyakov loop: $L_L \equiv \mathrm{Re}(\tilde{L})$
- Transverse Polyakov loop: $L_T \equiv \mathrm{Im}(\tilde{L})$

- Polyakov loop susceptibilities:

$$T^3 \chi_A = \frac{N_\sigma^3}{N_\tau^3} [\langle |L|^2 \rangle - \langle |L| \rangle^2],$$

$$T^3 \chi_L = \frac{N_\sigma^3}{N_\tau^3} [\langle (L_L)^2 \rangle - \langle L_L \rangle^2],$$

$$T^3 \chi_T = \frac{N_\sigma^3}{N_\tau^3} [\langle (L_T)^2 \rangle - \langle L_T \rangle^2]$$

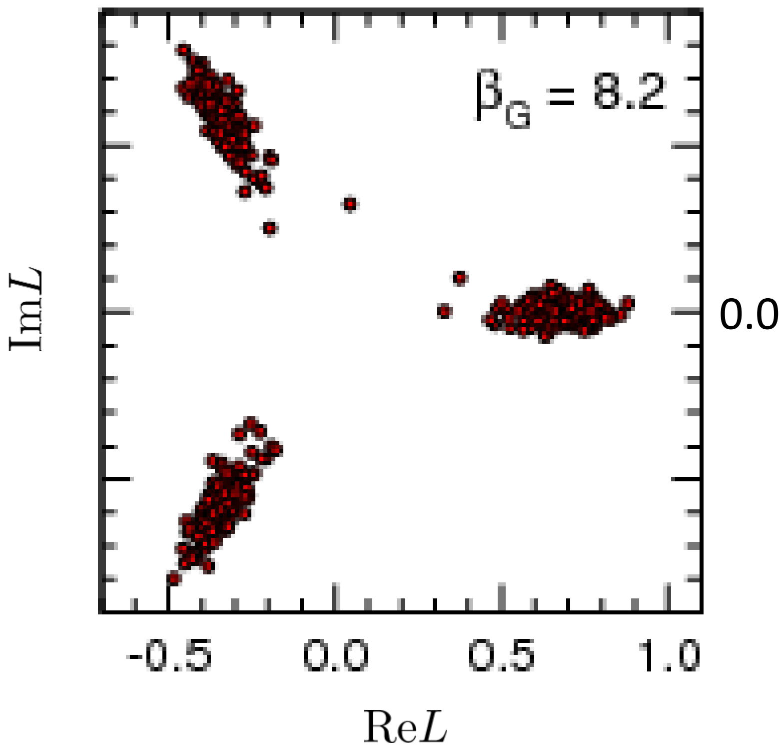
- Ratios of Polyakov loop susceptibilities:

$$R_A \equiv \frac{\chi_A}{\chi_L}, \qquad R_T \equiv \frac{\chi_T}{\chi_L}$$

T : temperature

$N_\sigma, \ N_\tau$: spatial and temporal lattice size

Scattered plot of the Polyakov loops



Polyakov loop fluctuations

P.M. Lo, B. Friman, O. Kaczmarek, K. Redlich and C. Sasaki,
Phys. Rev. D88, 014506 (2013); Phys. Rev. D88, 074502 (2013)

- Polyakov loop: $L \equiv \frac{1}{N_c V} \sum_s \mathrm{tr}_c \{ \prod_{i=0}^{N_\tau - 1} U_4(s + i\hat{4}) \}$

- Z3 rotated Polyakov loop: $\tilde{L} = L e^{2\pi k i/3}$
- longitudinal Polyakov loop: $L_L \equiv \mathrm{Re}(\tilde{L})$
- Transverse Polyakov loop: $L_T \equiv \mathrm{Im}(\tilde{L})$

- Polyakov loop susceptibilities:

$$T^3 \chi_A = \frac{N_\sigma^3}{N_\tau^3} [\langle |L|^2 \rangle - \langle |L| \rangle^2],$$

$$T^3 \chi_L = \frac{N_\sigma^3}{N_\tau^3} [\langle (L_L)^2 \rangle - \langle L_L \rangle^2],$$

$$T^3 \chi_T = \frac{N_\sigma^3}{N_\tau^3} [\langle (L_T)^2 \rangle - \langle L_T \rangle^2]$$

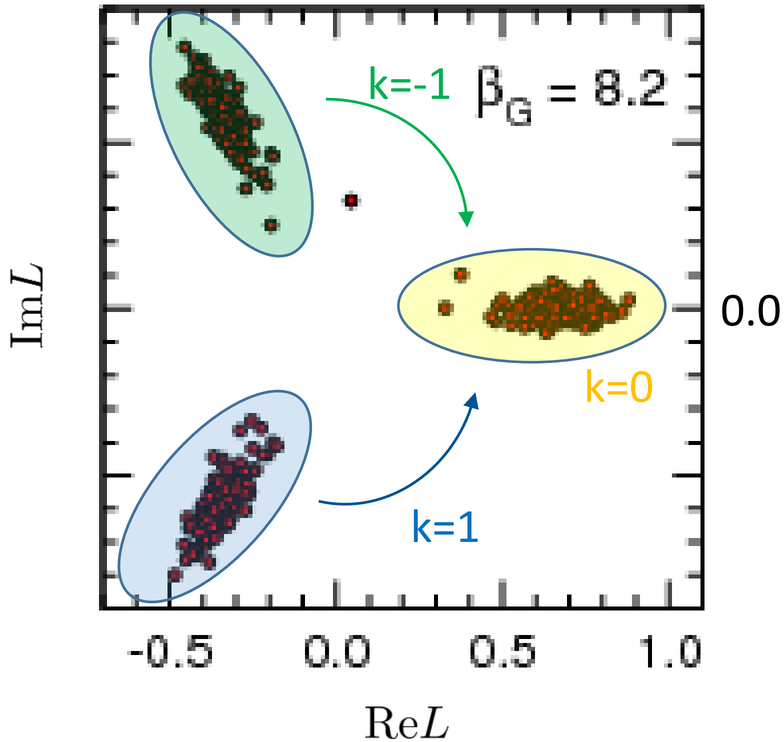
- Ratios of Polyakov loop susceptibilities:

$$R_A \equiv \frac{\chi_A}{\chi_L}, \qquad R_T \equiv \frac{\chi_T}{\chi_L}$$

T : temperature

$N_\sigma,\ N_\tau$: spatial and temporal lattice size

Scattered plot of the Polyakov loops



Polyakov loop fluctuations

P.M. Lo, B. Friman, O. Kaczmarek, K. Redlich and C. Sasaki,
Phys. Rev. D88, 014506 (2013); Phys. Rev. D88, 074502 (2013)

- Polyakov loop: $L \equiv \frac{1}{N_c V} \sum_s \mathrm{tr}_c \{ \prod_{i=0}^{N_\tau - 1} U_4(s + i\hat{4}) \}$

- Z3 rotated Polyakov loop: $\tilde{L} = L e^{2\pi k i/3}$
- longitudinal Polyakov loop: $L_L \equiv \mathrm{Re}(\tilde{L})$
- Transverse Polyakov loop: $L_T \equiv \mathrm{Im}(\tilde{L})$

- Polyakov loop susceptibilities:

$$T^3 \chi_A = \frac{N_\sigma^3}{N_\tau^3} [\langle |L|^2 \rangle - \langle |L| \rangle^2],$$

$$T^3 \chi_L = \frac{N_\sigma^3}{N_\tau^3} [\langle (L_L)^2 \rangle - \langle L_L \rangle^2],$$

$$T^3 \chi_T = \frac{N_\sigma^3}{N_\tau^3} [\langle (L_T)^2 \rangle - \langle L_T \rangle^2]$$

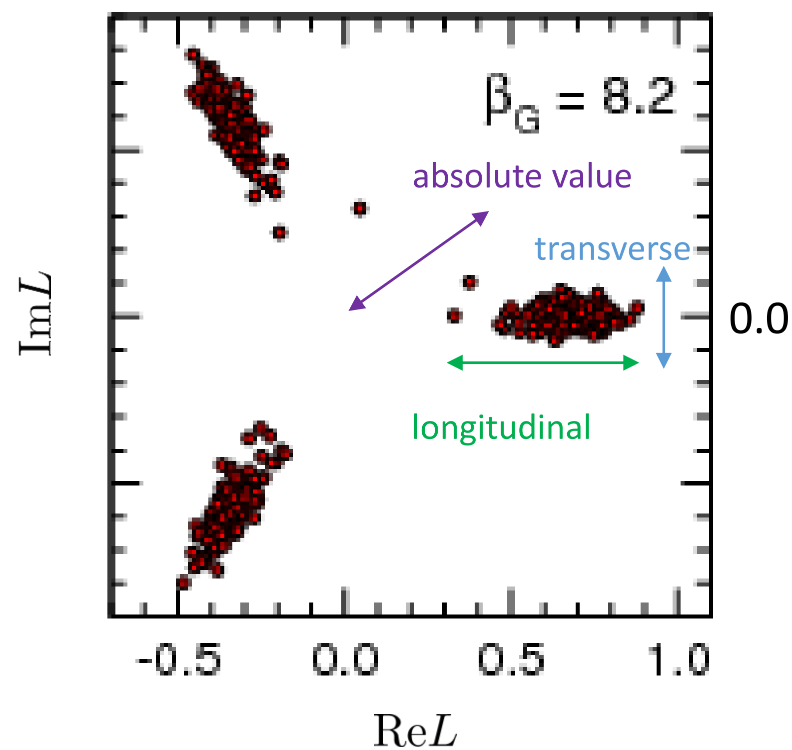
- Ratios of Polyakov loop susceptibilities:

$$R_A \equiv \frac{\chi_A}{\chi_L}, \qquad R_T \equiv \frac{\chi_T}{\chi_L}$$

T : temperature

$N_\sigma, \ N_\tau$: spatial and temporal lattice size

Scattered plot of the Polyakov loops



Polyakov loop fluctuations

P.M. Lo, B. Friman, O. Kaczmarek, K. Redlich and C. Sasaki,
Phys. Rev. D88, 014506 (2013); Phys. Rev. D88, 074502 (2013)

- Polyakov loop: $L \equiv \frac{1}{N_c V} \sum_s \mathrm{tr}_c \{ \prod_{i=0}^{N_\tau - 1} U_4(s + i\hat{4}) \}$
- Z3 rotated Polyakov loop: $\tilde{L} = L e^{2\pi k i/3}$
- longitudinal Polyakov loop: $L_L \equiv \mathrm{Re}(\tilde{L})$
- Transverse Polyakov loop: $L_T \equiv \mathrm{Im}(\tilde{L})$
- Polyakov loop susceptibilities:

$$T^3 \chi_A = \frac{N_\sigma^3}{N_\tau^3} [\langle |L|^2 \rangle - \langle |L| \rangle^2],$$

$$T^3 \chi_L = \frac{N_\sigma^3}{N_\tau^3} [\langle (L_L)^2 \rangle - \langle L_L \rangle^2],$$

$$T^3 \chi_T = \frac{N_\sigma^3}{N_\tau^3} [\langle (L_T)^2 \rangle - \langle L_T \rangle^2]$$

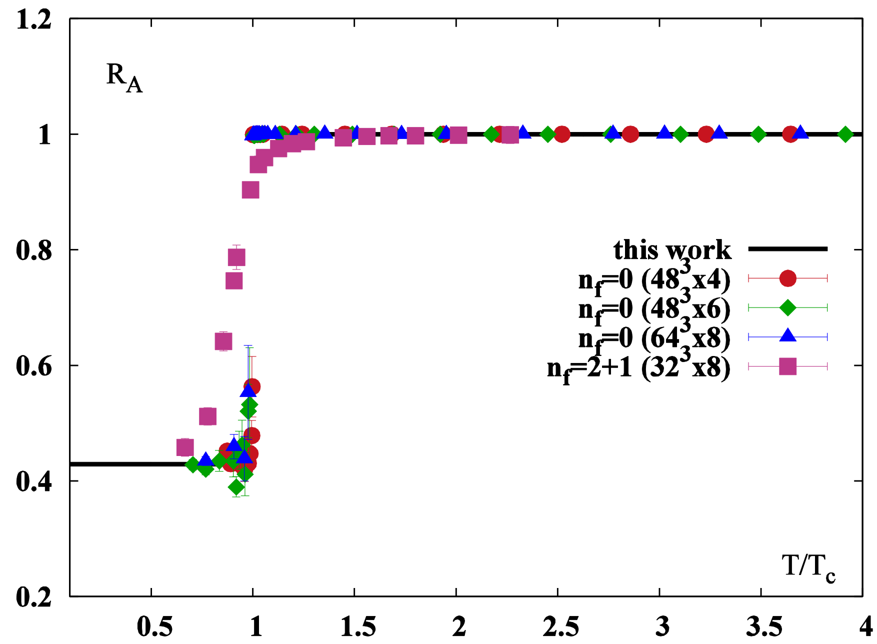
- Ratios of Polyakov loop susceptibilities:

$$R_A \equiv \frac{\chi_A}{\chi_L}, \qquad R_T \equiv \frac{\chi_T}{\chi_L}$$

T : temperature

$N_\sigma, \ N_\tau$: spatial and temporal lattice size

In particular,
R_A is a sensitive probe
for deconfinement transition



※nf=0: quenched level

nf=2+1: (2+1)flavor full QCD
(near physical point)

Polyakov loop fluctuations

P.M. Lo, B. Friman, O. Kaczmarek, K. Redlich and C. Sasaki,
Phys. Rev. D88, 014506 (2013); Phys. Rev. D88, 074502 (2013)

- Polyakov loop: $L \equiv \frac{1}{N_c V} \sum_s \mathrm{tr}_c \{ \prod_{i=0}^{N_\tau - 1} U_4(s + i\hat{4}) \}$
- Z3 rotated Polyakov loop: $\tilde{L} = L e^{2\pi k i/3}$
- longitudinal Polyakov loop: $L_L \equiv \mathrm{Re}(\tilde{L})$
- Transverse Polyakov loop: $L_T \equiv \mathrm{Im}(\tilde{L})$

- Polyakov loop susceptibilities:

$$T^3 \chi_A = \frac{N_\sigma^3}{N_\tau^3} [\langle |L|^2 \rangle - \langle |L| \rangle^2],$$

$$T^3 \chi_L = \frac{N_\sigma^3}{N_\tau^3} [\langle (L_L)^2 \rangle - \langle L_L \rangle^2],$$

$$T^3 \chi_T = \frac{N_\sigma^3}{N_\tau^3} [\langle (L_T)^2 \rangle - \langle L_T \rangle^2]$$

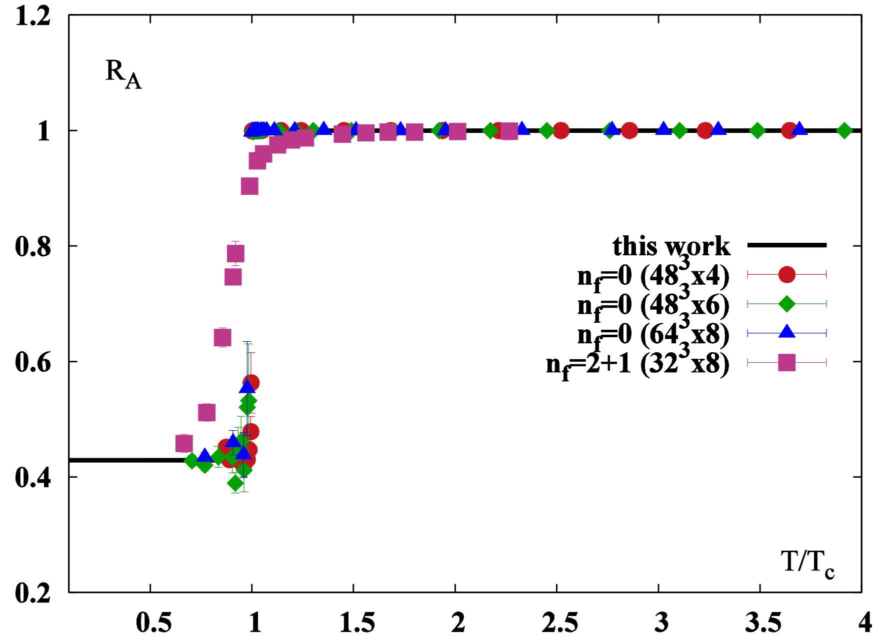
- Ratios of Polyakov loop susceptibilities:

$$R_A \equiv \frac{\chi_A}{\chi_L}, \qquad R_T \equiv \frac{\chi_T}{\chi_L}$$

T : temperature

$N_\sigma, \ N_\tau$: spatial and temporal lattice size

In particular,
R_A is a sensitive probe
for deconfinement transition



R_A is a good probe for deconfinement transition
even if considering light dynamical quarks.
And, another advantage of considering R_A is that
the ambiguity of the multiplicative renormalization
of the Polyakov loop can be avoided.

Dirac spectrum representation of the Polyakov loop fluctuations

TMD, K. Redlich, C. Sasaki and H. Suganuma, Phys. Rev. D92, 094004 (2015).

In particular, the ratio R_A can be represented using Dirac modes:

$$R_A = \frac{\left\langle \left| \sum_n \lambda_n^{N_\tau - 1} \langle n | \hat{U}_4 | n \rangle \right|^2 \right\rangle - \left\langle \left| \sum_n \lambda_n^{N_\tau - 1} \langle n | \hat{U}_4 | n \rangle \right| \right\rangle^2}{\left\langle \left(\sum_n \lambda_n^{N_\tau - 1} \mathrm{Re} \left(\mathrm{e}^{2\pi k i/3} \langle n | \hat{U}_4 | n \rangle \right) \right)^2 \right\rangle - \left\langle \sum_n \lambda_n^{N_\tau - 1} \mathrm{Re} \left(\mathrm{e}^{2\pi k i/3} \langle n | \hat{U}_4 | n \rangle \right) \right\rangle^2}$$

Note 1: The ratio R_A is a good “order parameter” for deconfinement transition.

Dirac spectrum representation of the Polyakov loop fluctuations

TMD, K. Redlich, C. Sasaki and H. Suganuma, Phys. Rev. D92, 094004 (2015).

In particular, the ratio R_A can be represented using Dirac modes:

$$R_A = \frac{\left\langle \left| \sum_n \lambda_n^{N_\tau - 1} \langle n|\hat{U}_4|n\rangle \right|^2 \right\rangle - \left\langle \left| \sum_n \lambda_n^{N_\tau - 1} \langle n|\hat{U}_4|n\rangle \right| \right\rangle^2}{\left\langle \left(\sum_n \lambda_n^{N_\tau - 1} \mathrm{Re}\left(\mathrm{e}^{2\pi k i/3} \langle n|\hat{U}_4|n\rangle \right) \right)^2 \right\rangle - \left\langle \sum_n \lambda_n^{N_\tau - 1} \mathrm{Re}\left(\mathrm{e}^{2\pi k i/3} \langle n|\hat{U}_4|n\rangle \right) \right\rangle^2}$$

Note 1: The ratio R_A is a good "order parameter" for deconfinement transition.

Note 2: Since the damping factor $\lambda_n^{N_\tau - 1}$ is very small with small $|\lambda_n| \simeq 0$,
low-lying Dirac modes (with small $|\lambda_n| \simeq 0$) are not important for R_A,
while these modes are important modes for chiral symmetry breaking.

Dirac spectrum representation of the Polyakov loop fluctuations

TMD, K. Redlich, C. Sasaki and H. Suganuma, Phys. Rev. D92, 094004 (2015).

In particular, the ratio R_A can be represented using Dirac modes:

$$R_A = \frac{\left\langle \left| \sum_n \lambda_n^{N_\tau - 1} \langle n | \hat{U}_4 | n \rangle \right|^2 \right\rangle - \left\langle \left| \sum_n \lambda_n^{N_\tau - 1} \langle n | \hat{U}_4 | n \rangle \right| \right\rangle^2}{\left\langle \left(\sum_n \lambda_n^{N_\tau - 1} \mathrm{Re} \left(\mathrm{e}^{2\pi k i/3} \langle n | \hat{U}_4 | n \rangle \right) \right)^2 \right\rangle - \left\langle \sum_n \lambda_n^{N_\tau - 1} \mathrm{Re} \left(\mathrm{e}^{2\pi k i/3} \langle n | \hat{U}_4 | n \rangle \right) \right\rangle^2}$$

Note 1: The ratio R_A is a good “order parameter” for deconfinement transition.

Note 2: Since the damping factor $\lambda_n^{N_\tau - 1}$ is very small with small $|\lambda_n| \simeq 0$,
low-lying Dirac modes (with small $|\lambda_n| \simeq 0$) are not important for R_A,
while these modes are important modes for chiral symmetry breaking.

Thus, the essential modes for chiral symmetry breaking in QCD are not important to quantify the Polyakov loop fluctuations, which are sensitive observables to confinement properties in QCD.

Our study

The relation btw quark-confinement and chiral symmetry breaking in QCD is investigated by deriving analytical formulae connecting the order parameters for confinement and Dirac eigenmodes with proper (anti)periodic boundary condition for temporal direction.

- Polyakov loop L

  TMD, H. Suganuma, T. Iritani, Phys. Rev. D90, 094505 (2014)

- Polyakov loop fluctuations

  TMD, K. Redlich, C. Sasaki, H. Suganuma, Phys. Rev. D92, 094004 (2015)

- Wilson loop

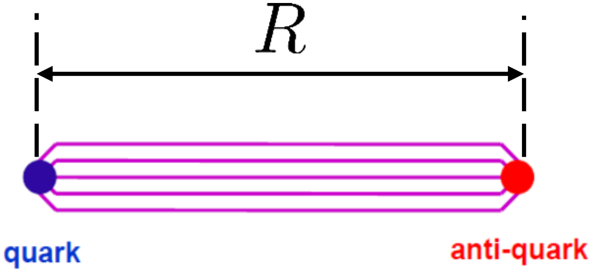
  H. Suganuma, TMD, T. Iritani, PTEP 2016, 013B06 (2016)

Dirac spectrum representation of the Wilson loop

H. Suganuma, TMD, T. Iritani, PTEP 2016, 013B06 (2016)

Interquark potential from the Wilson loop

$$V(R) = -\lim_{T\to\infty}\frac{1}{T}\ln\langle W(R,T)\rangle$$



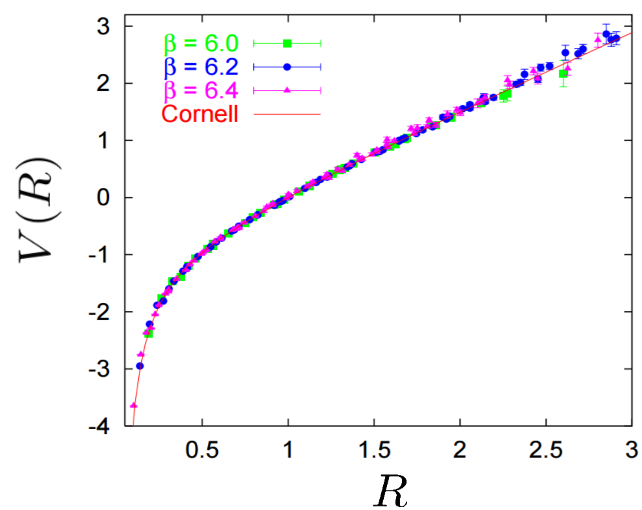
Cornell potential

$$V_{\mathrm{C}}(R) = -A/R + \underline{\sigma R} + C$$

String tension σ

$$\sigma = -\lim_{R,T\to\infty}\frac{1}{RT}\ln\langle W\rangle$$

G. S. Bali, Phys. Rept. 343, 1 (2001)



Dirac spectrum representation of string tension

$$\sigma = -\lim_{R,T\to\infty}\frac{1}{RT}\ln\left|\left\langle\sum_n \underline{(2a\lambda_n)^T}\langle n|\hat{S}|n\rangle\right\rangle\right|$$

Contents

- Introduction
  - Quark confinement and its order parameter
  - Chiral symmetry breaking and Dirac eigenmode
  - Deconfinement transition and chiral restoration at finite temperature
- Relation btw confinement and chiral symmetry breaking
  - Polyakov loop
  - Polyakov loop fluctuations
  - Wilson loop
- Generalization to chiral fermion on the lattice
- Discussion and Summary

Fermion-doubling problem and chiral symmetry on the lattice

Naive Dirac operator (So far)

$$\hat{\not{D}} = \frac{1}{2a}\sum_{\mu}\gamma_{\mu}(\hat{U}_{\mu} - \hat{U}_{-\mu})$$

- include fermion doubler
- exact chiral symmetry

Wilson-Dirac operator (Example to avoid the fermion-doubling problem)

$$D_{\mathrm{W}} = D_{\mu}\gamma_{\mu} - \frac{ar}{2}D_{\mu}D_{\mu}$$

- free from fermion-doubling problem
- explicit breaking of chiral symmetry e.g.) Nielsen-Ninomiya (1981)

Overlap-Dirac operator H. Neuberger, Phys. Lett. B417, 141 (1998)

- free from fermion-doubling problem
- exact chiral symmetry on the lattice

Overlap-Dirac operator

Overlap-Dirac operator H. Neuberger, Phys. Lett. B417, 141 (1998)

$$D_{\mathrm{ov}} = 1 + \frac{D_{\mathrm{W}}(-M_0)}{\sqrt{D_{\mathrm{W}}(-M_0)^\dagger D_{\mathrm{W}}(-M_0)}} \quad \text{(lattice unit)}$$

- Wilson-Dirac operator with negative mass ($-M_0$):

$$D_{\mathrm{W}}(-M_0) = D_\mu \gamma_\mu - \frac{ar}{2} D_\mu D_\mu - M_0$$

- The overlap-Dirac operator satisfies the Ginsparg-Wilson relation:

$$\{\gamma_5, D\} = D\gamma_5 D$$

P.H. Ginsparg and K.G. Wilson, PRD25, 2649 (1995)
H. Neuberger, Phys. Lett. B427, 353 (1998)

⇒ the overlap-Dirac operator solves the fermion doubling problem with the exact chiral symmetry on the lattice.

- The overlap-Dirac operator is non-local operator.

⇒ It is very difficult to directly derive the analytical relation between the Polyakov loop and the overlap-Dirac modes.

⇒ However, we can discuss the relation between the Polyakov loop and the overlap-Dirac modes. (next page~)

Polyakov loop and overlap-Dirac modes

Strategy

We discuss the relation btw the Polyakov loop and overlap-Dirac modes by showing the following facts:

① The low-lying overlap-Dirac modes are essential modes for chiral symmetry breaking.
② The low-lying eigenmodes of overlap-Dirac and Wilson-Dirac operators correspond.
③ The low-lying Wilson-Dirac modes have negligible contribution to the Polyakov loop.

$$D_{\rm ov}|n\rangle = \Lambda_n^{\rm ov}|n\rangle \qquad D_{\rm W}|n\rangle = \Lambda_n^{\rm W}|n\rangle$$

Low-lying overlap-Dirac modes

$|\Lambda_n^{\rm ov}| \sim 0$

② correspond

Low-lying Wilson-Dirac modes

$|\Lambda_n^{\rm W}| \sim 0$

① Essential modes for chiral symmetry breaking

③ Negligible contribution to the Polyakov loop

Polyakov loop and overlap-Dirac modes

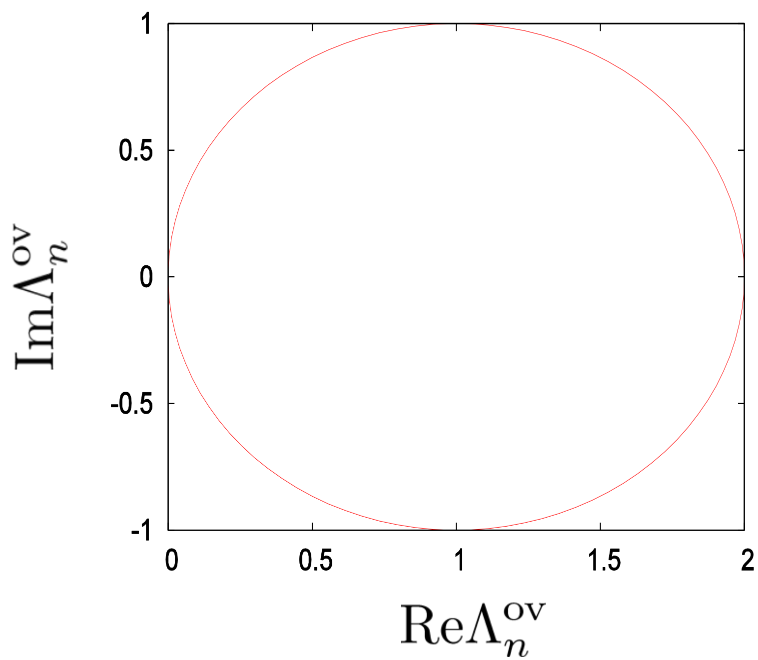
① The low-lying overlap-Dirac modes are essential modes for chiral symmetry breaking.
② The low-lying eigenmodes of Wilson-Dirac and overlap-Dirac operators correspond.
③ The low-lying Wilson-Dirac modes have negligible contribution to the Polyakov loop.

① The low-lying overlap-Dirac modes are essential modes for chiral symmetry breaking.

The chiral condensate in terms the overlap-Dirac eigenvalues:

$$\langle \bar{q}q \rangle \sim \sum_n \frac{1}{\Lambda_n^{\rm ov} + m}$$

$$D_{\rm ov}|n\rangle = \Lambda_n^{\rm ov}|n\rangle$$



The low-lying overlap-Dirac modes $(|\Lambda_n^{\rm ov}| \sim 0)$ have the dominant contribution to the chiral condensate $\langle \bar{q}q \rangle$.

(Recall the Banks-Casher relation)

$$\langle \bar{q}q \rangle = -\lim_{m\to 0}\lim_{V\to\infty} \pi\langle \rho(0) \rangle$$

Polyakov loop and overlap-Dirac modes

① The low-lying overlap-Dirac modes are essential modes for chiral symmetry breaking.
② The low-lying eigenmodes of Wilson-Dirac and overlap-Dirac operators correspond.
③ The low-lying Wilson-Dirac modes have negligible contribution to the Polyakov loop.

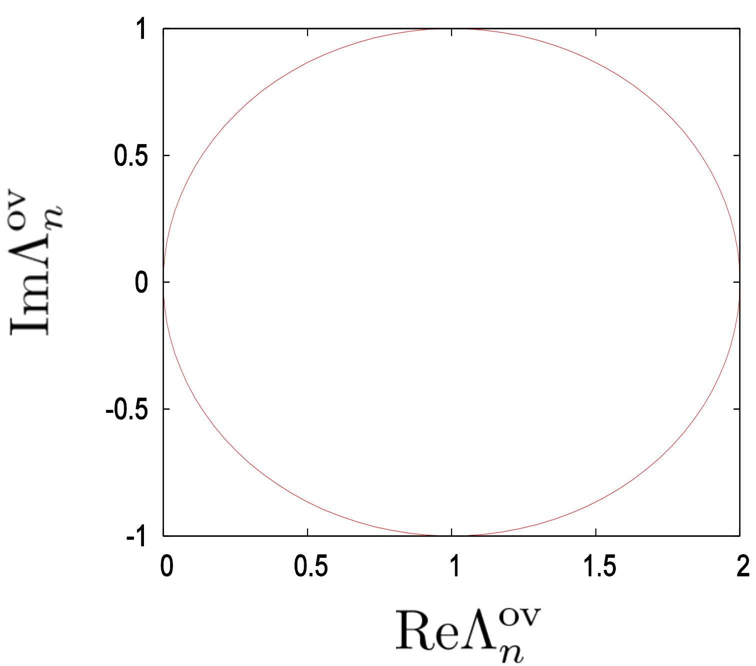
② The low-lying eigenmodes of Wilson-Dirac and overlap-Dirac operators correspond.

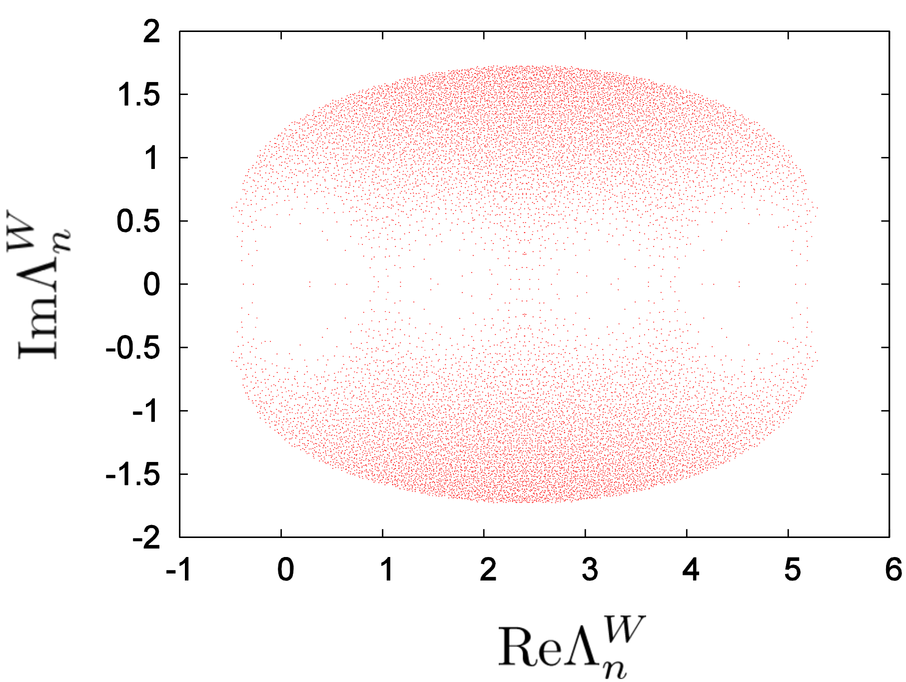
$$D_{\rm ov}|n\rangle = \Lambda_n^{\rm ov}|n\rangle$$

$$D_{\rm W}|n\rangle = \Lambda_n^{\rm W}|n\rangle$$

$$\Lambda_n^{\rm ov} = 1 + \frac{\Lambda_n^{\rm W}}{|\Lambda_n^{\rm W}|} + \mathcal{O}(a)$$

$$D_{\rm ov} = 1 + \frac{D_{\rm W}(-M_0)}{\sqrt{D_{\rm W}(-M_0)^\dagger D_{\rm W}(-M_0)}}$$





Polyakov loop and overlap-Dirac modes

① The low-lying overlap-Dirac modes are essential modes for chiral symmetry breaking.
② The low-lying eigenmodes of Wilson-Dirac and overlap-Dirac operators correspond.
③ The low-lying Wilson-Dirac modes have negligible contribution to the Polyakov loop.

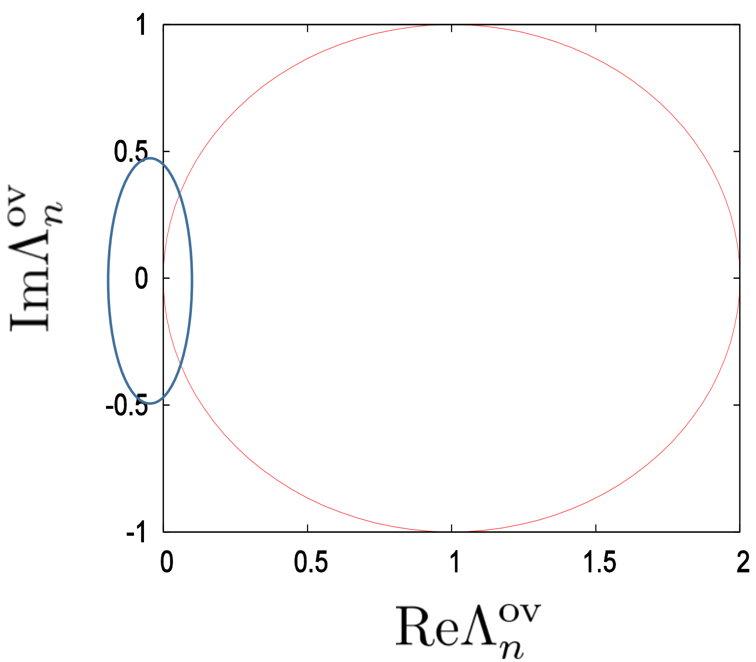
② The low-lying eigenmodes of Wilson-Dirac and overlap-Dirac operators correspond.

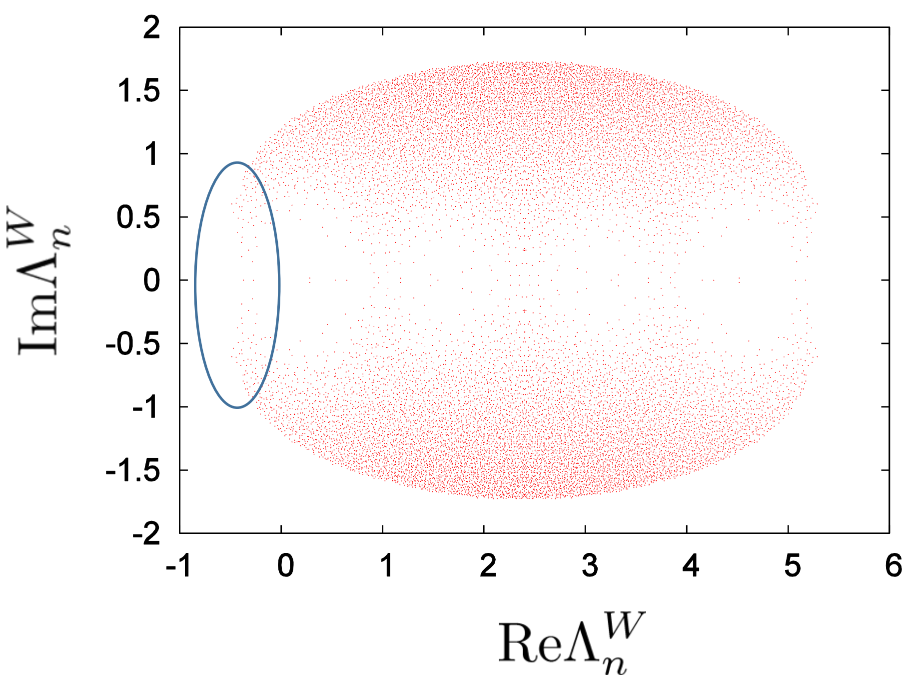
$$D_{\rm ov}|n\rangle = \Lambda_n^{\rm ov}|n\rangle$$

$$D_{\rm W}|n\rangle = \Lambda_n^{\rm W}|n\rangle$$

$$\Lambda_n^{\rm ov} = 1 + \frac{\Lambda_n^{\rm W}}{|\Lambda_n^{\rm W}|} + \mathcal{O}(a)$$

$$D_{\rm ov} = 1 + \frac{D_{\rm W}(-M_0)}{\sqrt{D_{\rm W}(-M_0)^\dagger D_{\rm W}(-M_0)}}$$





Polyakov loop and overlap-Dirac modes

① The low-lying overlap-Dirac modes are essential modes for chiral symmetry breaking.
② The low-lying eigenmodes of Wilson-Dirac and overlap-Dirac operators correspond.
③ The low-lying Wilson-Dirac modes have negligible contribution to the Polyakov loop.

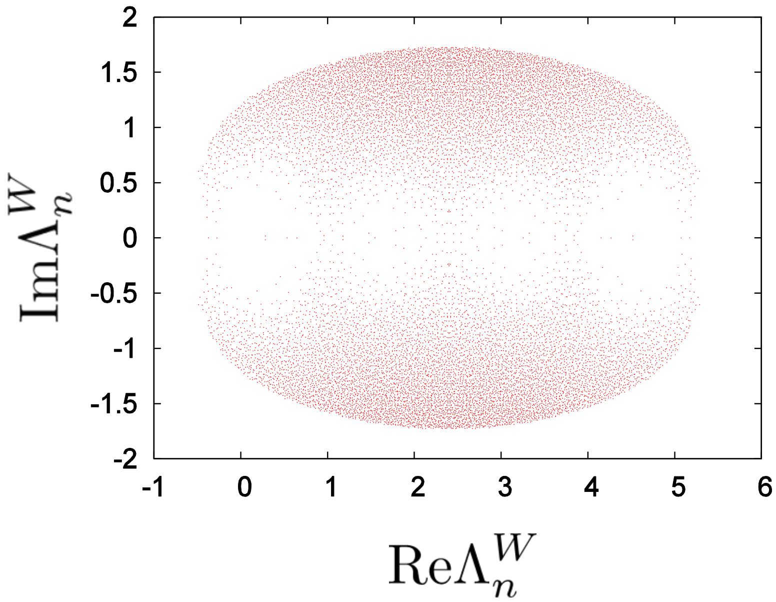
③ The low-lying Wilson-Dirac modes have negligible contribution to the Polyakov loop.

The analytical relation btw the Polyakov loop and the Wilson-Dirac modes

$$L = \frac{2a^N}{3V} \sum_n (\Lambda_n^{\rm W})^N \langle n | \hat{U}_4^{N+1} | n \rangle \quad (N_\tau = 2N+1)$$

$$D_{\rm W} |n\rangle = \Lambda_n^{\rm W} |n\rangle$$

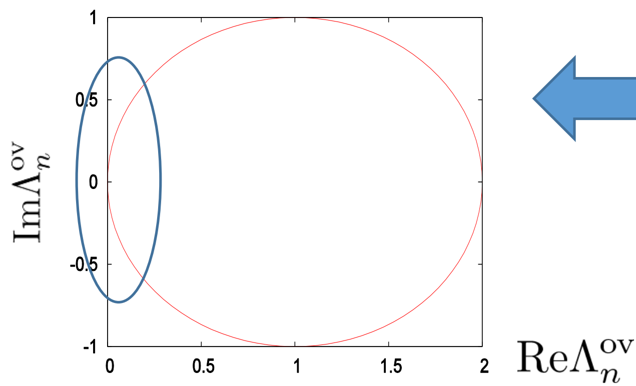
Due to the damping factor $(\Lambda_n^{\rm W})^N$,
the low-lying Wilson-Dirac modes (with $|\Lambda_n^{\rm W}| \sim 0$)
have negligible contribution to the Polyakov loop.



Polyakov loop and overlap-Dirac modes

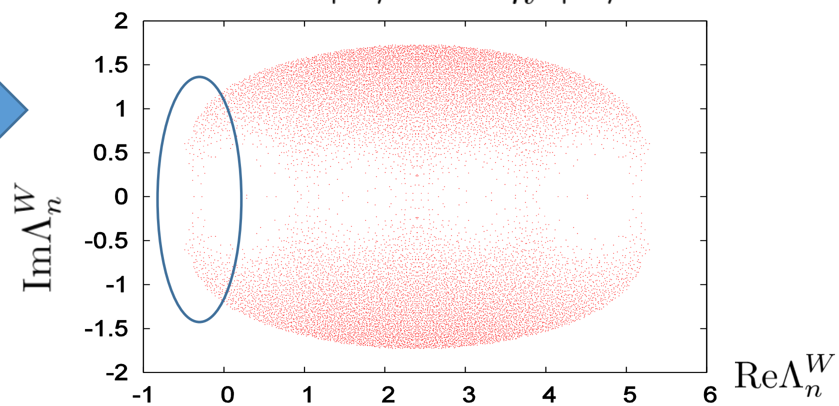
① The low-lying overlap-Dirac modes are essential modes for chiral symmetry breaking.
② The low-lying eigenmodes of Wilson-Dirac and overlap-Dirac operators correspond.
③ The low-lying Wilson-Dirac modes have negligible contribution to the Polyakov loop.

$$D_{\rm ov}|n\rangle = \Lambda_n^{\rm ov}|n\rangle$$



②
correspond

$$D_{\rm W}|n\rangle = \Lambda_n^{\rm W}|n\rangle$$



① $|\Lambda_n^{\rm ov}| \sim 0$

Essential modes for chiral symmetry breaking

③ $|\Lambda_n^{\rm W}| \sim 0$

Negligible contribution to the Polyakov loop

Therefore, the presence or absence of the low-lying overlap-Dirac modes is not related to the confinement properties such as the Polyakov loop while the chiral condensate is sensitive to the density of the low-lying modes.

Discussion

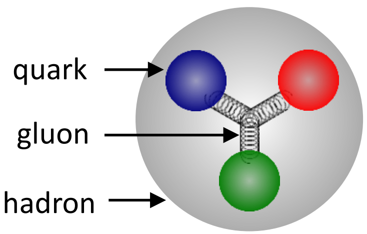
What we showed:

The important modes for chiral symmetry breaking are not important for confinement.

Expectation:

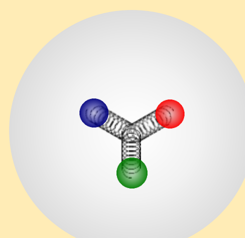
The deconfinement transition and chiral restoration can happen at different parameters, such as temperature and quark-chemical potential.
Therefore, for example, there might be new phases in QCD, where quarks are confined and chiral symmetry is restored.

Hadron phase



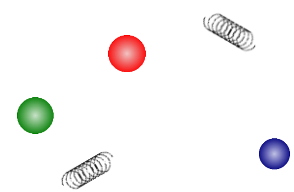
chiral symmetry: broken
quarks: confined

A new phase



chiral symmetry: restored
quarks: confined

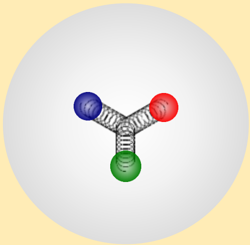
Quark-gluon plasma phase



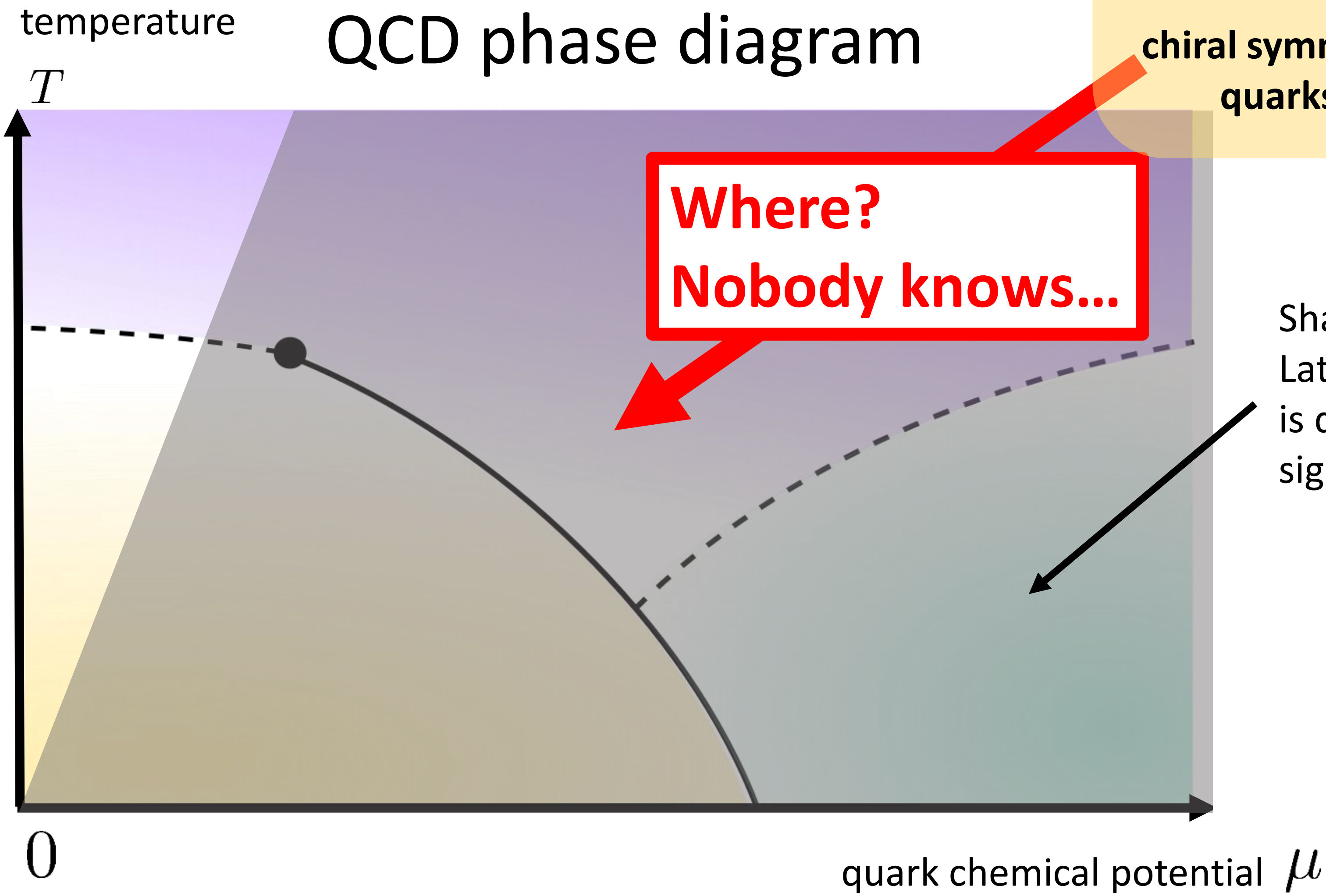
chiral symmetry: restored
quarks: deconfined

Discussion

A new phase



chiral symmetry: restored
quarks: confined



Shadowed region:
Lattice QCD calc.
is difficult due to
sign problem.

Summary

- Dirac-spectrum representations of several order parameters for quark-confinement are analytically derived in the lattice QCD.

  - Polyakov loop
  - Polyakov loop fluctuations
  - Wilson loop

- It is both analytically and numerically confirmed that Low-lying Dirac modes have negligible contribution to those order parameters.

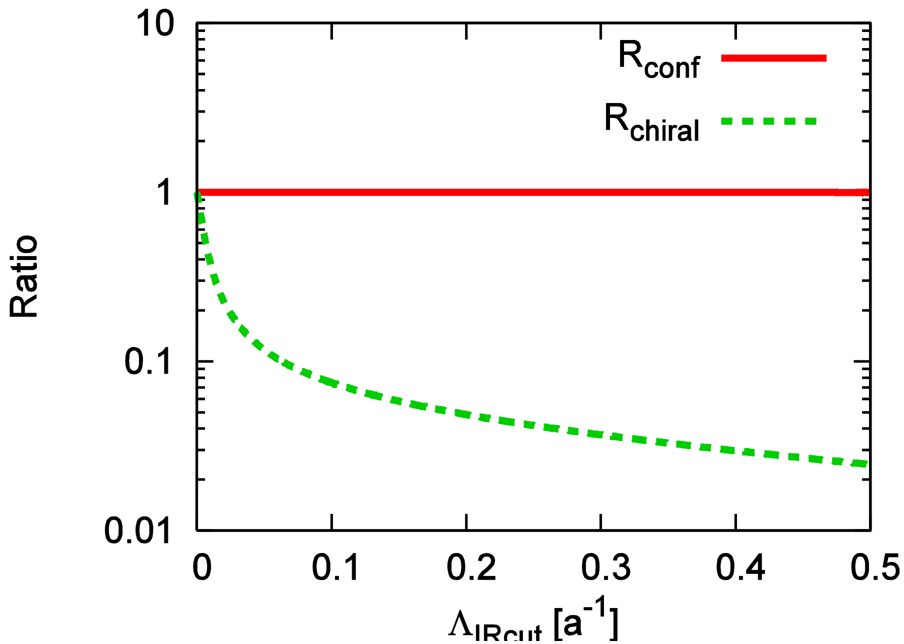
- The same result can be obtained within the overlap-fermion formalism, which solves the fermion-doubling problem with the exact chiral symmetry on the lattice.

- As a result, a new phase is expected to appear in QCD, for example, where quarks are confined and chiral symmetry is restored.

Appendix

Numerical analysis

$$R_{\rm conf} = \frac{(R_A)_\Lambda}{R_A}, \qquad R_{\rm chiral} = \frac{\langle\bar\psi\psi\rangle_\Lambda}{\langle\bar\psi\psi\rangle}$$



lattice setup:

- quenched SU(3) lattice QCD
- standard plaquette action
- gauge coupling: $\beta = \frac{2N_{\rm c}}{g^2} = 5.6$
- lattice size: $N_\sigma^3 \times N_\tau = 10^3 \times 5$

  ⇔ lattice spacing : $a \simeq 0.25$ fm
- periodic boundary condition
  for link-variables and Dirac operator

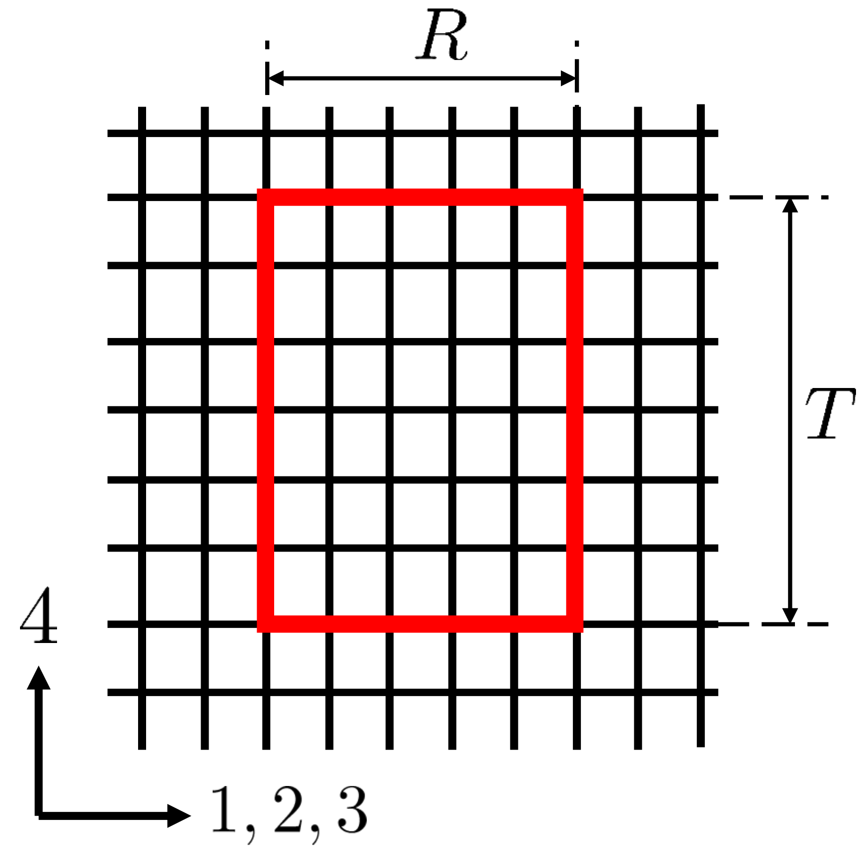
- $R_{\rm chiral}$ is strongly reduced by removing the low-lying Dirac modes.
- $R_{\rm conf}$ is almost unchanged.

⇒ It is also numerically confirmed that
low-lying Dirac modes are important for chiral symmetry breaking
and not important for quark confinement.

Dirac spectrum representation of the Wilson loop

H. Suganuma, TMD, T. Iritani, PTEP 2016, 013B06 (2016)

$$W(R,T) \equiv \mathrm{Tr}_c \left[\hat{U}_1^R \hat{U}_{-4}^T \hat{U}_{-1}^R \hat{U}_4^T \right]$$



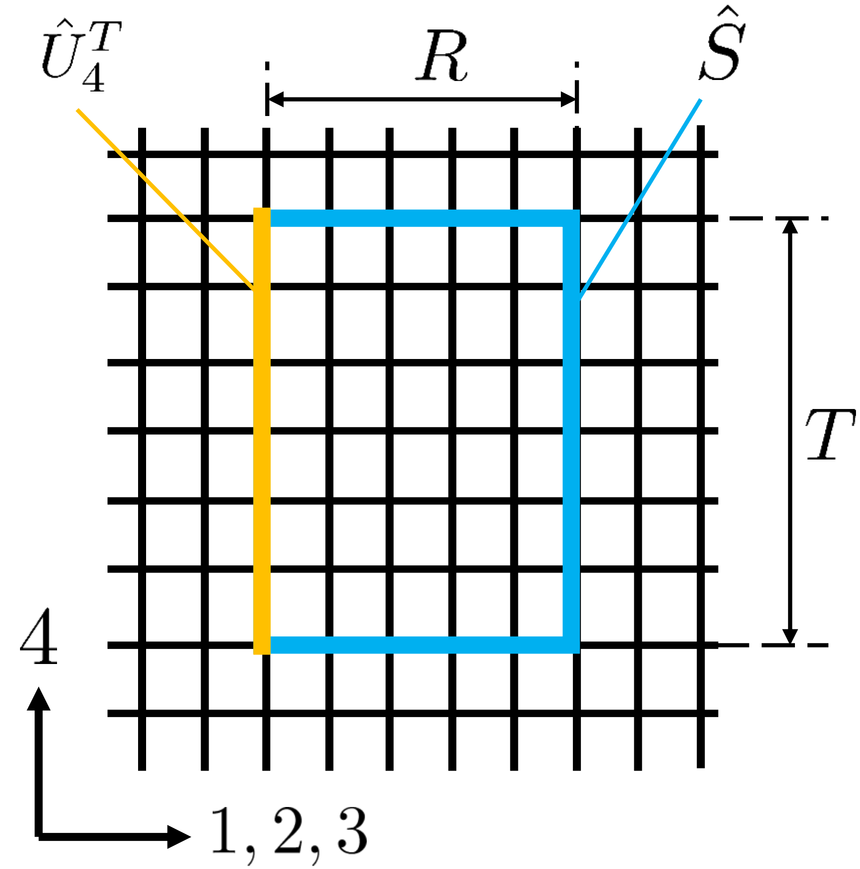
Dirac spectrum representation of the Wilson loop

H. Suganuma, TMD, T. Iritani, PTEP 2016, 013B06 (2016)

$$W(R,T) \equiv \mathrm{Tr}_c \left[\hat{U}_1^R \hat{U}_{-4}^T \hat{U}_{-1}^R \hat{U}_4^T \right]$$

$$= \mathrm{Tr}_c \left[\hat{S} \hat{U}_4^T \right]$$

(Staple operator: $\hat{S} \equiv \hat{U}_1^R \hat{U}_{-4}^T \hat{U}_{-1}^R$)



Dirac spectrum representation of the Wilson loop

H. Suganuma, TMD, T. Iritani, PTEP 2016, 013B06 (2016)

$$W(R,T) \equiv \mathrm{Tr}_c \left[\hat{U}_1^R \hat{U}_{-4}^T \hat{U}_{-1}^R \hat{U}_4^T \right]$$

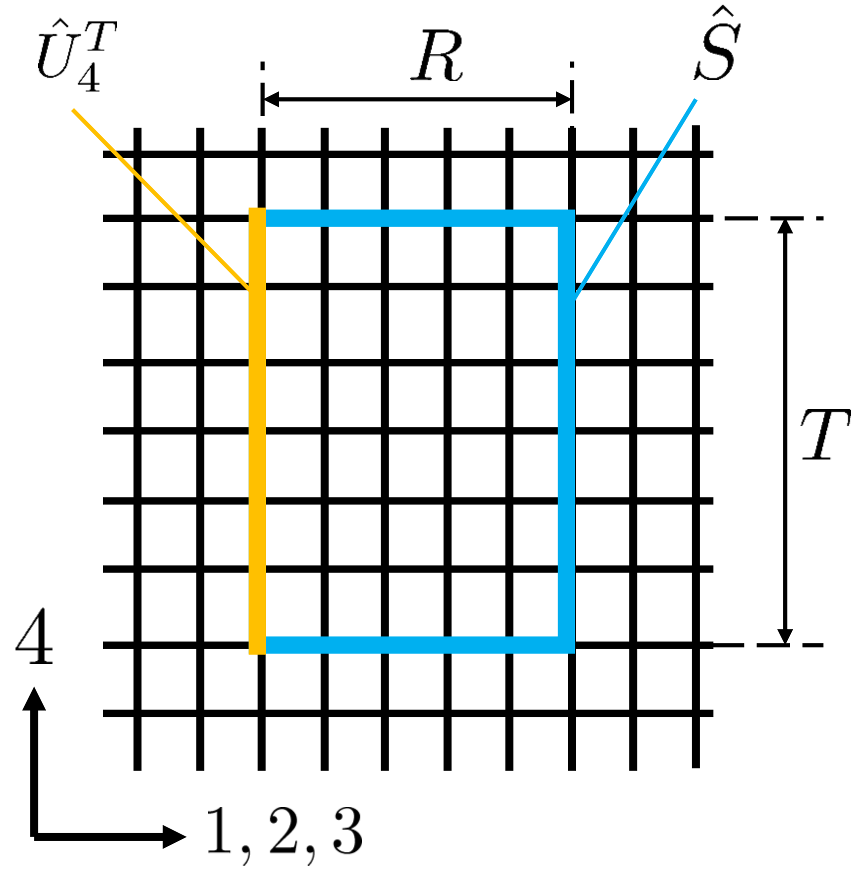
$$= \mathrm{Tr}_c \left[\hat{S} \hat{U}_4^T \right]$$

(Staple operator: $\hat{S} \equiv \hat{U}_1^R \hat{U}_{-4}^T \hat{U}_{-1}^R$)

$$J \equiv \mathrm{Tr}_{c,\gamma} \hat{S} \hat{\not{D}}^T \qquad \left(\hat{\not{D}} = \frac{1}{2} \sum_\mu \gamma_\mu (\hat{U}_\mu - \hat{U}_{-\mu}) \right)$$

$$= \frac{4}{(2a)^T} \mathrm{Tr}_c \hat{S} \hat{U}_4^T \qquad \text{(Consider even } T \text{)}$$

$$= \frac{4}{(2a)^T} W$$



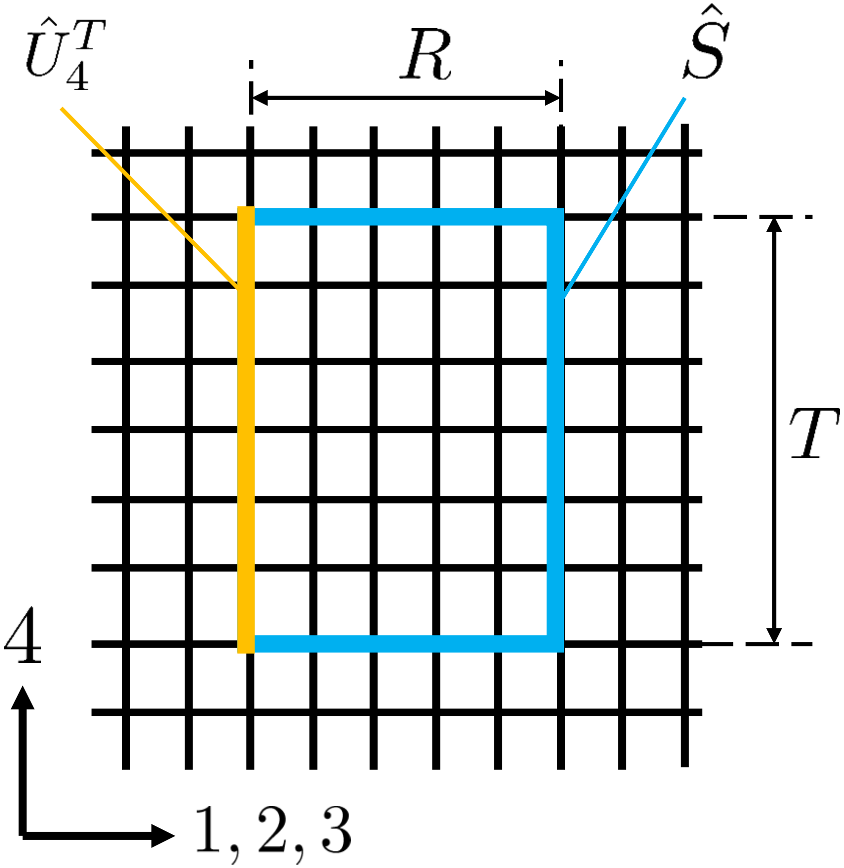
Dirac spectrum representation of the Wilson loop

H. Suganuma, TMD, T. Iritani, PTEP 2016, 013B06 (2016)

$$W(R,T) \equiv \mathrm{Tr}_c \left[\hat{U}_1^R \hat{U}_{-4}^T \hat{U}_{-1}^R \hat{U}_4^T \right]$$

$$= \mathrm{Tr}_c \left[\hat{S} \hat{U}_4^T \right]$$

(Staple operator: $\hat{S} \equiv \hat{U}_1^R \hat{U}_{-4}^T \hat{U}_{-1}^R$)



$$J \equiv \mathrm{Tr}_{c,\gamma} \hat{S} \hat{\not{D}}^T \qquad \left(\hat{\not{D}} = \frac{1}{2} \sum_\mu \gamma_\mu (\hat{U}_\mu - \hat{U}_{-\mu}) \right)$$

$$= \frac{4}{(2a)^T} \mathrm{Tr}_c \hat{S} \hat{U}_4^T \qquad \text{(Consider even } T \text{)}$$

$$= \frac{4}{(2a)^T} W$$

$$W(R,T) = \frac{(-)^{\frac{T}{2}} (2a)^T}{4} \sum_n \lambda_n^T \langle n | \hat{S} | n \rangle$$

Analytical relation btw
Wilson loop and Dirac modes

Outlook

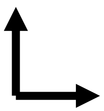
- Investigate the origin of the positive/negative symmetry.
  Is it related to the center symmetry?

- Study the other quantities which are important for confinement.
  In particular, the nucleon scalar density is interesting because it is quark bilinear.

- Find a new phase where quarks are confined but the chiral symmetry is restored.

Introduction – Lattice QCD

4 (imaginary time)

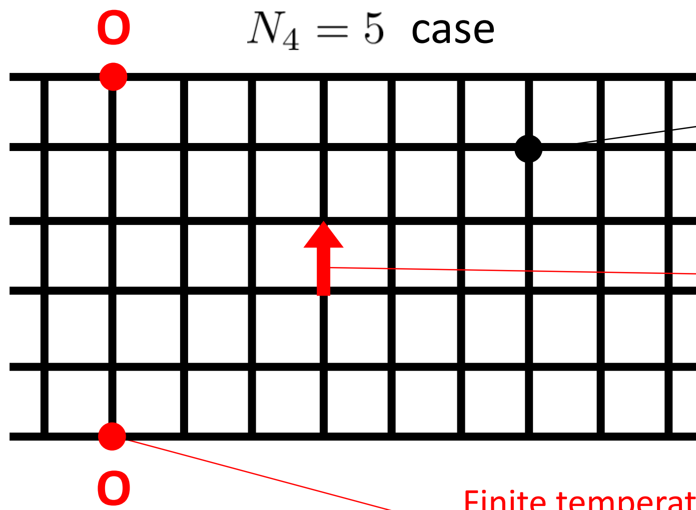


1,2,3
(space)

Site: $s = (s_1, s_2, s_3, s_4)$

$(s_\mu = 1, \cdots, N_\mu)$

$N_4 = 5$ case



quark field: $q(s)$

link variable:
$U_4(\mathbf{s}, s_4) = \mathrm{e}^{igA_4(\mathbf{s}, s_4)}$

Finite temperature :
(anti) periodic boundary condition for time direction

Partition function: $Z = \int \mathcal{D}\bar{q}\mathcal{D}q\mathcal{D}U \mathrm{e}^{-S_{\mathrm{G}}[U] + \bar{q}K[U]q}$

Observable:

$$\langle \mathcal{O}(U) \rangle = \frac{1}{Z} \int \mathcal{D}U \mathrm{e}^{-S_{\mathrm{G}}[U]} \mathrm{det} K[U] \mathcal{O}(U)$$

$$\simeq \frac{1}{N_{\mathrm{conf}}} \sum_{n=1}^{N_{\mathrm{conf}}} \mathcal{O}(\{U\}_n)$$

N_{conf} : Configuration number

$\{U\}_n$: Gauge configurations generated by Monte Carlo simulation

Representatives for the QCD vacuum

Why Polyakov loop fluctuations?

P.M. Lo, B. Friman, O. Kaczmarek, K. Redlich and C. Sasaki,
Phys. Rev. D88, 014506 (2013); Phys. Rev. D88, 074502 (2013)

Ans. 1: Avoiding ambiguities of the Polyakov loop renormalization

$$L^{\mathrm{ren}} = Z(g^2) L^{\mathrm{bare}}, \qquad L^{\mathrm{bare}} \equiv \frac{1}{N_c V} \sum_s \mathrm{tr}_c \{ \prod_{i=0}^{N_\tau - 1} U_4(s + i\hat{4}) \}$$

$Z(g^2)$: renormalization function for the Polyakov loop, which is still unknown

Avoid the ambiguity of renormalization function by considering the ratios of Polyakov loop susceptibilities:

$$R_A \equiv \frac{\chi_A}{\chi_L}, \qquad R_T \equiv \frac{\chi_T}{\chi_L}$$

Dirac spectrum representation of the Polyakov loop fluctuations

TMD, K. Redlich, C. Sasaki and H. Suganuma, Phys. Rev. D92, 094004 (2015).

$$L = -\frac{(2ai)^{N_\tau - 1}}{12V}\sum_n \lambda_n^{N_\tau - 1}\langle n|\hat{U}_4|n\rangle$$

Definition of the Polyakov loop fluctuations

- Polyakov loop: $L \equiv \frac{1}{N_c V}\sum_s \mathrm{tr}_c\{\prod_{i=0}^{N_\tau - 1} U_4(s+i\hat{4})\}$
- Z3 rotated Polyakov loop: $\tilde{L} = L\mathrm{e}^{2\pi k i/3}$
- longitudinal Polyakov loop: $L_L \equiv \mathrm{Re}(\tilde{L})$
- Transverse Polyakov loop: $L_T \equiv \mathrm{Im}(\tilde{L})$
- Polyakov loop susceptibilities:

$$T^3\chi_A = \frac{N_\sigma^3}{N_\tau^3}[\langle |L|^2\rangle - \langle |L|\rangle^2],$$

$$T^3\chi_L = \frac{N_\sigma^3}{N_\tau^3}[\langle (L_L)^2\rangle - \langle L_L\rangle^2],$$

$$T^3\chi_T = \frac{N_\sigma^3}{N_\tau^3}[\langle (L_T)^2\rangle - \langle L_T\rangle^2]$$

- Ratios of Polyakov loop susceptibilities:

$$R_A \equiv \frac{\chi_A}{\chi_L}, \qquad R_T \equiv \frac{\chi_T}{\chi_L}$$

Λ-dependent Polyakov loop fluctuations

Infrared cutoff of the Dirac eigenvalue: Λ

- Λ-dependent (IR-cut) Z3 rotated Polyakov loop:

$$\tilde{L}_\Lambda = -\frac{(2ai)^{N_\tau - 1}}{12V}\sum_{|\lambda_n|>\Lambda} \lambda_n^{N_\tau - 1}\mathrm{e}^{2\pi k i/3}\langle n|\hat{U}_4|n\rangle$$

- Λ-dependent Polyakov loop susceptibilities:

$$(\chi)_\Lambda = \frac{1}{T^3}\frac{N_\sigma^3}{N_\tau^3}[\langle Y_\Lambda^2\rangle - \langle Y_\Lambda\rangle^2],$$

$$Y \equiv |L|,\ L_L,\ L_T$$

- Λ-dependent ratios of susceptibilities:

$$(R_A)_\Lambda = \frac{(\chi_A)_\Lambda}{(\chi_L)_\Lambda}$$

Dirac spectrum representation of the Polyakov loop fluctuations

TMD, K. Redlich, C. Sasaki and H. Suganuma, Phys. Rev. D92, 094004 (2015).

Definition of the Polyakov loop fluctuations

- Polyakov loop: $L \equiv \frac{1}{N_c V} \sum_s \mathrm{tr}_c \{ \prod_{i=0}^{N_\tau - 1} U_4(s + i\hat{4}) \}$
- Z3 rotated Polyakov loop: $\tilde{L} = L e^{2\pi k i/3}$
- longitudinal Polyakov loop: $L_L \equiv \mathrm{Re}(\tilde{L})$
- Transverse Polyakov loop: $L_T \equiv \mathrm{Im}(\tilde{L})$
- Polyakov loop susceptibilities:

$$T^3 \chi_A = \frac{N_\sigma^3}{N_\tau^3} [\langle |L|^2 \rangle - \langle |L| \rangle^2],$$

$$T^3 \chi_L = \frac{N_\sigma^3}{N_\tau^3} [\langle (L_L)^2 \rangle - \langle L_L \rangle^2],$$

$$T^3 \chi_T = \frac{N_\sigma^3}{N_\tau^3} [\langle (L_T)^2 \rangle - \langle L_T \rangle^2]$$

- Ratios of Polyakov loop susceptibilities:

$$R_A \equiv \frac{\chi_A}{\chi_L}, \qquad R_T \equiv \frac{\chi_T}{\chi_L}$$

Dirac spectrum representation of the Polyakov loop

$$L = -\frac{(2ai)^{N_\tau - 1}}{12V} \sum_n \lambda_n^{N_\tau - 1} \langle n | \hat{U}_4 | n \rangle$$

Polyakov loop : L

Dirac eigenmode : $\hat{\not{D}} |n\rangle = i\lambda_n |n\rangle$

link variable operator : $\langle s | \hat{U}_\mu | s' \rangle = U_\mu(s) \delta_{s+\hat{\mu}, s'}$

$$\langle N_\tau, \mathbf{x} | \hat{U}_4 | 1, \mathbf{x} \rangle = -U_4(N_\tau, \mathbf{x})$$

Dirac spectrum representation of the Polyakov loop fluctuations

TMD, K. Redlich, C. Sasaki and H. Suganuma, Phys. Rev. D92, 094004 (2015).

Definition of the Polyakov loop fluctuations

- Polyakov loop: $L \equiv \frac{1}{N_c V} \sum_s \mathrm{tr}_c\{ \prod_{i=0}^{N_\tau - 1} U_4(s + i\hat{4})\}$
- Z3 rotated Polyakov loop: $\tilde{L} = L e^{2\pi k i/3}$
- longitudinal Polyakov loop: $L_L \equiv \mathrm{Re}(\tilde{L})$
- Transverse Polyakov loop: $L_T \equiv \mathrm{Im}(\tilde{L})$
- Polyakov loop susceptibilities:

$$T^3 \chi_A = \frac{N_\sigma^3}{N_\tau^3}[\langle |L|^2\rangle - \langle |L|\rangle^2],$$

$$T^3 \chi_L = \frac{N_\sigma^3}{N_\tau^3}[\langle (L_L)^2\rangle - \langle L_L\rangle^2],$$

$$T^3 \chi_T = \frac{N_\sigma^3}{N_\tau^3}[\langle (L_T)^2\rangle - \langle L_T\rangle^2]$$

- Ratios of Polyakov loop susceptibilities:

$$R_A \equiv \frac{\chi_A}{\chi_L}, \qquad R_T \equiv \frac{\chi_T}{\chi_L}$$

Dirac spectrum representation of the Polyakov loop

$$L = -\frac{(2ai)^{N_\tau - 1}}{12V} \sum_n \lambda_n^{N_\tau - 1} \langle n | \hat{U}_4 | n \rangle$$

Polyakov loop : L

Dirac eigenmode : $\hat{\not{D}} |n\rangle = i\lambda_n |n\rangle$

link variable operator : $\langle s | \hat{U}_\mu | s' \rangle = U_\mu(s) \delta_{s+\hat{\mu}, s'}$

$\langle N_\tau, \mathbf{x} | \hat{U}_4 | 1, \mathbf{x} \rangle = -U_4(N_\tau, \mathbf{x})$

combine

Dirac spectrum representation of the Polyakov loop fluctuations

For example,

$$L_L = -\frac{(2ai)^{N_\tau - 1}}{12V} \sum_n \lambda_n^{N_\tau - 1} \mathrm{Re}\left(e^{2\pi k i/3} \langle n | \hat{U}_4 | n \rangle \right)$$

and...

Polyakov loop fluctuations

P.M. Lo, B. Friman, O. Kaczmarek, K. Redlich and C. Sasaki,
Phys. Rev. D88, 014506 (2013); Phys. Rev. D88, 074502 (2013)

- Polyakov loop: $L \equiv \frac{1}{N_c V} \sum_s \mathrm{tr}_c \{ \prod_{i=0}^{N_\tau - 1} U_4(s + i\hat{4}) \}$

- Z3 rotated Polyakov loop: $\tilde{L} = L \mathrm{e}^{2\pi k i/3}$

- longitudinal Polyakov loop: $L_L \equiv \mathrm{Re}(\tilde{L})$
- Transverse Polyakov loop: $L_T \equiv \mathrm{Im}(\tilde{L})$

- Polyakov loop susceptibilities:

$$T^3 \chi_A = \frac{N_\sigma^3}{N_\tau^3} [\langle |L|^2 \rangle - \langle |L| \rangle^2],$$

$$T^3 \chi_L = \frac{N_\sigma^3}{N_\tau^3} [\langle (L_L)^2 \rangle - \langle L_L \rangle^2],$$

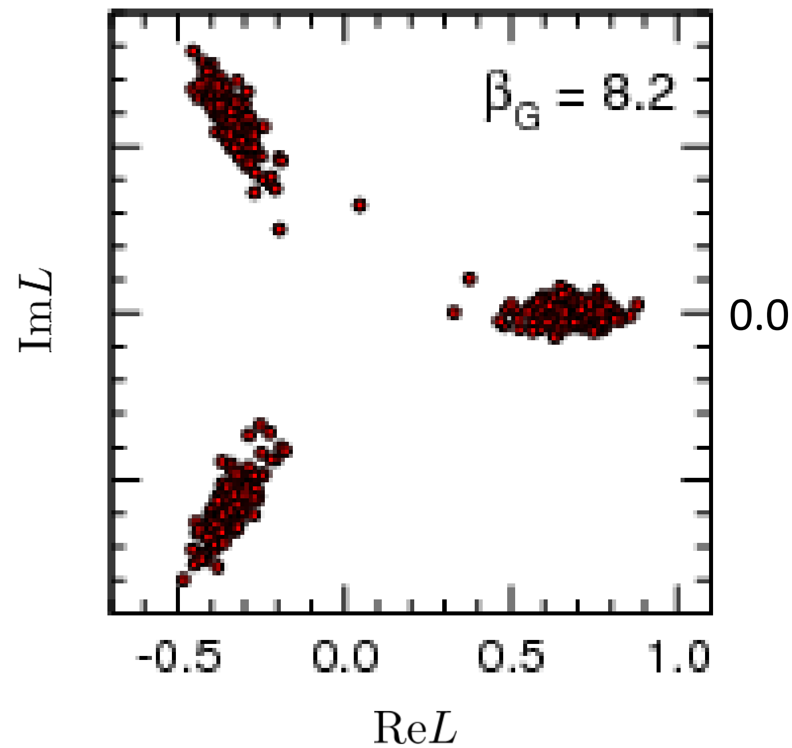
$$T^3 \chi_T = \frac{N_\sigma^3}{N_\tau^3} [\langle (L_T)^2 \rangle - \langle L_T \rangle^2]$$

- Ratios of Polyakov loop susceptibilities:

$$R_A \equiv \frac{\chi_A}{\chi_L}, \qquad R_T \equiv \frac{\chi_T}{\chi_L}$$

An example for the Polyakov loops

C. Gattringer et al., Phys.Lett. B697 (2011) 85



T : temperature

$N_\sigma, \ N_\tau$: spatial and temporal lattice size

Polyakov loop fluctuations

P.M. Lo, B. Friman, O. Kaczmarek, K. Redlich and C. Sasaki,
Phys. Rev. D88, 014506 (2013); Phys. Rev. D88, 074502 (2013)

- Polyakov loop: $L \equiv \frac{1}{N_c V} \sum_s \mathrm{tr}_c \{ \prod_{i=0}^{N_\tau - 1} U_4(s + i\hat{4}) \}$

- Z3 rotated Polyakov loop: $\tilde{L} = L e^{2\pi k i/3}$

- longitudinal Polyakov loop: $L_L \equiv \mathrm{Re}(\tilde{L})$
- Transverse Polyakov loop: $L_T \equiv \mathrm{Im}(\tilde{L})$

- Polyakov loop susceptibilities:

$$T^3 \chi_A = \frac{N_\sigma^3}{N_\tau^3} [\langle |L|^2 \rangle - \langle |L| \rangle^2],$$

$$T^3 \chi_L = \frac{N_\sigma^3}{N_\tau^3} [\langle (L_L)^2 \rangle - \langle L_L \rangle^2],$$

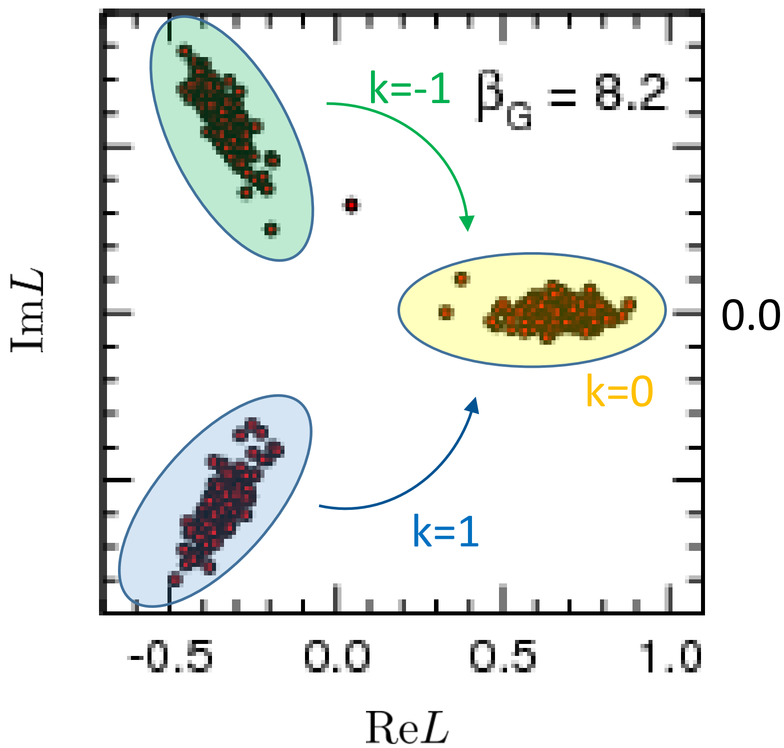
$$T^3 \chi_T = \frac{N_\sigma^3}{N_\tau^3} [\langle (L_T)^2 \rangle - \langle L_T \rangle^2]$$

- Ratios of Polyakov loop susceptibilities:

$$R_A \equiv \frac{\chi_A}{\chi_L}, \qquad R_T \equiv \frac{\chi_T}{\chi_L}$$

An example for the Polyakov loops

C. Gattringer et al., Phys.Lett. B697 (2011) 85



T : temperature

$N_\sigma, \ N_\tau$: spatial and temporal lattice size

Polyakov loop fluctuations

P.M. Lo, B. Friman, O. Kaczmarek, K. Redlich and C. Sasaki,
Phys. Rev. D88, 014506 (2013); Phys. Rev. D88, 074502 (2013)

- Polyakov loop: $L \equiv \frac{1}{N_c V} \sum_s \mathrm{tr}_c \{ \prod_{i=0}^{N_\tau - 1} U_4(s + i\hat{4}) \}$

- Z3 rotated Polyakov loop: $\tilde{L} = L e^{2\pi k i/3}$

- longitudinal Polyakov loop: $L_L \equiv \mathrm{Re}(\tilde{L})$
- Transverse Polyakov loop: $L_T \equiv \mathrm{Im}(\tilde{L})$

- Polyakov loop susceptibilities:

$$T^3 \chi_A = \frac{N_\sigma^3}{N_\tau^3} [\langle |L|^2 \rangle - \langle |L| \rangle^2],$$

$$T^3 \chi_L = \frac{N_\sigma^3}{N_\tau^3} [\langle (L_L)^2 \rangle - \langle L_L \rangle^2],$$

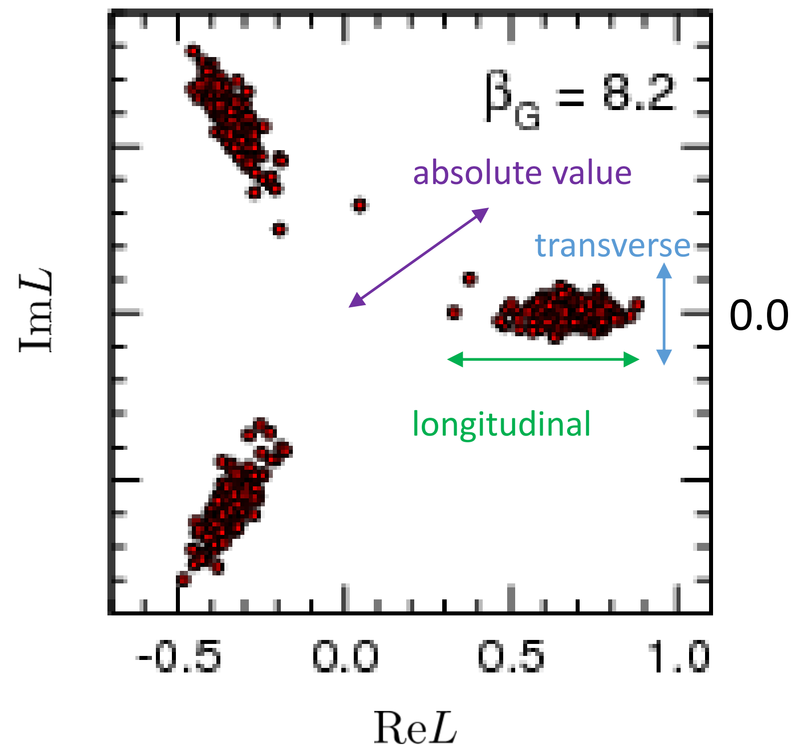
$$T^3 \chi_T = \frac{N_\sigma^3}{N_\tau^3} [\langle (L_T)^2 \rangle - \langle L_T \rangle^2]$$

- Ratios of Polyakov loop susceptibilities:

$$R_A \equiv \frac{\chi_A}{\chi_L}, \qquad R_T \equiv \frac{\chi_T}{\chi_L}$$

An example for the Polyakov loops

C. Gattringer et al., Phys.Lett. B697 (2011) 85



T : temperature

$N_\sigma,\ N_\tau$: spatial and temporal lattice size

Introduction of the Infrared cutoff for Dirac modes

TMD, K. Redlich, C. Sasaki and H. Suganuma, Phys. Rev. D92, 094004 (2015).

Define Λ-dependent (IR-cut) susceptibilities:

$$(\chi)_\Lambda = \frac{1}{T^3}\frac{N_\sigma^3}{N_\tau^3}[\langle Y_\Lambda^2\rangle - \langle Y_\Lambda\rangle^2], \quad Y \equiv |L|,\ L_L,\ L_T$$

where, for example, $(L_L)_\Lambda = C_\tau \sum_{|\lambda_n|>\Lambda} \lambda_n^{N_\tau-1} \mathrm{Re}\left(\mathrm{e}^{2\pi k i/3}(n|\hat{U}_4|n)\right)$

Define Λ-dependent (IR-cut) ratio of susceptibilities:

$$(R_A)_\Lambda = \frac{(\chi_A)_\Lambda}{(\chi_L)_\Lambda}$$

Define Λ-dependent (IR-cut) chiral condensate:

$$\langle\bar{\psi}\psi\rangle_\Lambda = -\frac{1}{V}\sum_{|\lambda_n|\geq\Lambda}\frac{2m}{\lambda_n^2+m^2}$$

Define the ratios, which indicate the influence of removing the low-lying Dirac modes:

$$R_{\mathrm{conf}} = \frac{(R_A)_\Lambda}{R_A}, \qquad R_{\mathrm{chiral}} = \frac{\langle\bar{\psi}\psi\rangle_\Lambda}{\langle\bar{\psi}\psi\rangle}$$

Numerical analysis for each Dirac-mode contribution to the Polyakov loop

Numerical analysis of this relation is important.

$$L_P = \frac{(2i)^{N_4-1}}{12V} \sum_n \lambda_n^{N_4-1} \langle n|\hat{U}_4|n\rangle \quad \cdots \text{(A)}$$

$$L_P = \frac{(2i)^{N_4-1}}{12V} \sum_n \lambda_n^{N_4-1} \sum_s \psi_n^\dagger(s) U_4(s) \psi_n(s+\hat{4})$$

$L_P,\ U_4(s)$: easily obtained

*This formalism is gauge invariant.

$\lambda_n,\ \psi_n^\dagger(s),\ \psi_n(s+\hat{4})$:are determined from $\hat{\not{D}}|n\rangle = i\lambda_n|n\rangle$

notation and coordinate representation

$$\langle s|\hat{U}_4|s'\rangle = U_4(s)\delta_{s+\hat{4},s'}$$

$$\langle s'|n\rangle = \psi_n(s')$$

$$\langle n|s\rangle = \psi_n^\dagger(s)$$

$$\hat{D}_\mu = \frac{1}{2}(\hat{U}_\mu - \hat{U}_{-\mu})$$

$$1 = \sum_s |s\rangle\langle s| \qquad |s\rangle \text{ : site}$$

explicit form of the Dirac eigenvalue equation

$$\sum_{s',j,\beta} \not{D}_{ss'}^{ij,\alpha\beta} \psi_n(s')^{j,\beta} = i\lambda_n \psi_n(s)^{i,\alpha}$$

where $\not{D}_{ss'}^{ij,\alpha\beta} = \frac{1}{2}\sum_{\mu=1}^{4} \gamma_\mu^{\alpha\beta}\left[U_\mu(s)^{ij}\delta_{s+\hat{\mu},s'} - U_{-\mu}(s)^{ij}\delta_{s-\hat{\mu},s'}\right]$

s, s' : site
i, j : color
α, β : spinor

$$U_{-\mu}(s) \equiv U_\mu(s-\hat{\mu})^\dagger$$

New Modified Kogut-Susskind Formalism on Temporally Odd Number Lattice

TMD, H. Suganuma, T. Iritani, Phys. Rev. D 90, 094505 (2014).

N_1, N_2, N_3 : even

N_4 : odd ← "temporally odd-number lattice"

$$M(s) \equiv \gamma_1^{s_1}\gamma_2^{s_2}\gamma_3^{s_3}\gamma_4^{s_1+s_2+s_3}$$

$$\Rightarrow M^\dagger(s)\gamma_\mu M(s\pm\hat{\mu}) = \eta_\mu(s)\gamma_4$$

We use Dirac representation(γ_4 is diagonalized)

$\not{D}$ is spin diagonalized

$$\Rightarrow M^\dagger \not{D} M \equiv \sum_\mu M^\dagger(s)\gamma_\mu D_\mu M(s+\hat{\mu}) = \begin{pmatrix} \eta\cdot D & 0 & 0 & 0 \\ 0 & \eta\cdot D & 0 & 0 \\ 0 & 0 & -\eta\cdot D & 0 \\ 0 & 0 & 0 & -\eta\cdot D \end{pmatrix}$$

where $(\eta\cdot D)_{ss'}^{ij} = (\eta_\mu D_\mu)_{ss'}^{ij} = \frac{1}{2a}\sum_{\mu=1}^{4}\eta_\mu(s)\left[U_\mu(s)^{ij}\delta_{s+\hat{\mu},s'} - U_{-\mu}(s)^{ij}\delta_{s-\hat{\mu},s'}\right]$

case of even lattice

N_1, N_2, N_3, N_4 : even

$$T(s) \equiv \gamma_1^{s_1}\gamma_2^{s_2}\gamma_3^{s_3}\gamma_4^{s_4}$$

$$\Rightarrow T^\dagger(s)\gamma_\mu T(s\pm\hat{\mu}) = \eta_\mu(s)\mathbf{1}_{\rm spinor}$$

$$\Rightarrow T^\dagger \not{D} T = \begin{pmatrix} \eta\cdot D & 0 & 0 & 0 \\ 0 & \eta\cdot D & 0 & 0 \\ 0 & 0 & \eta\cdot D & 0 \\ 0 & 0 & 0 & \eta\cdot D \end{pmatrix}$$

staggered phase: $\eta_\mu(s)$

$$\eta_\mu(s) = \begin{cases} 1 & (\mu=1) \\ (-1)^{s_1} & (\mu=2) \\ (-1)^{s_1+s_2} & (\mu=3) \\ (-1)^{s_1+s_2+s_3} & (\mu=4) \end{cases}$$

Numerical analysis for each Dirac-mode contribution to the Polyakov loop

$$L = \frac{(2i)^{N_4-1}}{12V} \sum_n \lambda_n^{N_4-1} \langle n|\hat{U}_4|n\rangle \quad \cdots (A)$$

Dirac eigenmode $|n\rangle$

$$\not{D}|n\rangle = i\lambda_n|n\rangle$$

$$L = \frac{(2i)^{N_4-1}}{3V} \sum_n \lambda_n^{N_4-1} (n|\hat{U}_4|n) \quad \cdots (A)'$$

KS Dirac eigenmode $|n)$

$$\eta \cdot D|n) = i\lambda_n|n)$$

(A)⇔(A)' relation (A)' is equivalent to (A)

lattice setup

- quenched SU(3) lattice QCD
- standard plaquette action
- gauge coupling: $\beta = \frac{2N_c}{g^2} = 5.6 \ , \ 6.0$ ⇔ lattice spacing : $a \simeq 0.25 \ , \ 0.10 \text{ fm}$
- lattice size: $N_{\rm space}^3 \times N_4 = 10^3 \times \underline{5}$ (odd)
- periodic boundary condition for link-variables